A method for characterizing graphs with specified throttling numbers

by

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DEDICATION

I dedicate this work to my amazing wife Janelle. Since I decided to start this journey, she has always been by my side helping me carry the weight. I could not have done this without her and I am looking forward to our next adventure.

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ABSTRACT

Zero forcing is a combinatorial game played on graphs in which a color change rule is used to progressively change the color of vertices from white to blue. The (standard) color change rule is that a blue vertex u can force a white vertex w to become blue if w is the only white vertex adjacent to u. When using the color change rule, the goal is to eventually change the color of every vertex in the graph to blue. Some interesting questions arise from this process that are heavily studied. What is the smallest possible size of an initial set of blue vertices that can eventually color the entire graph blue? How much time is required to complete this process? The answer to the first question is called the zero forcing number of a graph and the answer to the second question is called the propagation time of the initial set. A more recent area of study is throttling which balances the cost of the initial set with the cost of its propagation time in order to make the process as efficient as possible. Specifically, the (standard) throttling number of a graph is the minimum value of the sum of the size of an initial set and its propagation time taken over all possible initial sets. Many variations of the color change rule also lead to variations of propagation time and throttling. These include positive semidefinite (PSD) zero forcing, the minor monotone floor of zero forcing, and the minor monotone floor of PSD zero forcing. In this dissertation, general definitions are given that allow for the study of propagation and throttling for many variants of zero forcing. In addition, a technique is introduced that is used to characterize all graphs with specified throttling numbers. This technique is then generalized and applied to obtain similar characterizations for the variants of zero forcing mentioned above.

Keywords: Zero forcing, propagation time, throttling, positive semidefinite, minor monotone floor

CHAPTER 1. INTRODUCTION

Graphs are used to model various objects and the relationships between them (see Section 1.2 for precise definitions). An acquaintanceship graph can model a group of people and how they know each other. The internet graph can model websites and the links from one site to another. Graph theory provides tools that can be used to study a wide range of topics such as optimal travel in a transportation network and influence in a social network.

The graph parameters studied in this dissertation can be used to model concepts like spread or infection. For simple examples, consider a rumor spreading throughout a crowd or a computer virus infecting software. Zero forcing is an infection process that uses a color change rule to increase the number of blue vertices in a graph. The *(standard) color change rule* is that a blue vertex with a unique white neighbor can force that neighbor to become blue. If u is the blue vertex and w is the white vertex, the notation $u \to w$ is used to denote that u forces w to become blue. Suppose G is a graph with $B \subseteq V(G)$ colored blue and $V(G) \setminus B$ colored white. A set of blue vertices in G obtained by iteratively applying the color change rule until no more forces are possible is called a *final coloring* of B. If $V(G)$ is a final coloring of B, then B is called a *zero forcing set* of G. The size of the smallest possible zero forcing set in a graph G is the *zero forcing number* of G and is denoted by $\operatorname{Z}(G)$. The zero forcing number was introduced in [1] as a tool for bounding the minimum rank of a graph (see [9]). Zero forcing also has connections in physics and is used in [5] to model the control of a quantum system.

Let G be a graph and $B \subseteq V(G)$ be a zero forcing set of G. Start with B colored blue and count one time step by performing as many independent forces simultaneously as possible. Starting with the new set of blue vertices, repeat this process to count another time step. Continue in this fashion until all vertices in $V(G)$ are blue. The *(standard) propagation time of B in G*, denoted $\operatorname{pt}(G; B)$, is the number of time steps required to color $V(G)$ blue starting with B as the initial set of blue vertices. Propagation time is particularly important in the context of quantum control and is studied in [11]. The propagation time of minimum zero forcing sets is studied in [10].

If the size of the initial set of blue vertices is larger than necessary, it is possible to dramatically speed up (or throttle) the propagation process. However, it is still undesirable for the initial set of blue vertices to be too large. The goal is to find the ideal zero forcing set in a given graph that makes the zero forcing process as effective as possible. The *(standard) throttling number* of a graph G is the minimum value of $|B| + \mathrm{pt}(G; B)$ over all zero forcing sets B of G. In [6], upper bounds for the throttling number of a graph are given in terms of the order of the graph and its zero forcing number.

There are many forms of zero forcing that use modified color change rules. The *Positive semidefinite (PSD) color change rule* is a variant introduced in [2] that breaks a graph into components and applies standard zero forcing in each component. *Hopping* is a color change rule that allows a blue vertex u with no white neighbors to force any white vertex to become blue if u has not yet performed a force. The $\lfloor \mathrm{Z} \rfloor$ *color change rule* is to either apply the standard color change rule or apply the hopping rule. Analogous to PSD zero forcing, the $\lfloor \mathrm{Z}_+ \rfloor$ *color change rule* breaks a graph into components and applies $\lfloor \mathrm{Z} \rfloor$ forcing in each component. Both the $\lfloor \mathrm{Z} \rfloor$ and $\lfloor \mathrm{Z}_+ \rfloor$ forcing rules were introduced in [3] and they have connections to graph parameters that measure the width of trees and paths.

Every variation of zero forcing leads to an associated propagation time and throttling number. One contribution of this dissertation is to introduce a general theory of zero forcing, propagation time, and throttling. It is shown in [4] that for certain variants of zero forcing, determining whether a given graph has throttling number less than a given integer is NP-complete. Another main contribution of this dissertation is a technique for characterizing graphs with specified throttling numbers as certain minors of the Cartesian product of a complete graph and a tree.

1.1 Organization

This dissertation is organized in journal paper format. Some basic graph theory notation and definitions are outlined in Section 1.2. Section 1.3 familiarizes the reader with the research topic by reviewing previous literature on zero forcing, propagation time, and throttling.

Chapter 2 contains the paper "Throttling for Zero Forcing and Variants" which is currently under review for publication in *The Australasian Journal of Combinatorics*. Section 2.2 introduces general definitions and notation for studying multiple variations of propagation and throttling. In Section 2.3, throttling for the minor monotone floor of zero forcing (denoted $\lfloor Z \rfloor$) is studied and a tool is introduced that characterizes graphs with specified $\lfloor Z \rfloor$ throttling numbers. The same tool is used in Section 2.4 to obtain a similar characterization for standard throttling. Finally, these characterizations are used in Section 2.5 to classify graphs with extreme standard and $\lfloor Z \rfloor$ throttling numbers.

Chapter 3 contains the paper "Characterizations of Throttling for Positive Semidefinite Zero Forcing and its Minor Monotone Floor". This paper will be submitted for publication after further editing. In Section 3.2, the techniques in Chapter 2 are generalized to obtain a characterization of positive semidefinite (PSD) throttling numbers. Section 3.3 uses this characterization to prove a similar result for a variant of throttling called the minor monotone floor of PSD zero forcing. In Section 3.4, a different perspective on PSD zero forcing is described that is useful for further study of PSD throttling.

General conclusions are discussed in Chapter 4. This includes related research topics and directions for future work.

1.2 Basic graph theory

This section outlines some of the basic tools and definitions in graph theory. A *graph* is a pair $G = (V(G), E(G))$ where $V(G)$ is a set of vertices and $E(G)$ is a set of edges. The *order* of a graph G is the number of vertices in $V(G)$ and is denoted as $|G|$. In a *simple graph* G, the edge set $E(G)$

consists of two-element subsets of the vertex set. An edge $\{u,v\} \in E(G)$ is often written as uv and the vertices in $\{u,v\}$ are said to be *adjacent* (or *neighbors*). The *neighborhood* of a vertex $v \in V(G)$, denoted $N_G(v)$, is the set of vertices in G that are adjacent to v. The *closed neighborhood* of $v \in V(G)$ is $N_G[v] = \{v\} \cup N_G(v)$. The G is dropped from the neighborhood notation (e.g., $N(v)$ and $N[v]$) if the graph G is clear from context. A vertex $v \in V(G)$ is *incident to* (alternatively, an *endpoint* of) an edge $e \in E(G)$ if e contains v as an element. For each $v \in V(G)$, the *degree* of v is defined as $d(v) = |N(G)|$. The *maximum and minimum degree* over all vertices in $V(G)$ is denoted as $\Delta(G)$ and $\delta(G)$ respectively. The *degree sequence* of a graph G is the non-increasing sequence of degrees of the vertices in G. Graphs are often illustrated with vertices drawn as nodes and edges drawn as line segments that connect one node to another. There are many common families of graphs that are indexed by the order of the graph.

Example 1.2.1. The path P_n has vertex set $\{v_1, v_2, \dots v_n\}$ and edge set $\{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. The cycle C_n is a path on n vertices with the additional edge $v_1 v_n$. The complete graph K_n is the graph on n vertices with every possible edge. The star on n vertices is denoted $K_{1,n}$ and has one vertex of degree $n-1$ and every other vertex has degree 1. These graphs are illustrated left to right in Figure 1.1.

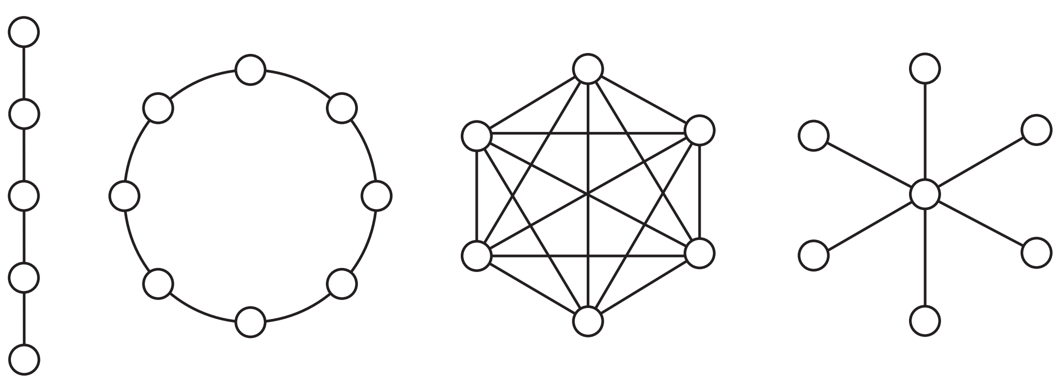

Figure 1.1 The path P_5, cycle C_8, complege graph K_6, and star $K_{1,6}$ shown left to right.

There are many important operations on graphs that change the structure of the vertices or the edges. Suppose G is a graph and S is a set of edges. Then $G - S$ is the graph with $V(G-S) = V(G)$

and $E(G - S) = E(G) \setminus S$. Whenever a vertex is deleted from a graph, all edges incident to that vertex are also deleted. If S is a set of vertices, $G - S$ is the graph obtained from G by deleting the vertices in S. A graph H is a *spanning subgraph* of a graph G if H can be obtained from G by deleting edges. In this case, G is a *spanning supergraph* of H. If H can be obtained from G by deleting edges and/or vertices from G, then H is a *subgraph* of G which is written as $H \leq G$. In the case that $H \leq G$, it is said that G is a *supergraph* of H. If $S \subseteq V(G)$, the subgraph *induced* by S is defined as $G[S] = G - (V(G) \setminus S)$. Let $e = uv$ be an edge in a graph G. The graph G/e is obtained from G by deleting the vertices $\{u, v\}$ and adding a vertex v_e such that $N_{G/e}(v_e) = N_G(u) \cup N_G(v)$. This operation is called *edge contraction* and G/e is read as "G contract e". If H can be obtained from G by deleting edges, deleting vertices, and/or contracting edges, then H is a *minor* of G which is denoted as $H \preceq G$. In this case, it is said that G is a *major* of H. Note that spanning subgraphs are subgraphs, induced subgraphs are subgraphs, and subgraphs are minors. However, there are minors that are not subgraphs, subgraphs that are not induced, and subgraphs that are not spanning.

Example 1.2.2. Let $G = C_8$ be the cycle on 8 vertices as depicted in Figure 1.1. Note that C_6 is a minor of G obtained by contracting any two edges. Since C_6 cannot be obtained from G by deleting edges and/or removing vertices, C_6 is not a subgraph of G. The path P_8 is a subgraph of G obtained by deleting an edge, but it is not induced. Finally, the graph P_5 is a subgraph of G that is not spanning because it is obtained from G by deleting vertices. All three of these graphs are minors of G and are illustrated in Figure 1.2.

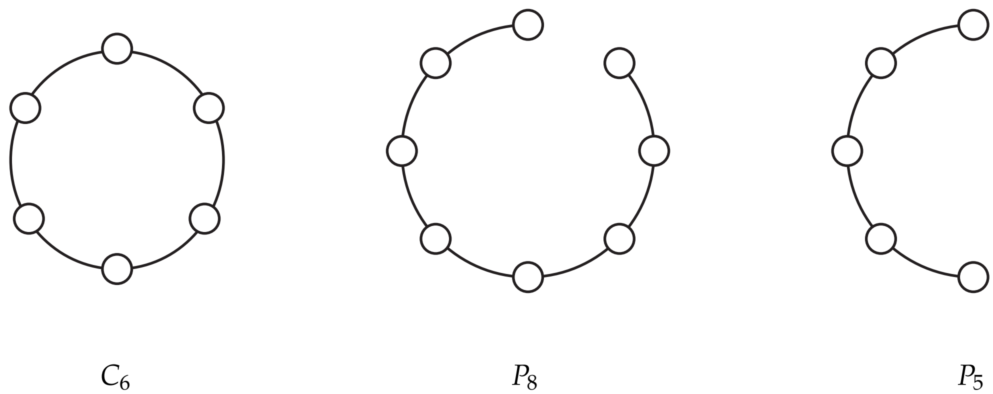

Figure 1.2 If $G = C_8$, $C_6 \preceq G$ is not a subgraph, $P_8 \leq G$ is not induced, and $P_5 \leq G$ is not spanning.

Some graph operations combine two different graphs to build a larger graph. If G and H are graphs, the *Cartesian product* of G and H is the graph $G \square H$ with vertex set $V(G \square H) = \{(u, v) \mid u \in V(G) \text{ and } v \in V(H)\}$ and edge set as follows. Vertex (a_1, a_2) is adjacent to vertex (b_1, b_2) in $G \square H$ if and only if either $a_1 = b_1$ and $a_2 \in N_H(b_2)$ or $a_2 = b_2$ and $a_1 \in N_G(b_1)$. For example, $P_4 \square P_6$ is a 4×6 rectangular grid. The *disjoint union* of G and H is denoted as $G \dot{\cup} H$ with $V(G \dot{\cup} H) = V(G) \dot{\cup} V(H)$ and $E(G \dot{\cup} H) = E(G) \dot{\cup} E(H)$. Note that if $u \in V(G)$ and $v \in V(H)$, then there is no path from u to v in $G \dot{\cup} H$.

A graph G is *connected* if for every $u, v \in V(G)$, there is a path from u to v in G. The *distance* betweeen two vertices $u, v \in V(G)$, denoted $d(u, v)$, is the number of edges in the shortest path in G from u to v. The *components* of G are the maximally connected subgraphs of G. In other words, if C is a component of G, then there is no connected subgraph of G other than C that is a supergraph of C. A graph T is a *tree* if T is connected and has no subgraphs that are cycles. If T is a tree and $v \in V(T)$ with $d(v) = 1$, then v is called a *leaf* of T. A *rooted tree* is a tree that is built from a single vertex v (the root) by iteratively adding leaves to the existing graph. In a rooted tree T, vertex u is a *parent* of vertex v if v was added as a leaf to u at some point in the construction of T. In this case, v is a *child* of u.

A common question in graph theory is to determine the maximum or minimum size of a subset of vertices with a given property. It is often the case that one of these extremes is easier to find than the other. Suppose G is a graph and $S \subseteq V(G)$. The set S is an *independent set* of G if the graph $G[S]$ has no edges. Of course, the smallest independent set in any graph is a single vertex. The maximum size of an independent set in G is the *independence number* of G and is denoted as $\alpha(G)$. The subset $S \subseteq V(G)$ is a *dominating set* of G if for every $u \in V(G)$, $u \in N[s]$ for some $s \in S$. Note that the largest dominating set in any graph G is the entire vertex set $V(G)$. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum size of a dominating set in G.

1.3 Literature review

This section reviews the literature on zero forcing, propagation time, and throttling.

1.3.1 Zero forcing parameters

Zero forcing is a process on graphs that was introduced in [1] using a color change rule. The colors used in this process were originally black and white, but the new convention in the literature is to use blue and white. In a graph whose vertices are colored blue or white, the *(standard) color change rule* is that a blue vertex u can force a white vertex w to become blue if w is the only white neighbor of u. Suppose G is graph with $B \subseteq V(G)$ colored blue and $V(G) \setminus B$ colored white. If it is possible to start with this coloring and repeatedly apply the color change rule in order to color the entire vertex set blue, then B is a *zero forcing set* of G. It is mentioned in Section 1.2 that finding the extreme sizes of particular subsets of vertices is usually easier for one extreme than the other. This is also true for zero forcing since the maximum zero forcing set in a graph G is always $V(G)$. It is much more interesting to try to find small zero forcing sets. The *zero forcing number* of a graph G, denoted $\operatorname{Z}(G)$, is the minimum size of a zero forcing set in G.

It is shown in [5] that zero forcing can be used to model control of a quantum system. Originally, zero forcing was introduced in [1] as a tool in combinatorial matrix theory. For an $n \times n$ real symmetric matrix $A = [a_{i,j}]$, $\mathcal{G}(A)$ is the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set

$\{ij \mid i \neq j \text{ and } a_{i,j} \neq 0\}$. If G is a graph of order n, $\mathcal{S}(G)$ is the family of $n \times n$ real symmetric matrices A such that $\mathcal{G}(A) = G$. The *maximum nullity* of a graph G, denoted $\mathrm{M}(G)$, is the maximum nullity of the matrices in $\mathcal{S}(G)$. The next proposition serves as one of the main motivations for studying zero forcing.

Proposition 1.3.1. [1] *If G is a graph, then* $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

The *minimum rank* of a graph G, denoted $\mathrm{mr}(G)$, is the minimum rank of the matrices in $\mathcal{S}(G)$. The problem of determining the minimum rank of a graph is a rich area of study (see [9]). As a consequence of the rank/nullity theorem, $\mathrm{mr}(G) = |G| - \mathrm{M}(G)$ for any graph G. The next result concerns the zero forcing number of Cartesian products.

Proposition 1.3.2. [1] *If G and H are graphs, then* $\mathrm{Z}(G \square H) \leq \min\{\mathrm{Z}(G)|H|,\ \mathrm{Z}(H)|G|\}$.

In a path, a single leaf is a zero forcing set which means that $\mathrm{Z}(P_t) = 1$ for all $t \geq 1$. Thus, Proposition 1.3.2 leads to an immediate corollary.

Corollary 1.3.3. [1] *For any graph G and integer $t > 0$,* $\mathrm{Z}(G \square P_t) \leq \min\{\mathrm{Z}(G)t,\ |G|\}$.

The *hypercube* of dimension n can be defined recursively as $Q_n = Q_{n-1} \square P_2$. Therefore, Corollary 1.3.3 can be used to give an upper bound for the zero forcing number of hypercubes.

Corollary 1.3.4. [1] *If Q_n is the n-dimensional hypercube, then* $\mathrm{Z}(Q_n) \leq 2^{n-1}$.

Sometimes for a graph G, the minimum rank is considered over certain subsets of the matrices in $\mathcal{S}(G)$. The *positive semidefinite minimum rank* of a graph G, denoted $\mathrm{mr}_+(G)$, is the minimum rank of the matrices in $\mathcal{S}(G)$ that are positive semidefinite. Likewise, the maximum nullity of the positive semidefinite matrices in $\mathcal{S}(G)$ is the *positive semidefinite maximum nullity* of G and is denoted $\mathrm{M}_+(G)$. The phrase "positive semidefinite" is often abbreviated to PSD for convenience.

In [2], it is shown that the PSD maximum nullity of a graph is also bounded above by a zero forcing parameter. Suppose G is a graph, $B \subseteq V(G)$ is the set of blue vertices in G, and there are $k \geq 1$ components of $G - B$. Let $W_1, W_2, \ldots W_k$ be the sets of white vertices in the components of $G - B$. The *PSD color change rule* is that if $u \in B$, $w \in W_i$ for some $1 \leq i \leq k$, and w is the only white neighbor of u in $G[B \cup W_i]$, then u can force w to become blue. Starting with $B \subseteq V(G)$ colored blue and $V(G) \setminus B$ colored white, if it is possible to repeatedly apply the PSD color change rule to color $V(G)$ blue, then B is a *PSD zero forcing set* of G. The minimum size of a PSD zero forcing set in a graph G is the *PSD zero forcing number* of G and is denoted as $\mathrm{Z}_+(G)$.

Theorem 1.3.5. [2] *For any graph G, $\mathrm{M}_+(G) \leq \mathrm{Z}_+(G)$.*

It is shown in [12] that $\mathrm{M}_+(G) = 1$ if and only if G is a tree. In [2], it is observed that if T is a tree, any single vertex of T is a PSD zero forcing set of T. This leads to the following characterization.

Corollary 1.3.6. [2, 12] *For a graph G, $\mathrm{Z}_+(G) = 1$ if and only if G is a tree.*

More extreme PSD zero forcing numbers are characterized in [8]. Also, a connection is made in [8] between PSD zero forcing and tree covers. The *tree cover number* of a graph G, denoted $\mathrm{T}(G)$, is the minimum number of vertex disjoint trees (as induced subgraphs of G) such that every vertex in $V(G)$ is in exactly one of the trees. Suppose G is a graph and $B \subseteq V(G)$ is a PSD zero forcing set of G. By keeping track of the pattern of forces that are performed in G as $V(G)$ is colored blue, G can be decomposed into *forcing trees*. (For a precise definition, see Chapter 3.) These forcing trees are used to prove the following theorem.

Theorem 1.3.7. [8] *For any graph G, $\mathrm{T}(G) \leq \mathrm{Z}_+(G)$.*

The zero forcing color change rule can be altered in many different ways. Certain variants of zero forcing are shown in [3] to have connections to parameters that measure the width a graph (i.e., tree-width and path-width). Suppose G is a graph and let $B \subseteq V(G)$ be an initial set of blue vertices in G. An *active* vertex is a vertex that is blue and has not yet performed a force. If u is active, the *CCR-$\lfloor \mathrm{Z} \rfloor$ color change rule* allows for the following two possibilities. First, if u has one white neighbor w, u can force w to become blue. Alternatively, if u has no white neighbors, then u can force any white vertex w to become blue. This second option is called *hopping*. For a graph G, *CCR-$\lfloor \mathrm{Z} \rfloor$ zero forcing sets* and the *CCR-$\lfloor \mathrm{Z} \rfloor$ zero forcing number*, denoted CCR-$\lfloor \mathrm{Z} \rfloor(G)$, are defined analagously to standard zero forcing sets and the standard zero forcing number. With the ability to hop, there are more choices involved in CCR-$\lfloor Z \rfloor$ zero forcing than standard zero forcing. In fact, it is possible to start with a CCR-$\lfloor Z \rfloor$ zero forcing set, but fail to color the entire graph due to poor hopping choices.

Example 1.3.8. Let G be the graph obtained by adding a leaf to C_4 (illustrated in Figure 1.3). Suppose u is the leaf, v is its neighbor, and $B = \{u, v\}$. Note that B is a CCR-$\lfloor Z \rfloor$ zero forcing set of G because u can force a white neighbor of w and that neighbor can force around the cycle. However, suppose the first force performed is the one depicted by the dashed arrow in Figure 1.3. Then the CCR-$\lfloor Z \rfloor$ forcing cannot progress because all active vertices have more than one white neighbor.

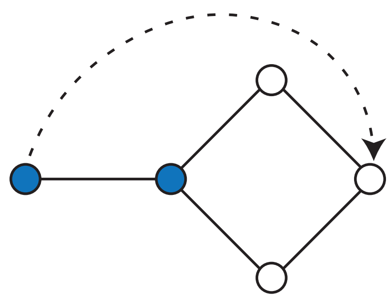

Figure 1.3 The dashed arrow in the above graph is a poor choice for hopping.

Suppose G is a graph and p is a graph parameter that is well-ordered. The *minor monotone floor* of p is defined in [3] as $\lfloor p \rfloor(G) = \min\{p(H) \mid G \preceq H\}$. Note that $\lfloor p \rfloor(G)$ gets its name because of the fact that it is minor monotone (i.e., $\lfloor p \rfloor(G) \leq \lfloor p \rfloor(H)$ if $G \preceq H$). To see why CCR-$\lfloor Z \rfloor$ gets its name, it is necessary to discuss linear k-trees. A *linear k-tree* is a graph that is built inductively from a K_{k+1} by repeatedly adding new vertices that connect to an existing K_{k+1} that contains a vertex of degree k. For example, a linear 1-tree is a path built by starting with K_2 and adding new leaves to the existing leaves. The *proper path-width* of a graph G, denoted $\mathrm{ppw}(G)$, is the smallest k such that G is a subgraph of a linear k-tree. The next theorem exhibits a connection between CCR-$\lfloor Z \rfloor$ zero forcing and linear k-trees.

Theorem 1.3.9. [3] *If G is a graph and $|E(G)| \geq 1$, then CCR-$\lfloor Z \rfloor(G) = \mathrm{ppw}(G)$.*

If B is a minimum CCR-$\lfloor Z \rfloor$ zero forcing set of a graph G, then B is a standard zero forcing set of the graph H obtained from G by adding the edges where hops occur. Since G is a minor of H, this means that $\lfloor Z \rfloor(G) \leq |B| =$ CCR-$\lfloor Z \rfloor(G)$. In addition, Theorem 1.3.9 is used in [3] to show that CCR-$\lfloor Z \rfloor(G) \leq \lfloor Z \rfloor(G)$ which leads to the following result.

Theorem 1.3.10. [3] *For any graph G, CCR-$\lfloor Z \rfloor(G) = \lfloor Z \rfloor(G)$.*

Therefore, CCR-$\lfloor Z \rfloor$ is the color change rule whose zero forcing parameter is $\lfloor Z \rfloor$ and CCR-$\lfloor Z \rfloor$ can be abbreviated to $\lfloor Z \rfloor$.

In the same way that hopping is added to standard zero forcing to create $\lfloor Z \rfloor$ forcing, hopping can be added to PSD zero forcing. In PSD zero forcing, standard zero forcing is applied in each component of the white vertices. The *CCR-$\lfloor Z_+ \rfloor$ color change rule* is to apply the $\lfloor Z \rfloor$ color change rule in each component of white vertices (i.e., hopping is allowed in each component). This leads to *CCR-$\lfloor Z_+ \rfloor$ zero forcing sets* and the *CCR-$\lfloor Z_+ \rfloor$ zero forcing number* of a graph G, denoted as CCR-$\lfloor Z_+ \rfloor(G)$. The next theorem states that the CCR-$\lfloor Z_+ \rfloor$ color change rule is the color change rule whose zero forcing parameter is $\lfloor Z_+ \rfloor$.

Theorem 1.3.11. [3] *For any graph G, CCR-$\lfloor Z_+ \rfloor(G) = \lfloor Z_+ \rfloor(G)$.*

In [3], the proof of Theorem 1.3.11 is analogous to the proof of Theorem 1.3.10 except that a generalization of linear k-trees is used (namely, *two-sided k-trees*).

1.3.2 Propagation time

The idea of zero forcing propagation is to partition a zero forcing process into time steps where every possible independent force is performed simultaneously in each time step. Suppose G is a graph and $B \subseteq V(G)$ is a zero forcing set of G. Partition the vertices of G as follows. Define $B^{(0)} = B$ and for each $t \geq 0$ and $w \in V(G)$, $w \in B^{(t+1)}$ if and only if there exists a vertex $b \in \bigcup_{i=0}^{t} B^{(i)}$ such that w is the only neighbor or b that is not in $\bigcup_{i=0}^{t} B^{(i)}$. Note that for each $t \geq 0$, $B^{(t)}$ is the set of vertices in G that become blue at time t if all possible forces are performed at each time step. The *propagation time of B in G*, denoted $\mathrm{pt}(G;B)$, is defined in [10] as the minimum t' such that $\bigcup_{i=0}^{t'} B^{(i)} = V(G)$.

Propagation time for zero forcing was introduced in [10] and it is used in [11] to measure the time required to gain control of quantum systems. There are many questions that concern uniqueness in the early literature on zero forcing and propagation. For example, it is shown in [2] that minimum zero forcing sets are not unique in connected graphs with at least two vertices. In [10], it is determined that the propagation time of minimum zero forcing sets in a graph is not unique.

Example 1.3.12. Once again, let G be the graph obtained from C_4 by adding a leaf. It is clear that since G has a cycle as a subgraph, $\mathrm{Z}(G) \geq 2$. Therefore, if $B \subseteq V(G)$ is a zero forcing set of G with $|B| = 2$, then B is a minimum zero forcing set of G. Suppose $B_1 = \{u_1, v_1\}$ is the subset of $V(G)$ shown on the left of Figure 1.4. Note that u_1 forces around the cycle in the first two time steps and v_1 cannot force the leaf until time step 3. Thus, $\mathrm{pt}(G;B_1) = 3$. Let $B_2 = \{u_2, v_2\}$ be the subset of $V(G)$ shown on the right of Figure 1.4. Now v_2 performs a force in the first time step and the two blue vertices in the cycle force the rest of $V(G)$ in the second time step. Therefore, $\mathrm{pt}(G;B_2) = 2 \neq 3 = \mathrm{pt}(G;B_1)$.

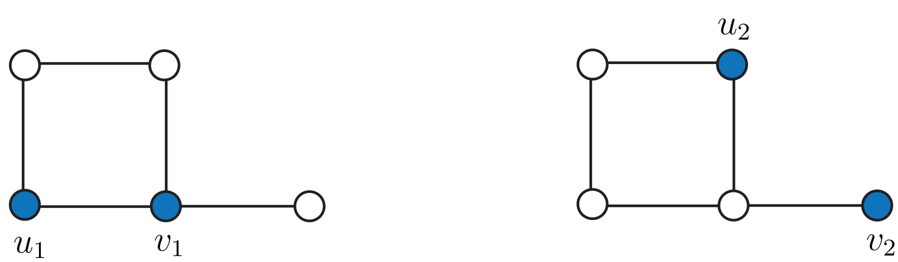

Figure 1.4 Two minimum zero forcing sets are shown with different propagation times.

Since two distinct minimum zero forcing sets in a given graph can have different propagation times, it is natural to minimize the propagation time over all such sets. The *minimum propagation* *time* of a graph G is defined in [10] as $\operatorname{pt}(G) = \min\{\operatorname{pt}(G;B) \mid B \text{ is a minimum zero forcing set of } G\}$. Since at least one vertex is forced in each time step, $\operatorname{pt}(G) \leq |G| - 1$ for any graph G. In [10], graphs with extreme minimum propagation times $|G| - 1$, $|G| - 2$, and 0 are characterized. An *efficient zero forcing set* of a graph G is a minimum zero forcing set $B \subseteq V(G)$ such that $\operatorname{pt}(G;B) = \operatorname{pt}(G)$. A reasonable question to ask is whether the efficient zero forcing sets of a given graph are unique. To answer this question, a few more definitions are needed.

A zero forcing set of a graph G is a subset $B \subseteq V(G)$ such that $V(G)$ is a final coloring of B. For standard zero forcing, final colorings of a given subset of vertices are shown in [1] to be unique. Consider a zero forcing set B of a graph G and write down each force that is performed (write $u \rightarrow w$ whenever u forces w to become blue) chronologically as the final coloring of B is obtained. This list of forces is called a *chronological list of forces* of B. The unordered set of forces in a chronological list of forces of B is called a *set of forces* of B. Sets of forces are used to study propagation because multiple vertices are forced at the same time.

The idea of propagation also applies to sets of forces $\mathcal{F}$ where as many forces in $\mathcal{F}$ as possible are performed simultaneously in each time step. If G is a graph, B is a zero forcing set of G, and $\mathcal{F}$ is a set of forces of B, define $\mathcal{F}^{(0)} = B$. Then for each $t \geq 0$, define $\mathcal{F}^{(t+1)}$ to be the set of vertices $w \in V(G) \setminus \bigcup_{i=0}^{t} \mathcal{F}^{(i)}$ such that $(u \rightarrow w) \in \mathcal{F}$ for some $u \in \bigcup_{i=0}^{t} \mathcal{F}^{(i)}$. The *propagation time of* $\mathcal{F}$ in G, denoted $\operatorname{pt}(G;\mathcal{F})$, is the minimum t' such that $\bigcup_{i=0}^{t'} \mathcal{F}^{(i)} = V(G)$. A set of forces $\mathcal{F}$ is *efficient* in G if $\operatorname{pt}(G;\mathcal{F}) = \operatorname{pt}(G)$. Note that for every $t \geq 0$, $\bigcup_{i=0}^{t} \mathcal{F}^{(i)} \subseteq \bigcup_{i=0}^{t} B^{(i)}$ which implies that $\operatorname{pt}(G;B) \leq \operatorname{pt}(G;\mathcal{F})$. This means that if $\mathcal{F}$ is an efficient set of forces of B, then B is an efficient zero forcing set of G. Since performing every possible force in G at each time step yields a valid set of forces of B,

$$\operatorname{pt}(G;B) = \min\{\operatorname{pt}(G;\mathcal{F}) \mid \mathcal{F} \text{ is a set of forces of } B\}.$$

Suppose G is a graph and $\mathcal{F}$ is a set of forces of a zero forcing set B of G. A *forcing chain* of $\mathcal{F}$ is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that for each $1 \leq i \leq k-1$, $(v_i \rightarrow v_{i+1}) \in \mathcal{F}$. The *reverse set of forces* of $\mathcal{F}$, denoted $\operatorname{Rev}(\mathcal{F})$, is the set of forces obtained from $\mathcal{F}$ by replacing each force $(u \rightarrow v) \in \mathcal{F}$ with the force $v \rightarrow u$. The set of vertices that do not perform a force in $\mathcal{F}$ is called the *terminus* of $\mathcal{F}$ and is denoted as $\text{Term}(\mathcal{F})$. The terminus and reverse forces were originally defined in [2] for chronological lists of forces. Using these ideas, it is observed in [10] that the reverse set of forces of $\mathcal{F}$ is a set of forces of $\text{Term}(\mathcal{F})$. Furthermore, an induction argument on the times steps of $\mathcal{F}$ is used to show that $\text{pt}(G; \text{Rev}(\mathcal{F})) \leq \text{pt}(G; \mathcal{F})$. This fact is used to determine that efficient zero forcing sets of a given graph are not unique.

Theorem 1.3.13. [10] *If G is a graph, $B \subseteq V(G)$ is an efficient zero forcing set of G, and $\mathcal{F}$ is an efficient set of forces of B, then* $\text{Rev}(\mathcal{F})$ *is an efficient set of forces of* $\text{Term}(\mathcal{F})$*.*

It is also natural to maximize the propagation time achieved by minimum zero forcing sets of a given graph. The *maximum propagation time* of a graph G is defined in [10] as

$$\text{PT}(G) = \max\{\text{pt}(G; B) \mid B \text{ is a minimum zero forcing set of } G\}.$$

The *propagation time interval* of a graph G, denoted $[\text{pt}(G), \text{PT}(G)]$, is the set of integers k such that $\text{pt}(G) \leq k \leq \text{PT}(G)$. The propagation time interval of G is said to be *full* if for each $k \in [\text{pt}(G), \text{PT}(G)]$, there exists a minimum zero forcing set B of G such that $\text{pt}(G; B) = k$. A *spider* is a tree T such that exactly one vertex $v \in V(T)$ has $d(v) \geq 3$. Certain spiders are shown in [10] to have propagation time intervals that are not full.

The concept of propagation time is extended to PSD zero forcing in [13]. For a graph G and PSD zero forcing set $B \subseteq V(G)$, the *PSD propagation time of B in G*, denoted $\text{pt}_+(G; B)$, is defined analogously to standard propagation time by performing every possible force at each time step. Note that in PSD propagation, a single vertex can force in multiple components of $G - B$ at the same time. If G is a graph, the *minimum PSD propagagation time* of G is denoted $\text{pt}_+(G)$ and the *maximum PSD propagation time* of G is denoted $\text{PT}_+(G)$. The *PSD propagation time interval* is the set of integers in $[\text{pt}_+(G), \text{PT}_+(G)]$ and this interval is *full* if every value in the interval is achieved as the PSD propagation time of some minimum PSD zero forcing set of G. The minimum and maximum PSD propagation times are examined in [13] for many families of graphs and extreme values of pt_+ and PT_+ are characterized. Many of the graphs studied in [13] are shown to have full PSD propagation time intervals and the author conjectures that this is true for all graphs. This is an interesting distinction between standard and PSD propagation.

1.3.3 Throttling

Propagation time is studied in [10] exclusively for minimum zero forcing sets. Suppose G is a graph and the cost of each vertex in an initial set $B \subseteq V(G)$ of blue vertices is comparable to the cost of each time step of propagation. In this case, if B is a zero forcing set of G that is not minimum and $|B| - \mathrm{Z}(G) < \mathrm{pt}(G) - \mathrm{pt}(G; B)$, then it is more cost-effective to start with B as the set of initial blue vertices than a minimum zero forcing set of G. In other words, by throttling the propagation time, it could be worth using more vertices than necessary in the initial set of blue vertices. The idea of balancing the size of a zero forcing set with its propagation time was first studied in [6]. The *(standard) throttling number* of graph G is defined as

$$\mathrm{th}(G) = \min\{|B| + \mathrm{pt}(G; B) \mid B \text{ is a zero forcing set of } G\}.$$

Let G be a graph of order n and $B \subseteq V(G)$ be a zero forcing set of G. At each time step in the propagation process of B, each blue vertex in G can force at most one white vertex to become blue. So at most $|B|$ vertices become blue at each time step. Since B is a zero forcing set, all vertices in $V(G)$ are blue after time step $\mathrm{pt}(G; B)$. This means that $|B|(1 + \mathrm{pt}(G; B)) \geq n$ for any graph G of order n and zero forcing set $B \subseteq V(G)$. A lower bound for the throttling number of a graph is obtained in [6] by minimizing $|B| + \mathrm{pt}(G; B)$ subject to the constraint that $|B|(1 + \mathrm{pt}(G; B)) \geq n$.

Proposition 1.3.14. [6] *If G is a graph and $|V(G)| = n$,* $\mathrm{th}(G) \geq \lceil 2\sqrt{n} - 1 \rceil$.

The bound in Proposition 1.3.14 is shown in [6] to be tight for paths by snaking the path into a square box. For example, consider the path P_n and let m be the largest integer such that $r = n - m^2 \geq 0$. Figure 1.5 illustrates snaking the path into as many rows as possible with m vertices in each row.

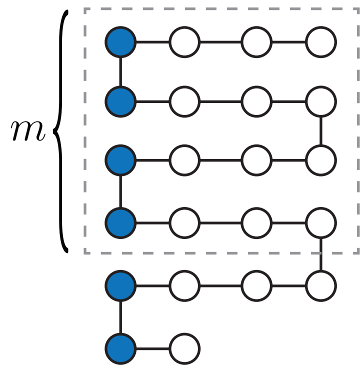

Figure 1.5 The path P_{22} is snaked in a 4×4 square box with some overhang.

Since m^2 is the largest perfect square less than or equal to n, $n < m^2 + 2m + 1 = (m+1)^2$. So the number of rows in the snaked P_n is at least m and at most $m + 2$. The set B of vertices in the left column (as shown in Figure 1.5) has $\text{pt}(P_n; B) = m - 1$ and $m \le |B| \le m + 2$. Therefore,

$$\text{th}(P_n) \le \begin{cases} 2m - 1 & \text{if } n = m^2; \\ 2m & \text{if } m^2 < n \le m^2 + m; \\ 2m + 1 & \text{if } m^2 + m < n \le m^2 + 2m. \end{cases}$$

It can be verified algebraically that the above upper bound for $\text{th}(P_n)$ is equal to $\lceil 2\sqrt{n} - 1 \rceil$. Since $\text{pt}(P_n) = n - 1$, it is significantly less efficient to force every vertex in a path to become blue by starting with a minimum zero forcing set.

Note that the snaking method constructs a zero forcing set that is guaranteed to have propagation time on the order of $\sqrt{n}$. In [6], such a zero forcing set is constructed in any graph by starting with a minimum zero forcing set and carefully choosing vertices along the forcing chains. This leads to the following theorem which shows that for graphs G with bounded zero forcing number, $\text{th}(G)$ is on the order of $\sqrt{n}$.

Theorem 1.3.15. [6] *If G is a graph of order n and $\text{Z}(G) \le k$, then $\text{th}(G) \le (2k+1)\lceil \sqrt{n} \rceil + k$.*

It is also interesting to consider changing the relative cost of vertices in a zero forcing set B and the propagation time of B. This idea is called *weighted throttling* and the proof of Theorem 1.3.15

can be used to obtain a similar result when the size of the zero forcing set and the propagation time of that set are given different weights.

Theorem 1.3.16. [6] *If G is a graph of order n, $\mathrm{Z}(G) \leq k$, and a_1 and a_2 are fixed weights, then*

$$\min\{a_1|B| + a_2 \operatorname{pt}(G; B) \mid B \text{ is a zero forcing set of } G\} \leq (2ka_1 + a_2)\lceil\sqrt{n}\rceil + ka_1.$$

In [7], the study of throttling is extended to PSD zero forcing. For a graph G and PSD zero forcing set $B \subseteq V(G)$, define $\operatorname{th}_+(G; B) = |B| + \operatorname{pt}(G; B)$. The *PSD throttling number* of a graph G is defined as

$$\operatorname{th}_+(G) = \min\{\operatorname{th}_+(G; B) \mid B \text{ is a PSD zero forcing set of } G\}.$$

Much of the work in [7] is analogous to the research that was done in [6]. For PSD throttling, the fact that a single vertex can simultaneously force in multiple components makes some of the results in [7] more complicated than their counterparts in [6]. For example, if B is a PSD zero forcing set of a graph G, then at most $|B|\Delta(G)$ vertices can become blue in the first time step. Then in each time step $t > 1$, a vertex that became blue in time step $t-1$ can force at most $\Delta(G) - 1$ new vertices to become blue. This is used to prove the following inequality.

Lemma 1.3.17. [7] *For every graph G of order n and PSD zero forcing set $B \subseteq V(G)$,*

$$n \leq \begin{cases} |B|(1 + 2\operatorname{pt}_+(G; B)) & \text{if } \Delta(G) = 2; \\ |B|\left(1 + \frac{\Delta(G)(\Delta(G)-1)^{\operatorname{pt}_+(G;B)} - \Delta(G)}{\Delta(G)-2}\right) & \text{if } \Delta(G) > 2. \end{cases}$$

In both of the cases $\Delta(G) = 2$ and $\Delta(G) > 2$, a lower bound for the PSD throttling number of a graph is obtained by minimizing $|B| + \operatorname{pt}_+(G; B)$ subject to the constraints in Lemma 1.3.17.

Theorem 1.3.18. [7] *For every graph G of order n,*

$$\operatorname{th}_+(G) \geq \begin{cases} \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil & \text{if } \Delta(G) = 2; \\ \left\lceil 1 + \log_{(\Delta(G)-1)}\left(\frac{(\Delta(G)-2)n+2}{\Delta(G)}\right) \right\rceil & \text{if } \Delta(G) > 2. \end{cases}$$

Similar to standard throttling, a snaking argument is used in [7] to show that the bound in Theorem 1.3.18 is tight for cycles of order at least 4 and paths. In the case that $\Delta(G) > 2$, Theorem 1.3.18 is shown to be tight for a certain family of rooted trees. It is also shown in [7] that given a tree T and a PSD zero forcing set $B \subseteq V(T)$, there exists a PSD zero forcing set B' of T such that $|B| = |B'|$, $\text{pt}_+(T; B') \leq \text{pt}_+(T; B)$, and B' contains no leaves. This fact is used in an induction argument to prove the following monotonicity result.

Theorem 1.3.19. [7] *If T and T' are trees with $T' \leq T$, then $\text{th}_+(T') \leq \text{th}_+(T)$.*

If G is a graph and $W \subseteq V(G)$ is an independent set of G, then every component of $G[W]$ is a single vertex. So in this case, $V(G) \setminus W$ is a PSD zero forcing set of G with $\text{pt}_+(G; V(G) \setminus W) = 1$ which means $\text{th}_+(G) \leq |G| - |W| + 1$. Choosing W to be a maximum independent set of G gives an upper bound for the PSD throttling number of a connected graph in terms of the independence number of G.

Proposition 1.3.20. [7] *For any connected graph G of order n, $\text{th}_+(G) \leq n - \alpha(G) + 1$.*

Proposition 1.3.20 is used in [7] to characterize all graphs G with $\text{th}_+(G) = |G| - 1$. Graphs with $\text{th}_+(G) \in \{1, 2, 3, |G|\}$ are also characterized and weighted PSD throttling is explored.

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CHAPTER 2. THROTTLING FOR ZERO FORCING AND VARIANTS

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Abstract

Zero forcing is a process on a graph in which the goal is to force all vertices to become blue by applying a color change rule. Throttling minimizes the sum of the number of vertices that are initially blue and the number of time steps needed to color every vertex. This paper provides a new general definition of throttling for variants of zero forcing and studies throttling for the minor monotone floor of zero forcing. The technique of using a zero forcing process to extend a given graph is introduced. For standard zero forcing and its floor, these extensions are used to characterize graphs with throttling number $\leq t$ as certain minors of cartesian products of complete graphs and paths. Finally, these characterizations are applied to determine graphs with extreme throttling numbers.

Keywords: Zero forcing, propagation time, throttling, minor monotone floor

AMS subject classification: 05C57, 05C15, 05C50

2.1 Introduction

Zero forcing is a process on graphs in which an initial set of vertices is colored blue (with the remaining vertices colored white) and vertices can force white vertices to become blue according to a color change rule. When using the color change rule, the goal is to eventually color every vertex in graph. Zero forcing can be used to model graph searching [11], the spread of information on graphs [5], and control of quantum systems [4, 9]. Naturally, it is useful to know the smallest possible size of an initial set that can be used to color all vertices in the graph blue. It is also useful to know the time it takes to complete this process (often called propagation time). The idea of throttling is to study the relationship between the size of the initial set and its propagation time. Richard Brualdi posed the problem of minimizing the sum of these two quantities in 2011 (see [5]).

Unless otherwise stated, the graphs in this paper are simple, undirected, and finite. For a graph G, $V(G)$ and $E(G)$ denote the sets of vertices and edges of G respectively. The cardinality of $V(G)$ is often denoted as $|G|$. The *(standard) color change rule* is that a blue vertex u can force a white vertex w to become blue if w is the only white neighbor of u. In this case, it is said that u forces w which is denoted as $u \to w$. A vertex is *active* if it is blue and has not yet performed a force. Note that in standard zero forcing, any vertex that performs a force becomes inactive and cannot perform another force. Let G be a graph with $B \subseteq V(G)$ colored blue and $V(G) \setminus B$ colored white. If every vertex in $V(G)$ can be forced to become blue by repeatedly applying the standard color change rule, then B is a *(standard) zero forcing set* of G. The *(standard) zero forcing number*, $\operatorname{Z}(G)$, is the minimum size of a standard zero forcing set of G. In [1], it is shown that the zero forcing number can be used to bound the minimum rank of a matrix associated with a graph.

Zero forcing propagation is studied in [8]. The idea is to simultaneously perform all possible forces at each time step. Define $B^{(0)} = B$ and for each $t \geq 0$, define $B^{(t+1)}$ to be the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^{t} B^{(s)}$ such that w is the only neighbor of b not in $\bigcup_{s=0}^{t} B^{(s)}$. The *(standard) propagation time of B in G*, denoted $\operatorname{pt}(G, B)$, is the smallest integer t' such that $V(G) = \bigcup_{t=0}^{t'} B^{(t)}$. Propagation time is particularly important in the control of quantum systems (see [9]).

Throttling for standard zero forcing was first studied by Butler and Young in [5]. If B is a zero forcing set of a graph G, the *throttling number of B in G* is $\operatorname{th}(G, B) = |B| + \operatorname{pt}(G, B)$. The *(standard) throttling number of G* is the minimum value of $\operatorname{th}(G, B)$ where B ranges over all zero forcing sets of

G. For a given graph G and an integer k, the ZERO FORCING THROTTLING problem is to determine if the standard throttling number of G is less than k. The many variations of zero forcing (see [2]) lead to many variations of throttling. In [3], it was shown that ZERO FORCING THROTTLING and other variants are NP-Complete.

Commonly studied variants of zero forcing include positive semidefinite zero forcing and loop zero forcing (see [2]). Let G be a graph. A connected component of G is a maximally connected subgraph of G. Suppose B is a set of blue vertices in G and $G - B$ has k separate connected components. Let $W_1, \ldots, W_k$ be the sets of (white) vertices of the connected components of $G - B$. The *positive semidefinite color change rule* applies the standard color change rule in $G[W_i \cup B]$ for any $1 \leq i \leq k$. The *positive semidefinite zero forcing number* of a graph G is denoted $\mathrm{Z}_+(G)$ and the positive semidefinite throttling number (studied in [6]) is defined analogously to standard throttling. Loop zero forcing (see [2]) arises by considering a graph where every vertex has a loop. The *loop color change rule* for simple graphs is to apply the standard color change rule, or if every neighbor of a white vertex w is blue, then w can force itself to become blue. The *loop zero forcing number* of a graph G is denoted $\mathrm{Z}_\ell(G)$.

If G and H are graphs and G is a subgraph of H, write $G \leq H$. If $G \leq H$ and $|V(G)| = |V(H)|$, G is a *spanning subgraph* of H and H is a *spanning supergraph* of G. If G is a minor of H, write $G \preceq H$. Note that this paper breaks the convention of using H to denote a minor or subgraph of a graph G because it considers many graph parameters that depend on majors or supergraphs of a given graph. For example, suppose p is a graph parameter whose range is well-ordered. The *minor monotone floor* of p is defined as $\lfloor p \rfloor(G) = \min\{p(H) \mid G \preceq H\}$. In [2], it was shown that $\lfloor \mathrm{Z} \rfloor$, $\lfloor \mathrm{Z}_+ \rfloor$, and $\lfloor \mathrm{Z}_\ell \rfloor$ are zero forcing parameters with their own unique color change rules. In particular, the $\lfloor \mathrm{Z} \rfloor$ *color change rule* is to either apply the standard color change rule, or alternatively if a vertex v is active and all neighbors of v are blue, then v can force any single white vertex w to become blue. The latter condition of the $\lfloor \mathrm{Z} \rfloor$ color change rule is called "hopping". If this condition is used, then it is said that v forces w by a hop. It was also shown in [2] that the minor monotone floors of various zero forcing parameters are related to tree-width, path-width, and proper path-width. In addition, the concepts of path-width and proper path-width were shown in [10] to have connections to search games on graphs.

In Section 2.2, a general definition of propagation and throttling is given that allows for the study of further variations. Throttling for $\lfloor Z \rfloor$ is studied in Section 2.3 and an "extension" technique that can be used to characterize graphs with $\lfloor Z \rfloor$ throttling number at most t for a fixed positive integer t is introduced. A similar characterization for standard throttling is given in Section 2.4. These characterizations are applied in Section 2.5 in order to quickly characterize graphs with extreme throttling numbers. Finally, in Section 3.5, an observation is made about proving the complexity of $\lfloor Z \rfloor$ throttling and possibilities for future work are given.

2.2 General propagation time and throttling

This section gives new general definitions of propagation time and throttling for color change rules. Define an *(abstract) color change rule* to be a set of conditions under which a vertex u can force a white vertex w to become blue in a graph whose vertices are colored white or blue. The notation $u \to w$ is used to indicate that vertex u forced vertex w to become blue. Let G be a graph with $B \subseteq V(G)$ colored blue and $V(G) \setminus B$ colored white. Let R be a given color change rule. Repeatedly apply R to G until it is no longer possible to do so and write down the forces $u \to w$ in the order in which they are performed. This list of forces is called a *chronological list of R forces of B* and the unordered set of forces that appear in the list is a *set of R forces of B*. Suppose G is a graph and $\mathcal{F}$ is a set of R forces of $B \subseteq V(G)$. An *R forcing chain of $\mathcal{F}$* is a sequence of vertices $(v_1, v_2, \dots, v_k)$ in G such that $(v_i \to v_{i+1}) \in \mathcal{F}$ for each $1 \le i \le k-1$. An R forcing chain of $\mathcal{F}$ is *maximal* if it is not properly contained in any other R forcing chain of $\mathcal{F}$. The set of vertices in G that are blue after all forces in $\mathcal{F}$ have been performed is an *R final coloring of B*.

Remark 2.2.1. Suppose B' is an R final coloring of a set $B \subseteq V(G)$ obtained by performing the forces in a chronological list of R forces of B (denoted by $\mathcal{L}$). Note that B' consists of the vertices in B together with all vertices that become forced in $\mathcal{L}$. Therefore, B' does not depend on the chronological ordering of $\mathcal{L}$. This means that R final colorings depend on sets of forces and not chronological lists of forces.

Let G be a graph and let R be a given color change rule. An *R forcing set of G* is a set $B \subseteq V(G)$ of vertices such that $V(G)$ is an R final coloring of B for some set of R forces. The *R forcing parameter*, $R(G)$, is the minimum size of an R forcing set of G. An R forcing set B is a *minimum R forcing set of G* if $|B| = R(G)$.

Note that the definition of standard propagation time of a set of vertices does not use sets of forces. This is because final colorings in standard zero forcing are unique and depend only on the initial set of blue vertices (see [1]). However, there are variants of zero forcing that do not have unique final colorings (e.g., $\lfloor Z \rfloor$ forcing). When performing a $\lfloor Z \rfloor$ force by hopping, there are many choices for the white vertex that gets forced. Example 2.36 in [2] illustrates that it is possible to start with a blue $\lfloor Z \rfloor$ forcing set B and fail to color every vertex in the graph due to poor hopping choices. In this case, B has at least two distinct sets of $\lfloor Z \rfloor$ forces with different propagation times. This motivates the following definitions.

For a set of R forces $\mathcal{F}$ of $B \subseteq V(G)$, define $\mathcal{F}^{(0)} = B$ and for $t \geq 0$, $\mathcal{F}^{(t+1)}$ is the set of vertices w such that the force $v \to w$ appears in $\mathcal{F}$ and w can be R forced by v if the vertices in $\bigcup_{i=0}^{t} \mathcal{F}^{(i)}$ are colored blue and the vertices in $V(G) \setminus \left(\bigcup_{i=0}^{t} \mathcal{F}^{(i)} \right)$ are colored white. The *R propagation time of $\mathcal{F}$ in G*, denoted $\operatorname{pt}_R(G; \mathcal{F})$, is the least t' such that $V(G) = \bigcup_{i=0}^{t'} \mathcal{F}^{(i)}$. If the R final coloring induced by $\mathcal{F}$ is not $V(G)$, then define $\operatorname{pt}_R(G; \mathcal{F}) = \infty$. Note that B is colored blue at time 0, and for each $1 \leq t \leq \operatorname{pt}_R(G; \mathcal{F})$, time step t takes place between time $t-1$ and time t in $\mathcal{F}$. A vertex in G is *active at time t* if it is blue at time t and has not performed a force in time step s for any $s \leq t$.

Definition 2.2.2. Let G be a graph with $B \subseteq V(G)$ and let R be a given color change rule. The *R propagation time of B* is defined as

$$\operatorname{pt}_R(G; B) = \min\{\operatorname{pt}_R(G; \mathcal{F}) \mid \mathcal{F} \text{ is set of } R \text{ forces of } B\}.$$

Note that Definition 2.2.2 doesn't require the set B to be an R forcing set of G. This is because a set $\mathcal{F}$ of R forces that fails to color every vertex in G has $\operatorname{pt}_R(G; \mathcal{F}) = \infty$. Therefore, such a set $\mathcal{F}$

does not realize $\mathrm{pt}_R(G;B)$ when B is an R forcing set of G. If B is not an R forcing set of G, then every set of R forces of B has infinite propagation time and $\mathrm{pt}_R(G;B) = \infty$. Another advantage of Definition 2.2.2 is that it is not required to prove that a subset of vertices is an R forcing set before discussing its propagation time. This is useful for proving Proposition 2.3.1 in the next section.

The (standard) propagation time of a graph (see [8]) considers the smallest propagation time among minimum zero forcing sets. The next definition generalizes this idea.

Definition 2.2.3. Let G be a graph and let R be a given color change rule. The R *propagation time of* G is defined as

$$\mathrm{pt}_R(G) = \min\{\mathrm{pt}_R(G;B) \mid B \text{ is a minimum } R \text{ forcing set of } G\}.$$

Definition 2.2.4. Let G be a graph with $B \subseteq V(G)$ and let R be a given color change rule. The R *throttling number of B in G* is

$$\mathrm{th}_R(G;B) = |B| + \mathrm{pt}_R(G;B).$$

Definition 2.2.5. Let G be a graph and let R be a given color change rule. The R *throttling number* of G is defined as

$$\mathrm{th}_R(G) = \min_{B \subseteq V(G)}\{\mathrm{th}_R(G;B)\}.$$

When comparing propagation time and throttling for various color change rules, Z is used to denote the standard zero forcing color change rule (i.e., pt_Z and th_Z).

2.3 Throttling for the minor monotone floor of Z

This section investigates propagation and throttling for the $\lfloor Z \rfloor$ color change rule. Definition 2.2.2 exhibits the connection between the $\lfloor Z \rfloor$ propagation time of a subset $B \subseteq V(G)$ and the $\lfloor Z \rfloor$ propagation time of a set of $\lfloor Z \rfloor$ forces of B. The following proposition shows that the $\mathrm{pt}_{\lfloor Z \rfloor}(G;B)$ can also be calculated by minimizing the standard zero forcing propagation time of B on spanning supergraphs of G.

Proposition 2.3.1. *If G is a graph and $B \subseteq V(G)$, then*

$$\mathrm{pt}_{\lfloor Z \rfloor}(G;B) = \min\{\mathrm{pt}_Z(H;B) \mid G \leq H \text{ and } |G| = |H|\}. \tag{2.1}$$

Proof. Let $\mathcal{F}$ be a set of $\lfloor Z \rfloor$ forces of B such that $\mathrm{pt}_{\lfloor Z \rfloor}(G;B) = \mathrm{pt}_{\lfloor Z \rfloor}(G;\mathcal{F})$. Note that every force in $\mathcal{F}$ is either a Z force or a force by a hop. Let G' be the graph obtained from G by adding the edges uw such that $u \to w$ appears in $\mathcal{F}$ and $u \to w$ by a hop. Note that for each edge $uw \in E(G') \setminus E(G)$, w is the only white neighbor of u in G' and u is active at the time that $u \to w$ in $\mathcal{F}$. This means that $u \to w$ is a valid Z force in G' for each such edge. Thus, $\mathcal{F}$ is a set of Z forces of B in G' and $\mathrm{pt}_Z(G';\mathcal{F}) = \mathrm{pt}_{\lfloor Z \rfloor}(G;\mathcal{F})$. Therefore,

$$\mathrm{pt}_{\lfloor Z \rfloor}(G;B) = \mathrm{pt}_Z(G';\mathcal{F}) \geq \min\{\mathrm{pt}_Z(H;B) \mid G \leq H \text{ and } |G| = |H|\}.$$

Now let H' be a spanning supergraph of G such that the right hand side of (2.1) is equal to $\mathrm{pt}_Z(H', B)$. Let $\mathcal{F}$ be a set of Z forces of B such that $\mathrm{pt}_Z(H', \mathcal{F}) = \mathrm{pt}_Z(H', B)$. Consider applying $\mathcal{F}$ to B in G and hopping when an edge is missing. If $(u \to w) \in \mathcal{F}$ and $uw \in E(H') \setminus E(G)$, then u can $\lfloor Z \rfloor$ force w in $H' - uw$ by a hop when $u \to w$ in $\mathcal{F}$. If $(u \to w) \in \mathcal{F}$ and $uw \notin E(H') \setminus E(G)$, then u will Z force w in G exactly the way $u \to w$ in H'. If $(u \to w) \notin \mathcal{F}$, then the propagation time of $\mathcal{F}$ does not change regardless of whether uw is removed from H' to obtain G. This means that $\mathcal{F}$ is a set of $\lfloor Z \rfloor$ forces of B in G with $\mathrm{pt}_{\lfloor Z \rfloor}(G;\mathcal{F}) = \mathrm{pt}_Z(H';\mathcal{F})$. Thus,

$$\mathrm{pt}_{\lfloor Z \rfloor}(G;B) \leq \mathrm{pt}_{\lfloor Z \rfloor}(G;\mathcal{F}) = \mathrm{pt}_Z(H', B) = \min\{\mathrm{pt}_Z(H;B) \mid G \leq H \text{ and } |G| = |H|\}. \square$$

By the definition of minor monotone floor given in Section 2.1, $\lfloor Z \rfloor$ is minor monotone (i.e., $\lfloor Z \rfloor(G) \leq \lfloor Z \rfloor(H)$ if $G \preceq H$). Since any Z forcing set of a graph G is also a $\lfloor Z \rfloor$ forcing set of G, $\lfloor Z \rfloor$ is bounded above by Z. These facts together with Definitions 2.2.3, 2.2.4, and 2.2.5 can be used to extend the above proposition and give similar results for the $\lfloor Z \rfloor$ propagation time of a graph and $\lfloor Z \rfloor$ throttling.

Corollary 2.3.2. *Let G be a graph. Then*

$$\mathrm{pt}_{\lfloor Z \rfloor}(G) = \min\{\mathrm{pt}_Z(H) \mid G \leq H \text{ with } |G| = |H| \text{ and } \lfloor Z \rfloor(G) = Z(H)\}.$$

Proof. Let H be a spanning supergraph of G with B a standard zero forcing set of H. Then, $\lfloor Z \rfloor(G) \leq \lfloor Z \rfloor(H) \leq Z(H) \leq |B|$. Therefore, assuming that $|B| = \lfloor Z \rfloor(G)$ gives $|B| = Z(H)$ which means that B is a minimum zero forcing set of H. By Proposition 2.3.1, it follows that

$$\begin{aligned}
\text{pt}_{\lfloor Z \rfloor}(G) &= \min\{\text{pt}_{\lfloor Z \rfloor}(G;B) \mid \lfloor Z \rfloor(G) = |B|\} \\
&= \min\{\min\{\text{pt}_Z(H;B) \mid G \leq H \text{ and } |G| = |H|\} \mid \lfloor Z \rfloor(G) = |B|\} \\
&= \min\{\text{pt}_Z(H;B) \mid G \leq H \text{ with } |G| = |H| \text{ and } \lfloor Z \rfloor(G) = |B|\} \\
&= \min\{\text{pt}_Z(H) \mid G \leq H \text{ with } |G| = |H| \text{ and } \lfloor Z \rfloor(G) = Z(H)\}. \square
\end{aligned}$$

Corollary 2.3.3. *If G is a graph and $B \subseteq V(G)$, then*

$$\text{th}_{\lfloor Z \rfloor}(G;B) = \min\{\text{th}_Z(H;B) \mid G \leq H \textit{ and } |G| = |H|\}.$$

Corollary 2.3.4. *Let G be a graph. Then*

$$\text{th}_{\lfloor Z \rfloor}(G) = \min\{\text{th}_Z(H) \mid G \leq H \textit{ and } |G| = |H|\}.$$

Theorem 2.3.5. *The $\lfloor Z \rfloor$ throttling number is subgraph monotone. In particular, if G and H are graphs with $G \leq H$, then $\text{th}_{\lfloor Z \rfloor}(G) \leq \text{th}_{\lfloor Z \rfloor}(H)$.*

Proof. Let H be a graph. By Corollary 2.3.4, $\text{th}_{\lfloor Z \rfloor}(G') \leq \text{th}_{\lfloor Z \rfloor}(H)$ for any spanning subgraph G' of H. Let $v \in V(H)$ and let $E(v)$ be the set of all edges in H incident with v. Define $G' = H - E(v)$. Note that $\text{th}_{\lfloor Z \rfloor}(G') \leq \text{th}_{\lfloor Z \rfloor}(H)$. Choose $B' \subseteq V(G')$ such that $\text{th}_{\lfloor Z \rfloor}(G';B') = \text{th}_{\lfloor Z \rfloor}(G')$. Let $\mathcal{F}'$ be a set of $\lfloor Z \rfloor$ forces of G' with $\text{pt}_{\lfloor Z \rfloor}(G';\mathcal{F}') = \text{pt}_{\lfloor Z \rfloor}(G';B')$. The goal is to produce a set $B \subseteq V(G' - v)$ and a set of $\lfloor Z \rfloor$ forces, $\mathcal{F}$, of B such that $|B| \leq |B'|$ and $\text{pt}_{\lfloor Z \rfloor}(G' - v, \mathcal{F}) \leq \text{pt}_{\lfloor Z \rfloor}(G';\mathcal{F}')$. Let $v_1 \to v_2 \to \cdots \to v_k$ be the maximal $\lfloor Z \rfloor$ forcing chain of $\mathcal{F}'$ that contains v. If $k = 1$, then it suffices to choose $B = B' \setminus \{v\}$ and $\mathcal{F} = \mathcal{F}'$. Now assume $k > 1$. Note that $v = v_i$ for some $1 \leq i \leq k$. Define B and $\mathcal{F}$ as

$$B = \begin{cases} (B' \setminus \{v_i\}) \cup \{v_{i+1}\} & \text{if } i = 1, \\ B' & \text{otherwise,} \end{cases}$$

and

$$\mathcal{F} = \begin{cases} \mathcal{F}' \setminus \{v_i \to v_{i+1}\} & \text{if } i = 1, \\ (\mathcal{F}' \setminus \{v_{i-1} \to v_i, v_i \to v_{i+1}\}) \cup \{v_{i-1} \to v_{i+1}\} & \text{if } 1 < i < k, \\ \mathcal{F}' \setminus \{v_{i-1} \to v_i\} & \text{if } i = k. \end{cases}$$

Recall that v is an isolated vertex in G'. So when $1 < i < k$, $v_{i-1} \to v_i$ and $v_i \to v_{i+1}$ by hopping in G'. This means at the time that $v_{i-1} \to v_i$ in G', v_{i-1} can force v_{i+1} by a hop in $G' - v$. In the other cases, simply remove the appropriate force from $\mathcal{F}'$. So in all cases, it is clear that $|B| \leq |B'|$ and $\text{pt}_{\lfloor Z \rfloor}(G' - v; \mathcal{F}) \leq \text{pt}_{\lfloor Z \rfloor}(G'; \mathcal{F}')$. Also note that $G' - v = H - v$. Thus, for all $1 \leq i \leq k$,

$$\text{th}_{\lfloor Z \rfloor}(H - v) \leq |B| + \text{pt}_{\lfloor Z \rfloor}(G' - v; \mathcal{F}) \leq |B'| + \text{pt}_{\lfloor Z \rfloor}(G'; \mathcal{F}') = \text{th}_{\lfloor Z \rfloor}(G') \leq \text{th}_{\lfloor Z \rfloor}(H).$$

Since v was chosen arbitrarily, it follows that removing vertices from H will not increase the $\lfloor Z \rfloor$ throttling number. □

Since $\lfloor Z \rfloor$ is minor monotone, it is natural to ask if Theorem 2.3.5 can be strengthened to say that $\text{th}_{\lfloor Z \rfloor}$ is minor monotone. This question is answered negatively (see Theorem 2.3.18) once a characterization of $\text{th}_{\lfloor Z \rfloor}$ is obtained. Note that Theorem 2.3.5 can be extended in other ways. For each $p \in \{Z_+, Z_\ell\}$, the color change rule for $\lfloor p \rfloor$ takes the color change rule for p and allows hopping. This leads to the following corollary.

Corollary 2.3.6. *Suppose G is a graph and $B \subseteq V(G)$. Then for each $p \in \{Z_+, Z_\ell\}$,*

$$\text{pt}_{\lfloor p \rfloor}(G; B) = \min\{\text{pt}_p(H; B) \mid G \leq H \text{ and } |G| = |H|\},$$

$$\text{th}_{\lfloor p \rfloor}(G; B) = \min\{\text{th}_p(H; B) \mid G \leq H \text{ and } |G| = |H|\},$$

and $\text{th}_{\lfloor p \rfloor}$ is subgraph monotone.

It is likely that Corollary 2.3.6 will hold for any graph parameter p such that $\lfloor p \rfloor$ has a corresponding color change rule that takes the color change rule for p and allows hopping. However, no other parameters p have been shown to have this property. Note that if B is a standard zero forcing set of a graph G, then B is also a $\lfloor Z \rfloor$ forcing set of G with $\text{pt}_{\lfloor Z \rfloor}(G; B) \leq \text{pt}_Z(G; B)$. Thus, it is immediate that for any graph G, $\text{th}_{\lfloor Z \rfloor}(G)$ is bounded above by $\text{th}_Z(G)$. Butler and Young showed in [5, page 66] that for any graph G of order n, $\text{th}_Z(G)$ is at least $\lceil 2\sqrt{n} - 1 \rceil$. By Corollary 2.3.4, this lower bound holds for $\text{th}_{\lfloor Z \rfloor}(G)$ as well.

Corollary 2.3.7. *If G is a graph of order n, then*

$$\text{th}_{\lfloor Z \rfloor}(G) = \min\{\text{th}_Z(H) \mid G \leq H \text{ and } |G| = |H|\} \geq \lceil 2\sqrt{n} - 1 \rceil .$$

Since the $\lfloor Z \rfloor$ throttling number is bounded above by the standard throttling number, any graph G that achieves $\text{th}_Z(G) = \lceil 2\sqrt{n} - 1 \rceil$ also achieves $\text{th}_{\lfloor Z \rfloor}(G) = \lceil 2\sqrt{n} - 1 \rceil$. It was shown in [5] that $\text{th}_Z(P_n) = \lceil 2\sqrt{n} - 1 \rceil$. Thus, it can be concluded that $\text{th}_{\lfloor Z \rfloor}(P_n) = \lceil 2\sqrt{n} - 1 \rceil$. The standard throttling number of a cycle was determined in [6] as follows.

Theorem 2.3.8. [6, Theorem 7.1] *Let C_n be a cycle on n vertices. Define m to be the largest integer such that $m^2 \leq n$ and $n = m^2 + r$. Then*

$$\text{th}_Z(C_n) = \begin{cases} 2m - 1 & \text{if } r = 0 \text{ and } m \text{ is even,} \\ 2m & \text{if } 0 < r \leq m \text{ or } (r = 0 \text{ and } m \text{ is odd}), \\ 2m + 1 & \text{if } m < r < 2m + 1. \end{cases}$$

Theorem 2.3.8 can be used to determine the $\lfloor Z \rfloor$ throttling number of a cycle.

Proposition 2.3.9. *Let C_n be a cycle on n vertices. Then $\text{th}_{\lfloor Z \rfloor}(C_n) = \lceil 2\sqrt{n} - 1 \rceil$.*

Proof. Define m to be the largest integer such that $m^2 \leq n$ and $n = m^2 + r$. Note that if m is even or $r > 0$, then the conditions in Theorem 2.3.8 are equivalent to the conditions for $\text{th}_Z(P_n)$ in [5]. So in this case, $\text{th}_{\lfloor Z \rfloor}(C_n) = \text{th}_Z(P_n) = \lceil 2\sqrt{n} - 1 \rceil$. Now suppose m is odd and $r = 0$. So $n = m^2$ and $\text{th}_Z(C_n) = 2m = \lceil 2\sqrt{n} - 1 \rceil + 1$. In this case, construct a $\lfloor Z \rfloor$ forcing set B with $|B| = m$ and $\text{pt}_{\lfloor Z \rfloor}(C_n; B) \leq m - 1$ as follows. Draw C_n by arranging the vertices in an m by m array and adding the edges as in Figure 2.1. Let B be the set of vertices in the left column. Note that in each time step, every active vertex can force the vertex to its right to become blue (sometimes by a hop), so every vertex becomes blue one column at a time. Let $\mathcal{F}$ be the set of $\lfloor Z \rfloor$ forces of B obtained by this process. Clearly $|B| = m$ and $\text{pt}_{\lfloor Z \rfloor}(C_n; B) \leq \text{pt}_{\lfloor Z \rfloor}(C_n; \mathcal{F}) = m - 1$. Thus $\text{th}_{\lfloor Z \rfloor}(C_n) \leq 2m - 1 = \lceil 2\sqrt{n} - 1 \rceil$. □

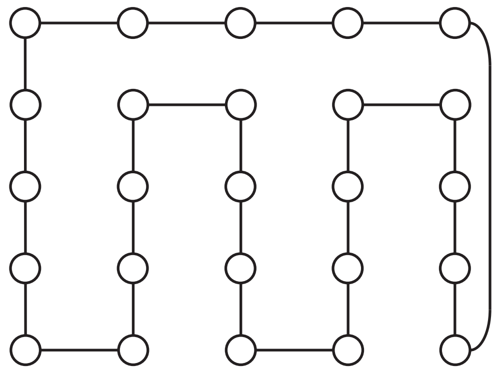
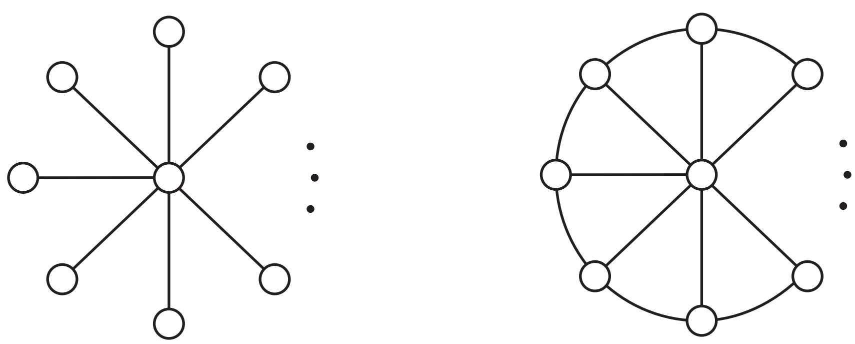


Figure 2.1 The cycle C_n with $n = m^2$ and $m = 5$.

Example 2.3.10 uses Theorem 2.3.5 to demonstrate that if $\text{th}_Z(G) > \lceil 2\sqrt{n} - 1 \rceil$, then $\text{th}_{\lfloor Z \rfloor}(G)$ can differ greatly from $\text{th}_Z(G)$.



$K_{1,n-1}$ W_{n-1}

Figure 2.2 The star on n vertices alongside the wheel as a spanning supergraph.

Example 2.3.10. Let G be the star $K_{1,n-1}$ on n vertices as shown on the left in Figure 2.2. Since $\mathrm{Z}(G) = n-2$, it can be verified by inspection that $\mathrm{th}_{\mathrm{Z}}(G) = n$. Consider the wheel W_{n-1} on n vertices as a spanning supergraph of G (shown on the right of Figure 2.2). Obtain $B \subseteq V(W_{n-1})$ by choosing the center vertex of the wheel and a set of vertices on the outside cycle that achieves optimal $\lfloor \mathrm{Z} \rfloor$ throttling for a cycle of order $n-1$. By Theorem 2.3.5, $\mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(G) \leq \mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(W_{n-1}) \leq \mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(C_{n-1}) + 1 \leq \lceil 2\sqrt{n-1} - 1 \rceil + 1$. Recall that $\mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(G) \geq \lceil 2\sqrt{n} - 1 \rceil$. Note that there are infinitely many integers n such that $\lceil 2\sqrt{n-1} - 1 \rceil + 1 = \lceil 2\sqrt{n} - 1 \rceil$. So in these cases, $\mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(G) = \lceil 2\sqrt{n} - 1 \rceil$.

The *largeur d'arborescence* of a graph was defined by Colin de Verdière in [7] to measure the width of trees. Note that largeur d'arborescence is french for tree width. The *largeur de chemin* of G, denoted by $\mathrm{lc}(G)$, was introduced in [2] as the analog of largeur d'arborescence that measures the width of paths. Formally, $\mathrm{lc}(G)$ is defined as the minimum k for which G is a minor of the Cartesian product $K_k \square P$ of a complete graph on k vertices with a path. The *proper path width* of a graph G, $\mathrm{ppw}(G)$, is the smallest k such that G is a partial linear k-tree (see [2]). These parameters are connected to $\lfloor \mathrm{Z} \rfloor$ by the following theorem.

Theorem 2.3.11. [2, Theorems 2.18 and 2.39] *For every graph G having at least one edge,* $\mathrm{lc}(G) = \mathrm{ppw}(G) = \lfloor \mathrm{Z} \rfloor(G)$.

It is known that proper path-width is equivalent to the mixed search number of a graph (see [10]). Since $\mathrm{ppw}(G) = \lfloor \mathrm{Z} \rfloor(G) \leq \mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(G)$ for any graph G, Theorem 2.3.11 connects $\lfloor \mathrm{Z} \rfloor$ throttling to mixed searching. Theorem 2.3.11 also exhibits a relationship between $\lfloor \mathrm{Z} \rfloor$ and graphs of the form $K_k \square P$. It is useful to capitalize on this relationship in order to characterize $\mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(G)$. For a given a graph G, the idea is to extend G by using a set of forces in G. The next definition constructs a graph from a given graph G, a standard zero forcing set $B \subseteq V(G)$, and a set of standard forces $\mathcal{F}$. This construction is illustrated in Figure 2.3.

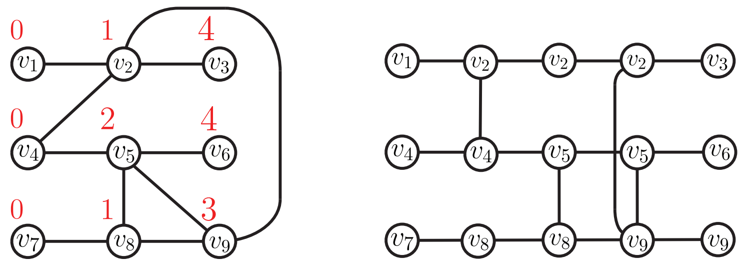
Definition 2.3.12. Let G be a graph and let $B \subseteq V(G)$ be a standard zero forcing set of G. Suppose $\mathcal{F}$ is a set of Z forces of B with $\mathrm{pt}_{\mathrm{Z}}(G; B) = \mathrm{pt}_{\mathrm{Z}}(G; \mathcal{F})$. Let $P_1, P_2, \ldots, P_{|B|}$ be the induced paths in

G formed by the maximal forcing chains of $\mathcal{F}$. For each vertex $v \in V(G)$, consider the path P_i that contains v and let $\tau(v)$ be the number of times in the propagation process of $\mathcal{F}$ at which v is active (possibly including time 0). Define the *(zero forcing) extension of G with respect to B and $\mathcal{F}$*, denoted $\mathcal{E}(G, B, \mathcal{F})$, to be the graph obtained by the following procedure.

1. From each path P_i in G, construct a new path P_i' so that for each $v \in P_i$, there are $\tau(v)$ copies of v in P_i', and for each pair $v_a, v_b \in P_i$ such that v_a is forced before v_b in P_i, every copy of v_a is to the left of every copy of v_b in P_i'. Note that for each $1 \leq i \leq |B|$, $|V(P_i')| = \text{pt}_Z(G; B) + 1$ and the paths $\{P_1', P_2', \ldots, P_{|B|}'\}$ can be arranged into a $|B|$ by $\text{pt}(G; B) + 1$ array of vertices.

2. For each edge $uv \in E(G) \setminus \bigcup_{i=1}^{|B|} E(P_i)$, suppose P_q and P_r are the paths that contain u and v respectively. Since u and v must both be active before u or v can perform a force in G, there is at least one column in the $|B|$ by $\text{pt}(G; B) + 1$ array such that a copy of u and a copy of v appear in that column. Draw an edge connecting the copy of u in P_q' and the copy of v in P_r' that are in the least such column.

Example 2.3.13. Let G be the graph shown on the left in Figure 2.3. Choose $B = \{v_1, v_4, v_7\}$ and let $\mathcal{F}$ be the set of standard forces $\mathcal{F} = \{v_1 \rightarrow v_2, v_2 \rightarrow v_3, v_4 \rightarrow v_5, v_5 \rightarrow v_6, v_7 \rightarrow v_8, v_8 \rightarrow v_9\}$. Note that the forces in $\mathcal{F}$ correspond to the horizontal edges in G as shown in Figure 2.3. The numbers above the vertices of G indicate the time step in $\mathcal{F}$ when that vertex is forced (making that vertex active at the next time in the propagation process). For example, $v_7 \rightarrow v_8$ in time step 1 and $v_8 \rightarrow v_9$ in time step 3. Since there are two times in $\mathcal{F}$ at which v_8 active, there are two copies of v_8 in $\mathcal{E}(G; B; \mathcal{F})$, which is shown on the right in Figure 2.3.



G $\qquad$ $\mathcal{E}(G;B;\mathcal{F})$

Figure 2.3 G, B, and $\mathcal{F}$ are illustrated alongside the extension $\mathcal{E}(G;B;\mathcal{F})$.

Consider the graph $G = K_a \square P_b$. Define the *path edges* of G to be the edges in each copy of P_b in the Cartesian product. Likewise, define the *complete edges* of G to be the edges in each copy of K_a in the Cartesian product. For example, if G is drawn so that $V(G)$ is arranged as an a by b array where each column induces a K_a and each row induces a P_b, then the path edges of G are the horizontal edges and the complete edges of G are the vertical edges. Given a graph G, an edge $e \in E(G)$, a standard zero forcing set $B \subseteq V(G)$, and a set $\mathcal{F}$ of standard forces in G that uses e to perform a force, the following definition constructs a standard zero forcing set in G/e and a set of standard forces in G/e that mimic B and $\mathcal{F}$ respectively.

Definition 2.3.14. Let G be a graph with standard zero forcing set $B \subseteq V(G)$ and suppose $\mathcal{F}$ is a set of forces of B. Let $e \in E(G)$ be an edge that is used to perform a force in $\mathcal{F}$. Define $v_1 \to v_2 \to \cdots \to v_k$ to be the maximal forcing chain of $\mathcal{F}$ that contains e. Note that $k \geq 2$. For each $1 \leq j \leq k-1$, let e_j be the edge $v_j v_{j+1}$ and let $\vec{e}_j$ denote the force $v_j \to v_{j+1}$. So $e = e_i$ for some $1 \leq i \leq k-1$. Define v_e to be the vertex in G/e obtained by contracting e in G and define the sets B/e and $\mathcal{F}/e$ as follows.

$$B/e = \begin{cases} (B \setminus \{v_i\}) \cup \{v_e\} & \text{if } i = 1, \\ B & \text{if } i > 1, \end{cases}$$

and

$$\mathcal{F}/e = \begin{cases} (\mathcal{F} \setminus \{\vec{e}_{i-1}, \vec{e}_i, \vec{e}_{i+1}\}) \cup \{v_{i-1} \to v_e, v_e \to v_{i+2}\} & \text{if } k > 2 \text{ and } 1 < i < k-1, \\ (\mathcal{F} \setminus \{\vec{e}_i, \vec{e}_{i+1}\}) \cup \{v_e \to v_{i+2}\} & \text{if } k > 2 \text{ and } i = 1, \\ (\mathcal{F} \setminus \{\vec{e}_{i-1}, \vec{e}_i\}) \cup \{v_{i-1} \to v_e\} & \text{if } k > 2 \text{ and } i = k-1, \\ \mathcal{F} \setminus \{\vec{e}_i\} & \text{if } k = 2. \end{cases}$$

Lemma 2.3.15 is used to prove Theorem 2.3.16 which exhibits a relationship between $\text{th}_{\lfloor Z \rfloor}$ and graphs of the form $K_a \square P_{b+1}$.

Lemma 2.3.15. *Let G be a graph. Suppose $B \subseteq V(G)$ is a standard zero forcing set of G with a set of standard forces $\mathcal{F}$. If $e = uv$ is an edge in $E(G)$ and $(u \to v) \in \mathcal{F}$, then $\mathcal{F}/e$ is a set of standard forces of B/e in G/e such that $\text{pt}_Z(G/e, \mathcal{F}/e) \leq \text{pt}_Z(G; \mathcal{F})$. Furthermore, if $\mathcal{F}$ and B satisfy $\text{pt}_Z(G; \mathcal{F}) = \text{pt}_Z(G; B)$ and $\text{th}_Z(G) = \text{th}_Z(G; B)$, then $\text{th}_Z(G/e) \leq \text{th}_Z(G)$.*

Proof. Let G be a graph with standard zero forcing set $B \subseteq V(G)$. Let $\mathcal{F}$ be a set of forces of B and suppose $e = uv \in E(G)$ is an edge that is used to perform a force in $\mathcal{F}$. Assume without loss of generality that $(u \to v) \in \mathcal{F}$. Proceed by induction on $\text{pt}_Z(G; \mathcal{F})$. If $\text{pt}_Z(G; \mathcal{F}) = 0$, then $B = V(G)$ and no such edge e exists and there is nothing to prove. Suppose $\text{pt}_Z(G; \mathcal{F}) = 1$. In this case, it is clear that $\mathcal{F}/e$ is a set of forces of B/e in G/e and $\text{pt}_Z(G/e; \mathcal{F}/e) \leq 1 = \text{pt}_Z(G; \mathcal{F})$.

Now suppose that for some $k \geq 1$, the result is true for any graph H and set of forces $\mathcal{Q}$ with $\text{pt}_Z(H; \mathcal{Q}) \leq k$. Again, let G be a graph with standard zero forcing set $B \subseteq V(G)$. Now, suppose $\mathcal{F}$ is a set of standard forces of B with $\text{pt}_Z(G; \mathcal{F}) = k + 1$. Let $e = uv$ be a given edge in G such that $(u \to v) \in \mathcal{F}$. Define $T(\mathcal{F})$ to be all vertices in G that are forced last in $\mathcal{F}$ (at time step $k + 1$). For all vertices $q \in T(\mathcal{F})$, let q' be the vertex in G that forces q at time step $k + 1$. Note that for any $q \in T(\mathcal{F})$ and any neighbor y of q in G with $y \neq q'$, y is also in $T(\mathcal{F})$. This is because if $y \notin T(\mathcal{F})$, then y cannot perform a force until q is forced. However, q is forced in time step $k + 1$ which implies that y forces in a time step greater than $\text{pt}_Z(G; \mathcal{F})$, and this is a contradiction. Suppose $uv = q'q$ for some $q \in T(\mathcal{F})$. Since $N(v) \setminus \{u\} \subseteq T(\mathcal{F})$, $\mathcal{F}/e$ is a set of forces of B/e in G/e such that $\text{pt}_Z(G/e; \mathcal{F}/e) \leq k + 1 = \text{pt}_Z(G; \mathcal{F})$.

Finally, suppose $u \to v$ in $\mathcal{F}$ at a time step less than $k+1$. Construct G/e by the following process. First, remove $T(\mathcal{F})$ from G to obtain $H = G - T(\mathcal{F})$. Next, contract e in H to obtain H/e. Finally, add $T(\mathcal{F})$ to H so that the neighborhood in H of each $q \in T(\mathcal{F})$ is the same as the neighborhood of q in G (except that there may be a $q \in T(\mathcal{F})$ such that $v_e \sim q$ in G/e whereas $v \sim q$ in G). Let $\mathcal{F}' = \mathcal{F} \setminus \{q' \to q \mid q \in T(\mathcal{F})\}$. Clearly $\mathrm{pt}_{\mathrm{Z}}(H;\mathcal{F}') \leq k$. So by the induction hypothesis, $\mathrm{pt}_{\mathrm{Z}}(H/e;\mathcal{F}'/e) \leq \mathrm{pt}_{\mathrm{Z}}(H;\mathcal{F}') \leq k$. When $T(\mathcal{F})$ is added to H/e and the set of forces $\mathcal{F}/e$ is considered instead of $\mathcal{F}'/e$, the propagation time will increase by at most 1. Thus, $\mathrm{pt}_{\mathrm{Z}}(G/e;\mathcal{F}/e) \leq \mathrm{pt}_{\mathrm{Z}}(H/e;\mathcal{F}'/e) + 1 \leq k+1 = \mathrm{pt}_{\mathrm{Z}}(G;\mathcal{F})$. Note that if $\mathcal{F}$ and B are chosen such that $\mathrm{pt}_{\mathrm{Z}}(G;\mathcal{F}) = \mathrm{pt}_{\mathrm{Z}}(G;B)$ and $\mathrm{th}_{\mathrm{Z}}(G) = \mathrm{th}_{\mathrm{Z}}(G;B)$, then

$$\mathrm{th}_{\mathrm{Z}}(G/e) \leq |B/e| + \mathrm{pt}_{\mathrm{Z}}(G/e;\mathcal{F}/e) \leq |B| + \mathrm{pt}_{\mathrm{Z}}(G;\mathcal{F}) = |B| + \mathrm{pt}_{\mathrm{Z}}(G;B) = \mathrm{th}_{\mathrm{Z}}(G). \square$$

Theorem 2.3.16. *Given a graph G and a positive integer t, $\mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(G) \leq t$ if and only if there exists integers $a \geq 1$ and $b \geq 0$ such that $a + b = t$ and G can be obtained from $K_a \square P_{b+1}$ by contracting path edges and deleting edges.*

Proof. First suppose $\mathrm{th}_{\lfloor \mathrm{Z} \rfloor}(G) \leq t$. Let H be a spanning supergraph of G such that H has a standard zero forcing set B with $\mathrm{th}_{\mathrm{Z}}(G;B) \leq t$. Let $\mathcal{F}$ be a set of Z forces of B in H such that $\mathrm{pt}_{\mathrm{Z}}(H;\mathcal{F}) = \mathrm{pt}_{\mathrm{Z}}(H;B)$. Let $a = |B|$, $b' = \mathrm{pt}_{\mathrm{Z}}(H;B) = \mathrm{th}_{\mathrm{Z}}(G;B) - a$, and $b = t - a$. Then $b' \leq b$ and

$$G \leq H \preceq \mathcal{E}(H, B, \mathcal{F}) \leq K_a \square P_{b'+1} \leq K_a \square P_{b+1}.$$

Note that by the construction of H and $\mathcal{E}(H, B, \mathcal{F})$, H can be obtained from $K_a \square P_{b+1}$ by contracting path edges. Then G can be obtained from H by deleting edges.

For the other direction, suppose $G' = K_a \square P_{b+1}$ with $a + b = t$ and G can be obtained from G' by contracting path edges and deleting edges. Choose $B' \subseteq V(G')$ such that B' induces a copy of K_a in G' that corresponds to an endpoint of P_{b+1}. Note that B' is a standard zero forcing set of G' with set of forces $\mathcal{F}'$ such that the set $\{uv \mid (u \to v) \in \mathcal{F}'\}$ is the set of path edges in G'. In other words, $\mathcal{F}'$ propagates along the path edges of G'. Also note that $\mathrm{pt}_{\mathrm{Z}}(G';\mathcal{F}') = b$ and $|B| = a$. Let D be a set of edges and let C be a set of path edges in G' such that G can be obtained from G' by first contracting the edges in C, then deleting the edges in D. Let H' be the graph obtained from

G' by contracting the edges in C. Note that $D \subseteq E(H')$. By repeated applications of Lemma 2.3.15, it is possible to obtain a standard zero forcing set $B \subseteq V(H')$ with set of forces $\mathcal{F}$ such $|B| \leq |B'|$ and $\text{pt}_Z(H';\mathcal{F}) \leq \text{pt}_Z(G';\mathcal{F}') = b$. Thus,

$$\text{th}_{\lfloor Z \rfloor}(H') \leq \text{th}_Z(H') \leq |B| + \text{pt}_Z(H';\mathcal{F}) \leq |B'| + \text{pt}_Z(G';\mathcal{F}') = a + b = t.$$

By Theorem 2.3.5, $\text{th}_{\lfloor Z \rfloor}(G) \leq \text{th}_{\lfloor Z \rfloor}(H') \leq t$. □

Note that if a fixed integer $t \geq 1$ is given, then the graphs that have $\lfloor Z \rfloor$ throttling number at most t are exactly the graphs given by Theorem 2.3.16. The following corollary is immediate from this observation.

Corollary 2.3.17. *If t is a fixed positive integer, then there are finitely many graphs with $\lfloor Z \rfloor$ throttling number equal to t.*

The next theorem uses Theorem 2.3.16 to show that $\text{th}_{\lfloor Z \rfloor}$ does not inherit the property of minor monotonicity from $\lfloor Z \rfloor$. Recall that the maximum degree of a graph G is denoted as $\Delta(G)$.

Theorem 2.3.18. *The $\lfloor Z \rfloor$ throttling number of a graph is not minor monotone.*

Proof. Consider the graph $K_3 \square P_3$ and let $B \subseteq V(K_3 \square P_3)$ be the three vertices in a copy of K_3 that corresponds to an endpoint of P_3. Since $\text{pt}_{\lfloor Z \rfloor}(K_3 \square P_3; B) \leq 2$, $\text{th}_{\lfloor Z \rfloor}(K_3 \square P_3) \leq 5$. Let G be the minor of $K_3 \square P_3$ shown on the left in Figure 2.4. The following argument shows that G cannot be obtained from $K_a \square P_{b+1}$ with $a + b = 5$ by contracting path edges and/or deleting edges. Since $|V(K_1 \square P_5)| = |V(K_5 \square K_1)| = 5 < 8 = |V(G)|$, G cannot be obtained from $K_1 \square P_5$ or $K_5 \square P_1$ without adding vertices. Note that $|V(K_2 \square P_4)| = |V(K_4 \square P_2)| = 8$ which means that contractions are not allowed in order to obtain G from those graphs. Since $\Delta(K_2 \square P_4) = 3$, $\Delta(K_4 \square P_2) = 4$, and $\Delta(G) = 5$, G cannot be obtained from those graphs by deleting edges. To obtain G from $K_3 \square P_3$ using the operations in Theorem 2.3.16, exactly one contraction of a path edge is required since $|V(G)| = 8$ and $|V(K_3 \square P_3)| = 9$. Note that by the symmetry of $K_3 \square P_3$, contracting any single path edge yields the same graph. Let G' be the graph obtained by contracting a path edge of $K_3 \square P_3$ shown in the middle of Figure 2.4. The degree sequences of G' and G are $(5, 4, 4, 3, 3, 3, 3, 3)$ and

$(5,3,3,3,3,3,3,3)$ respectively. Thus, the only possible way to delete edges in G' and obtain G is by deleting the edge between the two vertices of degree 4. Delete this edge from G' and let H be the resulting graph shown on the right in Figure 2.4. If v_1 and v_2 are the vertices of degree 5 in G and H respectively, then $H - v_2$ contains a 6-cycle and $G - v_1$ does not. Therefore, G is not isomorphic to H and G cannot be obtained from $K_a \square P_{b+1}$ with $a + b = 5$ by contracting path edges and/or deleting edges. By Theorem 2.3.16, this means that $\text{th}_{\lfloor Z \rfloor}(G) \geq 6$. Since $\text{th}_{\lfloor Z \rfloor}(K_3 \square P_3) \leq 5$, it follows that $\text{th}_{\lfloor Z \rfloor}$ is not minor monotone. $\square$

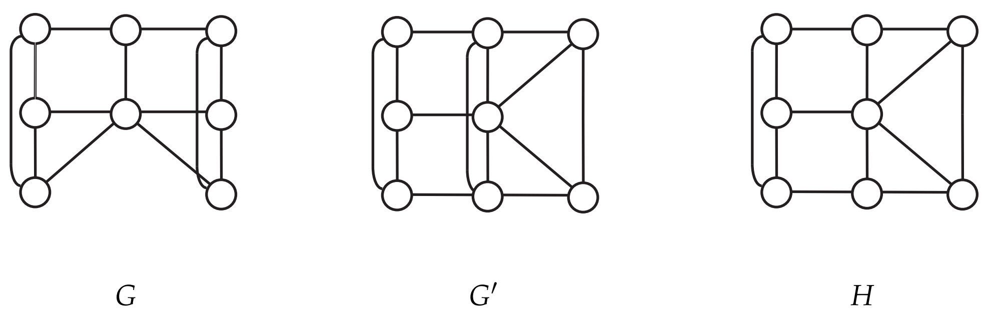


Figure 2.4 The graphs G, G', and H are minors of $K_3 \square P_3$ used in the proof of Theorem 2.3.18.

In the next section, the proof of Theorem 2.3.16 is modified in order to characterize standard throttling.

2.4 A characterization for standard throttling

Since there are graphs (e.g., stars) for which $\text{th}_Z \neq \text{th}_{\lfloor Z \rfloor}$, it is clear that the characterization in Theorem 2.3.16 does not also characterize th_Z. However, the only part of this characterization that does not work for standard throttling is the deletion of edges. In fact, Example 2.3.10 demonstrates that standard throttling is not spanning subgraph monotone. The next theorem shows how th_Z can be characterized by being more careful about which edges can be deleted.

Theorem 2.4.1. *Given a graph G and a positive integer t, $\operatorname{th}_Z(G) \le t$ if and only if there exists integers $a \ge 0$ and $b \ge 1$ such that $a + b = t$ and G can be obtained from $K_a \Box P_{b+1}$ by contracting path edges and deleting complete edges.*

Proof. First suppose $\operatorname{th}_Z(G) \le t$. Let $B \subseteq V(G)$ be a standard zero forcing set of G with $\operatorname{th}_Z(G; B) \le t$ and let $\mathcal{F}$ be a set of standard forces of B in G with $\operatorname{pt}_Z(G; \mathcal{F}) = \operatorname{pt}_Z(G; B)$. Let $a = |B|$, $b' = \operatorname{pt}_Z(G; B) = \operatorname{th}_Z(G; B) - a$, and $b = t - a$. Then $b' \le b$ and

$$G \preceq \mathcal{E}(G, B, \mathcal{F}) \le K_a \Box P_{b'+1} \le K_a \Box P_{b+1}.$$

Note that by the construction of $\mathcal{E}(G, B, \mathcal{F})$, G can be obtained from $K_a \Box P_{b+1}$ by contracting path edges and deleting complete edges.

For the other direction, suppose $G' = K_a \Box P_{b+1}$ with $a + b = t$ and G can be obtained from G' by contracting path edges and deleting complete edges. Choose $B' \subseteq V(G')$ such that B' induces a copy of K_a in G' that corresponds to an endpoint of P_{b+1}. Note that B' is a standard zero forcing set of G' with set of forces $\mathcal{F}'$ such that the set $\{uv \mid (u \to v) \in \mathcal{F}'\}$ is the set of path edges in G'. In other words, $\mathcal{F}'$ propagates along the path edges of G'. Also note that $\operatorname{pt}_Z(G'; \mathcal{F}') = b$ and $|B'| = a$. Let D be a set of complete edges in G' and let C be a set of path edges in G' such that G can be obtained from G' by first deleting the edges in D, then contracting the edges in C. Let H' be the graph obtained from G' by deleting the edges in D. Since no edge in D is used to perform a force in $\mathcal{F}'$, $\mathcal{F}'$ is still a set of forces of B' in H' with $\operatorname{pt}_Z(H'; \mathcal{F}') \le \operatorname{pt}_Z(G'; \mathcal{F}') = b$. Also, G can be obtained from H' by contracting the edges in C. By repeated applications of Lemma 2.3.15, it is possible to obtain a standard zero forcing set $B \subseteq V(G)$ with set of forces $\mathcal{F}$ such $|B| \le |B'|$ and $\operatorname{pt}_Z(G; \mathcal{F}) \le \operatorname{pt}_Z(H'; \mathcal{F}') \le b$. Thus,

$$\operatorname{th}_Z(G) \le |B| + \operatorname{pt}_Z(G; \mathcal{F}) \le |B'| + \operatorname{pt}_Z(H'; \mathcal{F}') \le |B'| + \operatorname{pt}_Z(G'; \mathcal{F}') = a + b = t. \square$$

Corollary 2.4.2. *If t is a fixed positive integer, then there are finitely many graphs G with standard throttling number equal to t.*

Suppose G is a graph on n vertices and t is a postive integer with $\text{th}_Z(G) \leq t$. Note that t can be used to bound the number of vertices in G. Since $\lceil 2\sqrt{n} - 1 \rceil \leq \text{th}_Z(G) \leq t$, $|V(G)| = n \leq \frac{(t+1)^2}{4}$. By Corollary 2.3.7, this bound still holds when $\text{th}_{\lfloor Z \rfloor}(G) \leq t$.

In order to construct forcing sets in paths and cycles that are optimal for throttling, it has been useful to "snake" the graph in some way. This idea was used for $\text{th}_Z(P_n)$ in [5], and again for $\text{th}_Z(C_n)$ in [6]. A "snaking" construction was also used for $\text{th}_{\lfloor Z \rfloor}(C_n)$ in Proposition 2.3.9 (see Figure 2.1). Note that in most of these cases, the "snaked" graph is a spanning subgraph or a minor of a graph of the form $K_a \square P_{b+1}$. It is interesting to observe that the "snaking" method is present in Theorems 2.3.16 and 2.4.1.

2.5 Extreme throttling

This section uses Theorems 2.3.16 and 2.4.1 to quickly characterize graphs with low throttling numbers. The connection between $\text{th}_{\lfloor Z \rfloor}$ and the independence number of a graph is also investigated. This connection is used to give a necessary condition for graphs G with $\text{th}_{\lfloor Z \rfloor}(G) = n$.

For a fixed positive integer t, Theorem 2.3.16 characterizes all graphs G with $\text{th}_{\lfloor Z \rfloor}(G) \leq t$. Clearly $\text{th}_{\lfloor Z \rfloor}(G) = t$ if and only if $\text{th}_{\lfloor Z \rfloor}(G) \leq t$ and $\text{th}_{\lfloor Z \rfloor}(G) \nleq t-1$. So all graphs with $\text{th}_{\lfloor Z \rfloor}(G) = t$ can be characterized by applying Theorem 2.3.16 and removing the graphs with $\lfloor Z \rfloor$ throttling number at most $t-1$. This is done by hand for $t \leq 3$ as follows.

Observation 2.5.1. *The graph $G = K_1$ is the only graph with $\text{th}_{\lfloor Z \rfloor}(G) = 1$.*

Proposition 2.5.2. *For a graph G, $\text{th}_{\lfloor Z \rfloor}(G) = 2$ if and only if $G = K_2$ or $G = 2K_1$.*

Proof. By Theorem 2.3.16, $\text{th}_{\lfloor Z \rfloor}(G) \leq 2$ if and only if G can be obtained from $K_1 \square P_2 = K_2$ or $K_2 \square P_1 = K_2$ by deleting edges and contracting path edges. Thus, $\text{th}_{\lfloor Z \rfloor}(G) \leq 2$ if and only if $G \in \{K_1, K_2, 2K_1\}$. Since $G = K_1$ is the only graph that satisfies $\text{th}_{\lfloor Z \rfloor}(G) = 1$, $\text{th}_{\lfloor Z \rfloor}(G) = 2$ if and only if $G \in \{K_2, 2K_1\}$. □

Proposition 2.5.3. *For a graph G, $\text{th}_{\lfloor Z \rfloor}(G) = 3$ if and only if $G \in \mathcal{G}$ where*

$$\mathcal{G} = \{C_4, P_4, 2K_2, K_1 \dot{\cup} P_3, K_2 \dot{\cup} 2K_1, 4K_1, K_3, P_3, K_1 \dot{\cup} K_2, 3K_1\}.$$

Proof. By Theorem 2.3.16, $\text{th}_{\lfloor Z \rfloor}(G) \leq 3$ if and only if G can be obtained from $K_3 \square P_1 = K_3$, $K_2 \square P_2 = C_4$, or $K_1 \square P_3 = P_3$ by deleting edges and contracting path edges. Let $\mathcal{H}$ be the set of all subgraphs of C_4 and K_3. It is clear that $\text{th}_{\lfloor Z \rfloor}(G) \leq 3$ if and only if $G \in \mathcal{H}$. Note that $\mathcal{G} = \mathcal{H} \setminus \{K_1, K_2, 2K_1\}$. □

Theorems 2.3.16 and 2.4.1 reinforce the fact that for any graph G, $\text{th}_{\lfloor Z \rfloor}(G) \leq \text{th}_Z(G)$. Let G be a graph. Since th_Z is bounded below by $\text{th}_{\lfloor Z \rfloor}$, if there is a subset $B \subseteq V(G)$ with $\text{th}_Z(G; B) = \text{th}_{\lfloor Z \rfloor}(G)$, then $\text{th}_Z(G) = \text{th}_{\lfloor Z \rfloor}(G)$.

Corollary 2.5.4. *If $t \in \{1, 2, 3\}$ and $G \notin \{K_1 \dot{\cup} P_3, K_2 \dot{\cup} 2K_1, 4K_1\}$, then $\text{th}_Z(G) = t$ if and only if $\text{th}_{\lfloor Z \rfloor}(G) = t$.*

Proof. Let $\mathcal{J} = \{K_1 \dot{\cup} P_3, K_2 \dot{\cup} 2K_1, 4K_1\}$. For each graph G with $\text{th}_{\lfloor Z \rfloor}(G) \leq 3$ and $G \notin \mathcal{J}$, it is possible to produce a standard zero forcing set $B \subseteq V(G)$ with $\text{th}_Z(G; B) = \text{th}_{\lfloor Z \rfloor}(G)$. If $G \in \mathcal{J}$, then $\text{th}_{\lfloor Z \rfloor}(G) = 3$, but $\text{th}_Z(G) = 4$ because forcing by a hop is no longer allowed. □

High $\lfloor Z \rfloor$ throttling values are harder to characterize. Clearly, $\text{th}_{\lfloor Z \rfloor}(K_n) = \text{th}_Z(K_n) = n$. Let $(K_n)_e$ be the complete graph on n vertices minus a single edge. It is also clear that $\text{th}_{\lfloor Z \rfloor}((K_n)_e) = \text{th}_Z((K_n)_e) = n$. More generally, $\text{th}_{\lfloor Z \rfloor}(G) = n$ implies that $\text{th}_Z(G) = n$. For a given graph G, the following proposition gives an upper bound for $\text{th}_{\lfloor Z \rfloor}(G)$ in terms of the independence number, $\alpha(G)$.

Proposition 2.5.5. *If G is a graph of order n, then $\text{th}_{\lfloor Z \rfloor}(G) \leq n - \alpha(G) + \left\lceil 2\sqrt{\alpha(G)} - 1 \right\rceil$.*

Proof. Suppose G is a graph with independent set $A \subseteq V(G)$. Let $B = V(G) \setminus A$. Note that $G - B$ has no edges and by Theorem 2.3.5, $\text{th}_{\lfloor Z \rfloor}(G - B) \leq \text{th}_{\lfloor Z \rfloor}(C_{|A|}) = \left\lceil 2\sqrt{|A|} - 1 \right\rceil$. Choose $C \subseteq A$ such that $\text{th}_{\lfloor Z \rfloor}(G - B, C) = \left\lceil 2\sqrt{|A|} - 1 \right\rceil$. Then $B \cup C$ is a $\lfloor Z \rfloor$ forcing set of G with $\text{pt}_{\lfloor Z \rfloor}(G; B \cup C) \leq \text{pt}_{\lfloor Z \rfloor}(G - B, C)$. Thus, $\text{th}_{\lfloor Z \rfloor}(G) \leq n - |A| + \left\lceil 2\sqrt{|A|} - 1 \right\rceil$. If A satisfies $|A| = \alpha(G)$, the desired result is obtained. □

Since $\alpha(K_{1,n-1}) = n - 1$, Example 2.3.10 shows that the bound in Proposition 2.5.5 is tight.

Corollary 2.5.6. *If G is a graph with $\text{th}_{\lfloor Z \rfloor}(G) = n$, then $\alpha(G) \leq 3$.*

Proof. Let G be a graph and define $f(x) = x - \lceil 2\sqrt{x} - 1 \rceil$. So Proposition 2.5.5 says that $\mathrm{th}_{\lfloor Z \rfloor}(G) \leq n - f(\alpha(G))$. If $x \geq 4$ is an integer, then $f(x) \geq 1$. So if $\alpha(G) \geq 4$, then $\mathrm{th}_{\lfloor Z \rfloor}(G) \leq n - f(\alpha(G)) \leq n - 1$. □

Note that the converse of Corollary 2.5.6 is false. For example, let $G = P_6$. Then $\alpha(G) = 3$ and $\mathrm{th}_{\lfloor Z \rfloor}(G) = \left\lceil 2\sqrt{6} - 1 \right\rceil = 4 < 6 = n$.

2.6 Concluding remarks

For a graph G and an integer k, define the Z FLOOR THROTTLING problem as the decision problem of determining whether $\mathrm{th}_{\lfloor Z \rfloor}(G) < k$. The complexity of Z FLOOR THROTTLING is an interesting question. Recall that for two graphs G_1 and G_2, the graph $G_1 \dot{\cup} G_2$ has vertex set and edge set equal to $V(G_1) \dot{\cup} V(G_2)$ and $E(G_1) \dot{\cup} E(G_2)$ respectively. For any graph G, let $X(G)$ be the set of subsets of $V(G)$ that are $\lfloor Z \rfloor$ forcing sets of G. A list of conditions is given in [3, Theorem 1] that would guarantee that Z FLOOR THROTTLING is NP-Complete. One of these conditions is that $X(G_1 \dot{\cup} G_2) = \{S_1 \dot{\cup} S_2 \mid S_1 \in X(G_1) \text{ and } S_2 \in X(G_2)\}$ for any two graphs G_1 and G_2. Due to hopping, this condition is not satisfied for $\lfloor Z \rfloor$ forcing sets. For example, let G_1 and G_2 each be the graph consisting of a single vertex labeled v_1 and v_2 respectively. Let $S_1 = \emptyset$ and $S_2 = \{v_2\}$. Note that $S_1 \dot{\cup} S_2$ is a $\lfloor Z \rfloor$ forcing set of $G_1 \dot{\cup} G_2$ since v_2 can force v_1 by a hop. However, S_1 is not a $\lfloor Z \rfloor$ forcing set of G_1. So the conditions given in [3, Theorem 1] cannot be used to prove that Z FLOOR THROTTLING is NP-Complete. It would be useful to have other tools to help determine the complexity of the Z FLOOR THROTTLING problem.

Corollary 2.5.6 states that a low independence number is necessary in order to achieve a maximum $\lfloor Z \rfloor$ throttling number. Another possible direction for future work is to completely characterize high $\lfloor Z \rfloor$ throttling numbers. It would also be interesting to determine the exact relationship between α and $\mathrm{th}_{\lfloor Z \rfloor}$. It is noted in [2, Remark 2.47] that for any graph G, $\lfloor Z_\ell \rfloor(G) \leq \lfloor Z \rfloor(G) \leq \lfloor Z_\ell \rfloor(G) + 1$. This motivates a comparison of $\mathrm{th}_{\lfloor Z \rfloor}$ and $\mathrm{th}_{\lfloor Z_\ell \rfloor}$. If $\mathrm{th}_{\lfloor Z \rfloor}$ and $\mathrm{th}_{\lfloor Z_\ell \rfloor}$ can be arbitrarily far apart, then studying $\lfloor Z_\ell \rfloor$ throttling may be of interest on its own.

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CHAPTER 3. CHARACTERIZATIONS OF THROTTLING FOR POSITIVE SEMIDEFINITE ZERO FORCING AND ITS MINOR MONOTONE FLOOR

A paper to be submitted for publication after further editing.

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Abstract

Zero forcing can be described as a combinatorial game that uses a color change rule to change the color of the vertices in a graph to blue. The throttling number of a graph minimizes the sum of the number of vertices initially colored blue and the number of time steps required to color the entire graph. Positive semidefinite (PSD) zero forcing is a commonly studied variant of standard zero forcing that alters the color change rule. This paper introduces a method for extending a graph using a PSD zero forcing process. Using this extension method, graphs with PSD throttling number at most t are characterized as specific minors of the Cartesian product of complete graphs and trees. A similar characterization is obtained for the minor monotone floor of PSD zero forcing. Finally, a new perspective on PSD zero forcing is described that aids the study of PSD extensions and throttling.

Keywords: Zero forcing, propagation time, throttling, minor monotone floor, positive semidefinite

AMS subject classification: 05C57, 05C15, 05C50

3.1 Introduction

Consider a process that requires initial resources and suppose that changing the initial resources can change the time it takes to complete the process. For a simple example, consider the process of spreading a rumor. The set of people who know the rumor initially are the initial resources and the time it takes for everyone to know the rumor is the completion time. The general idea of throttling is to balance the amount of initial resources with the completion time in order to make the process as effective as possible. Many of these kinds of processes can be described in the context of graph theory. An example of this is zero forcing in which an initial set of blue vertices and a color change rule are used to progressively change the color of all vertices in the graph to blue. Zero forcing was introduced in [1] as a way to bound the maximum nullity of a family of matrices corresponding to a given graph. Throttling for zero forcing was first studied by Butler and Young in [5]. Recently, the study of throttling has been expanded to include many variations of zero forcing in [4, 6, 7] and cops and robbers in [3].

The graphs in this paper are simple, finite, and undirected. If G is a graph, $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The edges of a graph can be denoted as subsets or by juxtaposition of the endpoints (i.e., uv is an edge if $\{u, v\} \in E(G)$). The order of a graph G is $|G| = |V(G)|$. The notation $G \leq H$ is used if G is a subgraph of H. If $G \leq H$ and $|G| = |H|$, then G is a spanning subgraph of H. If G is a minor of H, write $G \preceq H$. In the case that $H \leq G$, $H \leq G$ and $|H| = |G|$, or $H \preceq G$, it is said that G is a supergraph, spanning supergraph, or major of H, respectively. For a graph parameter p whose range is well-ordered, the *minor monotone floor of* p is defined as $\lfloor p \rfloor(G) = \min\{p(H) \mid G \preceq H\}$.

In [6], definitions are given that generalize throttling for zero forcing and many of its variants. In a graph whose vertices are white or blue, an *(abstract) color change rule* for zero forcing is a set of conditions that allow a vertex u to force a white vertex w to become blue. If R is the color change rule, it is said that u R forces w to become blue. The R can be dropped if the rule is clear from context, and forces are denoted by $u \to w$. Let R be a given color change rule and let G be a graph with $B \subseteq V(G)$ colored blue and $V(G) \setminus B$ colored white. A *chronological list of* R *forces of* B is an ordered list of valid R forces that can be performed consecutively in G resulting in a coloring in which no more R forces are possible. After all forces in a chronological list have been performed, the resulting set of blue vertices in G is an *R final coloring of B*. The set B is an *R forcing set of G* if $V(G)$ is an R final coloring of G for some chronological list of forces of B. The minimum size of an R forcing set of G is the *R forcing parameter* of G and is denoted by $R(G)$.

The set of forces in a particular chronological list of R forces of B is called a *set of R forces of B*. Sets of forces are used to define propagation time. If $\mathcal{F}$ is a set of R forces of B, $\mathcal{F}^{(0)} = B$ and for each $t \geq 0$, $\mathcal{F}^{(t+1)}$ is the set of vertices w such that $(u \to w) \in \mathcal{F}$ for some $u \in V(G)$ and u can R force w in G if $\bigcup_{i=0}^{t} \mathcal{F}^{(i)}$ is colored blue and $V(G) \setminus \bigcup_{i=0}^{t} \mathcal{F}^{(i)}$ is colored white. The smallest t' such that $\bigcup_{i=0}^{t'} \mathcal{F}^{(i)} = V(G)$ is the *R propagation time of $\mathcal{F}$ in G* and is denoted by $\mathrm{pt}_R(G; \mathcal{F})$. Note that if the forces in $\mathcal{F}$ do not eventually color every vertex in $V(G)$ blue, then $\mathrm{pt}_R(G; \mathcal{F}) = \infty$. The propagation process of $\mathcal{F}$ breaks $\mathcal{F}$ into time steps that transition from one time to the next by progressively changing the color of vertices in G to blue. At time $t = 0$, B is blue and $V(G) \setminus B$ is white. For each $t \geq 1$, time step t starts at time $t - 1$ with $\bigcup_{i=0}^{t-1} \mathcal{F}^{(i)}$ colored blue and performs every possible force in $\mathcal{F}$ at that time (transitioning to time t by coloring $\mathcal{F}^{(t)}$ blue). For a set $B \subseteq V(G)$, the *R propagation time of B in G* is defined as $\mathrm{pt}_R(G; B) = \min\{\mathrm{pt}_R(G; \mathcal{F}) \mid \mathcal{F} \text{ is a set of } R \text{ forces of } B\}$. The *$R$ throttling number of B* is $\mathrm{th}_R(G; B) = |B| + \mathrm{pt}_R(G; B)$ and the *R throttling number of a graph G* is $\mathrm{th}_R(G) = \min\{\mathrm{th}_R(G; B) \mid B \subseteq V(G)\}$.

The *(standard) zero forcing color change rule*, denoted Z, is that a blue vertex u can force a white vertex w to become blue if w is the only white neighbor of u. If G is a graph, $\mathrm{Z}(G)$ is the zero forcing parameter for Z and Z forces are also called "standard" forces. It is shown in [2] that the minor monotone floor of Z, $\lfloor \mathrm{Z} \rfloor$, can be described as a zero forcing parameter. In this variant, vertices becomes *active* when they become blue and they become *inactive* after they perform a force. The *$\lfloor \mathrm{Z} \rfloor$ color change rule* is that an active blue vertex $u \in V(G)$ can force a white vertex $w \in V(G)$ to become blue if u has no white neighbors in $V(G) \setminus \{w\}$. Note that if w is the only white neighbor of u in G, then the force $u \to w$ is a standard force. If u has no white neighbors in G, u is said to force w by *hopping*. So a $\lfloor \mathrm{Z} \rfloor$ force is either a Z force or a force by hopping. In [6], certain minors of the Cartesian product of a complete graph and a path are shown to characterize the graphs G that have $\text{th}_Z(G)$ at most t for any fixed positive integer t. An analogous characterization is also shown for $\lfloor Z \rfloor$ throttling. The proofs of these characterizations use a method of extending a given graph G into a major of G by considering a set of forces.

Suppose G is a graph and $B \subset V(G)$ is the set of vertices in G that are colored blue. Let $W_1, W_2, \ldots, W_k$ be the sets of white vertices in the k connected components of $G - B$. The *positive semidefinite (PSD) zero forcing color change rule*, denoted Z_+, is that a blue vertex $u \in B$ can force a white vertex w to become blue if w is the only white neighbor of u in $G[B \cup W_i]$ for some $1 \leq i \leq k$. PSD zero forcing can be thought of as standard zero forcing in each $G[B \cup W_i]$ and Z_+ forces are also called *PSD forces*. PSD propagation and throttling were studied in [9] and [7] before the introduction of the general definitions in [6]. Consistent with the original literature, Z_+ propagation and throttling are denoted by pt_+ and th_+, respectively.

If $\mathcal{F}$ is a set of PSD forces of B, then for each $u \in B$, V_u is the set of vertices w such that there is a sequence of vertices $u = v_1, v_2, \ldots, v_\ell = w$ with $(v_i \to v_{i+1}) \in \mathcal{F}$ for each $1 \leq i \leq \ell - 1$. The induced subgraph $T_u(\mathcal{F}) = G[V_u]$ is a *forcing tree of* $\mathcal{F}$. If k is a positive integer, a *k-ary tree* is a rooted tree such that every vertex either has k children or is a leaf. In Section 2, an extension technique is defined for PSD zero forcing and it is used to characterize all graphs G with $\text{th}_+(G) \leq t$ as certain minors of the Cartesian product of a complete graph and a k-ary tree. Section 3 gives a similar characterization for a variant of PSD zero forcing that uses hopping (called the *minor monotone floor of* Z_+). A more natural perspective of PSD zero forcing is introduced in Section 3.4 and directions for future work are outlined in Section 3.5.

3.2 Throttling positive semidefinite zero forcing

In this section, a technique is given for extending a graph using a PSD zero forcing process that generalizes the extension for standard zero forcing in [6, Definition 3.12]. This extension is used to obtain a characterization for all graphs G that satisfy $\text{th}_+(G) \leq t$ for any fixed positive integer t. The following definitions are useful for describing the PSD extension process.

Suppose G is a graph and $B \subseteq V(G)$ is a PSD zero forcing set of G. Let $\mathcal{F}$ be a set of PSD forces of B with $\text{pt}_+(G;\mathcal{F}) = \text{pt}_+(G;B)$ and for each $0 \leq t \leq \text{pt}_+(G;\mathcal{F})$, let $B^{[t]} = \bigcup_{i=0}^{t} \mathcal{F}^{(i)}$ be the set of blue vertices in G at time t. Also for each $1 \leq i \leq \text{pt}_+(G;\mathcal{F})$, define $C_i(\mathcal{F})$ to be the set of components of $G - B^{[i-1]}$ and label the components in $C_i(\mathcal{F})$ as $W_{i,1}, W_{i,2}, \ldots, W_{i,|C_i(\mathcal{F})|}$. For each $1 \leq i \leq \text{pt}_+(G;B)$ and $1 \leq j \leq |C_i(\mathcal{F})|$, define $G_{i,j}(\mathcal{F})$ to be the graph $G[B^{[i]} \cup W_{i,j}]$. Note that a Z_+ force that occurs in the j^{th} component of $C_i(\mathcal{F})$ during the i^{th} time step is a Z force in $G_{i,j}(\mathcal{F})$. The next example illustrates these definitions and is used throughout this paper.

Example 3.2.1. Let G be the graph shown in the top-left of Figure 3.1 with $V(G) = \{1, 2, \ldots, 7\}$. Choose $B = \{1, 2\}$ and $\mathcal{F} = \{1 \to 7, 2 \to 5, 2 \to 3, 1 \to 4, 5 \to 6\}$. Note that B is PSD zero forcing set of G and $\mathcal{F}$ is a set of PSD forces of B with $\text{pt}_+(G;\mathcal{F}) = 2$. The components in $C_1(\mathcal{F})$ are labeled as $W_{1,1}$ and $W_{1,2}$. The graph $G_{1,1}(\mathcal{F})$ is shown in the top-middle of Figure 3.1 and $G_{1,2}(\mathcal{F})$ is shown in the top-right. In the first time step of $\mathcal{F}$, the forces $2 \to 5$, $1 \to 7$, and $2 \to 3$ are performed simultaneously. This is illustrated in the bottom-left of Figure 3.1 alongside the graphs $G_{2,1}(\mathcal{F})$ and $G_{2,2}(\mathcal{F})$ in the bottom-middle and bottom-right, respectively. Note that in the second time step of $\mathcal{F}$, $1 \to 4$ and $5 \to 6$.

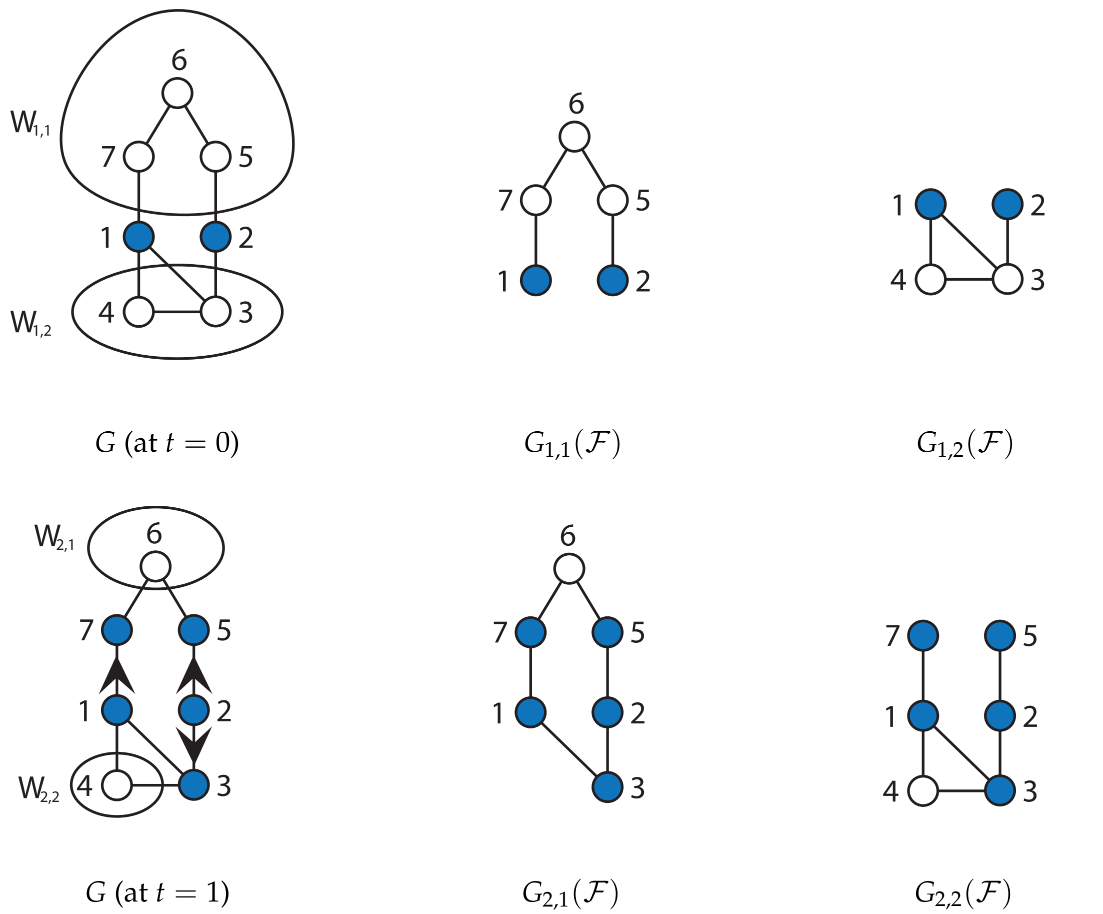

Figure 3.1 The graph G is shown at time $t = 0$ and $t = 1$ alongside the graphs $\{G_{i,j}(\mathcal{F})\}_{i,j=1}^{2}$.

If $\{a_k\}_{k=1}^{m}$ is a sequence of positive integers, let $T(a_1, a_2, \ldots, a_m)$ denote the rooted tree such that for each $0 \leq k \leq m-1$, the vertices at distance k from the root each have a_{k+1} children. The next definition (illustrated in Example 3.2.3) constructs a copy of $T(|C_1(\mathcal{F})|, |C_2(\mathcal{F})|, \ldots, |C_{\mathrm{pt}_+(G;\mathcal{F})}(\mathcal{F})|)$ from a forcing tree of $\mathcal{F}$.

Definition 3.2.2. Suppose G is a graph and $\mathcal{F}$ is a set of PSD forces of a PSD zero forcing set B of G with $\mathrm{pt}_+(G;\mathcal{F}) = \mathrm{pt}_+(G;B)$. For each $u \in B$, define $\mathcal{E}_u(\mathcal{F})$ to be a copy of the graph $T(|C_1(\mathcal{F})|, |C_2(\mathcal{F})|, \ldots, |C_{\mathrm{pt}_+(G;\mathcal{F})}(\mathcal{F})|)$ whose vertices and edges are labeled as follows.

1. Let $\mathcal{E}_u(\mathcal{F}) = T(|C_1(\mathcal{F})|, |C_2(\mathcal{F})|, \ldots, |C_{\mathrm{pt}_+(G;\mathcal{F})}(\mathcal{F})|)$ and let D_i denote the vertices that are distance i from the root of $\mathcal{E}_u(\mathcal{F})$. For each $1 \leq i \leq \mathrm{pt}_+(G;\mathcal{F})$ and vertex $v \in D_{i-1}$, label the edges of $\mathcal{E}_u(\mathcal{F})$ that connect v to its children as ordered pairs $(i,1), (i,2), \ldots (i, |C_i(\mathcal{F})|)$.

2. Label the root of $\mathcal{E}_u(\mathcal{F})$ as u. Suppose $v \in V(T_u(\mathcal{F}))$ is forced in $\mathcal{F}$ during time step $i > 0$. Let $\{c_n\}_{n=1}^{i}$ be a sequence of indices such that for each $1 \leq n \leq i$, the graph $G_{n,c_n}(\mathcal{F})$ contains v. Label as v the unique vertex in $\mathcal{E}_u(\mathcal{F})$ that is distance i from u and is obtained by starting at u and following the edges labeled $(1, c_1), (2, c_2), \ldots, (i, c_i)$.

3. Finally, give each unlabeled vertex in $\mathcal{E}_u(\mathcal{F})$ the label of its parent recursively.

Example 3.2.3. Let G, B, and $\mathcal{F}$ be given as in Example 3.2.1. Since $|C_1(\mathcal{F})| = |C_2(\mathcal{F})| = 2$, the edges of $\mathcal{E}_1(\mathcal{F})$ and $\mathcal{E}_2(\mathcal{F})$ are labeled as shown in the left column of Figure 3.2. Vertex 7 is forced in the first time step and is contained in $G_{1,1}(\mathcal{F})$. Vertex 4 is forced in the second time step and is contained chronologically in $G_{1,2}(\mathcal{F})$ and $G_{2,2}(\mathcal{F})$. Vertices 5 and 3 are forced in the first time step and are contained in $G_{1,1}(\mathcal{F})$ and $G_{1,2}(\mathcal{F})$, respectively. Vertex 6 is forced in the second time step of $\mathcal{F}$ and is contained chronologically in $G_{1,1}(\mathcal{F})$ and $G_{2,1}(\mathcal{F})$. Steps 2 and 3 of Definition 3.2.2 are shown in the middle and right columns of Figure 3.2.

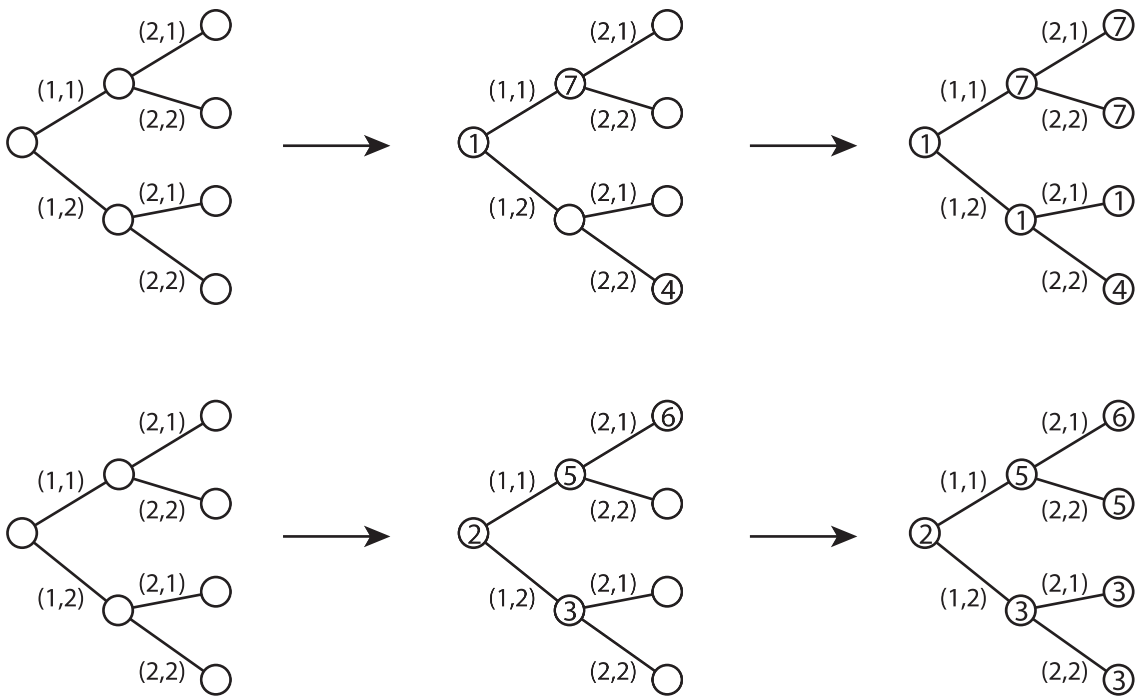

Figure 3.2 The trees $\mathcal{E}_1(\mathcal{F})$ and $\mathcal{E}_2(\mathcal{F})$ are constructed in the top and bottom row, respectively.

The next proposition concerns edges that are not contained in the forcing trees of a PSD zero forcing process.

Proposition 3.2.4. *Let G be a graph with PSD zero forcing set $B \subseteq V(G)$ and let $\mathcal{F}$ be a set of PSD forces of B with $\operatorname{pt}_+(G;\mathcal{F}) = \operatorname{pt}_+(G;B)$. Suppose $v_1v_2 \in E(G)$ is not in any forcing tree of $\mathcal{F}$ and choose $u_1, u_2 \in B$ such that $v_1 \in T_{u_1}(\mathcal{F})$ and $v_2 \in T_{u_2}(\mathcal{F})$. If $\{v_1, v_2\} \not\subseteq B$, then there exists a sequence $\{(a_i, b_i)\}_{i=1}^{j}$ of edge labels such that for each $k \in \{1, 2\}$, the path in $\mathcal{E}_{u_k}(\mathcal{F})$ obtained by starting at the root and following the edges labeled $(a_1, b_1), (a_2, b_2), \ldots, (a_j, b_j)$ leads to a copy of vertex v_k.*

Proof. Since $\{v_1, v_2\} \not\subseteq B$, at least one of v_1 and v_2 is white at time $t = 0$. Suppose the first time at which both v_1 and v_2 are blue is time $t = j > 0$. Without loss of generality, suppose v_2 is forced during time step j. Let u_2' be the copy of u_2 that is the root of $\mathcal{E}_{u_2}(\mathcal{F})$ and let v_2' be the copy of v_2 in $\mathcal{E}_{u_2}(\mathcal{F})$ that is distance j from u_2' (i.e., closest to u_2'). Suppose $\{(a_i, b_i)\}_{i=1}^{j}$ is the sequence of edge labels in $\mathcal{E}_{u_2}(\mathcal{F})$ along the path from u_2' to v_2'. Since $v_1v_2 \in E(G)$ and v_2 remains white in G during the first $j-1$ time steps of $\mathcal{F}$, v_1 and v_2 are both contained chronologically in the graphs $G_{a_1,b_1}(\mathcal{F}), G_{a_1,b_1}(\mathcal{F}), \ldots G_{a_j,b_j}(\mathcal{F})$. Note that by the PSD color change rule, v_1 cannot perform a force in $G_{a_j,b_j}(\mathcal{F})$ until both v_1 and v_2 are colored blue, which happens first in time step j. Therefore, starting at the root u_1 in $\mathcal{E}_{u_1}(\mathcal{F})$ and following the edges labeled $(a_1,b_1),(a_2,b_2),\ldots,(a_j,b_j)$ leads to a copy of v_1. □

The next definition gives the full description of the extension of a graph using a set of PSD forces. Propositon 3.2.4 guarantees that the definition is well-defined.

Definition 3.2.5. Suppose G is a graph and $\mathcal{F}$ is a set of PSD forces of a PSD zero forcing set $B \subseteq V(G)$ such that $\text{pt}_+(G;\mathcal{F}) = \text{pt}_+(G;B)$. For each edge $uv \in E(G)$, let $t(uv)$ denote the earliest time in $\mathcal{F}$ at which both u and v are blue. Define the *(PSD) extension of G with respect to B and $\mathcal{F}$*, denoted $\mathcal{E}_+(G;B;\mathcal{F})$, to be the graph obtained by the following procedure.

1. Start with $T_1 = \dot{\bigcup}\{\mathcal{E}_b(\mathcal{F}) \mid b \in B\}$. For each $v \in V(T_1)$, let $r(v)$ be the root of the tree in T_1 that contains v.

2. For each edge $v_1v_2 \in E(G)$ with $v_1, v_2 \in B$, add to T_1 the edge that connects the root of $\mathcal{E}_{v_1}(\mathcal{F})$ to the root of $\mathcal{E}_{v_2}(\mathcal{F})$. Call the resulting graph T_2.

3. For each edge $v_1v_2 \in E(G)$ with $\{v_1,v_2\} \nsubseteq B$ that is not in any forcing tree of $\mathcal{F}$, add to T_2 the edge that connects the copies of v_1 and v_2 that are distance $t(v_1v_2)$ away from the roots in $\mathcal{E}_{r(v_1)}(\mathcal{F})$ and $\mathcal{E}_{r(v_2)}(\mathcal{F})$, respectively.

Example 3.2.6. Let G, B, and $\mathcal{F}$ be given as in Example 3.2.1. The forcing tree $T_1(\mathcal{F})$ has vertex 1 as a root and vertex 1 has two children (namely, vertices 4 and 7). The forcing tree $T_2(\mathcal{F})$ has vertex 2 as a root with children $\{5,3\}$ and vertex 5 is the parent of vertex 6. On the left of Figure 3.3, G is shown with $T_1(\mathcal{F})$ drawn above $T_2(\mathcal{F})$. The sequence of edge labels guaranteed by Proposition 3.2.4 for the set of vertices $\{3,4\} \nsubseteq B$ is $\{(1,2),(2,2)\}$. For $\{6,7\} \nsubseteq B$, the sequence of edge labels is $\{(1,1),(2,1)\}$ and for $\{1,3\} \nsubseteq B$, the sequence of edge labels is $\{(1,2)\}$. The extension $\mathcal{E}_+(G;B;\mathcal{F})$ is shown on the right of Figure 3.3.

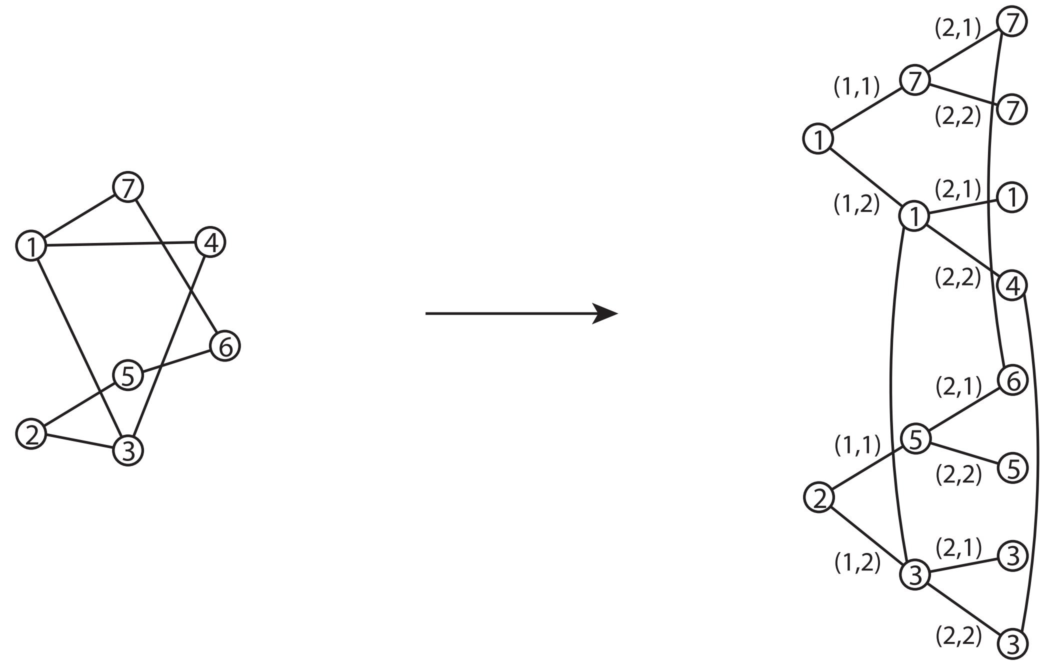

Figure 3.3 A graph G (left) is shown before and after its extension $\mathcal{E}_+(G;B;\mathcal{F})$ (right).

Note that for any graph G, PSD zero forcing set $B \subseteq V(G)$, and set of PSD forces of $\mathcal{F}$ of B with $\mathrm{pt}_+(G;\mathcal{F}) = \mathrm{pt}_+(G;B)$, $\mathcal{E}_+(G;B;\mathcal{F})$ is a spanning subgraph of the Cartesian product of $K_{|B|}$ and $T(|C_1(\mathcal{F})|, |C_2(\mathcal{F})|, \ldots, |C_{\mathrm{pt}_+(G;\mathcal{F})}(\mathcal{F})|)$. This observation is useful for proving Theorem 3.2.9.

Remark 3.2.7. In the case where $|C_i(\mathcal{F})| = 1$ for each $1 \leq i \leq \mathrm{pt}_+(G;\mathcal{F})$, $\mathcal{F}$ is a set of standard forces. In this case, $\mathcal{E}_+(G, B, \mathcal{F})$ is equal to the extension $\mathcal{E}(G, B, \mathcal{F})$ that is defined for standard zero forcing in [6, Definition 3.12]. Thus, Definition 3.2.5 generalizes the extension given in [6] to PSD zero forcing.

Lemma 3.2.8. *If G is a graph, $B \subseteq V(G)$ is as PSD zero forcing set of G, and $\mathcal{F}$ is a set of PSD forces of B with* $\mathrm{pt}_+(G;\mathcal{F}) = \mathrm{pt}_+(G;B)$, *then contracting an edge in a forcing tree of $\mathcal{F}$ does not increase the PSD propagation time of $\mathcal{F}$.*

Proof. Consider induction on $\mathrm{pt}_+(G;\mathcal{F})$. If $\mathrm{pt}_+(G;\mathcal{F}) = 0$, then there are no edges in any forcing tree of $\mathcal{F}$ and Lemma 3.2.8 is vacuously true. Assume Lemma 3.2.8 holds for any G' and $\mathcal{F}'$ with

$0 \leq \mathrm{pt}_+(G'; \mathcal{F}') \leq t-1$ and suppose that G and $\mathcal{F}$ satisfy $\mathrm{pt}_+(G; \mathcal{F}) = t$. It is shown in the proof of [6, Lemma 3.15] that in standard zero forcing, a vertex v that is forced in the last time step can only be adjacent to the vertex that forced v and vertices that do not perform a force. Therefore, if $1 \leq j \leq |C_t(\mathcal{F})|$ and $v \in G_{t,j}(\mathcal{F})$ is forced during time step t in the component $W_{t,j}$, then $N(v)$ consists entirely of vertices in $W_{t,j}$ that are leaves of a forcing tree of $\mathcal{F}$. So if e is an edge that is used to perform a PSD force in $\mathcal{F}$ during time step t, then contracting e does not increase the PSD propagation time of $\mathcal{F}$.

Now suppose $e = uv$ is an edge such that $u \to v$ in time step i of $\mathcal{F}$ for some $i < t$. Label the vertics of G as $v_1, v_2, \ldots, v_{|G|}$ and let G/e be the graph obtained from G by contracting e and labeling as v the new vertex that is formed as a result of the contraction. Let S be the set of vertices in G that are forced last in $\mathcal{F}$. Obtain the graph G/e as follows. First, delete the vertices in S from G. Next, contract the edge e. Finally, add the vertices in S back to the graph preserving the original neighborhood of each vertex in S. Note that $\mathrm{pt}_+(G - S; \mathcal{F}) \leq t-1$. So by the induction hypothesis, the PSD propagation time of $\mathcal{F}$ after contracting e is also at most $t-1$. The vertices in S are added back to the graph at the end of the forcing trees and each vertex in S will be come blue simultaneously in the final time step. Therefore, $\mathrm{pt}_+(G/e; \mathcal{F}) \leq t - 1 + 1 = t$. □

Recall that the depth of a vertex v in a rooted tree T is the distance from v to the root and the height of T is the maximum depth of the vertices in T. For integers $k > 0$ and $b \geq 0$, let $T_{k,b}$ denote the rooted tree of height b such that every vertex of depth less than b has k children. If G is a graph of the form $K_a \square T_{k,b}$, define the *tree edges* of G to be the edges in each copy of $T_{k,b}$ in the Cartesian product. Likewise, define the *complete edges* of G to be the edges in each copy of K_a in the Cartesian product. Similar to standard throttling, the extension in Definition 3.2.5 can be used to give a structural characterization of graphs with a given PSD throttling number.

Theorem 3.2.9. *Suppose G is a graph and t is a fixed positive integer. Then $\mathrm{th}_+(G) \leq t$ if and only if there exists integers $a, k > 0$ and $b \geq 0$ such that $a + b = t$ and G can be obtained from $K_a \square T_{k,b}$ by contracting tree edges and/or deleting complete edges.*

Proof. Suppose $\text{th}_+(G) = t' \leq t$. Let $\mathcal{F}$ be a set of PSD forces of a PSD zero forcing set $B \subseteq V(G)$ with $\text{pt}_+(G;\mathcal{F}) = \text{pt}_+(G;B) = b'$. Choose $k = \max\{|C_i(\mathcal{F})| \mid 1 \leq i \leq b'\}$ and $a = |B|$. Then $\mathcal{E}_+(G;B;\mathcal{F})$ can be obtained from $K_a \square T_{k,b'}$ by contracting tree edges and/or deleting complete edges. Note that G can be obtained from $\mathcal{E}_+(G;B;\mathcal{F})$ by contracting the tree edges whose endpoints have the same label. Finally, if $b = t - a$, then $K_a \square T_{k,b'}$ can be obtained from $K_a \square T_{k,b}$ by contracting tree edges and $a + b = t$.

Now suppose G can be obtained from $K_a \square T_{k,b}$ by contracting tree edges and/or deleting complete edges. Let B be the vertices in the copy of K_a that corresponds to the root of $T_{k,b}$. Choose $\mathcal{F}$ to be the set of PSD forces of B obtained by having each vertex in every copy of $T_{k,b}$ force each of its children in that copy. Note that $\text{pt}_+(G;\mathcal{F}) = b$ because no vertex is required to wait for multiple time steps in order to perform a force. This means that $\text{th}_+(K_a \square T_{k,b}) \leq a + b$. The tree edges of $K_a \square T_{k,b}$ are exactly the edges used in the forcing trees of $\mathcal{F}$. By Lemma 3.2.8, contracting these edges does not increase the PSD propagation time of $\mathcal{F}$. Since the complete edges of $K_a \square T_{k,b}$ are not in any forcing tree of $\mathcal{F}$, deleting these edges does not increase the PSD propagation time of $\mathcal{F}$. Thus, if G is obtained from $K_a \square T_{k,b}$ by contracting tree edges and/or deleting complete edges, then $\text{th}_+(G) \leq a + b$. □

It is shown in [7] that if T' and T are trees with $T' \leq T$, then $\text{th}_+(T') \leq \text{th}_+(T)$ (i.e., the PSD throttling number is subtree monotone). This result can be extended to minors of trees as an immediate consequence of Theorem 3.2.9.

Corollary 3.2.10. *If T' and T are trees with $T' \preceq T$, then* $\text{th}_+(T') \leq \text{th}_+(T)$.

In Section 3.3, Theorem 3.2.9 is used to quickly obtain a similar characterization for a variant of PSD throttling.

3.3 Throttling the minor monotone floor of PSD zero forcing

This section considers throttling for a variant of PSD zero forcing that allows hopping in each component. Let G be a graph with $B \subseteq V(G)$ colored blue and $V(G) \setminus B$ colored white. Let

$W_1, W_2, \ldots, W_k$ be the sets of white vertices in each connected component of $G - B$. For each $1 \leq i \leq k$, let $A_i \subseteq B$ be the set of vertices that are considered "active" with respect to W_i. The $\lfloor Z_+ \rfloor$ *color change rule* is that if $u \in A_i$, $w \in W_i$, and every neighbor of u in $G[W_i \cup B] - w$ is blue, then u can force w to become blue. (Note that if w is the only white neighbor of u in $G[B \cup W_i]$, then $u \to w$ is a Z_+ force. Otherwise, u has no white neighbors in $G[B \cup W_i]$ and $u \to w$ by hopping.) After $u \to w$, u is removed from A_i and w becomes active with respect to W_i.

It is shown in [2] that the minor monotone floor of Z_+ of a graph G (denoted $\lfloor Z_+ \rfloor(G)$) can be defined as the R forcing parameter, $R(G)$, where R is the $\lfloor Z_+ \rfloor$ color change rule. This allows for the study of $\lfloor Z_+ \rfloor$ propagation time and $\lfloor Z_+ \rfloor$ throttling. Since every PSD zero forcing set B of a graph G is also a $\lfloor Z_+ \rfloor$ forcing set of G with $\mathrm{pt}_{\lfloor Z_+ \rfloor}(G; B) \leq \mathrm{pt}_+(G; B)$, $\mathrm{th}_{\lfloor Z_+ \rfloor}(G) \leq \mathrm{th}_+(G)$. In [6, Corollary 3.6], it is shown that for a graph G and subset $B \subseteq V(G)$, $\mathrm{th}_{\lfloor Z_+ \rfloor}(G; B) = \min\{\mathrm{th}_+(H; B)\}$ where H ranges over all spanning supergraphs of G. This leads to an analogous fact for the $\lfloor Z_+ \rfloor$ throttling number of a graph.

Corollary 3.3.1. *If G is a graph, then* $\mathrm{th}_{\lfloor Z_+ \rfloor}(G) = \min\{\mathrm{th}_+(H) \mid G \leq H \text{ and } |G| = |H|\}$.

Proof. Choose a subset $B \subseteq V(G)$ and a set $\mathcal{F}$ of $\lfloor Z_+ \rfloor$ forces of B such that $\mathrm{pt}_{\lfloor Z_+ \rfloor}(G; \mathcal{F}) = \mathrm{pt}_{\lfloor Z_+ \rfloor}(G; B)$ and $\mathrm{th}_{\lfloor Z_+ \rfloor}(G) = \mathrm{th}_{\lfloor Z_+ \rfloor}(G; B)$. Then

$$\begin{aligned} \min\{\mathrm{th}_+(H) \mid G &\leq H \text{ and } |G| = |H|\} \leq \min\{\mathrm{th}_+(H; B) \mid G \leq H \text{ and } |G| = |H|\} \\ &= \mathrm{th}_{\lfloor Z_+ \rfloor}(G; B) = \mathrm{th}_{\lfloor Z_+ \rfloor}(G). \end{aligned}$$

Let H' be a spanning supergraph of G such that $\mathrm{th}_+(H') \leq \mathrm{th}_+(H)$ for any spanning supergraph H of G. Suppose $B' \subseteq V(H')$ with $\mathrm{th}_+(H') = \mathrm{th}_+(H'; B')$. Now suppose $\mathcal{F}'$ is a set of PSD forces of B' such that $\mathrm{pt}_+(H'; B') = \mathrm{pt}_+(H'; \mathcal{F}')$. The next step is to show that $\mathcal{F}'$ is a set of $\lfloor Z_+ \rfloor$ forces of B' in G with $\mathrm{pt}_{\lfloor Z_+ \rfloor}(G; \mathcal{F}') \leq \mathrm{pt}_+(H'; \mathcal{F}')$. Choose an edge $uw \in E(H') \setminus E(G)$ and suppose $(u \to w) \in \mathcal{F}'$. In the component where $u \to w$, w is the only white neighbor of u. So if the edge uw is removed from $E(G)$, u is allowed to force w by a hop. If $(u \to w) \notin \mathcal{F}'$, then removing uw does not slow down the propagation time of $\mathcal{F}'$. Note that removing edges from H' may increase the number of components at each time step when the blue vertices are removed. However, due to hopping, every force in $\mathcal{F}'$ is still a valid $\lfloor Z_+ \rfloor$ force in G and $\text{pt}_{\lfloor Z_+ \rfloor}(G; \mathcal{F}') \leq \text{pt}_+(H'; \mathcal{F}')$. Thus,

$$\begin{aligned} \text{th}_{\lfloor Z_+ \rfloor}(G) &\leq \text{th}_{\lfloor Z_+ \rfloor}(G; B') \leq \text{th}_{\lfloor Z_+ \rfloor}(G; \mathcal{F}') \leq \text{th}_+(H'; \mathcal{F}') \\ &= \text{th}_+(H') = \min\{\text{th}_+(H) \mid G \leq H \text{ and } |G| = |H|\}.\square \end{aligned}$$

Theorem 3.2.9 and Corollary 3.3.1 can be used to characterize graphs G with $\text{th}_{\lfloor Z_+ \rfloor}(G) \leq t$ for any positive integer t. This characterization is also in terms of specified minors of the Cartesian product of a tree and a complete graph.

Theorem 3.3.2. *Suppose G is a graph and t is a fixed positive integer. Then $\text{th}_{\lfloor Z_+ \rfloor}(G) \leq t$ if and only if there exists integers $a, k > 0$ and $b \geq 0$ such that $a + b = t$ and G can be obtained from $K_a \Box T_{k,b}$ by contracting tree edges and/or deleting edges.*

Proof. Suppose $\text{th}_{\lfloor Z_+ \rfloor}(G) \leq t$. By Corollary 3.3.1, there exists a spanning supergraph H of G such that $\text{th}_+(H) = \text{th}_{\lfloor Z_+ \rfloor}(G) \leq t$. Clearly G can be obtained from H by removing edges. By Theorem 3.2.9, there exists integers $a, k > 0$ and $b \geq 0$ such that $a + b = t$ and H can be obtained from $K_a \Box T_{k,b}$ by contracting tree edges and/or deleting complete edges. Thus, G can be obtained from $K_a \Box T_{k,b}$ by contracting tree edges and/or deleting edges.

Let $T = K_a \Box T_{k,b}$ for some integers $a, k > 0$ and $b \geq 0$. Suppose $D \subseteq E(T)$ and C is a set of tree edges of T such that $C \cap D = \emptyset$ and G can be obtained from T by contracting the edges in C and deleting the edges in D. Let T' be the graph obtained from T by contracting the tree edges in C. By Theorem 3.2.9, $\text{th}_+(T') \leq a + b$. Note that G can be obtained from T' by deleting the edges in D. By Corollary 3.3.1, $\text{th}_{\lfloor Z_+ \rfloor}(G) \leq \text{th}_+(T') \leq a + b$. □

The next section introduces a new perspective of the PSD zero forcing process that is more natural from a graph theoretical context.

3.4 A reduction perspective on PSD zero forcing

Suppose G is a graph, $\mathcal{F}$ is a set of PSD forces of a PSD zero forcing set $B \subseteq V(G)$, and $\text{pt}_+(G; B) = \text{pt}_+(G; \mathcal{F})$. The color change rule for PSD zero forcing requires breaking G into components and performing forces in each component individually. Throughout the literature on PSD zero forcing (see [7, 9]), each time step t in the PSD propagation process of B is visualized as follows. Start by removing the current set $B^{[t-1]}$ of blue vertices in G. Then for each $1 \leq j \leq |C_t(\mathcal{F})|$, force new vertices to become blue in $G_{t,j}(\mathcal{F})$. Finally, update the set of blue vertices in G to $B^{[t]}$. Removing all blue vertices in G at each time step can be misleading because it seems like the vertices that became blue in the previous time step could potentially perform a force in any of the white components. Example 3.4.1 shows that this is not the case.

Example 3.4.1. Let $G = P_5$ be the path with vertices labeled in order as v_1, v_2, v_3, v_4, and v_5. Consider the set of PSD forces $\mathcal{F} = \{v_3 \to v_4, v_3 \to v_2, v_4 \to v_5, v_2 \to v_1\}$. The forces $v_3 \to v_2$ and $v_3 \to v_4$ occur in the first time step. In the second time step, the two components are $W_{2,1} = \{v_1\}$ and $W_{2,2} = \{v_5\}$. Note that in $G_{2,1}(\mathcal{F})$, v_4 has no white neighbors. Also v_2 has no white neighbors in $G_{2,2}(\mathcal{F})$. So v_4 cannot force in $W_{2,1}$ and v_2 cannot force in $W_{2,2}$.

In general, if a blue vertex v is forced in the component $W_{i,j}$, then v cannot perform a force in any future component that is not contained in $W_{i,j}$. This means that it is more natural to think of the PSD propagation process in the following way. In the first time step, the set B of blue vertices is removed from the graph G, a copy of $G[B]$ is re-attached to each component of $G - B$, and one time step of standard zero forcing is applied to each of the resulting graphs. Then, in each subsequent time step, this process is repeated on each of the smaller graphs. Note that each time step can be thought of as applying the first time step to a reduced version (i.e., induced subgraphs) of the graphs obtained in the previous step.

Figure 3.4 illustrates this process when it is applied to the graph G and set of PSD forces $\mathcal{F}$ from Example 3.2.1. In the first time step, G breaks into two components. However, each of those components only break into one component in time step 2.

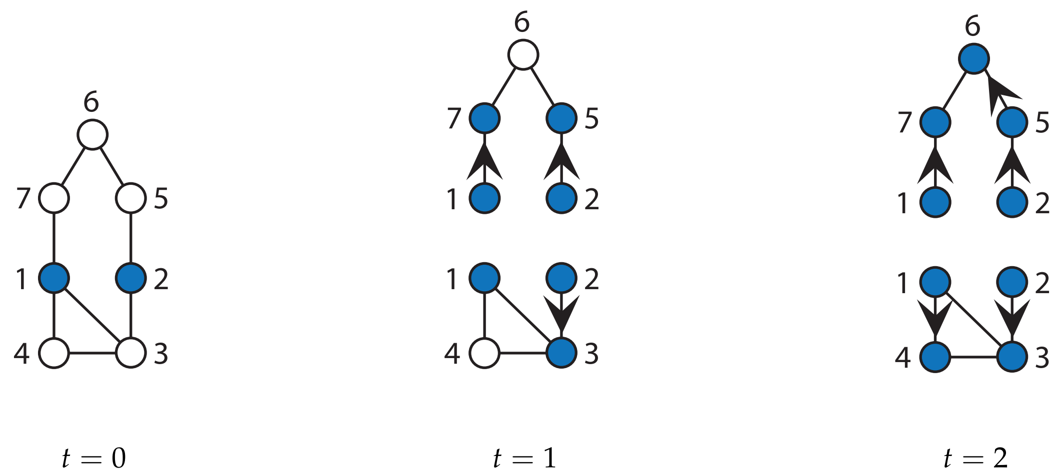

Figure 3.4 The PSD zero forcing process as seen from the reduction perspective.

This reduction perspective of PSD zero forcing leads to a better way to extend a graph using a set of PSD forces (see Definition 3.4.2 and Example 3.4.3). Suppose G is a graph, $B \subseteq V(G)$ is a PSD zero forcing set of G, and $\mathcal{F}$ is a set of PSD forces of B with $\text{pt}_+(G; \mathcal{F}) = \text{pt}_+(G; B)$. Define $T^*(G; B; \mathcal{F})$ to be the rooted tree that represents the breakdown of components throughout the PSD reduction process where the edges of the tree are labeled by the components. Note that if two vertices u and v have the same depth in $T^*(G; B; \mathcal{F})$, then u can have a different number of children than v. For example, suppose G breaks into two components W_1 and W_2 in the first time step. In the second time step, suppose W_1 breaks into one component W_3 and suppose W_2 breaks into two components W_4 and W_5. In this case, $T^*(G; B; \mathcal{F})$ is the tree illustrated in Figure 3.5.

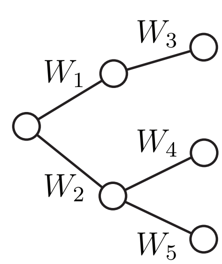

Figure 3.5 The component W_2 breaks into two components, but W_1 only breaks into one.

For each $b \in B$, define $\mathcal{E}_b^*(\mathcal{F})$ to be the copy of $T^*(G;B;\mathcal{F})$ whose vertices are labeled as follows.

1. Label the root of $T^*(G;B;\mathcal{F})$ as b.

2. Suppose u is a vertex in the forcing tree $T_b(\mathcal{F})$ and u becomes blue at time t in component W. Label as u the vertex in $T^*(G;B;\mathcal{F})$ that is distance t from b and is incident to the edge labeled W.

3. Give each remaining unlabeled vertex the label of its parent recursively.

Let uv be an edge in G that is not in any forcing tree of $\mathcal{F}$. Choose $u', v' \in B$ such that $u \in T_{u'}(\mathcal{F})$ and $v \in T_{v'}(\mathcal{F})$. Suppose u becomes blue at time i and v becomes blue at time j with $i \leq j$. Let $W_1, W_2, \ldots, W_j$ be the sequence of components that contain v during the first j time steps of $\mathcal{F}$. Therefore, the path in $\mathcal{E}_{v'}^*(\mathcal{F})$ obtained by starting at the root v' and following the edges labeled $W_1, W_2, \ldots, W_j$ leads to a vertex labeled v. Since $uv \in E(G)$, u is in components $W_1, W_2, \ldots W_i$ and u cannot force in a future component contained in W_i until v becomes blue in time step j. Once u becomes blue in component W_i, u remains in the set of blue vertices that are attached to every future component that is contained in W_i. So u is a blue vertex in the graph in which v is forced in time step j. Thus, the path in $\mathcal{E}_{u'}^*(\mathcal{F})$ obtained by starting at the root u' and following the edges labeled $W_1, W_2, \ldots W_j$ leads to a copy of u. This fact is used in the next definition.

Definition 3.4.2. Suppose G is a graph, $B \subseteq V(G)$ is a PSD zero forcing set of G, and $\mathcal{F}$ is a set of PSD forces of B with $\operatorname{pt}_+(G;B) = \operatorname{pt}_+(G;\mathcal{F})$. Let $t : E(G) \to \mathbb{N}$ and $r : V(G) \to V(G)$ be the functions defined in Definition 3.2.5. The extension $\mathcal{E}_+^*(G;B;\mathcal{F})$ is the graph obtained by the following procedure.

1. Construct the graph $G_1 = \dot{\bigcup}\{\mathcal{E}_b^*(\mathcal{F}) \mid b \in B\}$.

2. For each edge $v_1v_2 \in E(G)$ with $v_1, v_2 \in B$, add to G_1 the edge that connects the root of $\mathcal{E}_{r(v_1)}^*(\mathcal{F})$ to the root of $\mathcal{E}_{r(v_2)}^*(\mathcal{F})$. Call the resulting graph G_2.

3. Let $E^*(G)$ be the set of edges $v_1v_2 \in E(G)$ such that $\{v_1, v_2\} \nsubseteq B$ and v_1v_2 is not contained in any forcing tree of $\mathcal{F}$. For each $e = v_1v_2 \in E^*(G)$, add to G_2 the edge that connects the copy of v_1 in $\mathcal{E}^*_{r(v_1)}(\mathcal{F})$ to the copy of v_2 in $\mathcal{E}^*_{r(v_2)}(\mathcal{F})$ obtained in the following way. Suppose v_2 is the last endpoint of e to become blue (at time $t(e)$). Start at the root $r(v_1)$ (respectively $r(v_2)$) and follow the sequence of $t(e)$ edges that correspond to the components that contain v_2 until v_2 becomes blue.

Example 3.4.3. Let G, B, and $\mathcal{F}$ be the graph, PSD zero forcing set, and set of PSD forces illustrated in Figure 3.4. In the first time step, let W_1 and W_2 be the components of $G - \{1, 2\}$ that that contain the vertices $\{5, 6, 7\}$ and $\{3, 4\}$, respectively. In the second time step, let $W_3 = \{6\}$ be the component contained in W_1 and let $W_4 = \{4\}$ be the component contained in W_2. Note that $T^*(G; B; \mathcal{F})$ is a path on 5 vertices with the center vertex as the root. The graph G is shown alongside the extension $\mathcal{E}^*_+(G; B; \mathcal{F})$ in Figure 3.6. The edge $\{3, 4\}$ is in $E^*(G)$, vertex 4 becomes blue after vertex 3, and vertex 4 is contained consecutively in the components W_2 and W_4. Therefore, there is an edge in $\mathcal{E}^*_+(G; B; \mathcal{F})$ that connects the vertices obtained by starting at the roots of $\mathcal{E}^*_1(\mathcal{F})$ and $\mathcal{E}^*_2(\mathcal{F})$ and following the edges labeled W_2 and W_4.

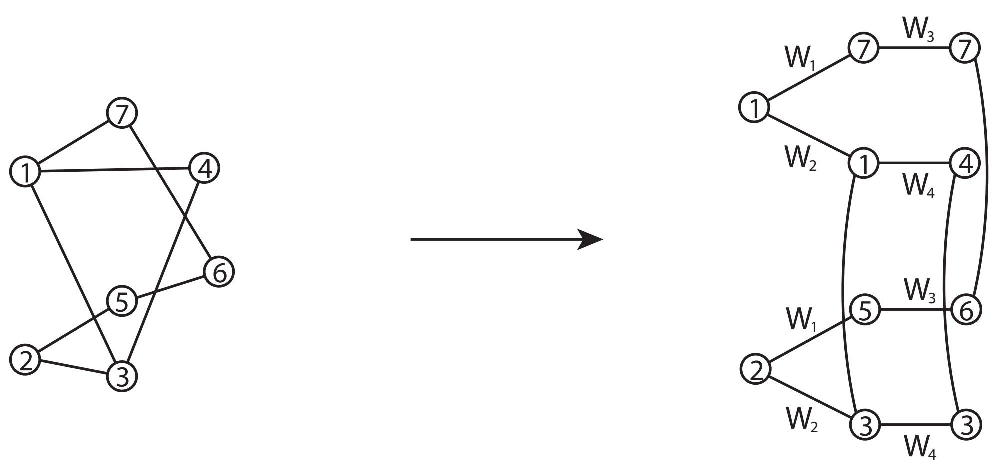

Figure 3.6 A graph G (left) is shown before and after its extension $\mathcal{E}^*_+(G; B; \mathcal{F})$ (right).

One of the advantages of the extension $\mathcal{E}_+^*$ in Definition 3.4.2 is that it has fewer vertices than $\mathcal{E}_+$. The extension definitions are motivated by their use to prove Theorems 3.2.9 and 3.3.2. Suppose $a', k' > 0$ and $b' \geq 0$ are integers such that G is a subgraph of $K_{a'} \square T_{k',b'}$ obtained by deleting complete edges. In this case, an extension is not necessary to see that G can be obtained from a graph of the form $K_a \square T_{k,b}$ by contracting tree edges and/or deleting complete edges. However, the extension $\mathcal{E}_+$ can be much larger than $K_{a'} \square T_{k',b'}$. For example, let B be the vertices of the copy of $K_{a'}$ in G that corresponds to the root of $T_{k',b'}$ and suppose $\mathcal{F}$ is the set of PSD forces of B that occur along the tree edges of G (directed away from the roots). Since $G - B$ has at least $a'k'$ components, the trees in $\{\mathcal{E}_b(\mathcal{F}) \mid b \in B\}$ are supergraphs of a $(a'k')$-ary tree. In contrast, $\mathcal{E}_+^*(G; B; \mathcal{F}) = G$ which reflects that no extension is needed.

3.5 Concluding remarks

For a graph G on n vertices, $\text{th}_+(G; V(G)) = n + 0 = n$. Thus, the PSD throttling number of a graph G is trivially bounded above by n. In [7], all graphs G with $\text{th}_+(G) \geq n-1$ are characterized using a connection between th_+ and the independence number α. One direction for future work is to attempt to use Theorem 3.2.9 to help characterize graphs with $\text{th}_+(G) = n-2$. It may be useful to study how certain graph properties change (such as independence number) as a graph G is replaced with a PSD extension of G. Such information could be used to specify the graphs of the form $K_a \square T_{k,b}$ that contain G as a minor if it is known that $\text{th}_+(G) \leq a + b$ and G has a given property.

Another direction for future work is to further study the $\lfloor Z_+ \rfloor$ throttling number of a graph. Since $\text{th}_{\lfloor Z_+ \rfloor}(G) \leq \text{th}_+(G)$ for any graph G, a better understanding of $\lfloor Z_+ \rfloor$ throttling would be useful in obtaining lower bounds for the PSD throttling number of a graph. The *largeur d'arborescence of a graph* G (denoted $la(G)$) is defined in [8] as the minimum k such that G is a minor of the Cartesian product of a complete graph on k vertices and a tree. In [2], it is shown that for any graph G, $la(G) = \lfloor Z_+ \rfloor(G)$. Further research on the types of trees that can show up in the definition of largeur d'arborescence could be useful for studying $\lfloor Z_+ \rfloor$ propagation and throttling.

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CHAPTER 4. GENERAL CONCLUSION

Section 1.2 provided the basic graph theory tools and notations that were used in this thesis. In Section 1.3, a survey was given of previous literature on zero forcing parameters, propagation, and throttling. General definitions of propagation and throttling were given in Chapter 2 and throttling was explored for the CCR-$\lfloor Z \rfloor$ color change rule that was introduced in [2]. Chapter 2 also introduced a method of extending graphs using zero forcing chains which was used to characterize all graphs with specified standard and $\lfloor Z \rfloor$ throttling numbers. In Chapter 3, the extension technique from Chapter 2 was generalized to positive semidefinite zero forcing and forcing trees. In addition, PSD extensions were used to characterize all graphs with specified PSD and $\lfloor Z_+ \rfloor$ throttling numbers.

Some of the work in this thesis has had an impact on other areas of study. Perhaps most notable is the topic of power domination. *Power domination* is a variant of zero forcing in which all white vertices that are adjacent to a blue vertex in the initial set are forced to become blue in the first time step. Each subsequent time step is identical to standard zero forcing. The power domination analog to the zero forcing number is called the *power domination number* and is studied in [6]. Power domination models the supervision of an electrical power system and the power domination number is tied to the phasor measurement unit placement problem (see [5]). Throttling for power domination is studied in [4] and the extension technique in Chapter 2 is used to characterize all graphs with specified power domination throttling numbers.

Another topic that has been influenced by throttling is the game of cops and robbers. *Cops and robbers* (introduced independently in [1, 7, 8]) is a game played on graphs in which a cop team and a robber choose initial placements on the vertices and then take turns moving along the edges. Of course, the cops try to capture the robber and the robber tries to evade the cops. Throttling for this game balances the number of cops used with the length of the game and is studied in [3]. In particular, the cop-throttling number is introduced and shown to have strong connections to the PSD throttling number.

For future research, there are many questions related to throttling that remain unanswered. Throttling for the minor monotone floor of positive semidefinite zero forcing is still largely unexplored. There are still many extreme values of standard and PSD throttling that are uncharacterized. Can the extension techniques in this thesis be used to aid the characterization of extreme throttling numbers? Is there a structural characterization of cop-throttling numbers? A variation of cops and robbers is shown in [2] to be connected to the tree-width of a graph. Can throttling be applied to this variation in order to bound tree-width?

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