Throttling for Cops & Robbers, zero forcing, and power domination

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Outline

Cops & Robbers, zero forcing and power domination

Cops and Robbers, zero forcing and power domination (and their variants) are games played on graphs.

Cops and Robbers:

- In (standard) Cops and Robbers, the cops and one robber alternate turns moving along the edges of the graph.
- The cops capture the robber when a cop occupies the robber's vertex.
- Cops and Robbers is a form of graph searching with applications to computer science.

Zero forcing and power domination

Zero forcing is a coloring game in which each vertex is initially blue or white and the goal is to color all vertices blue.

- The standard color change rule for zero forcing on a graph G is that a blue vertex v can change the color of a white vertex w to blue if w is the only white neighbor of v in G.
- There are many variants of zero forcing, each of which uses a different color change rule.
- Zero forcing has applications to combinatorial matrix theory and mathematical physics.

Power domination is zero forcing applied to the set of initial vertices and all their neighbors.

- Power domination was defined before zero forcing.
- A minimum power dominating set gives the optimal placement of monitoring units in an electric network.

Overview of throttling

Throttling minimizes a combination of the resources used to accomplish a task and the time needed to accomplish the task.

- Throttling originated with a question of Richard Brualdi to Michael Young in a talk about zero forcing and propagation time at the 2011 International Linear Algebra Society Conference in Brauschweig, Germany.
- Butler and Young initiated the study of throttling, for (standard) zero forcing in 2013.
- [Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019] studied throttling for positive semidefinite (PSD) zero forcing.

Overview of throttling (continued)

- [Breen, Brimkov, Carlson, H, Perry, Reinhart, 2019] and [Bonato, Breen, Brimkov, Carlson, English, Geneson, H, Perry, Reinhart, 2019+] studied cop throttling.
- [Carlson, 2018+] introduced a universal theory of of throttling for variants of zero forcing.
- [Brimkov, Carlson, Hicks, Patel, Smith, 2019+] studied power domination throttling.

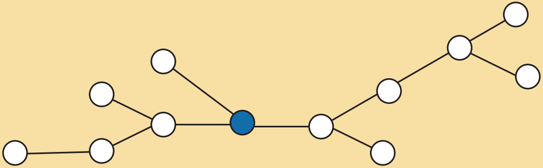
Cops and Robbers

- Cops and Robbers is a game played on a graph.
- Cops are placed on vertices first (they may share a vertex) and then the one robber chooses a vertex.
- Cops and the robber can all see each other's positions.
- Cops and robber alternate turns, with each staying put or moving along one edge (as many cops as desired may move in one cop turn).
- Robber is caught when a cop occupies the robber's vertex.
- A round is one turn for cops followed by one turn for robber.

Cops and Robbers Example

Example

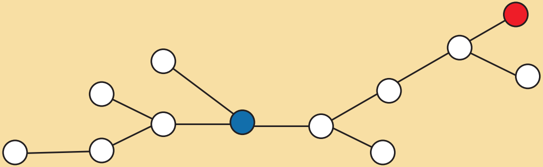
Round 0: Cop is placed



Cops and Robbers Example

Example

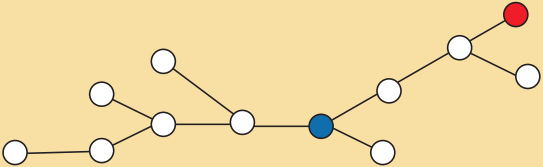
Round 0: Robber is placed



Cops and Robbers Example

Example

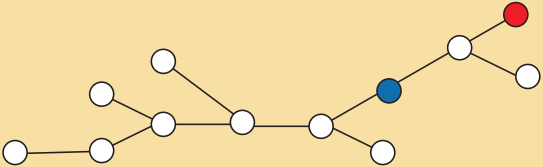
Round 1



Cops and Robbers Example

Example

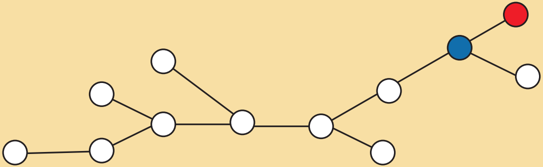
Round 2



Cops and Robbers Example

Example

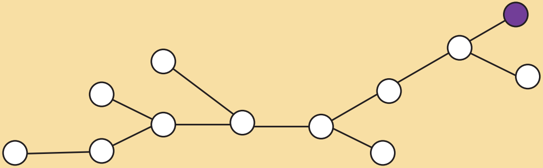
Round 3



Cops and Robbers Example

Example

Round 4: Robber is caught



Cop number

- Minimum number of cops needed to capture the robber is the cop number, $c(G)$, of the graph G.
- Cop number of any tree is 1.
- Cop number of any cycle is 2.
- A set S of vertices dominates G if every vertex of G is adjacent to a vertex in S.
- The domination number of G is

$$\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}.$$

- $c(G) \leq \gamma(G)$.

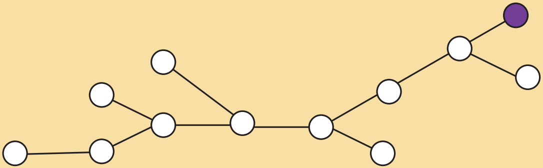
Capture time

- The capture time of G is the number of rounds needed to capture the robber using $c(G)$ cops, with each side playing optimally.

Example

The capture time of this tree is 4.

Round 4



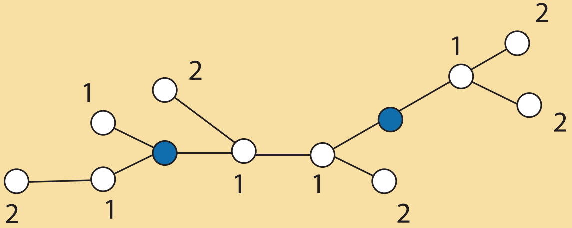
Throttling involves minimizing the sum of the number of resources used to accomplish a task (e.g., cops) and the time needed to accomplish the task (e.g., capture time).

- Capture time of a multiset S, $\mathrm{capt}(G;S)$, is the number of rounds needed to capture the robber (playing optimally) when the cops initially occupy vertices in S.
- If $|S| < c(G)$, then $\mathrm{capt}(G;S) = \infty$.
- Cops and Robbers throttling number of G is $\mathrm{th}_c(G) = \min_{S \subseteq V(G)} (|S| + \mathrm{capt}(G;S))$.

Cops and Robbers throttling example

Example

$\text{th}_c(T) = 2 + 2 = 4.$



k-capture time and k-throttling

- The k-capture time of graph G is $\mathrm{capt}_k(G) = \min_{|S|=k} \mathrm{capt}(G; S)$.
- The k-throttling number of graph G is $\mathrm{th}_c(G, k) = \min_{|S|=k} \mathrm{th}_c(G; S) = k + \mathrm{capt}_k(G)$.
- $\mathrm{th}_c(G) = \min_k \mathrm{th}_c(G, k) = \min_k (k + \mathrm{capt}_k(G))$.

The *k-center radius* of a graph G is $\mathrm{rad}_k(G) = \min_{|S|=k} \max_{v \in V} \mathrm{dist}(v, S)$.

Theorem (Bonato, Pérez-Gimnez, Pralat, Reiniger, 2017)

For a tree T, $\mathrm{capt}_k(T) = \mathrm{rad}_k(T)$.

Theorem (Breen, Brimkov, Carlson, H, Perry, Reinhart, 2019)

Let T denote a tree and P_n denote the path on n vertices.

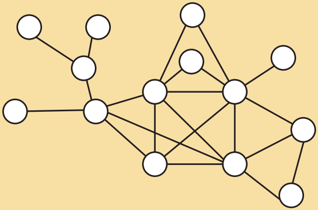
- $\mathrm{th}_c(T) = \min_k(k + \mathrm{rad}_k(T))$.
- $\mathrm{th}_c(T) \le 2\left\lfloor\sqrt{n}\right\rfloor$.
- $\mathrm{th}_c(P_n) = \left\lceil\sqrt{2n} - \frac{1}{2}\right\rceil$.

Chordal graphs

- A graph is chordal if every cycle of length 4 or more has a chord.
- A chordal graphs can be built from cliques.
- $c(H) = 1$ for a chordal graph H.

Example

A chordal graph



Cops and Robbers throttling for chordal graphs

Theorem (Bonato, Breen, Brimkov, Carlson, English, Geneson, H, Perry, Reinhart, 2019+)

For a chordal graph H,

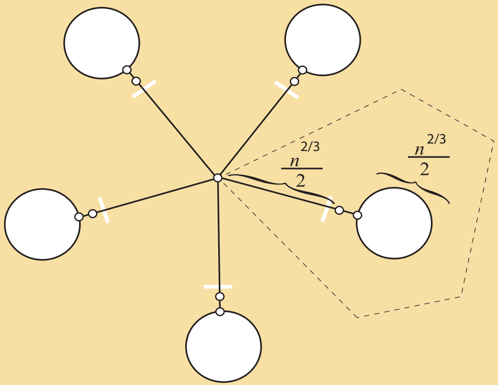
- $\mathrm{capt}_k(H) = \mathrm{rad}_k(H)$.
- $\mathrm{th}_c(H) \leq 2\sqrt{n}$.

Let G be a connected graph of order n

- Meyniel's Conjecture states that $c(G) = O(\sqrt{n})$, i.e., there exists b such that $c(G) \leq b\sqrt{n}$ for graphs G of order n.
- It was asked in [Breen, Brimkov, Carlson, H, Perry, Reinhart, 2019] whether $\mathrm{th}_c(G) = O(\sqrt{n})$.
- $\mathrm{th}_c(G) = O(\sqrt{n})$ would imply Meyniel's Conjecture, since $c(G) \leq \mathrm{th}_c(G)$.
- It was shown in [BBCHPR2019] that $\mathrm{th}_c(G) = O(\sqrt{n})$ for incidence graphs of finite projective planes, a family of cop-win graphs with maximum capture time, grids, hypercubes, and unicyclic graphs, in addition to trees.
- However it is not true in general.

$\mathrm{th}_c(H_n) = \Omega(n^{2/3})$, i.e., $\mathrm{th}_c(H_n) \geq bn^{2/3}$

Example (Bonato, Breen, Brimkov, Carlson, English, Geneson, H, Perry, Reinhart, 2019+)



Product throttling

- For $S \subseteq V(G)$, $\text{th}_c^{\times}(G;S) = |S|(1 + \text{capt}(G;S))$
- The product throttling number is

$$\text{th}_c^{\times}(G) = \min_{S \subseteq V(G)} \{\text{th}_c^{\times}(G;S)\} = \min_k \{k\,(1 + \text{capt}_k(G))\}$$

- For $S \subseteq V(G)$, $\text{th}_c(G;S) \leq \text{th}_c^{\times}(G;S)$. Thus

$$\text{th}_c(G) \leq \text{th}_c^{\times}(G).$$

- If $\text{th}_c(G) = \text{th}_c(G,1)$, then $\text{th}_c^{\times}(G) = \text{th}_c(G)$.

Product throttling for chordal graphs

Theorem (Bonato, Breen, Brimkov, Carlson, English, Geneson, H, Perry, Reinhart, 2019+)

For a chordal graph H,

$$\mathrm{th}_c^{\times}(H) = 1 + \mathrm{rad}(H).$$

Zero forcing and its variants

- Each type of zero forcing is a coloring game in which each vertex is initially blue or white.
- A color change rule allows white vertices to be colored blue under certain conditions.
- The type of zero forcing is determined by the color change rule used.

Let R be a color change rule.

- The set of initially blue vertices is $B^{[0]} = B$.
- The set of blue vertices $B^{[t]}$ after time step t (under R) is the set of blue vertices in G after the color change rule is applied in $B^{[t-1]}$ to every white vertex independently.
- An initial set of blue vertices $B = B^{[0]}$ is an R zero forcing set if there exists a t such that $B^{[t]} = V(G)$ using the R color change rule.

Let R be a color change rule.

- The R-propagation time for a set $B = B^{[0]}$ of vertices, $\mathrm{pt}_R(G; B)$, is the smallest t such that $B^{[t]} = V(G)$ using the R color change rule (and is infinity if this never happens).
- The R-propagation time of G is

$$\mathrm{pt}_R(G) = \min\{\mathrm{pt}_R(G; B) : B \text{ is a minimum } R\text{-forcing set.}\}$$

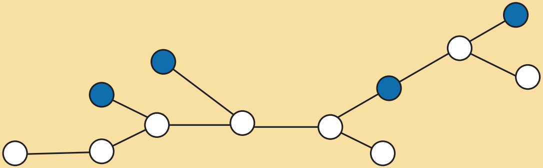
- The R-throttling number of G for zero forcing is $\mathrm{th}_R(G) = \min_{B \subseteq V(G)} (|B| + \mathrm{pt}_R(G; B))$.

(Standard) zero forcing color change rule

Let W be the set of (currently) white vertices. A blue vertex v can change the color of vertex $w \in W$ to blue if

$$N_G(v) \cap W = \{w\}.$$

Example ($B = B^{[0]}$)



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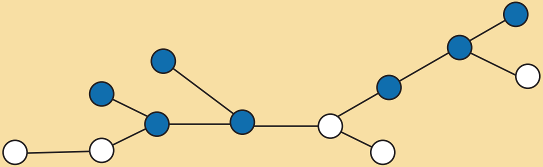
Example ($B^{[1]}$)



(Standard) zero forcing color change rule

Let W be the set of (currently) white vertices. A blue vertex v can change the color of vertex $w \in W$ to blue if

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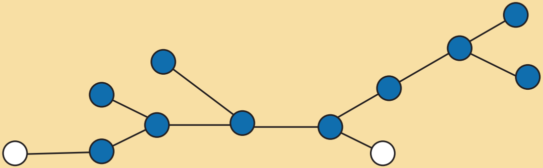
Example ($B^{[2]}$)



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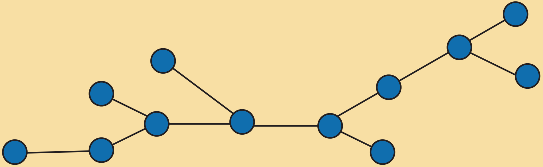
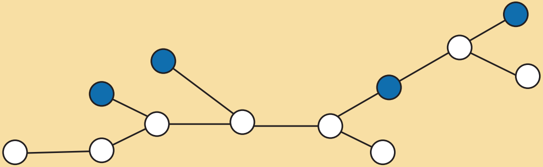
Example ($B^{[3]}$)



(Standard) zero forcing, propagation, and throttling

- The propagation time for a set $B = B^{[0]}$ of vertices, $\mathrm{pt}(G; B)$, is the smallest t such that $B^{[t]} = V(G)$ using the (standard) zero forcing color change rule.
- The propagation time of G is $\mathrm{pt}(G) = \min\{\mathrm{pt}(G; B) : B \text{ is a minimum zero forcing set}\}$.
- The throttling number of G for zero forcing is $\mathrm{th}(G) = \min_{B \subseteq V(G)}(|B| + \mathrm{pt}(G; B))$.

Example $(\mathrm{Z}(T) = 4, \mathrm{pt}(T) = 3, \mathrm{th}(T) = 7)$



Lower bounds on th(G)

Theorem (Butler, Young, 2013)

Let G be a graph of order n. Then

$$\operatorname{th}(G) \geq \left\lceil 2\sqrt{n} - 1 \right\rceil$$

and this bound is tight.

Paths and cycles

Theorem

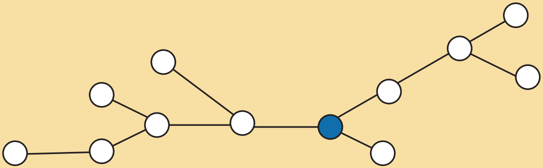
- [Butler, Young, 2013] $\mathrm{th}(P_n) = \left\lceil 2\sqrt{n} - 1 \right\rceil$.
- [Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019]
$$\mathrm{th}(C_n) = \begin{cases} \lceil 2\sqrt{n} - 1 \rceil & \textit{unless } n = (2k+1)^2 \\ 2\sqrt{n} & \textit{if } n = (2k+1)^2 \end{cases}.$$

PSD color change rule

PSD color change rule: Delete the currently blue vertices from the graph G and determine the components of the resulting graph; let W_i be the set of vertices of the ith component. A blue vertex v can change the color of a white vertex w to blue if

$$N_G(v) \cap W_i = \{w\}.$$

Example ($B = B^{[0]}$)



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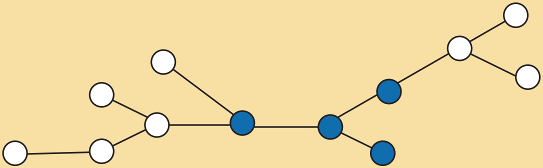

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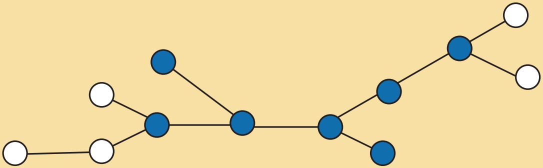

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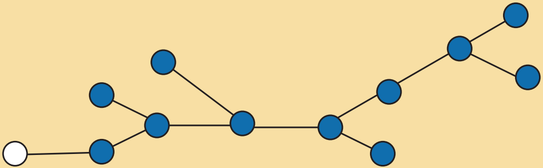

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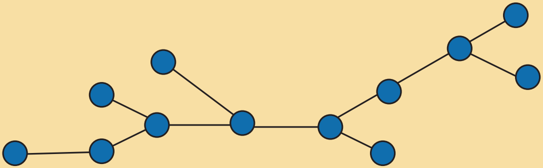
Example ($B^{[4]}$)



PSD propagation and throttling

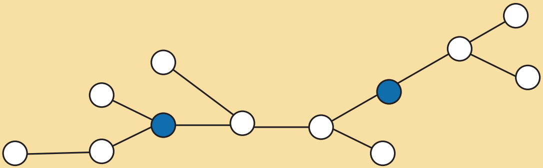
PSD color change rule: Delete the currently blue vertices from the graph G and determine the components of the resulting graph; let W_i be the set of vertices of the ith component. A blue vertex v can change the color of a white vertex w to blue if

$$N_G(v) \cap W_i = \{w\}.$$

- The PSD propagation time for a set $B = B^{[0]}$ of vertices, $\mathrm{pt}_+(G; B)$, is the smallest t such that $B^{[t]} = V(G)$ using the PSD color change rule.
- The PSD propagation time of G is $\mathrm{pt}_+(G) = \min\{\mathrm{pt}_+(G; B) : B \text{ is a minimum PSD forcing set}\}$.
- The PSD throttling number of G for zero forcing is the $\mathrm{th}_+(G) = \min_{B \subseteq V(G)}(|B| + \mathrm{pt}_+(G; B))$.

Example

$Z_+(T) = 1$, but using a PSD zero forcing set B of 2 vertices, $\mathrm{pt}_+(G; B) = 2$ and $\mathrm{th}_+(T) = 2 + 2 = 4$.



PSD throttling

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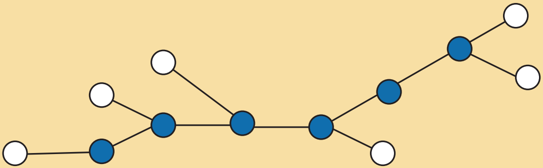

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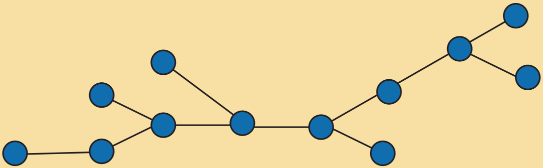

Lower bounds on $\mathrm{th}_+(G)$

Proposition (Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019)

Let $\Delta(G) = 2$. *Then*

$$\mathrm{th}_+(G) \geq \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$$

and this bound is tight.

Theorem (Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019)

Let G *be a graph of order* n *with* $\Delta(G) \geq 3$. *Then*

$$\mathrm{th}_+(G) \geq \left\lceil 1 + \log_{(\Delta(G)-1)} \left(\frac{(\Delta(G)-2)n+2}{\Delta(G)} \right) \right\rceil$$

and this bound is tight.

Paths & cycles

Theorem (Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019)

$$\mathrm{th}_+(P_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil.$$

$$\mathrm{th}_+(C_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil.$$

Power domination

[Brimkov, Carlson, Hicks, Patel, Smith, 2019+] initiated the study of power domination throttling.

- A power dominating set is a set $S \subseteq V(G)$ such that $N[S]$ is a zero forcing set for G.
- The power domination number $\gamma_P(G)$ of G is the minimum size of a power dominating set.
- The power propagation time of S is $\text{pt}_{\gamma_P}(G; S) = 1 + \text{pt}(G; N[S])$
- The power propagation time of G is $\text{pt}_{\gamma_P}(G) = \min\{\text{pt}_{\gamma_P}(G; B) : B \text{ is a minimum power dominating set.}\}$
- The power domination throttling number of G is $\text{th}_{\gamma_P}(G) = \min_{S\subseteq V(G)}(|S| + \text{pt}_{\gamma_P}(G; S))$.

(Standard) throttling and PSD throttling

Observation

Let $B \subseteq V(G)$ be a zero forcing set. Then,

- *B is a PSD zero forcing set.*
- $\mathrm{Z}_+(G) \leq \mathrm{Z}(G)$
- $\mathrm{pt}_+(G; B) \leq \mathrm{pt}(G; B)$
- $\mathrm{th}_+(G; B) \leq \mathrm{th}(G; B)$.
- $\mathrm{th}_+(G) \leq \mathrm{th}(G)$.
- $\mathrm{pt}_+(G)$ *and* $\mathrm{pt}(G)$ *are noncomparable (minimum values can differ).*

Differences between $\mathrm{th}_+(G)$ and $\mathrm{th}(G)$

Example

$\mathrm{th}(K_{1,n}) = n$ but $\mathrm{th}_+(K_{1,n}) = 2$.

(Standard) throttling and power domination

Observation

Let $S \subseteq V(G)$ be a zero forcing set. Then,

- *S is a power dominating set.*
- $\gamma_P(G) \leq \mathrm{Z}(G)$
- $\mathrm{pt}_{\gamma_P}(G; S) \leq \mathrm{pt}(G; S)$
- $\mathrm{th}_{\gamma_P}(G; S) \leq \mathrm{th}(G; S)$.
- $\mathrm{th}_{\gamma_P}(G) \leq \mathrm{th}(G)$
- $\mathrm{pt}_{\gamma_P}(G)$ *and* $\mathrm{pt}(G)$ *are noncomparable (minimum values can differ).*

Differences between $\mathrm{th}_{\gamma_P}(G)$ and $\mathrm{th}(G)$

Example

$\mathrm{th}(K_n) = n$ but $\mathrm{th}_{\gamma_P}(K_n) = 2$.

PSD throttling and Cops & Robbers throttling

Theorem (Breen, Brimkov, Carlson, H, Perry, Reinhart, 19)

Let $S \subseteq V(G)$ be a PSD zero forcing set. Then,

- *S is a capture set.*
- $c(G) \leq \mathrm{Z}_+(G)$.
- $\mathrm{capt}(G; S) \leq \mathrm{pt}_+(G; S)$.
- $\mathrm{th}_c(G; S) \leq \mathrm{th}_+(G; S)$.
- $\mathrm{th}_c(G) \leq \mathrm{th}_+(G)$.

Theorem (Breen, Brimkov, Carlson, H, Perry, Reinhart, 19)

Suppose T is a tree. Then for $S \subseteq V(T)$,

- $\mathrm{capt}(T; S) = \mathrm{pt}_+(T; S)$.
- $\mathrm{th}_c(T; S) = \mathrm{th}_+(T; S)$.

Furthermore, $\mathrm{th}_c(T) = \mathrm{th}_+(T)$.

Differences between $\mathrm{th}_c(G)$ and $\mathrm{th}_+(G)$

Observation

$\mathrm{th}_c(G) \leq \gamma(G) + 1$.

Example

$\mathrm{th}_c(K_n) = 2$ but $\mathrm{th}_+(K_n) = n$.

Observation

$\mathrm{pt}_+(G)$ *and* $\mathrm{capt}(G)$ *are noncomparable (minimum values can differ).*

Thank you!

References

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