Throttling for Cops & Robbers, zero forcing, and power domination

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Outline

Cops & Robbers, zero forcing, power domination

Cops and Robbers Throttling
  Cops and Robbers
  Capture time and throttling
  Trees and chordal graphs
  Meyniel’s conjecture and throttling bounds
  Product throttling

Zero forcing throttling
  Standard zero forcing
  PSD zero forcing

Power domination throttling

Relationships between throttling numbers
Cops and Robbers, zero forcing and power domination (and their variants) are games played on graphs.

Cops and Robbers:
- In (standard) Cops and Robbers, the cops and one robber alternate turns moving along the edges of the graph.
- The cops capture the robber when a cop occupies the robber’s vertex.
- Cops and Robbers is a form of graph searching with applications to computer science.
Zero forcing is a coloring game in which each vertex is initially blue or white and the goal is to color all vertices blue.

- The standard color change rule for zero forcing on a graph $G$ is that a blue vertex $v$ can change the color of a white vertex $w$ to blue if $w$ is the only white neighbor of $v$ in $G$.
- There are many variants of zero forcing, each of which uses a different color change rule.
- Zero forcing has applications to combinatorial matrix theory and mathematical physics.

Power domination is zero forcing applied to the set of initial vertices and all their neighbors.

- Power domination was defined before zero forcing.
- A minimum power dominating set gives the optimal placement of monitoring units in an electric network.
Overview of throttling

Throttling minimizes a combination of the resources used to accomplish a task and the time needed to accomplish the task.

- Throttling originated with a question of Richard Brualdi to Michael Young in a talk about zero forcing and propagation time at the 2011 International Linear Algebra Society Conference in Braunschweig, Germany.
- Butler and Young initiated the study of throttling, for (standard) zero forcing in 2013.
- [Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019] studied throttling for positive semidefinite (PSD) zero forcing.
[Breen, Brimkov, Carlson, H, Perry, Reinhart, 2019] and

[Carlson, 2018+] introduced a universal theory of throttling for variants of zero forcing.

[Brimkov, Carlson, Hicks, Patel, Smith, 2019+] studied power domination throttling.
Cops and Robbers is a game played on a graph.

Cops are placed on vertices first (they may share a vertex) and then the one robber chooses a vertex.

Cops and the robber can all see each other’s positions.

Cops and robber alternate turns, with each staying put or moving along one edge (as many cops as desired may move in one cop turn).

Robber is caught when a cop occupies the robber’s vertex.

A round is one turn for cops followed by one turn for robber.
Example

Round 0: Cop is placed
Example

Round 0: Robber is placed
Example

Round 1
Cops and Robbers Example

Example

Round 2
Example

Round 3
Example

Round 4: Robber is caught
Minimum number of cops needed to capture the robber is the cop number, $c(G)$, of the graph $G$.

Cop number of any tree is 1.

Cop number of any cycle is 2.

A set $S$ of vertices dominates $G$ if every vertex of $G$ is adjacent to a vertex in $S$.

The domination number of $G$ is

$$\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}.$$
The capture time of $G$ is the number of rounds needed to capture the robber using $c(G)$ cops, with each side playing optimally.

Example

The capture time of this tree is 4.

Round 4
Throttling involves minimizing the sum of the number of resources used to accomplish a task (e.g., cops) and the time needed to accomplish the task (e.g., capture time).

- Capture time of a multiset $S$, $\text{capt}(G; S)$, is the number of rounds needed to capture the robber (playing optimally) when the cops initially occupy vertices in $S$.
- If $|S| < c(G)$, then $\text{capt}(G; S) = \infty$.
- Cops and Robbers throttling number of $G$ is $\text{th}_c(G) = \min_{S \subseteq V(G)} (|S| + \text{capt}(G; S))$. 

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Example

\[ \text{th}_c(T) = 2 + 2 = 4. \]
The $k$-capture time of graph $G$ is

$$\text{capt}_k(G) = \min_{|S|=k} \text{capt}(G; S).$$

The $k$-throttling number of graph $G$ is

$$\text{th}_c(G, k) = \min_{|S|=k} \text{th}_c(G; S) = k + \text{capt}_k(G).$$

$$\text{th}_c(G) = \min_k \text{th}_c(G, k) = \min_k (k + \text{capt}_k(G)).$$
The \textit{k-center radius} of a graph \(G\) is \(\text{rad}_k(G) = \min_{|S|=k} \max_{v \in V} \text{dist}(v, S)\).

\begin{theorem*}[Bonato, Pérez-Gimnez, Pralat, Reiniger, 2017]
For a tree \(T\), \(\text{capt}_k(T) = \text{rad}_k(T)\).
\end{theorem*}

\begin{theorem*}[Breen, Brimkov, Carlson, H, Perry, Reinhart, 2019]
Let \(T\) denote a tree and \(P_n\) denote the path on \(n\) vertices.
\begin{itemize}
  \item \(\text{th}_c(T) = \min_k (k + \text{rad}_k(T))\).
  \item \(\text{th}_c(T) \leq 2 \left\lfloor \sqrt{n} \right\rfloor\).
  \item \(\text{th}_c(P_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil\).
\end{itemize}
\end{theorem*}
A graph is chordal if every cycle of length 4 or more has a chord.

A chordal graphs can be built from cliques.

\[ c(H) = 1 \] for a chordal graph \( H \).
Theorem (Bonato, Breen, Brimkov, Carlson, English, Geneson, H, Perry, Reinhart, 2019+)

For a chordal graph $H$,

- $\text{capt}_k(H) = \text{rad}_k(H)$.
- $\text{th}_c(H) \leq 2\sqrt{n}$. 
Throttling and Meyniel’s Conjecture

Let $G$ be a connected graph of order $n$

- **Meyniel’s Conjecture** states that $c(G) = O(\sqrt{n})$, i.e., there exists $b$ such that $c(G) \leq b\sqrt{n}$ for graphs $G$ of order $n$.

- It was asked in [Breen, Brimkov, Carlson, H, Perry, Reinhart, 2019] whether $th_c(G) = O(\sqrt{n})$.

- $th_c(G) = O(\sqrt{n})$ would imply Meyniel’s Conjecture, since $c(G) \leq th_c(G)$.

- It was shown in [BBCHPR2019] that $th_c(G) = O(\sqrt{n})$ for incidence graphs of finite projective planes, a family of cop-win graphs with maximum capture time, grids, hypercubes, and unicyclic graphs, in addition to trees.

- However it is not true in general.
\[ t_h c(H_n) = \Omega(n^{2/3}), \text{ i.e., } t_h c(H_n) \geq b n^{2/3} \]

Example (Bonato, Breen, Brimkov, Carlson, English, Geneson, H, Perry, Reinhart, 2019+)
Product throttling

For $S \subseteq V(G)$, $\text{th}^\times_c (G; S) = |S|(1 + \text{capt}(G; S))$

- The product throttling number is

\[
\text{th}^\times_c (G) = \min_{S \subseteq V(G)} \{\text{th}^\times_c (G; S)\} = \min_{k} \{k (1 + \text{capt}_k (G))\}
\]

- For $S \subseteq V(G)$, $\text{th}_c (G; S) \leq \text{th}^\times_c (G; S)$. Thus

\[
\text{th}_c (G) \leq \text{th}^\times_c (G).
\]

- If $\text{th}_c (G) = \text{th}_c (G, 1)$, then $\text{th}^\times_c (G) = \text{th}_c (G)$. 

Theorem (Bonato, Breen, Brimkov, Carlson, English, Geneson, H, Perry, Reinhart, 2019+)

For a chordal graph $H$,

$$\text{th}_c^x(H) = 1 + \text{rad}(H).$$
Zero forcing and its variants

- Each type of zero forcing is a coloring game in which each vertex is initially blue or white.
- A color change rule allows white vertices to be colored blue under certain conditions.
- The type of zero forcing is determined by the color change rule used.

Let \( R \) be a color change rule.

- The set of initially blue vertices is \( B[0] = B \).
- The set of blue vertices \( B[t] \) after time step \( t \) (under \( R \)) is the set of blue vertices in \( G \) after the color change rule is applied in \( B^{t-1} \) to every white vertex independently.
- An initial set of blue vertices \( B = B[0] \) is an \( R \) zero forcing set if there exists a \( t \) such that \( B[t] = V(G) \) using the \( R \) color change rule.
Let $R$ be a color change rule.

- The $R$-propagation time for a set $B = B^{[0]}$ of vertices, $\text{pt}_R(G; B)$, is the smallest $t$ such that $B^{[t]} = V(G)$ using the $R$ color change rule (and is infinity if this never happens).

- The $R$-propagation time of $G$ is

$$\text{pt}_R(G) = \min\{\text{pt}_R(G; B) : B \text{ is a minimum } R\text{-forcing set.}\}$$

- The $R$-throttling number of $G$ for zero forcing is

$$\text{th}_R(G) = \min_{B \subseteq V(G)} (|B| + \text{pt}_R(G; B)).$$
Let $W$ be the set of (currently) white vertices. A blue vertex $v$ can change the color of vertex $w \in W$ to blue if

$$N_G(v) \cap W = \{w\}.$$
(Standard) zero forcing color change rule

Let $\mathcal{W}$ be the set of (currently) white vertices. A blue vertex $v$ can change the color of vertex $w \in \mathcal{W}$ to blue if

$$\mathcal{N}_G(v) \cap \mathcal{W} = \{w\}.$$
(Standard) zero forcing color change rule

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(Standard) zero forcing color change rule

Let $W$ be the set of (currently) white vertices. A blue vertex $v$ can change the color of vertex $w \in W$ to blue if

$$N_G(v) \cap W = \{w\}.$$
The propagation time for a set $B = B^0$ of vertices, $\text{pt}(G; B)$, is the smallest $t$ such that $B^t = V(G)$ using the (standard) zero forcing color change rule.

The propagation time of $G$ is $\text{pt}(G) = \min \{ \text{pt}(G; B) : B \text{ is a minimum zero forcing set} \}$.

The throttling number of $G$ for zero forcing is $\text{th}(G) = \min_{B \subseteq V(G)} (|B| + \text{pt}(G; B))$.

Example ($\text{Z}(T) = 4$, $\text{pt}(T) = 3$, $\text{th}(T) = 7$)
Theorem (Butler, Young, 2013)

Let $G$ be a graph of order $n$. Then

$$\text{th}(G) \geq \left\lceil 2\sqrt{n} - 1 \right\rceil$$

and this bound is tight.
Theorem

- [Butler, Young, 2013] $\text{th}(P_n) = \lceil 2\sqrt{n} - 1 \rceil$.
- [Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019]

$$\text{th}(C_n) = \begin{cases} 
\lceil 2\sqrt{n} - 1 \rceil & \text{unless } n = (2k + 1)^2 \\
2\sqrt{n} & \text{if } n = (2k + 1)^2
\end{cases}.$$
PSD color change rule: Delete the currently blue vertices from the graph $G$ and determine the components of the resulting graph; let $W_i$ be the set of vertices of the $i$th component. A blue vertex $v$ can change the color of a white vertex $w$ to blue if

$$N_G(v) \cap W_i = \{w\}.$$
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PSD color change rule: Delete the currently blue vertices from the graph $G$ and determine the components of the resulting graph; let $W_i$ be the set of vertices of the $i$th component. A blue vertex $v$ can change the color of a white vertex $w$ to blue if

$$N_G(v) \cap W_i = \{w\}.$$ 

- The PSD propagation time for a set $B = B^{[0]}$ of vertices, $pt_+(G; B)$, is the smallest $t$ such that $B^{[t]} = V(G)$ using the PSD color change rule.
- The PSD propagation time of $G$ is $pt_+(G) = \min \{pt_+(G; B) : B$ is a minimum PSD forcing set$\}$.
- The PSD throttling number of $G$ for zero forcing is the $th_+(G) = \min_{B \subseteq V(G)}(|B| + pt_+(G; B))$. 

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PSD throttling

Example

$Z_+(T) = 1$, but using a PSD zero forcing set $B$ of 2 vertices, $\text{pt}_+(G; B) = 2$ and $\text{th}_+(T) = 2 + 2 = 4$. 

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Example

$Z_+(T) = 1$, but using a PSD zero forcing set $B$ of 2 vertices, $pt_+(G; B) = 2$ and $\text{th}_+(T) = 2 + 2 = 4$. 
PSD throttling

Example

\[ Z_+(T) = 1, \text{ but using a PSD zero forcing set } B \text{ of 2 vertices}, \]
\[ \text{pt}_+(G; B) = 2 \text{ and } \text{th}_+(T) = 2 + 2 = 4. \]
Lower bounds on $th_+(G)$

**Proposition (Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019)**

Let $\Delta(G) = 2$. Then

$$th_+(G) \geq \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$$

and this bound is tight.

**Theorem (Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019)**

Let $G$ be a graph of order $n$ with $\Delta(G) \geq 3$. Then

$$th_+(G) \geq \left\lceil 1 + \log(\Delta(G) - 1) \left( \frac{(\Delta(G) - 2)n + 2}{\Delta(G)} \right) \right\rceil$$

and this bound is tight.
Paths & cycles

Theorem (Carlson, H, Kritschgau, Lorenzen, Ross, Selken, Valle Martinez, 2019)

\[
\text{th}_+(P_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil.
\]

\[
\text{th}_+(C_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil.
\]
Power domination

[Brimkov, Carlson, Hicks, Patel, Smith, 2019+] initiated the study of power domination throttling.

- A power dominating set is a set $S \subseteq V(G)$ such that $N[S]$ is a zero forcing set for $G$.
- The power domination number $\gamma_P(G)$ of $G$ is the minimum size of a power dominating set.
- The power propagation time of $S$ is $\text{pt}_{\gamma_P}(G; S) = 1 + \text{pt}(G; N[S])$.
- The power propagation time of $G$ is $\text{pt}_{\gamma_P}(G) = \min \{\text{pt}_{\gamma_P}(G; B) : B \text{ is a minimum power dominating set} \}$.
- The power domination throttling number of $G$ is $\text{th}_{\gamma_P}(G) = \min_{S \subseteq V(G)}(|S| + \text{pt}_{\gamma_P}(G; S))$. 
(Standard) throttling and PSD throttling

**Observation**

Let $B \subseteq V(G)$ be a zero forcing set. Then,

- $B$ is a PSD zero forcing set.
- $Z_+(G) \leq Z(G)$
- $pt_+(G; B) \leq pt(G; B)$
- $th_+(G; B) \leq th(G; B)$.
- $th_+(G) \leq th(G)$.
- $pt_+(G)$ and $pt(G)$ are noncomparable (minimum values can differ).
Differences between $\text{th}_+(G)$ and $\text{th}(G)$

Example

$\text{th}(K_{1,n}) = n$ but $\text{th}_+(K_{1,n}) = 2$. 
Observation

Let $S \subseteq V(G)$ be a zero forcing set. Then,

- $S$ is a power dominating set.
- $\gamma_P(G) \leq Z(G)$
- $\text{pt}_{\gamma_P}(G; S) \leq \text{pt}(G; S)$
- $\text{th}_{\gamma_P}(G; S) \leq \text{th}(G; S)$
- $\text{th}_{\gamma_P}(G) \leq \text{th}(G)$
- $\text{pt}_{\gamma_P}(G)$ and $\text{pt}(G)$ are noncomparable (minimum values can differ).
Differences between $\text{th}_{\gamma_P}(G)$ and $\text{th}(G)$

Example

$\text{th}(K_n) = n$ but $\text{th}_{\gamma_P}(K_n) = 2$. 
**Theorem (Breen, Brimkov, Carlson, H, Perry, Reinhart, 19)**

Let $S \subseteq V(G)$ be a PSD zero forcing set. Then,
- $S$ is a capture set.
- $c(G) \leq Z_+(G)$.
- $\text{capt}(G; S) \leq \text{pt}_+(G; S)$.
- $\text{th}_c(G; S) \leq \text{th}_+(G; S)$.
- $\text{th}_c(G) \leq \text{th}_+(G)$.

**Theorem (Breen, Brimkov, Carlson, H, Perry, Reinhart, 19)**

Suppose $T$ is a tree. Then for $S \subseteq V(T)$,
- $\text{capt}(T; S) = \text{pt}_+(T; S)$.
- $\text{th}_c(T; S) = \text{th}_+(T; S)$.

Furthermore, $\text{th}_c(T) = \text{th}_+(T)$. 
Differences between $\text{th}_c(G)$ and $\text{th}_+(G)$

**Observation**

$\text{th}_c(G) \leq \gamma(G) + 1$.

**Example**

$\text{th}_c(K_n) = 2$ but $\text{th}_+(K_n) = n$.

**Observation**

$\text{pt}_+(G)$ and $\text{capt}(G)$ are noncomparable (minimum values can differ).
Thank you!

