Principal rank characteristic sequences

by

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DEDICATION

I dedicate this to my mother, Cieni, to my stepfather, Hiraís, and to my grandmother, Risalina.

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ABSTRACT

The necessity to know certain information about the principal minors of a given/desired matrix is a situation that arises in several areas of mathematics. As a result, researchers associated two sequences with an $n \times n$ symmetric, complex Hermitian, or skew-Hermitian matrix B. The first of these is the principal rank characteristic sequence (abbreviated pr-sequence). This sequence is defined as $r_0]r_1 \cdots r_n$, where, for $k \geq 1$, $r_k = 1$ if B has a nonzero order-k principal minor, and $r_k = 0$, otherwise; $r_0 = 1$ if and only if B has a 0 diagonal entry.

The second sequence, one that "enhances" the pr-sequence, is the enhanced principal rank characteristic sequence (epr-sequence), denoted by $\ell_1 \ell_2 \cdots \ell_n$, where ℓ_k is either `A`, `S`, or `N`, based on whether all, some but not all, or none of the order-k principal minors of B are nonzero.

In this dissertation, restrictions for the attainability of epr-sequences by real symmetric matrices are established. These restrictions are then used to classify two related families of sequences that are attainable by real symmetric matrices: the family of pr-sequences not containing three consecutive 1s, and the family of epr-sequences containing an `N` in every subsequence of length 3.

The epr-sequences that are attainable by symmetric matrices over fields of characteristic 2 are considered: For the prime field of order 2, a complete characterization of these epr-sequences is obtained; and for more general fields of characteristic 2, some restrictions are also obtained.

A sequence that refines the epr-sequence of a Hermitian matrix B, the signed enhanced principal rank characteristic sequence (sepr-sequence), is introduced. This sequence is defined as $t_1t_2\cdots t_n$, where t_k is either $\mathtt{A}^*$, $\mathtt{A}^+$, $\mathtt{A}^-$, $\mathtt{N}$, $\mathtt{S}^*$, $\mathtt{S}^+$, or $\mathtt{S}^-$, based on the following criteria: $t_k = \mathtt{A}^*$ if B has both a positive and a negative order-k principal minor, and each order-k principal minor is nonzero; $t_k = \mathtt{A}^+$ (respectively, $t_k = \mathtt{A}^-$) if each order-k principal minor is positive (respectively, negative); $t_k = \mathtt{N}$ if each order-k principal minor is zero; $t_k = \mathtt{S}^*$ if B has each a positive, a negative, and a zero order-k principal minor; $t_k = \mathtt{S}^+$ (respectively, $t_k = \mathtt{S}^-$) if B has both a zero and a nonzero order-k principal minor, and each nonzero order-k principal minor is positive (respectively, negative). The unattainability of various sepr-sequences is established. Among other results, it is shown that subsequences such as $\mathtt{A}^*\mathtt{N}$ and $\mathtt{N}\mathtt{A}^*$ cannot occur in the sepr-sequence of a Hermitian matrix. The notion of a nonnegative and nonpositive subsequence is introduced, leading to a connection with positive semidefinite matrices. Moreover, restrictions for sepr-sequences attainable by real symmetric matrices are established.

CHAPTER 1. GENERAL INTRODUCTION

1.1 Introduction

The necessity to know certain information about the principal minors of a given/desired matrix is a situation that arises in several areas of mathematics: As stated by Griffin and Tsatsomeros [8], instances where the principal minors of a matrix are of interest include the detection of P-matrices in the study of the complementarity problem, Cartan matrices in Lie algebras, univalent differentiable mappings, self-validating algorithms, interval matrix analysis, counting of spanning trees of a graph using the Laplacian, D-nilpotent automorphisms, and in the solvability of the inverse multiplicative eigenvalue problem.

A matrix is called *(positive) stable* if the real part of each of its eigenvalues is positive [9]. Stable matrices play an important role when studying the asymptotic stability of solutions of differential systems. A class of matrices that are stable are the Hermitian positive definite matrices —their eigenvalues are real and positive; they happen to possess two special properties: They are P-matrices, meaning that each principal minor is positive, and are weakly sign-symmetric (for a definition of the latter term, see [9] and the references therein). This led to the study of GKK matrices, which are the weakly sign-symmetric P-matrices [9]. The Gantmacher-Krein-Carlson theorem [5, 7], which states that a P-matrix is GKK if and only if certain principal minors satisfy the generalized Hadamard-Fischer inequality [9], led to the following question:

Question 1.1.1. Given a list of $2^n - 1$ real numbers, when can one find an $n \times n$ matrix whose principal minors are these numbers?

Given a matrix B, a *principal* submatrix of B is a submatrix lying in the same set of rows and columns of B; a minor of B is *principal* if it is the determinant of a principal submatrix; a principal minor of B is said to have *order* k if it is the determinant of a $k \times k$ submatrix. Let us illustrate Question 1.1.1 with two examples.

Example 1.1.2. With the assigned orders, can the entries of the vector

$$[\underbrace{1,\ 4,\ 6}_{\text{Order1}},\ \underbrace{0,\ -3,\ -1}_{\text{Order2}},\ \underbrace{-1}_{\text{Order3}}]^T \in \mathbb{R}^{2^3-1}$$

be realized as the principal minors of some 3×3 Hermitian matrix? The answer is affirmative: Consider the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Obviously, the principal minors of B of order 1 (i.e., the diagonal entries of B) are 1, 4 and 6. The order-2 principal minors of B are

$$\det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0,\ \det \begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix} = -3,\ \det \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} = -1.$$

Finally, it is easy to check that $\det B = -1$, which is the only principal minor of order 3.

Although the answer was affirmative in the example above, that is not always the case:

Example 1.1.3. Consider the same question as in Example 1.1.2, but for the vector

$$[\underbrace{0,\ a,\ b}_{\text{Order1}},\ \underbrace{0,\ 0,\ 0}_{\text{Order2}},\ \underbrace{c}_{\text{Order3}}]^T \in \mathbb{R}^{2^3-1},$$

where a, b and c are nonzero. The restrictions imposed by the order-1 and order-2 principal minors require the desired Hermitian matrix to have a zero row, meaning that its determinant is $0 \neq c$. Hence, the desired matrix does not exist.

Question 1.1.1 is known as the *principal minor assignment problem*, and has been answered set-theoretically by Oeding [15] in the case where the desired matrix is complex symmetric. This question serves to illustrate the aforementioned interest in the principal minors of a matrix. We note that Question 1.1.1 remains open for the case when the desired matrix is real symmetric or Hermitian, for example.

In this dissertation, we confine our attention to the study of the principal minors of symmetric matrices over a given field, and of Hermitian matrices. The principal minors of symmetric and Hermitian matrices have attracted considerable attention (see [1, 2, 3, 10, 11, 15, 16], for example). The focus of this dissertation is on studying certain sequences associated with a given/desired matrix, where the terms of a sequence collect certain information about the principal minors of the matrix. The first sequence, introduced by Brualdi et al. [2], was defined as follows: Given an $n \times n$ symmetric matrix $B \in F^{n\times n}$ (or Hermitian matrix $B \in \mathbb{C}^{n\times n}$), *the principal rank characteristic sequence* (abbreviated pr-sequence) of B is defined as $\mathrm{pr}(B) = r_0]r_1 \cdots r_n$, where, for $k \geq 1$,

$$r_k = \begin{cases} 1 & \text{if } B \text{ has a nonzero principal minor of order } k, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

while $r_0 = 1$ if and only if B has a 0 diagonal entry [2]. We note that the original definition of the pr-sequence was for real symmetric, complex symmetric and Hermitian matrices only; Barrett et al. [1] later extended it to symmetric matrices over any field.

For a given $n \times n$ matrix B, $B[\alpha]$ denotes the (principal) submatrix lying in rows and columns indexed by $\alpha \subseteq \{1, 2, \ldots, n\}$.

Example 1.1.4. Consider the real symmetric matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix},$$

where $\mathrm{pr}(B) = r_0]r_1r_2r_3r_4$. Since B does not have a 0 diagonal entry, $r_0 = 0$. Because B contains at least one principal minor of order 1 that is nonzero (i.e., because it contains a nonzero diagonal entry), $r_1 = 1$. Note that all the order-2 principal minors are zero; thus, $r_2 = 0$. Since $\det(B[\{2,3,4\}]) \neq 0$, $r_3 = 1$. Finally, as $\det(B) = 0$, $r_4 = 0$. It follows that $\mathrm{pr}(B) = 0]1010$.

The second sequence is one that was introduced by Butler et al. [3] as an "enhancement" of the pr-sequence: Given an $n \times n$ symmetric matrix $B \in F^{n\times n}$ (or Hermitian matrix $B \in \mathbb{C}^{n\times n}$), the *enhanced principal rank characteristic sequence* (abbreviated epr-sequence) of B is defined as $\mathrm{epr}(B) = \ell_1\ell_2\cdots\ell_n$, where

$$\ell_k = \begin{cases} \mathtt{A} & \text{if all of the principal minors of order } k \text{ are nonzero;} \\ \mathtt{S} & \text{if some but not all of the principal minors of order } k \text{ are nonzero;} \\ \mathtt{N} & \text{if none of the principal minors of order } k \text{ are nonzero, i.e., if all are zero.} \end{cases}$$

Example 1.1.5. Consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

from Example 1.1.4. Since all order-1 principal minors of B are nonzero, $\ell_1 = \mathtt{A}$. Note that all the order-2 principal minors are zero; thus, $\ell_2 = \mathtt{N}$. Since $\det(B[\{1,2,3\}]) = 0$ and $\det(B[\{2,3,4\}]) \neq 0$, $\ell_3 = \mathtt{S}$. Finally, as $\det(B) = 0$, $\ell_4 = \mathtt{N}$. Hence, $\mathrm{epr}(B) = \mathtt{ANSN}$.

It is known that if the epr-sequence of a Hermitian matrix begins with `SN`, then it cannot contain an `A` (see [3, Proposition 2.5]); this result justifies the negative answer obtained in Example 1.1.3, which serves to illustrate the usefulness of epr-sequences in the study of principal minors.

This dissertation is devoted to studying the pr- and epr-sequences of symmetric and complex Hermitian matrices, and to introducing a third sequence:

Definition 1.1.6. [14] Let $B \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. The *signed enhanced principal rank characteristic sequence* (abbreviated sepr-sequence) of B is the sequence $\operatorname{sepr}(B) = t_1t_2\cdots t_n$, where

$$t_k = \begin{cases} \mathtt{A}^* & \text{if } \ell_k = \mathtt{A} \text{ and } B \text{ has both a positive and a negative order-}k\text{ principal minor;} \\ \mathtt{A}^+ & \text{if each order-}k\text{ principal minor of } B \text{ is positive;} \\ \mathtt{A}^- & \text{if each order-}k\text{ principal minor of } B \text{ is negative;} \\ \mathtt{N} & \text{if each order-}k\text{ principal minor of } B \text{ is zero;} \\ \mathtt{S}^* & \text{if } \ell_k = \mathtt{S} \text{ and } B \text{ has both a positive and a negative order-}k\text{ principal minor;} \\ \mathtt{S}^+ & \text{if } \ell_k = \mathtt{S} \text{ and each order-}k\text{ principal minor of } B \text{ is nonnegative;} \\ \mathtt{S}^- & \text{if } \ell_k = \mathtt{S} \text{ and each order-}k\text{ principal minor of } B \text{ is nonpositive.} \end{cases}$$

For example, the sepr-sequence of the matrix B from Examples 1.1.4 and 1.1.5 is $\operatorname{sepr}(B) = \mathtt{A}^+\mathtt{N}\mathtt{S}^-\mathtt{N}$.

A (pr-, epr- or sepr-) sequence is said to be *attainable* by a class of matrices provided that there exists a matrix in the class that attains it; otherwise, we say that it is *unattainable* (by the given class). For any sepr-sequence σ, the epr-sequence resulting from removing the superscripts of each term in σ is called the *underlying* epr-sequence of σ.

When the pr-sequence was refined (or "enhanced"), the number of potential sequences involved in the case of an $n \times n$ matrix increased from $3(2)^{n-1}$ to $2(3)^{n-1}$. Now, after the second refinement, which leads to the sepr-sequence, the potential number of sequences increases to $3(7)^{n-1}$. Although these increments obviously make the determination of all the sequences that are attainable by an $n \times n$ matrix harder, the refinements are worthwhile, since they reveal more information about the principal minors of a matrix, while also remaining tractable.

1.2 Dissertation Organization

The format adopted for this dissertation presents it as a collection of research papers published or submitted to journals. The present chapter provides the main definitions, outlines some of the applications of the present work, and provides a short literature review.

Chapter 2 contains the paper [12], entitled "Classification of families of pr- and epr-sequences," which has been published in the journal *Linear and Multilinear Algebra*. In this paper, restrictions for the attainability of epr-sequences by real symmetric matrices are established. These restrictions are then used to classify two related families of sequences that are attainable by real symmetric matrices: the family of pr-sequences not containing three consecutive 1s, and the family of epr-sequences containing an `N` in every subsequence of length 3.

In Chapter 3, the paper [13], entitled "The enhanced principal rank characteristic sequence over a field of characteristic 2," is presented; this paper has been submitted to *Electronic Journal of Linear Algebra*. The focus of this paper is on the epr-sequences that are attainable by symmetric matrices over fields of characteristic 2. Its main result is the complete characterization of the epr-sequences that are attainable by symmetric matrices over the prime field of order 2; for more general fields of characteristic 2, some restrictions are also obtained.

Chapter 4 presents the paper [14], entitled "The signed enhanced principal rank characteristic sequence," which has been submitted to *Linear and Multilinear Algebra*. This paper introduces the sepr-sequence of a Hermitian matrix (which was defined in the previous section). There, the unattainability of various sepr-sequences is established; among other results, it is shown that subsequences such as $\mathtt{A}^*\mathtt{N}$ and $\mathtt{N}\mathtt{A}^*$ cannot occur in the sepr-sequence of a Hermitian matrix. Moreover, the notion of a nonnegative and nonpositive subsequence is introduced, leading to a connection with positive semidefinite matrices. For Hermitian matrices of orders $n = 1, 2, 3$, all attainable sepr-sequences are classified. And for real symmetric matrices, a complete characterization of the attainable sepr-sequences whose underlying epr-sequence contains `ANA` as a non-terminal subsequence is established.

Chapter 5 summarizes the results established.

1.3 Literature Review

The purpose of this section is to list/discuss a small selection of known results about pr- and epr-sequences that have appeared on the literature. It should be noted that there are no results to list about sepr-sequences, since this sequence was introduced in [14], which is the subject of Chapter 4.

The study of pr-sequences was started by Brualdi et al. [2], with the focus on real symmetric matrices. However, their original definition of the pr-sequence was the following, which justifies its name:

Definition 1.3.1. [2, Definition 1.1] The *principal rank characteristic sequence* of an $n \times n$ real symmetric matrix is defined to be $\mathrm{pr}(B) = r_0 r_1 \cdots r_n$, where, for $0 \le k \le n$,

$$r_k = \begin{cases} 1 & \text{if } B \text{ has a principal submatrix of rank } k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For practical purposes, the equivalent definition of the pr-sequence that is used in the literature, and the one used here, is the one given in Section 1.1. The equivalence of these definitions is a consequence of a well-known result, which is stated in [1], and by virtue of which the rank of a symmetric (or Hermitian) matrix is called *principal*:

Theorem 1.3.2. [1, Theorem 1.1] *If B is a symmetric matrix over a field F or a complex Hermitian matrix, then* $\mathrm{rank}(B) = \max\{|\alpha| : \det(B[\alpha]) \neq 0\}$ *(where the maximum over the empty set is defined to be* 0*).*

In [2], it was established that the occurrence of two consecutive 0s in the pr-sequence of a real symmetric matrix implies that the sequence can only contain 0s from that point forward (see [2, Theorem 4.4]). This result was later generalized for symmetric matrices over any field and for complex Hermitian matrices (see [1, Theorem 2.1]), which led to the following important result about epr-sequences:

Theorem 1.3.3. [3, Theorem 2.3] *Suppose that B is a symmetric matrix over a field F or a complex Hermitian matrix, that* $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$*, and that* $\ell_k = \ell_{k+1} = \mathtt{N}$ *for some k. Then $\ell_i = \mathtt{N}$ for all $i \geq k$.*

Brualdi et al. [2, Theorem 2.7] applied Jacobi's determinantal identity to obtain results about the pr-sequence of the inverse of a matrix. These ideas were extended in [3] to epr-sequences:

Theorem 1.3.4. [3, Theorem 2.4] (Inverse Theorem.) *Suppose that B is a symmetric matrix over a field F or a complex Hermitian matrix. If* $\text{epr}(B) = \ell_1\ell_2\cdots\ell_{n-1}\mathtt{A}$*, then* $\text{epr}(B^{-1}) = \ell_{n-1}\ell_{n-2}\cdots\ell_1\mathtt{A}$.

Given a sequence $t_{i_1}t_{i_2}\cdots t_{i_k}$, the notation $\overline{t_{i_1}t_{i_2}\cdots t_{i_k}}$ indicates that the sequence may be repeated as many times as desired (or it may be omitted entirely).

The use of probabilistic methods in [3] led to the next result.

Theorem 1.3.5. [3, Theorem 4.4 and Theorem 4.6] *Any sequence of the form $\ell_1\ell_2\cdots\ell_m\overline{\mathtt{N}}$ not ending in $\mathtt{S}$, with $\ell_k \in \{\mathtt{A},\mathtt{S}\}$ for $k = 1, 2, \ldots, m$ and $t \geq 0$ copies of $\mathtt{N}$, is attainable by a symmetric matrix over a field of characteristic* 0.

By Theorem 1.3.5, any epr-sequence not containing $\mathtt{A}$s or $\mathtt{S}$s after the occurrence of an $\mathtt{N}$ is attainable by a symmetric matrix over a field of characteristic 0. However, we do not know as much about the attainability of epr-sequences containing the subsequence $\mathtt{NA}$ or $\mathtt{NS}$. The next three results made contributions in this direction.

Theorem 1.3.6. [3, Corollary 2.7] *No symmetric matrix over any field (or complex Hermitian matrix) can have* `NSA` *in its epr-sequence.*

Theorem 1.3.7. [3, Theorem 2.14] *Neither the epr-sequences* `NAN` *nor* `NAS` *can occur as a subsequence of the epr-sequence of a symmetric matrix over a field of characteristic not* 2*.*

Theorem 1.3.8. [3, Theorem 2.15] *In the epr-sequence of a symmetric matrix over a field of characteristic not* 2*, the subsequence* `ANS` *can occur only as the initial subsequence.*

Although Theorems 1.3.6, 1.3.7 and 1.3.8 provide some insight for understanding epr-sequences containing the subsequence `NA` or `NS`, this is far from enough for arriving at a result analogous to Theorem 1.3.5 —which establishes the attainability of a large class of sequences— for sequences that are allowed to contain `NA` or `NS` as subsequences. However, we will see in Chapter 2 that there is a class of epr-sequences that allow the occurrence of `NA` or `NS` that can be completely characterized.

As implied above, obtaining a complete characterization of all the epr-sequences that are attainable by real symmetric matrices (or symmetric matrices over any field) is a difficult problem. This problem is not as difficult if it is instead considered for real skew-symmetric matrices, as was done by Fallat et al. in [6], where the following characterization was established:

Theorem 1.3.9. [6, Theorem 3.3] *Suppose $\ell_1\ell_2\cdots\ell_n$ is a given sequence from $\{\mathtt{A},\mathtt{S},\mathtt{N}\}$. Then $\ell_1\ell_2\cdots\ell_n$ is the epr-sequence of a real skew-symmetric matrix if and only if the following conditions hold:*

- *(i)* $\ell_j = \mathtt{N}$ *for* j *odd;*
- *(ii)* *if* $\ell_k = \ell_{k+1} = \mathtt{N}$, *then* $\ell_j = \mathtt{N}$ *for all* $j \geq k$*;*
- *(iii)* $\ell_n \neq \mathtt{S}$.

One of the reasons that allowed the characterization for real skew-symmetric matrices is the fact that when these matrices have odd order, their determinant is zero, which automatically means that their epr-sequences must contain an `N` in every odd position. Unlike real skew-symmetric matrices, symmetric and Hermitian matrices do not impose such a severe a constraint on their epr-sequences, which is one reason we are still far away from a similar characterization for these classes of matrices. However, we show in Chapter 3 that if one considers the epr-sequences of symmetric matrices over the prime field of order 2, then such a characterization is achievable. Our characterization in Chapter 3 is inspired by a result of Barrett et al. [1] that completely characterizes the pr-sequences that can be attained by symmetric matrices over a field of characteristic 2:

Theorem 1.3.10. [1, Theorem 3.1] *A pr-sequence of order $n \geq 2$ is attainable by an $n \times n$ symmetric matrix over a field of characteristic 2 if and only if it has one of the following forms:*

$$0]1\,\overline{1}\,\overline{0}, \qquad 1]\overline{01}\,\overline{0}, \qquad 1]1\,\overline{1}\,\overline{0}.$$

Although the study of epr-sequences has focused primarily on symmetric matrices, it was not until recently, in the paper [4], that the epr-sequences of Hermitian matrices received the attention they deserve. This paper and [1] show that there is a drastic difference between the epr-sequences attainable by real symmetric matrices and those attainable by Hermitian matrices. For example, the sequences containing the subsequence `NAN`, which cannot occur in the epr-sequence of a real symmetric matrix (see Theorem 1.3.7), can in fact occur in the epr-sequence of a Hermitian matrix. The following conjecture and theorems provide further illustration of these differences.

Conjecture 1.3.11. [4] *If the epr-sequence of a Hermitian matrix contains* `NAN` *as a subsequence, then the sequence is attainable by a real skew-symmetric matrix.*

The following two cases of Conjecture 1.3.11 suggest that the answer may be affirmative:

Theorem 1.3.12. [4] *If the epr-sequence of a Hermitian matrix starts with* `NAN`*, then it is attainable by a real skew-symmetric matrix.*

Theorem 1.3.13. [4] *If the epr-sequence of a Hermitian matrix contains* `NANA` *as a subsequence, then it is attainable by a real skew-symmetric matrix.*

Conjecture 1.3.11 is very interesting, and provides another incentive for studying the epr-sequences of Hermitian matrices, which is done in Chapter 4 in the context of sepr-sequences.

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CHAPTER 2. CLASSIFICATION OF FAMILIES OF PR- AND EPR-SEQUENCES

A paper published in the journal *Linear and Multilinear Algebra*

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Abstract

This paper establishes new restrictions for attainable enhanced principal rank characteristic sequences (epr-sequences). These results are then used to classify two related families of sequences that are attainable by a real symmetric matrix: the family of principal rank characteristic sequences (pr-sequences) not containing three consecutive 1s and the family of epr-sequences which contain an `N` in every subsequence of length 3.

Keywords: Principal rank characteristic sequence; enhanced principal rank characteristic sequence; minor; rank; symmetric matrix

AMS Subject Classifications: 15A15; 15A03; 15B57.

2.1 Introduction

Given an $n \times n$ symmetric matrix B over a field F, *the principal rank characteristic sequence* (abbreviated pr-sequence) of B is defined as $\operatorname{pr}(B) = r_0]r_1 \cdots r_n$, where, for

$k \geq 1$,

$$r_k = \begin{cases} 1 & \text{if } B \text{ has a nonzero principal minor of order } k\text{, and} \\ 0 & \text{otherwise,} \end{cases}$$

while $r_0 = 1$ if and only if B has a 0 diagonal entry [2]; the *order* of a minor is k if it is the determinant of a $k \times k$ submatrix.

The *principal minor assignment problem*, introduced in [5], asks the following question: Can we find an $n \times n$ matrix with prescribed principal minors? As a simplification of the principal minor assignment problem, Brualdi et al. [2] introduced the pr-sequence of a real symmetric matrix as defined above. An attractive result obtained in [2] is the requirement that a pr-sequence that can be realized by a real symmetric matrix cannot contain the subsequence 001, meaning that in the pr-sequence of such matrix, the presence of the subsequence 00 forces 0s from that point forward. This result was later generalized by Barrett et al. [1] for symmetric matrices over any field; this led them to the study of symmetric matrices over various fields, where, among other results, a characterization of the pr-sequences that can be realized by a symmetric matrix over a field of characteristic 2 was obtained. Although not deeply studied, the family of pr-sequences not containing three consecutive 1s were of interest in [2], since the pr-sequences of the principal submatrices of a matrix realizing a pr-sequence not containing three consecutive 1s possess the rare property of being able to inherit the majority of the 1s of the original sequence; this family will be one of the central themes of this paper.

Due to the limitations of the pr-sequence, which only records the presence or absence of a full-rank principal submatrix of each possible order, Butler et al. [3] introduced the the *enhanced principal rank characteristic sequence* (abbreviated epr-sequence) of an

$n \times n$ symmetric matrix B over a field F, denoted by $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$, where

$$\ell_k = \begin{cases} \texttt{A} & \text{if all the principal minors of order } k \text{ are nonzero;} \\ \texttt{S} & \text{if some but not all the principal minors of order } k \text{ are nonzero;} \\ \texttt{N} & \text{if none of the principal minors of order } k \text{ are nonzero, i.e., all are zero.} \end{cases}$$

A (pr- or epr-) sequence is said to be *attainable* over a field F provided that there exists a symmetric matrix $B \in F^{n\times n}$ that attains it; otherwise, we say that it is *unattainable*. Among other results, techniques to construct attainable epr-sequences were presented in [3], as well as necessary conditions for an epr-sequence to be attainable by a symmetric matrix, with many of them asserting that subsequences such as `NSA`, `NAN` and `NAS`, among others, cannot occur in epr-sequences over certain fields. Continuing the study of epr-sequences, Fallat et al. [4] characterized all the epr-sequences that are attainable by skew-symmetric matrices.

In this paper, the study of pr- and epr-sequences of symmetric matrices is continued. Section 2.2 establishes new restrictions for epr-sequences to be attainable over certain fields. The results from Section 2.2 are then implemented in Section 2.3, where, for real symmetric matrices, we classify all the attainable pr-sequences not containing three consecutive 1s. Using this classification, in Section 2.4, a related family of attainable epr-sequences is classified, namely those that contain an `N` in every subsequence of length 3. We then conclude with Proposition 2.4.6, where we highlight an interesting property exhibited by the vast majority of attainable pr-sequences not containing three consecutive 1s; that is, the property of being associated with a unique attainable epr-sequence.

A pr-sequence and an epr-sequence are *associated* with each other if a matrix (which may not exist) attaining the epr-sequence also attains the pr-sequence. A subsequence that does not appear in an attainable sequence is *forbidden* (and we may also say that it is *prohibited*). Moreover, a sequence is said to have *order* n if it corresponds to a matrix of order n, while a subsequence has *length* n if it consists of n terms.

Let $B = [b_{ij}]$ and let $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$. Then the submatrix lying in rows indexed by α, and columns indexed by β, is denoted by $B[\alpha, \beta]$; if $\alpha = \beta$, then $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$. The matrices 0_n, I_n and J_n are the matrices of order n denoting the zero matrix, the identity matrix and the all-1s matrix, respectively. The direct sum of two matrices B and C is denoted by $B \oplus C$. Given a graph G, $A(G)$ denotes the adjacency matrix of G, while P_n and C_n denote the path and cycle, respectively, on n vertices.

2.1.1 Results cited

The purpose of this section is to list results we will cite frequently, and assign abbreviated nomenclature to some of them.

Theorem 2.1.1. [2, Theorem 2.7] Suppose B is a nonsingular real symmetric matrix with $\mathrm{pr}(B) = r_0]r_1 \cdots r_n$. Let $\mathrm{pr}(B^{-1}) = r'_0]r'_1 \cdots r'_n$. Then $r'_n = r_n = 1$, while for each i with $1 \leq i \leq n-1$, $r'_i = r_{n-i}$. Finally, $r'_0 = 1$ if and only if B has some principal minor of order $n-1$ that is zero.

Theorem 2.1.2. [2, Theorem 4.4] (00 Theorem) *Let B be a real symmetric matrix. Let pr$(B) = r_0]r_1 \cdots r_n$ and suppose that, for some k with $0 \leq k \leq n-2$, $r_{k+1} = r_{k+2} = 0$. Then $r_i = 0$ for all $i \geq k+1$. In particular, $r_n = 0$, so that B is singular.*

Theorem 2.1.3. [2, Theorem 6.5] (0110 Theorem) *Suppose $n \geq 4$ and pr$(B) = r_0]r_1 \cdots r_n$. If, for some k with $1 \leq k \leq n-3$, $r_k = r_{k+3} = 0$, then $r_i = 0$ for all $k+3 \leq i \leq n$. In particular, B is singular.*

A generalization of Theorem 2.1.2 in [1] led to an analogous result for epr-sequences over any field:

Theorem 2.1.4. [3, Theorem 2.3] (`NN` Theorem) *Suppose B is a symmetric matrix over a field F, $\mathrm{epr}(B) = \ell_1\ell_2 \cdots \ell_n$, and $\ell_k = \ell_{k+1} = \mathtt{N}$ for some k. Then $\ell_i = \mathtt{N}$ for all $i \geq k$.*

(That is, if an epr-sequence of a matrix ever has `NN`*, then it must have* `N`*s from that point forward.)*

Theorem 2.1.5. [3, Theorem 2.4] (Inverse Theorem) *Suppose B is a nonsingular symmetric matrix over a field F. If* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_{n-1}\mathtt{A}$*, then* $\operatorname{epr}(B^{-1}) = \ell_{n-1}\ell_{n-2}\cdots\ell_1\mathtt{A}$*.*

Each instance of $\cdots$ below is permitted to be empty.

Proposition 2.1.6. [3, Proposition 2.5] *The epr-sequence* $\mathtt{SN}\cdots\mathtt{A}\cdots$ *is forbidden for symmetric matrices over any field.*

We say that $\mathtt{SN}\cdots\mathtt{A}\cdots$ is prohibited when referencing Proposition 2.1.6.

Theorem 2.1.7. [3, Theorem 2.6] (Inheritance Theorem) *Suppose that B is a symmetric matrix over a field F, $m \leq n$, and $1 \leq i \leq m$.*

1. *If* $[\operatorname{epr}(B)]_i = \mathtt{N}$*, then* $[\operatorname{epr}(C)]_i = \mathtt{N}$ *for all $m \times m$ principal submatrices C.*
2. *If* $[\operatorname{epr}(B)]_i = \mathtt{A}$*, then* $[\operatorname{epr}(C)]_i = \mathtt{A}$ *for all $m \times m$ principal submatrices C.*
3. *If* $[\operatorname{epr}(B)]_m = \mathtt{S}$*, then there exist $m \times m$ principal submatrices C_A and C_N of B such that* $[\operatorname{epr}(C_A)]_m = \mathtt{A}$ *and* $[\operatorname{epr}(C_N)]_m = \mathtt{N}$*.*
4. *If $i < m$ and* $[\operatorname{epr}(B)]_i = \mathtt{S}$*, then there exists an $m \times m$ principal submatrix C_S such that* $[\operatorname{epr}(C_S)]_i = \mathtt{S}$*.*

Corollary 2.1.8. [3, Corollary 2.7] *No symmetric matrix over any field can have* `NSA` *in its epr-sequence. Further, no symmetric matrix over any field can have the epr-sequence* $\cdots\mathtt{ASN}\cdots\mathtt{A}\cdots$*.*

Corollary 3.1.7 will be invoked by just stating that `NSA` or $\cdots\mathtt{ASN}\cdots\mathtt{A}\cdots$ is prohibited.

If B is a matrix with a nonsingular principal submatrix $B[\alpha]$, $B/B[\alpha]$ denotes the Schur complement of $B[\alpha]$ in B [6].

Theorem 2.1.9. [3, Proposition 2.13] (Schur Complement Theorem) *Suppose B is a symmetric matrix over a field of characteristic not 2 with $\operatorname{rank} B = m$. Let $B[\alpha]$ be a nonsingular principal submatrix of B with $|\alpha| = k \leq m$, and let $C = B/B[\alpha]$. Then the following results hold.*

1. *C is an $(n-k)\times(n-k)$ symmetric matrix.*
2. *Assuming the indexing of C is inherited from B, any principal minor of C is given by*
$$\det C[\gamma] = \det B[\gamma \cup \alpha]/\det B[\alpha].$$
3. *$\operatorname{rank} C = m - k$.*
4. *Any nonsingular principal submatrix of B of order at most m is contained in a nonsingular principal submatrix of order m.*

Theorem 2.1.10. [3, Theorem 2.14] *Neither the epr-sequences* `NAN` *nor* `NAS` *can occur as a subsequence of the epr-sequence of a symmetric matrix over a field of characteristic not* 2.

We will refer to Theorem 2.1.10 by simply stating that `NAN` or `NAS` is prohibited, while Theorem 4.3.12 below is referenced by stating that `ANS` 'must be initial.'

Theorem 2.1.11. [3, Theorem 2.15] *In the epr-sequence of a symmetric matrix over a field of characteristic not* 2*, the subsequence* `ANS` *can only occur as the initial subsequence.*

2.2 Restrictions on attainable epr-sequences

In this section, we establish new restrictions on attainable epr-sequences. We begin with restrictions that apply to fields of characteristic not 2. For convenience, given a matrix B, we adopt some of the notation in [2], and denote with $B_{i_1 i_2 \ldots i_k}$ the principal minor $\det(B[\{i_1, i_2, \ldots, i_k\}])$.

Proposition 2.2.1. *Let $n \geq 6$. Then no $n \times n$ symmetric matrix over a field of characteristic not 2 has an epr-sequence starting* `NSNA`$\cdots$.

Proof. Let $B = [b_{ij}]$ be an $n \times n$ symmetric matrix over a field of characteristic not 2 and let $\operatorname{epr}(B) = \ell_1\ell_2 \cdots \ell_n$. Suppose to the contrary that $\operatorname{epr}(B) = \mathtt{NSNA}\cdots$. Since $\ell_3 = \mathtt{N}$, and because $B_{pqr} = 2b_{pq}b_{pr}b_{qr}$ for any distinct $p, q, r \in \{1, 2, \ldots, n\}$, $B[\{1, 2, 3\}]$ and $B[\{4, 5, 6\}]$ must each contain a zero off-diagonal entry. Moreover, since $\ell_4 = \mathtt{A}$, 0_3 is not a principal submatrix of B, implying that $B[\{1, 2, 3\}]$ and $B[\{4, 5, 6\}]$ must each contain a nonzero off-diagonal entry. Since $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are disjoint, and because a simultaneous permutation of the rows and columns of a matrix has no effect on its determinant, we may assume, without loss of generality, that $b_{12} = b_{56} = 0$ and that b_{13}, b_{46} are nonzero. Similarly, since $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are disjoint, and because multiplication of any row and column of a matrix by a nonzero constant preserves symmetry and the rank of every submatrix, we may also assume, without loss of generality, that $b_{13} = b_{46} = 1$. We consider two cases.

Case 1: $b_{14} = 0$. Since $\ell_4 = \mathtt{A}$, $(b_{15}b_{24})^2 = B_{1245} \neq 0$; it follows that b_{15} and b_{24} are nonzero. Since $\ell_3 = \mathtt{N}$, $B_{135} = 2b_{15}b_{35} = 0$; hence, $b_{35} = 0$. Since $B[\{3, 5, 6\}] \neq 0_3$, $b_{36} \neq 0$. Since $2b_{16}b_{36} = B_{136} = 0$, $b_{16} = 0$. Then, as $B[\{1, 2, 6\}] \neq 0_3$, $b_{26} \neq 0$. It follows that $B_{246} = 2b_{24}b_{26} \neq 0$, a contradiction to $\ell_3 = \mathtt{N}$, implying that it is impossible to have $b_{14} = 0$.

Case 2: $b_{14} \neq 0$. Since $2b_{14}b_{34} = B_{134} = 0$, and because $2b_{14}b_{16} = B_{146} = 0$, $b_{34} = b_{16} = 0$. Since $B[\{1, 2, 6\}] \neq 0_3$, $b_{26} \neq 0$. Since $2b_{24}b_{26} = B_{246} = 0$, $b_{24} = 0$. Since $(b_{14}b_{23})^2 = B_{1234} \neq 0$, $b_{23} \neq 0$. Then, as $2b_{23}b_{26}b_{36} = B_{236} = 0$, $b_{36} = 0$. It follows that $B_{1356} = 0$, a contradiction to $\ell_4 = \mathtt{A}$. □

It should be noted that `NSNA` and `NSNAA` are attainable by $A(P_4)$ and $A(C_5)$, respectively [3], but this does not contradict Proposition 2.2.1, which requires $n \geq 6$.

Proposition 2.2.2. *Let B be a symmetric matrix over a field of characteristic not 2 and* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Then* `NSNA` *cannot occur as a subsequence of* $\ell_1\ell_2\cdots\ell_{n-2}$.

Proof. If $n \leq 5$, the result follows vacuously. So, assume $n \geq 6$. Suppose to the contrary that `NSNA` occurs as a subsequence of $\ell_1\ell_2\cdots\ell_{n-2}$ and that $\ell_k\ell_{k+1}\ell_{k+2}\ell_{k+3} = \mathtt{NSNA}$, for some k with $1 \leq k \leq n-5$. By Proposition 2.2.1, $k \geq 2$, and, by the `NN` Theorem, $\ell_{k-1} \neq \mathtt{N}$; it follows that B has a $(k-1)\times(k-1)$ nonsingular principal submatrix, say $B[\alpha]$. By the Schur Complement Theorem, $B/B[\alpha]$ has an epr-sequence starting $\mathtt{NXNAYZ}\cdots$, where $\mathtt{X}, \mathtt{Y}, \mathtt{Z} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$. The `NN` Theorem and the fact that `NAN` is prohibited imply that $\mathtt{X} = \mathtt{S}$; hence, $\operatorname{epr}(B)$ starts $\mathtt{NSNAYZ}\cdots$, a contradiction to Proposition 2.2.1.
□

With the next result, we generalize (and provide a simpler proof of) [3, Proposition 2.11].

Proposition 2.2.3. *Suppose B is a symmetric matrix over a field of characteristic not 2,* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$ *and* $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{SAN}$ *for some k. Then* $\ell_j = \mathtt{N}$ *for all* $j \geq k+2$.

Proof. If $n = 3$, we are done. Suppose $n > 3$. Suppose that $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{SAN}$ for some k with $1 \leq k \leq n-2$. If $k = n-2$, we are done. Suppose $k < n-2$. By [3, Corollary 2.10], which prohibits `SANA`, $\ell_{k+3} \neq \mathtt{A}$. Since `ANS` must be initial, $\ell_{k+3} \neq \mathtt{S}$. Hence, $\ell_{k+3} = \mathtt{N}$. The desired conclusion now follows from the `NN` Theorem. □

We now confine our attention to real symmetric matrices. The next result is immediate from Theorem 2.1.3.

Proposition 2.2.4. *Let B be a real symmetric matrix and* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Suppose* $\ell_k = \ell_{k+3} = \mathtt{N}$ *for some* $k \geq 1$. *Then* $\ell_i = \mathtt{N}$ *for all* $i \geq k+3$. *In particular, B is singular.*

We emphasize that Proposition 2.2.4 asserts that a sequence of the form $\cdots\mathtt{NXYN}\cdots\mathtt{Z}\cdots$, with $\mathtt{X}, \mathtt{Y} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$ and $\mathtt{Z} \in \{\mathtt{A}, \mathtt{S}\}$, is unattainable by a real symmetric matrix.

Given a sequence $t_{i_1}t_{i_2}\cdots t_{i_k}$, $\overline{t_{i_1}t_{i_2}\cdots t_{i_k}}$ indicates that the sequence may be repeated as many times as desired (or it may be omitted entirely). According to [3, Proposition 2.17], the sequence $\mathtt{ANA}\overline{\mathtt{A}}$ is attainable by a symmetric matrix over a field of characteristic 0. [3, Table 1] raises the following question: Does a real symmetric matrix, with an epr-sequence starting $\mathtt{ANA}\cdots$, always have epr-sequence $\mathtt{ANA}\overline{\mathtt{A}}$? The answer is affirmative; what follows makes this precise.

Proposition 2.2.5. *Any $n \times n$ real symmetric matrix with an epr-sequence starting $\mathtt{ANA}\cdots$ is conjugate by a nonsingular diagonal matrix to one of $\pm(J_n - 2I_n)$. Furthermore, its epr-sequence is $\mathtt{ANA}\overline{\mathtt{A}}$.*

Proof. Let $B = [b_{ij}]$ be an $n \times n$ real symmetric matrix with an epr-sequence starting $\mathtt{ANA}\cdots$. Notice that all the diagonal entries of B must have the same sign, as otherwise there would be a principal minor of order 2 that is nonzero. Let $C = [c_{ij}]$ be the matrix among B and $-B$ with all diagonal entries negative. Let $D = [d_{ij}]$ be the $n \times n$ diagonal matrix with $d_{11} = 1/\sqrt{-c_{11}}$ and $d_{jj} = \operatorname{sign}(c_{1j})/\sqrt{-c_{jj}}$ for $j \geq 2$. Now, notice that every entry of DCD is ± 1, every diagonal entry is -1 and every off-diagonal entry in the first row and the first column is 1. We now show that $DCD = J_n - 2I_n$. Since multiplication of any row and column of a matrix by a nonzero constant preserves the rank of every submatrix, $\operatorname{epr}(DCD) = \operatorname{epr}(C) = \operatorname{epr}(B)$. Let $i, j \in \{2, 3, \ldots, n\}$ be distinct, $\alpha = \{1, i, j\}$ and let a be the (i, j)-entry of DCD. A simple computation shows that $\det((DCD)[\alpha]) = (a + 1)^2$. Since every principal minor of order 3 of DCD is nonzero, $a = 1$. Then, as i and j were arbitrary, $DCD = J_n - 2I_n$. Then, as $C = B$ or $C = -B$, it follows that B is conjugate by a nonsingular diagonal matrix to one of $\pm(J_n - 2I_n)$, and that $\operatorname{epr}(B) = \operatorname{epr}(J_n - 2I_n) = \mathtt{ANA}\overline{\mathtt{A}}$ (see [3, Proposition 2.17]). □

We are now in position to prove the following result.

Theorem 2.2.6. *Any epr-sequence of a real symmetric matrix containing $\mathtt{ANA}$ as a non-terminal subsequence is of the form $\overline{\mathtt{A}}\mathtt{ANAA}\overline{\mathtt{A}}$.*

Proof. Let B be a real symmetric matrix containing $\mathtt{ANA}$ as a non-terminal subsequence. Let $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_{k+1}\ell_{k+2}\ell_{k+3} = \mathtt{ANA}$ for some k with $0 \leq k \leq n-4$. Since $\mathtt{NAN}$ and $\mathtt{NAS}$ are prohibited, $\ell_{k+4} = \mathtt{A}$. If $k = 0$, the conclusion follows from Proposition 2.2.5; so, assume $k > 0$. Suppose $\ell_i \neq \mathtt{A}$ for some i with $i < k+1$. By the Inheritance Theorem, B has a (nonsingular) $(k+4)\times(k+4)$ principal submatrix B' whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_{k+4}$ ends with $\mathtt{ANAA}$ and has $\ell'_i \neq \mathtt{A}$. Then, by the Inverse Theorem, $\operatorname{epr}((B')^{-1})$ starts with $\mathtt{ANA}$ and $\operatorname{epr}((B')^{-1}) \neq \mathtt{ANA}\overline{\mathtt{A}}$, a contradiction to Proposition 2.2.5. Thus, $\operatorname{epr}(B) = \overline{\mathtt{A}}\mathtt{AANAA}\ell_{k+5}\cdots\ell_n$, where $\ell_{k+5}\cdots\ell_n$ may not exist.

We now show that $\ell_{k+5}\cdots\ell_n = \overline{\mathtt{A}}$. If $n = k+4$, we are done; so, suppose $n > k+4$. We proceed by contradiction, and consider two cases.

Case 1: $\ell_j = \mathtt{N}$ for some $j > k+4$. Since $\ell_k = \mathtt{A}$, there exists a $k \times k$ principal submatrix of B, say $B[\alpha]$, that is nonsingular. Let $C = B/B[\alpha]$. By the Schur Complement Theorem, C has order $n-k$, $\operatorname{epr}(C)$ starts $\mathtt{ANA}\cdots$ and $\operatorname{epr}(C)$ has an $\mathtt{N}$ in the $(j-k)$-th position; hence, $\operatorname{epr}(C) \neq \mathtt{ANA}\overline{\mathtt{A}}$, a contradiction to Proposition 2.2.5. It follows that a sequence containing $\mathtt{ANA}$ as a non-terminal subsequence cannot contain an $\mathtt{N}$ from that point forward, implying that any real symmetric matrix with an epr-sequence containing $\mathtt{ANA}$ is nonsingular.

Case 2: $\ell_j = \mathtt{S}$ for some $j > k+4$. By the Inheritance Theorem, B has a singular $j\times j$ principal submatrix whose epr-sequence contains $\mathtt{ANA}$, which contradicts the assertion above.

We conclude that we must have $\ell_{k+5}\cdots\ell_n = \overline{\mathtt{A}}$, which completes the proof. □

It is natural to now ask, does Theorem 2.2.6 hold if $\mathtt{ANA}$ occurs at the end of the sequence? According to [3, Table 5], $\mathtt{SAANA}$ is attainable, answering the question negatively.

Theorem 2.2.7. *Let B be a real symmetric matrix with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Then* $\mathtt{SNA}$ *cannot occur as a subsequence of* $\ell_1\ell_2\cdots\ell_{n-2}$.

Proof. If $n \leq 4$, the result follows vacuously. So, assume $n > 4$. Suppose to the contrary that $\mathtt{SNA}$ occurs as a subsequence of $\ell_1\ell_2\cdots\ell_{n-2}$, and that $\ell_{k+1}\ell_{k+2}\ell_{k+3} = \mathtt{SNA}$ for some k with $0 \leq k \leq n-5$. Since $\mathtt{SN}\cdots\mathtt{A}\cdots$ is prohibited, $k \geq 1$. Since $\mathtt{NAN}$ and $\mathtt{NAS}$ are prohibited, $\ell_{k+4} = \mathtt{A}$. Then, as $\mathtt{ASNA}$ is prohibited, $\ell_k \neq \mathtt{A}$. And, by Proposition 2.2.2, $\ell_k \neq \mathtt{N}$; it follows that $\ell_k = \mathtt{S}$. Thus, we have $\ell_k \cdots \ell_{k+4} = \mathtt{SSNAA}$. We examine the three possibilities for ℓ_{k+5} .

Case 1: $\ell_{k+5} = \mathtt{A}$. Now we have $\ell_k \cdots \ell_{k+5} = \mathtt{SSNAAA}$. By the Inheritance Theorem, B has a $(k+5) \times (k+5)$ principal submatrix B' whose epr-sequence ends with $\mathtt{SXNAAA}$, where $\mathtt{X} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$. By the $\mathtt{NN}$ Theorem, $\mathtt{X} \neq \mathtt{N}$; and, by Proposition 2.2.3, $\mathtt{X} \neq \mathtt{A}$; it follows that $\mathtt{X} = \mathtt{S}$. By the Inverse Theorem, $\operatorname{epr}((B')^{-1})$ contains $\mathtt{ANS}$ as a non-initial subsequence, a contradiction, since $\mathtt{ANS}$ must be initial. We conclude that $\ell_{k+5} \neq \mathtt{A}$.

Case 2: $\ell_{k+5} = \mathtt{N}$. Now we have $\ell_k \cdots \ell_{k+5} = \mathtt{SSNAAN}$. Since $\ell_k = \mathtt{S}$, B has a $k \times k$ nonsingular principal submatrix, say $B[\alpha]$. By the Schur Complement Theorem, $B/B[\alpha]$ has an epr-sequence starting $\mathtt{YNAAN}\cdots$, where $\mathtt{Y} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$. By Theorem 2.2.6, $\mathtt{Y} \neq \mathtt{A}$; since $\mathtt{SN}\cdots\mathtt{A}\cdots$ is prohibited, $\mathtt{Y} \neq \mathtt{S}$; and, by the $\mathtt{NN}$ Theorem, $\mathtt{Y} \neq \mathtt{N}$. It follows that we must have $\ell_{k+5} \neq \mathtt{N}$.

From Cases 1 and 2 we can deduce that the subsequence $\mathtt{SSNAAZ}$, where $\mathtt{Z} \in \{\mathtt{A}, \mathtt{N}\}$, cannot occur in the epr-sequence of a real symmetric matrix.

Case 3: $\ell_{k+5} = \mathtt{S}$. Now we have $\ell_k \cdots \ell_{k+5} = \mathtt{SSNAAS}$. By the Inheritance Theorem, B has a $(k+5) \times (k+5)$ principal submatrix with an epr-sequence ending with $\mathtt{SXNAAY}$, where $\mathtt{X} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$ and $\mathtt{Y} \in \{\mathtt{A}, \mathtt{N}\}$. By the $\mathtt{NN}$ Theorem, $\mathtt{X} \neq \mathtt{N}$; and, by Proposition 2.2.3, $\mathtt{X} \neq \mathtt{A}$. It follows that $\mathtt{X} = \mathtt{S}$, which contradicts the assertion above. □

As $\mathtt{NAN}$ is prohibited, we have the following corollary to Theorem 4.3.9.

Corollary 2.2.8. *The only way* $\mathtt{SNA}$ *can occur in the epr-sequence of a real symmetric matrix is in one of the two terminal sequences* $\mathtt{SNA}$ *or* $\mathtt{SNAA}$.

We note that the epr-sequences `ANSSSNA` and `ANSSSNAA` are attainable [3, Table 1], implying that `SNA` is not completely prohibited in the epr-sequence of a real symmetric matrix. Theorem 2.2.6 and Corollary 2.2.8 lead to the following observation.

Observation 2.2.9. *Any epr-sequence of a real symmetric matrix that contains* `NA` *as a non-initial subsequence is of the form* $\cdots \mathtt{NA}\overline{\mathtt{A}}$.

The following results in this section will be of particular relevance to the main results in Sections 2.3 and 2.4.

Lemma 2.2.10. *Let n be even and B be a nonsingular $n \times n$ real symmetric matrix. Then $J_{\frac{n}{2}+1}$ is not a principal submatrix of B.*

Lemma 2.2.11. *Let $n \geq 8$ be even. Let B be an $n \times n$ nonsingular real symmetric matrix with every entry ± 1 and all entries in the first row, the first column, and the diagonal equal to 1. Suppose that* $\operatorname{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$ *and that* $\ell_4 = \mathtt{N}$. *Then every row and column of B has at most $\frac{n}{2} - 1$ negative entries. Equivalently, every row and column of B has at least $\frac{n}{2} + 1$ positive entries.*

Proof. Suppose $B = [b_{ij}]$ contains a row with $\frac{n}{2}$ negative entries. Let $U = \{3, 4, \ldots, \frac{n}{2} + 2\}$. Without loss of generality, suppose $b_{2j} = -1$ for all $j \in U$. We claim that $B[\{1\} \cup U] = J_{\frac{n}{2}+1}$. Suppose to the contrary that $B[\{1\} \cup U]$ contains a negative entry; without loss of generality, we may assume that this entry is b_{34}. It follows that $B[\{1, 2, 3, 4\}]$ is nonsingular, a contradiction to $\ell_4 = \mathtt{N}$; hence, $B[\{1\} \cup U] = J_{\frac{n}{2}+1}$. By Lemma 2.2.10, B is singular, a contradiction to the nonsingularity of B. We conclude that every row and column of B has at most $\frac{n}{2} - 1$ negative entries. □

Theorem 2.2.12. *Let $n \geq 8$ be even and B be an $n \times n$ real symmetric matrix. Suppose that* $\operatorname{epr}(B) = \mathtt{ANSNSN}\cdots$. *Then B is singular.*

Proof. Suppose to the contrary that B is nonsingular. Let $B = [b_{ij}]$. By [2, Proposition 8.1], we may assume that every entry of B is ± 1 and all entries in the first row, the first column, and the diagonal are equal to 1. By Lemma 2.2.11, every row and column of B has at least $\frac{n}{2}+1$ positive entries. Because a simultaneous permutation of the rows and columns of a matrix has no effect on its determinant, we may assume, without loss of generality, that the first $\frac{n}{2}+1$ entries in the second row (and column) are positive. Let

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Since M_1 and M_2 are nonsingular, they are not principal submatrices of B. We now show by induction on the number of negative entries in the second row that B contains a row with $\frac{n}{2}$ negative entries. For the base case, first notice that the nonsingularity of B implies that B must have a row with at least one negative entry, as otherwise it will have a repeated row; without loss of generality, we assume that $b_{2n} = -1$. By Lemma 2.2.10, $B[\{1,\dots,\frac{n}{2}+1\}]$ has a negative entry; without loss of generality, suppose $b_{34} = -1$. Then, as $B[\{2,3,4,n\}] \neq M_2$, either b_{3n} or b_{4n} is negative, implying that either the third or fourth row contains two negative entries. It follows that B must contain a row with two negative entries.

Now, for the inductive step, suppose the second row contains $2 \leq k \leq \frac{n}{2}-1$ negative entries. Without loss of generality, suppose $b_{2j} = -1$ for $j \in U = \{n-k+1,\dots,n\}$. As in the base case, Lemma 2.2.10 implies that $B[\{1,\dots,\frac{n}{2}+1\}]$ has a negative entry, and, again, without loss of generality, we may assume that $b_{34} = -1$. Since $B[\{1,2,p,q\}] \neq M_1$ for $p,q \in U$, $b_{pq} = 1$ for all $p,q \in U$. Similarly, $B[\{1,3,4,j\}] \neq M_1$ and $B[\{2,3,4,j\}] \neq M_2$ for $j \in U$, implying that $b_{3j} \neq b_{4j}$ for all $j \in U$; so, suppose $b_{3j} = x_j$ and $b_{4j} = -x_j$ for all $j \in U$. Then, as $\ell_6 = \mathtt{N}$, $-16(x_p - x_q)^2 = \det B[\{1,2,3,4,p,q\}] = 0$ for all $p,q \in U$; hence, $x_p = x_q$ for all $p,q \in U$. It follows that either the third or the fourth row contains $(n-(n-k+1)+1)+1 = k+1$ negative entries. Hence, by induction, B most have a row with $\frac{n}{2}$ negative entries; by Lemma 2.2.11, B is singular, a contradiction. $\square$

We note that Theorem 2.2.12 cannot be generalized for n odd, since, by the Inverse Theorem, $\mathtt{AN}\overline{\mathtt{SN}}\mathtt{A}$ is attained by $(A(C_n))^{-1}$ (see [3, Observation 3.1]).

Proposition 2.2.13. *No real symmetric matrix has an epr-sequence starting* SSNSNSS$\cdots$.

Proof. Let $B = [b_{ij}]$ be a real symmetric with an epr-sequence starting SSNSNSS$\cdots$. By the Inheritance Theorem, B has a 7×7 principal submatrix $B[\alpha]$ with epr-sequence $\ell_1'\ell_2'\mathtt{N}\ell_4'\mathtt{N}\ell_6'\mathtt{A}$. Without loss of generality, suppose $\alpha = \{2, 3, \ldots, 8\}$. By the NN Theorem, ℓ_2', ℓ_4', ℓ_6' are not N. Since NAN and NSA are prohibited, $\ell_4' = \mathtt{S}$ and $\ell_6' = \mathtt{A}$. Since ANS must be initial, $\ell_2' = \mathtt{S}$. Hence, $\operatorname{epr}(B[\alpha]) = \ell_1'\mathtt{SNSNAA}$. Since ASN$\cdots$A is prohibited, $\ell_1' \neq \mathtt{A}$. Then, as the epr-sequence SSNSNAA is associated with the pr-sequence 1]1101011, which is unattainable by [1, Proposition 4.1], $\ell_1' \neq \mathtt{S}$; hence, $\ell_1' = \mathtt{N}$, so that $\operatorname{epr}(B[\alpha]) = \mathtt{NSNSNAA}$. We note that a simultaneous permutation of the rows and columns of a matrix has no effect on its determinant; thus, since all diagonal entries of $B[\alpha]$ are zero, and because B contains a nonzero diagonal entry, we may assume, without loss of generality, that $b_{11} \neq 0$.

Let $C = B[\{1\} \cup \alpha]$ and $C = [c_{ij}]$. Then $\operatorname{epr}(C)$ starts with S and $\operatorname{epr}(C[\alpha]) = \operatorname{epr}(B[\alpha]) = \mathtt{NSNSNAA}$. Since every 6×6 principal submatrix of $C[\alpha]$ is nonsingular, $C[\alpha]$ contains at least two nonzero entries in each row (and column), as otherwise $C[\alpha]$ contains a 6×6 principal submatrix with a row (and column) consisting of only zeros. Moreover, we note that $c_{11} = b_{11} \neq 0$; because multiplication of any row and column of a matrix by a nonzero constant preserves the rank of every submatrix, we may assume without loss of generality that $c_{11} = 1$. Since $C[\alpha]$ contains a nonzero principal minor of order 2, we may assume, without loss of generality, that $\det((C[\alpha])[\{1, 2\}]) \neq 0$; thus, $-(c_{23})^2 = C_{23} = (C[\alpha])[\{1, 2\}] \neq 0$; hence, $c_{23} \neq 0$, and, without loss of generality, we may assume that $c_{23} = 1$. Since $C[\alpha]$ contains at least two nonzero entries in each row and column, $c_{2j} \neq 0$ for some $j \in \{4, 5, 6, 7, 8\}$; so, we may assume that $c_{24} = 1$. It follows that $2c_{34} = C_{234} = 0$, and so $c_{34} = 0$. Then, as $C[\alpha]$ contains at least two nonzero entries in each row and column, $c_{3j} \neq 0$ for some $j \in \{5,6,7,8\}$; thus, suppose $c_{35} = 1$. It follows that $2c_{25} = C_{235} = 0$, and so $c_{25} = 0$. Now we have $-1 + 2c_{12}c_{13} = C_{123} = 0$, $-1 + 2c_{12}c_{14} = C_{124} = 0$ and $-1 + 2c_{13}c_{15} = C_{135} = 0$; it follows that c_{12}, c_{13}, c_{14} and c_{15} are nonzero. Let $c_{12} = x$; then $c_{13} = c_{14} = 1/2x$ and $c_{15} = x$. We now show that each of c_{16}, c_{17} and c_{18} is nonzero. Suppose to the contrary that $c_{1j} = 0$ for some $j \in \{6,7,8\}$; then $-(c_{ij})^2 = C_{1ij} = 0$ for all $i \in \{3,4,\ldots,8\} \setminus \{j\}$; hence, $c_{ij} = 0$ for all $i \in \{3,4,\ldots,8\}$, implying that $C[\alpha]$ contains a row with only one nonzero entry, which is a contradiction. Without loss of generality, we may assume that $c_{16} = c_{17} = c_{18} = 1$. Now, observe that $C_{145} = c_{45}(1 - c_{45})$; since all the principal minors of order 3 are zero, it follows that $c_{45} = 0$ or $c_{45} = 1$. Besides for the $(1,2)$-entry x, we have similar restrictions for all the remaining unknown entries of C; notice that, for $j \in \{6,7,8\}$, $C_{12j} = c_{2j}(2x - c_{2j})$, $C_{13j} = c_{3j}(1/x - c_{3j})$, $C_{14j} = c_{4j}(1/x - c_{4j})$ and $C_{15j} = c_{5j}(2x - c_{5j})$. Similarly, for $k \in \{7,8\}$, $C_{16k} = c_{6k}(2 - c_{6k})$. Lastly, $C_{178} = c_{78}(2 - c_{78})$. It is now clear that, besides the $(1,2)$-entry x, each unknown entry of C is restricted to exactly two values.

We now show that $c_{45} = 1$. Suppose to the contrary that $c_{45} = 0$. Since $C[\alpha]$ must contain at least two nonzero entries in each row and column, without loss of generality, we may assume that b_{56} is nonzero, implying that $c_{56} = 2x$. Then $4xc_{36} = C_{356} = 0$, and therefore $c_{36} = 0$. We proceed by examining the only two possibilities for the entry c_{26}. First, suppose $c_{26} = 0$. Since all the principal minors of order 5 of C are zero, $4xc_{46} = C_{23456} = 0$, implying that $c_{46} = 0$. Then $C_{12456} = -4x^2 \neq 0$, a contradiction. So, suppose $c_{26} = 2x$. Since $4xc_{46} = C_{246} = 0$, $c_{46} = 0$. Since $C[\alpha]$ must contain at least two nonzero entries in each row and column, suppose, without loss of generality, that $c_{47} \neq 0$; hence, $c_{47} = 1/x$. Since $2c_{27}/x = C_{247} = 0$, $c_{27} = 0$. Now, observe that $C_{13457} = (-2x + 2x^2c_{37} + c_{57} - xc_{37}c_{57})/2x^3$ and $C_{23457} = 2c_{57}/x - 2c_{37}c_{57}$; since $C_{13457} = 0$, at least one of c_{37} and c_{57} is nonzero; then, as $C_{23457} = 0$, $c_{37} \neq 0$, and so $c_{37} = 1/x$. It follows that $2c_{57}/x = C_{357} = 0$, and so $c_{57} = 0$. As $-4 + 2c_{67} = C_{14567} = 0$, $c_{67} = 2$.

Then we have $C_{234567} = 0$, implying that $C[\alpha]$ has a singular 6×6 principal submatrix, which is a contradiction. We conclude that $c_{45} \neq 0$; hence, $c_{45} = 1$.

Now, observe that at least one of c_{36}, c_{37}, c_{38}, c_{46}, c_{47} and c_{48} is nonzero, as otherwise $C[\alpha]$, which is nonsingular, would have two identical rows; thus, without loss of generality, we assume that $c_{36} \neq 0$; hence, $c_{36} = 1/x$. Similarly, at least one of c_{27}, c_{28}, c_{57} and c_{58} is nonzero, as otherwise $C[\{2,3,4,5,7,8\}] = (C[\alpha])[\{1,2,3,4,6,7\}]$, which is nonsingular, would have two identical rows; without loss of generality, we assume that $c_{27} \neq 0$; thus, $c_{27} = 2x$. Now the conditions $C_{236} = C_{237} = C_{247} = C_{356} = 0$ imply that $c_{26} = c_{37} = c_{47} = c_{56} = 0$.

Finally, we consider the only two possibilities for the entry c_{57}. First, suppose $c_{57} = 2x$. Then $C_{234567} = 0$, a contradiction. Now, suppose $c_{57} = 0$. Since $C_{234567} = -4x^2(c_{46} - 1/x)^2$ is nonzero, $c_{46} = 0$. Then $-2c_{67} = C_{14567} = 0$, and so $c_{67} = 0$. Since every row and column of $C[\alpha]$ must contain at least two nonzero entries, it follows that c_{68} and c_{78} are nonzero, implying that $c_{68} = c_{78} = 2$. The conditions $C_{278} = C_{368} = 0$ imply that $c_{28} = c_{38} = 0$. Hence, $C_{23678} = 16 \neq 0$, a contradiction. □

2.3 Pr-sequences not containing three consecutive 1s

We begin with results that forbid certain pr-sequences not containing three consecutive 1s; we then implement these in Theorem 2.3.10, where, for real symmetric matrices, we classify all the attainable pr-sequences not containing three consecutive 1s.

It is obvious from Theorem 2.1.1 that, with the exception of the 0th term r_0', we can explicitly determine each term in the pr-sequence of the inverse of a nonsingular real symmetric matrix B. The next result demonstrates that, when $n \geq 3$, r_0' can always be determined from $\mathrm{pr}(B)$ if this sequence does not end with 111.

Remark 2.3.1. Let $n \geq 3$, B be a nonsingular real symmetric matrix with $\mathrm{pr}(B) = r_0]r_1 \cdots r_{n-1}1$ and r_0' be the 0th term of $\mathrm{pr}(B^{-1})$.

(i) If $r_{n-1}r_n = 01$, then $r_0' = 1$.

(ii) If $r_{n-2}r_{n-1}r_n = 011$, then $r_0' = 0$.

(i) is immediate from Theorem 2.1.1, since B obviously has a principal minor of order $n-1$ that is zero. As for (ii), first, notice that the penultimate term of $\text{epr}(B)$ must be `A`, as `NSA` is prohibited; therefore, B does not have a principal minor of order $n-1$ that is zero, implying that $r_0' = 0$.

The next proposition generalizes a particular case of [2, Lemma 4.5].

Proposition 2.3.2. *Let B be a real symmetric matrix with $\text{pr}(B) = r_0]r_1 \cdots r_n$. Suppose that $\text{pr}(B)$ does not contain three consecutive 1s and that $r_0]r_1 \neq 1]1$. Then, for any m with $1 \leq m \leq n$, there exists a principal submatrix B' of B such that $\text{pr}(B') = r_0]r_1 \cdots r_m$.*

Proof. Let $1 \leq m \leq n$. By [2, Lemma 4.5], B has a principal submatrix B' with $\text{pr}(B') = r_0']r_1r_2 \cdots r_m$. Since B does not contain both a zero and a nonzero diagonal entry, it follows that $r_0']r_1 = r_0]r_1$, and therefore $\text{pr}(B') = r_0]r_1 \cdots r_m$. □

Corollary 2.3.3. *Let $\sigma = r_0]r_1 \cdots r_n$ be a pr-sequence not containing three consecutive 1s. Suppose $r_0]r_1 \neq 1]1$. If any initial subsequence of σ is unattainable, then σ is unattainable.*

It was shown in [2] that appending 0 to the end of an attainable pr-sequence results in a new attainable pr-sequence; but what if 0 is appended to an unattainable pr-sequence? For example, if we append 0 to $1]1011$, which is unattainable (see [2, Table 5.4]), we obtain the attainable pr-sequence $1]10110$ (see [2, Table 6.1]). However, there are some cases where appending 0 preserves unattainability. The next observation, a consequence of Corollary 2.3.3, illustrates this.

Observation 2.3.4. *Let $r_0]r_1 \cdots r_n$ be an unattainable pr-sequence not containing three consecutive 1s. Suppose $r_0]r_1 \neq 1]1$. Then $r_0]r_1 \cdots r_n0$ is also unattainable.*

Propositions 2.3.5 and 2.3.7 below are corollaries to Theorem 2.2.12.

Proposition 2.3.5. *Let B be a real symmetric matrix with* $\operatorname{epr}(B) = \mathtt{ANSNSN}\cdots$. *Then, for $k \geq 1$, $\ell_{2k} = \mathtt{N}$. Furthermore,* $\operatorname{epr}(B) = \mathtt{ANSNSN}\overline{\mathtt{SN}}\,\overline{\mathtt{N}}$ *or* $\operatorname{epr}(B) = \mathtt{ANSNSN}\overline{\mathtt{SN}}\mathtt{A}$.

Proof. Let $k \geq 1$. By hypothesis, the first assertion holds for $k \leq 3$. Suppose $\ell_{2k} \neq \mathtt{N}$ for some $k > 3$. By the Inheritance Theorem, B has a nonsingular $2k \times 2k$ principal submatrix with epr-sequence $\mathtt{ANXNYN}\cdots\mathtt{A}$, where $\mathtt{X}$, $\mathtt{Y}$, $\mathtt{Z} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$. By the $\mathtt{NN}$ Theorem, $\mathtt{X}$ and $\mathtt{Y}$ are not $\mathtt{N}$. Since $\mathtt{NAN}$ is prohibited, $\mathtt{X} = \mathtt{Y} = \mathtt{S}$, a contradiction to Theorem 2.2.12. The final assertion is immediate from the $\mathtt{NN}$ Theorem and the fact that $\mathtt{NAN}$ is prohibited. □

Corollary 2.3.6. *The pr-sequence* $0]1010101\overline{01}1\overline{0}$ *is unattainable by a real symmetric matrix.*

Proof. Since $0]1010101\overline{01}1$ satisfies the hypothesis of Observation 2.3.4, it suffices to show that this sequence is unattainable. Suppose that there is a real symmetric matrix B with $\operatorname{pr}(B) = 0]1010101\overline{01}1$ and $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Obviously, $\ell_1 = \ell_n = \mathtt{A}$ and $\ell_2 = \ell_4 = \ell_6 = \mathtt{N}$. Since $\mathtt{NAN}$ is prohibited, $\ell_3 = \ell_5 = \mathtt{S}$. Hence, $\operatorname{epr}(B) = \mathtt{ANSNSN}\cdots\mathtt{XA}$, where $\mathtt{X}$ is not $\mathtt{N}$, which contradicts Proposition 2.3.5. □

Proposition 2.3.7. *Let B be a real symmetric matrix with* $\operatorname{epr}(B) = \mathtt{SNSNSN}\cdots$. *Then, for $k \geq 1$, $\ell_{2k} = \mathtt{N}$. Furthermore,* $\operatorname{epr}(B) = \mathtt{SNSNSN}\overline{\mathtt{SN}}\,\overline{\mathtt{N}}$ *or* $\operatorname{epr}(B) = \mathtt{SNSNSN}\overline{\mathtt{SN}}\mathtt{A}$.

Proof. Let $k \geq 1$. By hypothesis, the first assertion holds for $k \leq 3$. Suppose $\ell_{2k} \neq \mathtt{N}$ for some $k > 3$. By the Inheritance Theorem, B has a nonsingular $2k \times 2k$ principal submatrix with an epr-sequence $\mathtt{XNYNZN}\cdots\mathtt{A}$, where $\mathtt{X}$, $\mathtt{Y}$, $\mathtt{Z} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$. By the $\mathtt{NN}$ Theorem, $\mathtt{X}$, $\mathtt{Y}$ and $\mathtt{Z}$ are not $\mathtt{N}$. Since $\mathtt{NAN}$ is prohibited, $\mathtt{Y} = \mathtt{Z} = \mathtt{S}$. Since $\mathtt{SN}\cdots\mathtt{A}\cdots$ is prohibited, $\mathtt{X} \neq \mathtt{S}$, and hence $\mathtt{X} = \mathtt{A}$, a contradiction to Theorem 2.2.12. As in Proposition 2.3.5, the final assertion follows from the $\mathtt{NN}$ Theorem and the fact that $\mathtt{NAN}$ is prohibited.

□

Corollary 2.3.8. *The pr-sequence* $1]1010101\overline{01}10\overline{0}$ *is unattainable by a real symmetric matrix.*

Proof. Suppose there is a real symmetric matrix B with $\mathrm{pr}(B) = 1]1010101\overline{01}10\overline{0}$. Let $\mathrm{epr}(B) = \ell_1\ell_2 \cdots \ell_n$. Obviously, $\ell_1 = \mathtt{S}$ and $\ell_2 = \ell_4 = \ell_6 = \mathtt{N}$. Since `NAN` is prohibited, $\ell_3 = \ell_5 = \mathtt{S}$. Hence, $\mathrm{epr}(B) = \mathtt{SNSNSN} \cdots \mathtt{XYN}\overline{\mathtt{N}}$, where `X` and `Y` are both not `N`, which contradicts Proposition 2.3.7. □

Before proving the main result of this section, we need a lemma.

Lemma 2.3.9. *Let* B *be a real symmetric matrix with* $\mathrm{pr}(B) = r_0]r_1 \cdots r_n$. *Suppose* $r_1r_2 \cdots r_n$ *does not contain three consecutive* 1*s. Let* $1 \le k \le \mathrm{rank}(B) - 2$. *If* $r_kr_{k+1} = 01$, *then either* $r_{k+2}r_{k+3} \cdots r_n = \overline{01}1\overline{0}$ *or* $r_{k+2}r_{k+3} \cdots r_n = \overline{01}01\overline{0}$

Proof. Suppose $r_kr_{k+1} = 01$. We proceed by examining the only two possibilities for r_{k+2}.

Case 1: $r_{k+2} = 1$. Now we have $r_kr_{k+1}r_{k+2} = 011$. If $n = k + 2$, then we are done. Now, suppose $n > k + 2$. By hypothesis, $r_{k+3} = 0$, and therefore, by the 0110 Theorem, $r_{k+2}r_{k+3} \cdots r_n = 1\overline{0}$, where $\overline{0}$ is non-empty.

Case 2: $r_{k+2} = 0$. Now we have $r_kr_{k+1}r_{k+2} = 010$. Then, as $\mathrm{rank}(B) \ge k + 2$, by the 00 Theorem, $r_{k+3} \ne 0$; hence, $r_{k+3} = 1$, and so $r_{k+2}r_{k+3} = 01$. If $n = k + 3$, then we are done. Suppose $n > k + 3$. If $\mathrm{rank}(B) = k + 3$, then we have $r_{k+2}r_{k+3} \cdots r_n = 01\overline{0}$, where $\overline{0}$ is non-empty. Suppose $\mathrm{rank}(B) > k + 3$, i.e., suppose $\mathrm{rank}(B) \ge k + 4$. Thus, so far we have $r_kr_{k+1}r_{k+2}r_{k+3} = 0101$, where $r_{k+2}r_{k+3} = 01$ and $1 \le k + 2 \le \mathrm{rank}(B) - 2$. Since n is finite, it is evident that reimplementing the steps above by replacing k with $k + 2$, and repeating this process until reaching the last term of the sequence, yields the desired conclusion. □

With the next theorem, we classify all the attainable pr-sequences of order $n \ge 3$ not containing three consecutive 1s.

Theorem 2.3.10. *Let $n \geq 3$. A pr-sequence of order n not containing three consecutive* 1*s is attainable by a real symmetric matrix if and only if it is one of the following sequences.*

1. $0]100\overline{0}$.
2. $0]1\overline{01}01\overline{0}$.
3. $0]1011\overline{0}$.
4. $0]101011\overline{0}$.
5. $0]110\overline{0}$.
6. $0]1101\overline{0}$.
7. $0]11011\overline{0}$.
8. $1]000\overline{0}$.
9. $1]010\overline{0}$.
10. $1]01\overline{01}01\overline{0}$.
11. $1]01\overline{01}1\overline{0}$.
12. $1]100\overline{0}$.
13. $1]1\overline{01}010\overline{0}$.
14. $1]10110\overline{0}$.
15. $1]1010110\overline{0}$.

Proof. Let B be a real symmetric matrix with $\text{pr}(B) = r_0]r_1 \cdots r_n$ not containing three consecutive 1s. Since $0]0\cdots$ is forbidden by definition, $r_0]r_1 \in \{0]1, 1]0, 1]1\}$. We proceed by examining all the possibilities for $r_0]r_1r_2$.

Case i: $r_0]r_1r_2 = 0]10$. If $r_3 = 0$, then, by the 00 Theorem, we have sequence (1). Suppose $r_3 = 1$. Hence, $\mathrm{pr}(B)$ starts $0]101\cdots$. If $\mathrm{rank}(B) = 3$, then $\mathrm{pr}(B) = 0]101\overline{0}$, which is sequence (2). Now, suppose $\mathrm{rank}(B) > 3$. Then $r_2r_3 = 01$ and $1 \leq 2 \leq \mathrm{rank}(B) - 2$; hence, by applying Lemma 2.3.9 to $\mathrm{pr}(B)$, starting with $k = 2$, we have either $\mathrm{pr}(B) = 0]101\overline{01}010\overline{0}$ or $\mathrm{pr}(B) = 0]101\overline{01}1\overline{0}$. Hence, by Corollary 2.3.6, $\mathrm{pr}(B)$ is one of the sequences (2), (3) and (4).

Case ii: $r_0]r_1r_2 = 0]11$. By hypothesis, $r_3 = 0$. If $\mathrm{rank}(B) = 2$, then $\mathrm{pr}(B) = 0]110\overline{0}$, which is sequence (5). Now suppose $\mathrm{rank}(B) > 2$. Then $n > 3$ and, by the 00 Theorem, $r_4 \neq 0$, implying that $r_4 = 1$. Hence, $\mathrm{pr}(B)$ starts $0]1101\cdots$. If $n = 4$, then we have sequence (6). Suppose $n > 4$. If $r_5 = 1$, then, by the 0110 Theorem, we must have sequence (7), where $\overline{0}$ may be empty. Now, suppose $r_5 = 0$. If $n = 5$, then we have sequence (6). Suppose $n > 5$. Thus far we have $\mathrm{pr}(B) = 0]11010\cdots$; it follows from [2, Theorem 7.2] that $r_6 = 0$, and therefore, by the 00 Theorem, we have sequence (6).

Case iii: $r_0]r_1 = 1]0$. If $r_2 = 0$, then, by the 00 Theorem, we have sequence (8). Now, suppose $r_2 = 1$. Hence, $\mathrm{pr}(B)$ starts $1]01\cdots$. If $\mathrm{rank}(B) = 2$, then $\mathrm{pr}(B) = 1]010\overline{0}$, which is sequence (9). Now, suppose $\mathrm{rank}(B) > 2$. Then $r_1r_2 = 01$ and $1 \leq 1 \leq \mathrm{rank}(B) - 2$; hence, by applying Lemma 2.3.9 to $\mathrm{pr}(B)$, starting with $k = 1$, we have either $\mathrm{pr}(B) = 1]01\overline{01}010\overline{0}$ or $\mathrm{pr}(B) = 1]01\overline{01}1\overline{0}$. Thus, $\mathrm{pr}(B)$ is either sequence (10) or (11).

Case iv: $r_0]r_1 = 1]1$. By hypothesis, $r_2 = 0$. If $r_3 = 0$, then the 00 Theorem implies that we have sequence (12). Now, suppose $r_3 = 1$. Hence, $\mathrm{pr}(B)$ starts $1]101\cdots$. Suppose $\mathrm{rank}(B) = 3$; then $\mathrm{pr}(B) = 1]101\overline{0}$, and, by [2, Theorem 4.1], $\overline{0}$ is non-empty, implying that $\mathrm{pr}(B) = 1]1010\overline{0}$, which is sequence (13). Now, suppose $\mathrm{rank}(B) > 3$. Then $r_2r_3 = 01$ and $1 \leq 2 \leq \mathrm{rank}(B) - 2$; hence, by applying Lemma 2.3.9 to $\mathrm{pr}(B)$, starting with $k = 2$, we have $\mathrm{pr}(B) = 1]101\overline{01}010\overline{0}$ or $\mathrm{pr}(B) = 1]101\overline{01}1\overline{0}$; again, it follows from [2, Theorem 4.1] that in either case $\overline{0}$ must be non-empty, and therefore

$\mathrm{pr}(B) = 1]101\overline{01}010\overline{0}$ or $\mathrm{pr}(B) = 1]101\overline{01}10\overline{0}$. Hence, by Corollary 2.3.8, $\mathrm{pr}(B)$ is one of the sequences (13), (14) and (15).

For the other direction, since appending 0 to the end of an attainable sequence results in another attainable sequence (see [2, Theorem 2.6]), it suffices to establish the attainability of each sequence when $\overline{0}$ is empty. We assume that the sequence under consideration has order $n \geq 3$ and provide an $n \times n$ real symmetric matrix that attains it.

1. $0]100\overline{0}$: $\mathrm{pr}(J_3) = 0]100$.

2. $0]1\overline{01}01\overline{0}$: $\mathrm{pr}((A(C_n))^{-1}) = 0]1\overline{01}01$, with n odd (see [2, Lemma 3.4] and Remark 2.3.1).

3. $0]1011\overline{0}$: $\mathrm{pr}(J_4 - 2I_4) = 0]1011$.

4. $0]101011\overline{0}$: $\mathrm{pr}(M_{0101011}) = 0]101011$, where $M_{0101011}$ appears in [2, p. 2153].

5. $0]110\overline{0}$: $\mathrm{pr}(J_1 \oplus J_2) = 0]110$.

6. $0]1101\overline{0}$: $\mathrm{pr}(J_4 - 3I_4) = 0]1101$.

7. $0]11011\overline{0}$: $\mathrm{pr}(J_5 - 3I_5) = 0]11011$.

8. $1]000\overline{0}$: $\mathrm{pr}(0_3) = 1]000$.

9. $1]010\overline{0}$: $\mathrm{pr}((J_2 - I_2) \oplus 0_1) = 1]010$.

10. $1]01\overline{01}01\overline{0}$: $\mathrm{pr}(A(P_n)) = 1]01\overline{01}01$, with n even (see [2, Lemma 3.3]).

11. $1]01\overline{01}1\overline{0}$: $\mathrm{pr}(A(C_n)) = 1]01\overline{01}1$, with n odd (see [2, Lemma 3.4]).

12. $1]100\overline{0}$: $\mathrm{pr}(J_1 \oplus 0_2) = 1]100$.

13. $1]1\overline{01}010\overline{0}$: $\mathrm{pr}((A(C_{n-1}))^{-1} \oplus 0_1) = 1]1\overline{01}010$, with n even (see [2, Lemma 3.4], Remark 2.3.1 and [2, Theorem 2.3]).

14. $1]10110\overline{0}$: $\mathrm{pr}((J_4 - 2I_4) \oplus 0_1) = 1]10110$.

15. $1]1010110\overline{0}$: $\mathrm{pr}(M_{0101011} \oplus 0_1) = 1]1010110$, where $M_{0101011}$ appears in [2, p. 2153].

That concludes the proof. □

We conclude this section with a classification of the attainable pr-sequences that only contain three consecutive 1s in the initial subsequence $1]11$. The primary motivation for including this result is its application in Section 2.4.

Proposition 2.3.11. *The epr-sequences* $\mathtt{SSNSNS\overline{NS}SN\overline{N}}$ *and* $\mathtt{SSNS\overline{NS}NAA}$ *are unattainable by a real symmetric matrix.*

Proof. Suppose to the contrary that there is a real symmetric matrix B with $\mathrm{epr}(B) = \mathtt{SSNSNS\overline{NS}SN\overline{N}}$. Notice that $\mathrm{rank}(B)$ is odd. If $\overline{\mathtt{NS}}$ is empty, then we have a contradiction to Proposition 2.2.13. So, suppose $\overline{\mathtt{NS}}$ is non-empty. Let $B[\alpha]$ be a nonsingular 1×1 principal submatrix of B. By the Schur Complement Theorem, $\mathrm{rank}(B/B[\alpha])$ is even, $\mathrm{rank}(B/B[\alpha]) \geq 8$, and $\mathrm{epr}(B/B[\alpha]) = \mathtt{XNYNZN}\cdots$, where $\mathtt{X}$, $\mathtt{Y}$, $\mathtt{Z} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$. Then, as $\mathrm{rank}(B/B[\alpha]) \geq 8$, by the $\mathtt{NN}$ Theorem, $\mathtt{X}$, $\mathtt{Y}$ and $\mathtt{Z}$ are not $\mathtt{N}$. Since $\mathtt{NAN}$ is prohibited, $\mathtt{Y} = \mathtt{Z} = \mathtt{S}$. Thus, we have $\mathrm{epr}(B/B[\alpha]) = \mathtt{XNSNSN}\cdots$, where $\mathtt{X}$ is not $\mathtt{N}$. It follows from Propositions 2.3.5 and 2.3.7 that $\mathrm{rank}(B/B[\alpha])$ is odd, a contradiction.

Now, suppose $\mathtt{SSNS\overline{NS}NAA}$ is attainable. Then applying [3, Observation 2.19(2)] to this sequence implies that $\mathtt{SSNSNS\overline{NS}SN}$ is attainable, a contradiction to the first assertion. □

Corollary 2.3.12. *The pr-sequence* $1]110101\overline{01}1\overline{0}$ *is unattainable by a real symmetric matrix.*

Proof. Suppose that there is a real symmetric matrix B with $\mathrm{pr}(B) = 1]110101\overline{01}1\overline{0}$ and $\mathrm{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Obviously, $\ell_1 = \mathtt{S}$ and $\ell_3 = \ell_5 = \mathtt{N}$. By the $\mathtt{NN}$ Theorem, and because $\mathtt{NAN}$ is prohibited, $\ell_4 = \mathtt{S}$. Since ℓ_2 is not $\mathtt{N}$, it follows from Proposition 2.2.3 that $\ell_2 = \mathtt{S}$. Hence, $\mathrm{epr}(B) = \mathtt{SSNSN}\cdots$. We examine two cases.

Case 1: $\overline{0}$ is empty. Notice that $\mathrm{pr}(B) = 1]110101\overline{01}1 = 1]1101\overline{01}011$. Moreover, $\ell_n = \mathtt{A}$ and $\ell_i = \mathtt{N}$ for all odd i with $3 \leq i \leq n-2$. Then, as $\mathtt{NAN}$ is prohibited, $\ell_j = \mathtt{S}$ for all even j with $4 \leq j \leq n-3$. Therefore, we have $\mathrm{epr}(B) = \mathtt{SSNS\overline{NS}NXA}$, where $\mathtt{X}$ is not $\mathtt{N}$. Since $\mathtt{NSA}$ is prohibited, $\mathtt{X} = \mathtt{A}$, which contradicts Proposition 2.3.11.

Case 2: $\overline{0}$ is non-empty. Thus, $\mathrm{pr}(B) = 1]110101\overline{01}10\overline{0} = 1]1101\overline{01}0110\overline{0}$. As in the preceding case, the fact that $\mathtt{NAN}$ is prohibited implies that $\mathrm{epr}(B) = \mathtt{SSNS\overline{NS}NXYN\overline{N}}$, where $\mathtt{X}$ and $\mathtt{Y}$ are not $\mathtt{N}$. By Theorem 4.3.9, $\mathtt{X} = \mathtt{S}$. Then, as $\mathtt{NSA}$ is prohibited, $\mathtt{Y} = \mathtt{S}$. Hence, $\mathrm{epr}(B) = \mathtt{SSNSNS\overline{NS}SN\overline{N}}$, a contradiction to Proposition 2.3.11. □

Proposition 2.3.13. *Let $n \geq 3$. A pr-sequence $r_0]r_1 \cdots r_n$, with $r_1 r_2 \cdots r_n$ not containing three consecutive* 1*s, is attainable by a real symmetric matrix if and only if it is one of the sequences in Theorem 2.3.10 or one of the following sequences.*

16. $1]110\overline{0}$.

17. $1]11\overline{01}01\overline{0}$.

18. $1]11011\overline{0}$.

Proof. Let B be a real symmetric matrix with $\mathrm{pr}(B) = r_0]r_1 \cdots r_n$. Suppose $r_1 r_2 \cdots r_n$ does not contain three consecutive 1s. If $r_0]r_1 r_2 \neq 1]11$, then $\mathrm{pr}(B)$ does not contain three consecutive 1s, and therefore it is one of the sequences listed in Theorem 2.3.10. Thus, suppose $r_0]r_1 r_2 = 1]11$. By hypothesis, $r_3 = 0$. If $n = 3$, then $\mathrm{pr}(B)$ is sequence (16). So, suppose $n > 3$. If $r_4 = 0$, then, by the 00 Theorem, $\mathrm{pr}(B)$ is sequence (16). Now, suppose $r_4 = 1$. Then $\mathrm{pr}(B)$ starts $1]1101 \cdots$. If $\mathrm{rank}(B) = 4$, then $\mathrm{pr}(B) = 1]1101\overline{0}$, which is sequence (17). Now, suppose $\mathrm{rank}(B) > 4$. Hence, $r_3 r_4 = 01$ and $1 \leq 3 \leq \mathrm{rank}(B) - 2$. It follows from applying Lemma 2.3.9 to $\mathrm{pr}(B)$, starting with $k = 3$, that $\mathrm{pr}(B) = 1]1101\overline{01}01\overline{0}$ or $\mathrm{pr}(B) = 1]1101\overline{01}1\overline{0}$. Hence, by Corollary 2.3.12, $\mathrm{pr}(B)$ is either sequence (17) or sequence (18).

For the other direction, as in Theorem 2.3.10, it suffices to show that each sequence is attainable when $\overline{0}$ is empty. By [2, Theorem 3.7], the sequences $1]110$ and $1]11011$ are attainable by $Q_{3,1}$ and $Q_{5,1}$, respectively. Finally, $1]11\overline{01}01$ is attained by $(A(F_n))^{-1}$ (see [2, Lemma 3.5]), where n is even and F_n is the graph on n vertices formed by adding a pendent edge to C_{n-1}. $\square$

2.4 Epr-sequences with an $\mathtt{N}$ in every subsequence of length 3

This section focuses on epr-sequences with an $\mathtt{N}$ in every subsequence of length 3, and culminates with a classification of all the attainable epr-sequences with this property.

The sequence accounted for in the next result is of particular relevance to the main result at the end of this section.

Proposition 2.4.1. *Let $n \geq 3$ and $B = [b_{ij}]$ be the $n \times n$ real symmetric matrix with $b_{ij} = (i-j)^2$. Then* $\operatorname{epr}(B) = \mathtt{NAA\overline{N}}$.

Proof. Suppose that $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. It is easy to verify the assertion for $n = 3$. Suppose $n > 3$. Obviously, $\ell_1 = \mathtt{N}$. Let $p, q, r, s \in \{1, 2, \dots, n\}$, where $p < q < r < s$. Since every off-diagonal entry of B is nonzero, we have $B_{pq} = -(b_{pq})^2 \neq 0$ and $B_{pqr} = 2b_{pq}b_{pr}b_{qr} \neq 0$. A simple computation reveals that the order-4 principal minor B_{pqrs} is given by

$$(b_{ps}b_{qr})^2 + (b_{pr}b_{qs})^2 + (b_{pq}b_{rs})^2 - 2b_{pr}b_{ps}b_{qr}b_{qs} - 2b_{pq}b_{ps}b_{qr}b_{rs} - 2b_{pq}b_{pr}b_{qs}b_{rs} =$$

$$((p-s)(q-r))^4 + ((p-r)(q-s))^4 + ((p-q)(r-s))^4$$

$$-2((p-r)(p-s)(q-r)(q-s))^2 - 2((p-q)(p-s)(q-r)(r-s))^2$$

$$-2((p-q)(p-r)(q-s)(r-s))^2 = 0.$$

Hence, we have $\ell_2 = \ell_3 = \mathtt{A}$ and $\ell_4 = \mathtt{N}$. The conclusion now follows from Proposition 2.2.4. $\square$

Observation 2.4.2. *If an attainable pr-sequence does not contain three consecutive 1s, then an attainable epr-sequence associated with it contains an* `N` *in every subsequence of length* 3.

Remark 2.4.3. The converse of Observation 2.4.2 is false. An attainable epr-sequence starting `SS`$\cdots$, or starting `SA`$\cdots$, with an `N` in every subsequence of length 3, provides a counterexample. It can be deduced that all counterexamples are of that form, and therefore that the converse of Observation 2.4.2 is true if additionally we assume that the pr-sequence does not start with 1]11.

Observation 2.4.4. *Let* $n \geq 3$ *and* B *be a real symmetric matrix with* $\mathrm{pr}(B) = r_0]r_1 \cdots r_n$. *Then* $\mathrm{epr}(B)$ *contains an* `N` *in every subsequence of length* 3 *if and only if* $r_1 r_2 \cdots r_n$ *does not contain three consecutive* 1*s*

Observation 2.4.4 suggests that we can use Theorem 2.3.10 and Proposition 2.3.13 to classify all the epr-sequences with an `N` in every subsequence of length 3, as the pr-sequences associated with these epr-sequences must be those listed on these results.

Theorem 2.4.5. *Let* $n \geq 3$. *An epr-sequence of order* n *with an* `N` *in every subsequence of length* 3 *is attainable by a real symmetric matrix if and only if it is one of the following sequences.*

1. $\mathtt{ANN\overline{N}}$.

2a. $\mathtt{A\overline{NS}NA}$.

2b. $\mathtt{A\overline{NS}NSN\overline{N}}$.

3a. $\mathtt{ANAA}$.

3b. $\mathtt{ANSSN\overline{N}}$.

4a. $\mathtt{ANSNAA}$.

4b. $\mathtt{ANSNSSN\overline{N}}$.

5a. $\mathtt{AAN\overline{N}}$.

5b. $\mathtt{ASN\overline{N}}$.

6a. $\mathtt{AANA}$.

6b. $\mathtt{ASNSN\overline{N}}$.

7a. $\mathtt{AANAA}$.

7b. $\mathtt{ASNSSN\overline{N}}$.

8. $\mathtt{NNN\overline{N}}$.

9. $\mathtt{NSN\overline{N}}$.

10a. $\mathtt{NS\overline{NS}NA}$.

10b. $\mathtt{NS\overline{NS}NSN\overline{N}}$.

11a. $\mathtt{N\overline{SN}AA}$.

11b. $\mathtt{N\overline{SN}SSN\overline{N}}$.

11c. $\mathtt{NAAN\overline{N}}$.

12. $\mathtt{SNN\overline{N}}$.

13. $\mathtt{S\overline{NS}NSN\overline{N}}$.

14. $\mathtt{SNSSN\overline{N}}$.

15. $\mathtt{SNSNSSN\overline{N}}$.

16a. $\mathtt{SAN\overline{N}}$.

16b. $\mathtt{SSN\overline{N}}$.

17a. $\mathtt{SS\overline{NS}NA}$.

17b. $\mathtt{SS\overline{NS}NSN\overline{N}}$.

18a. $\mathtt{SSNAA}$.

18b. $\mathtt{SSNSSN\overline{N}}$.

Proof. Let B be a real symmetric matrix with $\mathrm{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose that $\mathrm{epr}(B)$ contains an $\mathtt{N}$ in every subsequence of length 3. It follows from Observation 2.4.4 that $\mathrm{pr}(B)$ is one of the sequences listed in Theorem 2.3.10 or Proposition 2.3.13. We examine the 18 possible cases.

Case 1: $\mathrm{pr}(B) = 0]100\overline{0}$. Obviously, $\mathrm{epr}(B) = \mathtt{ANN\overline{N}}$, which is sequence (1).

Case 2: $\mathrm{pr}(B) = 0]1\overline{01}010\overline{0}$. First, suppose $\overline{0}$ is empty. Then, as $\mathtt{NAN}$ is prohibited, $\mathrm{epr}(B) = \mathtt{A\overline{NS}NA}$, which is sequence (2a). Now, suppose $\overline{0}$ is non-empty. Similarly, since $\mathtt{NAN}$ is prohibited, $\mathrm{epr}(B) = \mathtt{A\overline{NS}NSN\overline{N}}$, which is sequence (2b).

Case 3: $\mathrm{pr}(B) = 0]1011\overline{0}$. If $\overline{0}$ is empty, then, as $\mathtt{NSA}$ is prohibited, $\mathrm{epr}(B) = \mathtt{ANAA}$, which is sequence (3a). If $\overline{0}$ is non-empty, then, since $\mathtt{NSA}$ and $\mathtt{NAS}$ are prohibited, we must have $\mathtt{ANSSN\overline{N}}$ or $\mathtt{ANAAN\overline{N}}$; as the latter sequence is forbidden by Theorem 2.2.6, $\mathrm{epr}(B)$ is sequence (3b).

Case 4: $\mathrm{pr}(B) = 0]101011\overline{0}$. Suppose $\overline{0}$ is empty. Since $\mathtt{NAN}$ and $\mathtt{NSA}$ are prohibited, $\mathrm{epr}(B) = \mathtt{ANSNAA}$, which is sequence (4a). Now suppose $\overline{0}$ is non-empty. Then, as $\mathtt{NAN}$, $\mathtt{NAS}$ and $\mathtt{NSA}$ are prohibited, $\mathrm{epr}(B)$ is either $\mathtt{ANSNSSN\overline{N}}$ or $\mathtt{ANSNAAN\overline{N}}$; by Theorem 4.3.9, the latter sequence is forbidden, and thus we have sequence (4b).

Case 5: $\mathrm{pr}(B) = 0]110\overline{0}$. Clearly, $\mathrm{epr}(B) = \mathtt{AAN\overline{N}}$ or $\mathrm{epr}(B) = \mathtt{ASN\overline{N}}$, which are sequences (5a) and (5b), respectively.

Case 6: $\mathrm{pr}(B) = 0]1101\overline{0}$. If $\overline{0}$ is empty, then, as $\mathtt{ASNA}$ is forbidden, $\mathrm{epr}(B) = \mathtt{AANA}$, which is sequence (6a). Suppose $\overline{0}$ is non-empty. Since $\mathtt{NAN}$ is prohibited, and because $\mathtt{ANS}$ must be initial, $\mathrm{epr}(B) = \mathtt{ASNSN\overline{N}}$, which is sequence (6b).

Case 7: $\mathrm{pr}(B) = 0]11011\overline{0}$. Suppose $\overline{0}$ is empty. Since $\mathtt{NSA}$ and $\mathtt{ASN}\cdots\mathtt{A}$ are prohibited, $\mathrm{epr}(B) = \mathtt{AANAA}$, which is sequence (7a). Suppose $\overline{0}$ is non-empty. Moreover, suppose $\ell_2 = \mathtt{A}$. Obviously, $\ell_n = \mathtt{N}$; but, as $\mathtt{ANS}$ must be initial, $\ell_4 = \mathtt{A}$, and therefore Theorem 2.2.6 implies that $\ell_n = \mathtt{A}$, a contradiction. It follows that we must have $\ell_2 = \mathtt{S}$. Since $\mathtt{ASN}\cdots\mathtt{A}\cdots$ is prohibited, $\mathrm{epr}(B) = \mathtt{ASNSSN\overline{N}}$, which is sequence (7b).

Case 8: $\mathrm{pr}(B) = 1]000\overline{0}$. Clearly, $\mathrm{epr}(B) = \mathtt{NNN\overline{N}}$, which is sequence (8).

Case 9: $\mathrm{pr}(B) = 1]010\overline{0}$. Since $\mathtt{NAN}$ is prohibited, $\mathrm{epr}(B) = \mathtt{NSN\overline{N}}$, which is sequence (9).

Case 10: $\mathrm{pr}(B) = 1]01\overline{01}01\overline{0}$. If $\overline{0}$ is empty, then, as $\mathtt{NAN}$ is prohibited, $\mathrm{epr}(B) = \mathtt{NS\overline{NS}NA}$, which is sequence (10a). Similarly, if $\overline{0}$ is non-empty, $\mathrm{epr}(B) = \mathtt{NS\overline{NS}NSN\overline{N}}$, which is sequence (10b).

Case 11: $\mathrm{pr}(B) = 1]01\overline{01}1\overline{0}$. First, observe that $\mathrm{pr}(B) = 1]0\overline{10}11\overline{0}$. Suppose $\overline{0}$ is empty. Since $\mathtt{NSA}$ and $\mathtt{NAN}$ are prohibited, $\mathrm{epr}(B) = \mathtt{N\overline{SN}AA}$, which is sequence (11a). Suppose $\overline{0}$ is non-empty. Moreover, suppose $\overline{10}$ is empty. Then, as $\mathtt{NAS}$ and $\mathtt{NSA}$ are prohibited, $\mathrm{epr}(B)$ is $\mathtt{NSSN\overline{N}}$ or $\mathtt{NAAN\overline{N}}$, which are sequences (11b) and (11c), respectively. Finally, suppose $\overline{10}$ is non-empty. Since $\mathtt{NAS}$, $\mathtt{NSA}$ and $\mathtt{NAN}$ are prohibited, $\mathrm{epr}(B)$ is either $\mathtt{NSN\overline{SN}SSN\overline{N}}$ or $\mathtt{NSN\overline{SN}AAN\overline{N}}$; by Theorem 4.3.9, the latter sequence is forbidden, and therefore $\mathrm{epr}(B)$ is sequence (11b), with $\mathtt{\overline{SN}}$ non-empty.

Case 12: $\mathrm{pr}(B) = 1]100\overline{0}$. Obviously, $\mathrm{epr}(B) = \mathtt{SNN\overline{N}}$, which is sequence (12).

Case 13: $\mathrm{pr}(B) = 1]1\overline{01}010\overline{0}$. Since $\mathtt{SN}\cdots\mathtt{A}\cdots$ is prohibited, it is immediate that $\mathrm{epr}(B) = \mathtt{S\overline{NS}NSN\overline{N}}$, which is sequence (13).

Case 14: $\mathrm{pr}(B) = 1]10110\overline{0}$. As in Case 13, since $\mathtt{SN}\cdots\mathtt{A}\cdots$ is prohibited, we must have $\mathrm{epr}(B) = \mathtt{SNSSN\overline{N}}$, which is sequence (14).

Case 15: $\mathrm{pr}(B) = 1]1010110\overline{0}$. Again, as $\mathtt{SN}\cdots\mathtt{A}\cdots$ is prohibited, we must have $\mathrm{epr}(B) = \mathtt{SNSNSSN\overline{N}}$, which is sequence (15).

Case 16: $\mathrm{pr}(B) = 1]110\overline{0}$. Clearly, $\mathrm{epr}(B)$ is either $\mathtt{SAN\overline{N}}$ or $\mathtt{SSN\overline{N}}$, which are sequences (16a) and (16b), respectively.

Case 17: $\mathrm{pr}(B) = 1]11\overline{01}01\overline{0}$. Since $\mathtt{SAN}\cdots\mathtt{A}\cdots$ and $\mathtt{SAN}\cdots\mathtt{S}\cdots$ are prohibited by Proposition 2.2.3, $\ell_2 = \mathtt{S}$. Suppose $\overline{0}$ is empty. Then, as `NAN` is prohibited, $\mathrm{epr}(B) = \mathtt{SS\overline{NS}NA}$, which is sequence (17a). Suppose $\overline{0}$ is non-empty. Similarly, since `NAN` is prohibited, $\mathrm{epr}(B) = \mathtt{SS\overline{NS}NSN\overline{N}}$, which is sequence (17b).

Case 18: $\mathrm{pr}(B) = 1]11011\overline{0}$. As in the preceding case, we must have $\ell_2 = \mathtt{S}$. Suppose $\overline{0}$ is empty. Since `NSA` is prohibited, $\mathrm{epr}(B) = \mathtt{SSNAA}$, which is sequence (18a). Suppose $\overline{0}$ is non-empty. Hence, the fact that `NAS` and `NSA` are prohibited implies that $\mathrm{epr}(B)$ is either $\mathtt{SSNSSN\overline{N}}$ or $\mathtt{SSNAAN\overline{N}}$; by Theorem 4.3.9, the latter sequence is forbidden, and thus $\mathrm{epr}(B)$ is sequence (18b).

For the other direction, we show that all the sequences listed are attainable, and assume that the sequence under consideration has order $n \geq 3$. Sequence (1) is attained by J_n. Sequence (2a) is attained by $A((C_n)^{-1})$ (see [3, Observation 3.1] and the Inverse Theorem), when $\mathtt{\overline{NS}}$ is non-empty, and by [3, Proposition 2.17], when $\mathtt{\overline{NS}}$ is empty. As for (2b), applying [3, Observation 2.19(1)] to (2a), results in this sequence. Sequence (3a) is attainable by [3, Proposition 2.17]. Sequence (3b) is attainable by applying [3, Observation 2.19(1)] to (3a). Sequence (4a) is attainable by [3, Table 1], and (4b) results from applying [3, Observation 2.19(1)] to (4a). Sequences (5a) and (5b) are attainable by [3, Theorem 4.6]. Sequence (6a) is attainable by [3, Proposition 2.17], and (6b) results from applying [3, Observation 2.19(1)] to (6a). Sequence (7a) is attainable by [3, Proposition 2.17], and (7b) results from applying [3, Observation 2.19(1)] to (7a). Sequence (8) is attained by 0_n. As for (9), applying [3, Observation 2.19(1)] to the sequence `NA`, which is attained by $J_2 - I_2$, results in this sequence. Sequence (10a) is attainable by [3, Observation 3.1], and (10b) results from applying [3, Observation

2.19(1)] to (10a). Sequence (11a) is attainable by [3, Observation 3.1], while (11b) is obtained from applying [3, Observation 2.19(1)] to (11a). Sequence (11c) is attainable by Proposition 2.4.1. Sequence (12) is attainable by [3, Theorem 4.6]. Sequences (13), (14) and (15) result from applying [3, Observation 2.19(2)] to (2a), (3a) and (4a), respectively. Sequences (16a) and (16b) are attainable by [3, Theorem 4.6]. According to Proposition 2.3.13, the sequence $1]11\overline{01}01$ is attainable; by Proposition 2.2.3, and because `NAN` is prohibited, an attainable epr-sequence associated with this pr-sequence, must be $\mathtt{SS\overline{NS}NA}$, which is sequence (17a). Sequence (17b) results from applying [3, Observation 2.19(2)] to (17a). Sequence (18a) is attainable by [3, Table 5], and (18b) is attainable by [3, Corollary 2.20(2)]. □

If an epr-sequence is attainable, then the pr-sequence associated with it must be attainable. The converse is not true; this is because an epr-sequence associated with a pr-sequence may not be unique, since a 1 in the pr-sequence can correspond to an `A` or `S` in the epr-sequence. For example, the epr-sequences `NSSN` and `NAAN`, which are associated with the pr-sequence 1]0110, are each attainable by a real symmetric matrix (see [3, Table 4]). We now show that, for real symmetric matrices, almost all attainable pr-sequences not containing three consecutive 1s are associated with a unique epr-sequence.

Proposition 2.4.6. *Let $n \geq 3$ and σ be a pr-sequence that is attainable by an $n \times n$ real symmetric matrix. Suppose σ does not contain three consecutive 1s, $\sigma \neq 0]110\overline{0}$ and that $\sigma \neq 1]0110\overline{0}$. Then there is a unique attainable epr-sequence associated with σ.*

Proof. Since the attainable epr-sequences associated with pr-sequences not containing three consecutive 1s are the epr-sequences (1a)–(15) listed in Theorem 2.4.5, an attainable epr-sequence associated with σ must be one of these sequences. Note that σ is not associated with any of the epr-sequences (16a)–(18b), as these are the epr-sequences that are associated with the pr-sequences listed in Proposition 2.3.13. We consider two cases.

Case 1: $\sigma = 1]010\overline{10}1100\overline{0}$. Observe that σ is associated with the epr-sequence (11b) in Theorem 2.4.5, with $\overline{\mathtt{SN}}$ non-empty. It is easy to see that σ is not associated with any of the other epr-sequences listed in Theorem 2.4.5, thereby establishing the uniqueness of the associated epr-sequence (11b).

Case 2: $\sigma \neq 1]010\overline{10}1100\overline{0}$. Then, as $\sigma \neq 1]0110\overline{0}$, the epr-sequences (11b) and (11c) in Theorem 2.4.5 are not associated with σ. Also, it is clear that σ is not associated with the epr-sequence (11a) in Theorem 2.4.5. Since $\sigma \neq 0]110\overline{0}$, the epr-sequences (5a) and (5b) in Theorem 2.4.5 are not associated with σ. Thus far we have that σ is not associated with any of the epr-sequences (5a), (5b), (11a), (11b) or (11c). Hence, σ must be one of the pr-sequences (1)–(4), (6)–(10) or (12)–(15) in Theorem 2.3.10. Now, by considering all the possible cases, one easily verifies that an attainable epr-sequence associated with σ, which must be listed in Theorem 2.4.5, is unique. $\square$

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CHAPTER 3. THE ENHANCED PRINCIPAL RANK CHARACTERISTIC SEQUENCE OVER A FIELD OF CHARACTERISTIC 2

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Abstract

The enhanced principal rank characteristic sequence (epr-sequence) of an $n \times n$ symmetric matrix over a field $\mathbb{F}$ was recently defined as $\ell_1\ell_2\cdots\ell_n$, where ℓ_k is either `A`, `S`, or `N` based on whether all, some (but not all), or none of the order-k principal minors of the matrix are nonzero. Here, a complete characterization of the epr-sequences that are attainable by symmetric matrices over the field $\mathbb{Z}_2$, the integers modulo 2, is established. Contrary to the attainable epr-sequences over a field of characteristic 0, our characterization reveals that the attainable epr-sequences over $\mathbb{Z}_2$ possess very special structures. For more general fields of characteristic 2, some restrictions on attainable epr-sequences are obtained.

Keywords. Principal rank characteristic sequence; enhanced principal rank characteristic sequence; minor; rank; symmetric matrix; finite field.

AMS Subject Classifications. 15A15, 15A03.

3.1 Introduction

For an $n \times n$ real symmetric matrix B, Brualdi et al. [2] introduced the *principal rank characteristic sequence* (abbreviated pr-sequence), which was defined as $\mathrm{pr}(B) = r_0]r_1 \cdots r_n$, where, for $k \geq 1$,

$$r_k = \begin{cases} 1 & \text{if } B \text{ has a nonzero principal minor of order } k \text{, and} \\ 0 & \text{otherwise,} \end{cases}$$

while $r_0 = 1$ if and only if B has a 0 diagonal entry. This definition was generalized for symmetric matrices over any field by Barrett et al. [1].

Our focus will be studying a sequence that was introduced by Butler et al. [4] as a refinement of the pr-sequence of an $n \times n$ symmetric matrix B over a field $\mathbb{F}$, which they called the *enhanced principal rank characteristic sequence* (abbreviated epr-sequence), and which was defined as $\mathrm{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$, where

$$\ell_k = \begin{cases} \mathtt{A} & \text{if all the principal minors of order } k \text{ are nonzero;} \\ \mathtt{S} & \text{if some but not all the principal minors of order } k \text{ are nonzero;} \\ \mathtt{N} & \text{if none of the principal minors of order } k \text{ are nonzero, i.e., all are zero.} \end{cases}$$

The definition of the epr-sequence was later extended to the class of real skew-symmetric matrices in [6], where a complete characterization of the epr-sequences realized by this class was presented. However, things are more subtle for the class of symmetric matrices over a field $\mathbb{F}$, and thus obtaining a similar characterization presents a difficult problem. When $\mathbb{F}$ is of characteristic 0, it is known that any epr-sequence of the form $\ell_1 \cdots \ell_{n-k}\overline{\mathtt{N}}$, with $\ell_i \in \{\mathtt{A}, \mathtt{S}\}$, is attainable by an $n \times n$ symmetric matrix over $\mathbb{F}$, where $\overline{\mathtt{N}}$ (which may be empty) is the sequence consisting of k consecutive $\mathtt{N}$s [4] – if $\overline{\mathtt{N}}$ is empty, note that we must have $\ell_n = \mathtt{A}$. In general, the subtlety for symmetric matrices becomes evident once the $\mathtt{N}$s are not restricted to occur consecutively at the end of the sequence: Sequences such as $\mathtt{NSA}$, $\mathtt{NNA}$ and $\mathtt{NNS}$ can never occur as a subsequence of the epr-sequence of a symmetric matrix over any field [4]; the same holds for the sequences `NAN` and `NAS` when the field is of characteristic not 2 [4]. Moreover, over fields of characteristic not 2, the sequence `ANS` can only occur at the start of the sequence [4]. Over the real field, `SNA` can only occur as a terminal subsequence, or in the terminal subsequence `SNAA` [10]. Furthermore, over the real field, we also know that when the subsequence `ANA` occurs as a non-terminal subsequence, it forces every other term of the sequence to be `A` [10]. However, it is unknown what kind of restrictions a subsequence such as `SNS` imposes on an attainable sequence (over any field); this is one of the difficulties in arriving at a complete characterization of the epr-sequences attainable by a symmetric matrix over a field $\mathbb{F}$. In order to simplify this problem, it is natural to consider the case when $\mathbb{F}$ is of characteristic 2. The analogous problem for pr-sequences was already settled in [1]:

Theorem 3.1.1. [1, Theorem 3.1] *A pr-sequence of order* $n \geq 2$ *is attainable by a symmetric matrix over a field of characteristic* 2 *if and only if it has one of the following forms:*

$$0]1\,\overline{1}\,\overline{0}, \qquad 1]\overline{01}\,\overline{0}, \qquad 1]1\,\overline{1}\,\overline{0}.$$

We see that for any two fields of characteristic 2, the class of pr-sequences attainable by symmetric matrices over each of the two fields is the same. This is not true in the case of epr-sequences: Consider an epr-sequence starting with `AA` over the field $\mathbb{Z}_2 = \{0, 1\}$, the integers modulo 2; over this field, any such sequence must be $\mathtt{AA}\overline{\mathtt{A}}$, since any symmetric matrix attaining this sequence must be the identity matrix. However, in Example 3.2.5 below, it is shown that the epr-sequence `AAN` is attainable over a field of characteristic 2, implying that not all fields of characteristic 2 give rise to the same class of attainable epr-sequences. In light of this difficulty, our main focus here will be on the field $\mathbb{F} = \mathbb{Z}_2$; after establishing some restrictions for the attainability of epr-sequences over a field of characteristic 2 at the beginning of Section 3.2, our main objective is a complete characterization of the epr-sequences that are attainable by symmetric matrices over $\mathbb{Z}_2$ (see Theorems 3.3.2, 3.3.8 and 3.3.11). We find that the attainable epr-sequences over $\mathbb{Z}_2$ possess very special structures, which is in contrast to the family of attainable epr-sequences over a field of characteristic 0, which was described above.

Another motivating factor for considering this problem is that it is a simplification of the *principal minor assignment problem* as stated in [8], which also served as motivation for the introduction of the pr-sequence in [2]. Note that epr-sequences provide more information than pr-sequences, and thus are a step closer to the principal minor assignment problem.

Extra motivation for this problem comes from the observation that there is a one-to-one correspondence between adjacency matrices of simple graphs and symmetric matrices over $\mathbb{Z}_2$ with zero diagonal, and, more generally, between adjacency matrices of loop graphs and symmetric matrices over $\mathbb{Z}_2$.

It should be noted that, although epr-sequences have received attention after their introduction in [4] (see [5], [6] and [10], for example), very little is known about epr-sequences of symmetric matrices over a field of characteristic 2, since the vast majority of what has appeared on the literature regarding epr-sequences has been focused on fields of characteristic *not* 2.

Although Theorem 3.1.1 sheds some light towards settling the problem under consideration, it does not render it trivial by any means; one reason is the observation that two symmetric matrices may have distinct epr-sequences while having the same pr-sequence: As it is shown in Theorem 3.3.8 below, the epr-sequences `ASAA` and `ASSA`, which are associated with the pr-sequence $0]1111$, are both attainable over $\mathbb{Z}_2$.

To highlight a second reason, we state the two results upon which Barrett et al. [1] relied in order to obtain Theorem 3.1.1 (the latter is a variation of a result of Friedland [7, p. 426]).

Lemma 3.1.2. [1, Lemma 3.2] *Let $\mathbb{F}$ be a field of characteristic 2, let B be a symmetric matrix over $\mathbb{F}$ with $\operatorname{pr}(B) = r_0]r_1 \cdots r_n$, and let E be an $n \times n$ invertible matrix over $\mathbb{F}$. Then $\operatorname{epr}(EBE^T) = r_0']r_1 r_2 \cdots r_n$ for some $r_0' \in \{0, 1\}$.*

In what follows, K_n denotes the complete graph on n vertices, and $A(K_n)$ denotes its adjacency matrix.

Lemma 3.1.3. [1, Lemma 3.3] *Let B be a symmetric matrix over a field $\mathbb{F}$ with characteristic 2. Then B is congruent to the direct sum of a (possibly empty) invertible diagonal matrix D, and a (possibly empty) direct sum of $A(K_2)$ matrices, and a (possibly empty) zero matrix.*

The two lemmas above permitted Barrett et al. [1] to arrive at their characterization for pr-sequences in Theorem 3.1.1 by restricting themselves to symmetric matrices that are in the canonical form described in Lemma 3.1.3. We cannot use this approach to obtain our desired characterization for epr-sequences: Suppose one tries to apply the congruence described in Lemma 3.1.2 to a symmetric matrix B with $\operatorname{epr}(B) = \mathtt{ASAN}$, which is shown to be attainable in Theorem 3.3.8. Then, because B is singular, and because multiplication by an invertible matrix preserves the rank of the original matrix, once B has been transformed into the canonical form described in Lemma 3.1.3, it must be the case that in this resulting matrix the zero summand is non-empty. Thus, the resulting matrix has a zero row (and zero column), which implies that it contains a principal minor of order 3 that is zero. Then, as the principal minors of order 3 of the original matrix B were all nonzero, the congruence performed did not preserve the third term of $\operatorname{epr}(B)$, which is in contrast to what happens to $\operatorname{pr}(B)$, which, with the exception of the zeroth term, must be preserved completely by Lemma 3.1.2.

We say that a (pr- or epr-) sequence is *attainable* over a field $\mathbb{F}$ provided that there exists a symmetric matrix $B \in \mathbb{F}^{n\times n}$ that attains it. A pr-sequence and an epr-sequence are *associated* with each other if a matrix (which may not exist) attaining the epr-sequence also attains the pr-sequence. A subsequence that does not appear in any attainable sequence is *prohibited*. We say that a sequence has *order n* if it corresponds to a matrix of order n. Let B be an $n \times n$ matrix, and let $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$; then the submatrix lying in rows indexed by α, and columns indexed by β, is denoted by $B[\alpha, \beta]$.

The matrix obtained by deleting the rows indexed by α, and columns indexed by β, is denoted by $B(\alpha, \beta)$. If $\alpha = \beta$, then the principal submatrix $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$, while $B(\alpha, \alpha)$ is abbreviated to $B(\alpha)$. The matrices O_n and I_n denote, respectively, the zero and identity matrix of order n. We denote by $J_{m,n}$ the $m \times n$ all-1s matrix, and, when $m = n$, $J_{n,n}$ is abbreviated to J_n. The block diagonal matrix formed from two square matrices B and C is denoted by $B \oplus C$. The matrices B and C are *permutationally* similar if there exists a permutation matrix P such that $C = P^T BP$. Given a graph G, $A(G)$ denotes the adjacency matrix of G.

3.1.1 Results cited

This section lists results that will be cited frequently, with some of them being assigned abbreviated nomenclature.

Theorem 3.1.4. [4, Theorem 2.3] ($\mathtt{NN}$ Theorem.) *Suppose B is a symmetric matrix over a field $\mathbb{F}$, $\operatorname{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$, and $\ell_k = \ell_{k+1} = \mathtt{N}$ for some k. Then $\ell_i = \mathtt{N}$ for all $i \geq k$.*

Theorem 3.1.5. [4, Theorem 2.4] (Inverse Theorem.) *Suppose B is a nonsingular symmetric matrix over a field $\mathbb{F}$. If $\operatorname{epr}(B) = \ell_1 \ell_2 \cdots \ell_{n-1}\mathtt{A}$, then $\operatorname{epr}(B^{-1}) = \ell_{n-1} \ell_{n-2} \cdots \ell_1 \mathtt{A}$.*

Given a matrix B, the ith term in its epr-sequence is denoted by $[\operatorname{epr}(B)]_i$.

Theorem 3.1.6. [4, Theorem 2.6] (Inheritance Theorem.) *Suppose that B is a symmetric matrix over a field $\mathbb{F}$, $m \leq n$, and $1 \leq i \leq m$.*

1. *If $[\operatorname{epr}(B)]_i = \mathtt{N}$, then $[\operatorname{epr}(C)]_i = \mathtt{N}$ for all $m \times m$ principal submatrices C.*

2. *If $[\operatorname{epr}(B)]_i = \mathtt{A}$, then $[\operatorname{epr}(C)]_i = \mathtt{A}$ for all $m \times m$ principal submatrices C.*

3. *If $[\operatorname{epr}(B)]_m = \mathtt{S}$, then there exist $m \times m$ principal submatrices C_A and C_N of B such that $[\operatorname{epr}(C_A)]_m = \mathtt{A}$ and $[\operatorname{epr}(C_N)]_m = \mathtt{N}$.*

4. *If $i < m$ and $[\operatorname{epr}(B)]_i = \mathtt{S}$, then there exists an $m \times m$ principal submatrix C_S such that $[\operatorname{epr}(C_S)]_i = \mathtt{S}$.*

In the rest of this paper, each instance of $\cdots$ is permitted to be empty.

Corollary 3.1.7. [4, Corollary 2.7] (`NSA` Theorem.) *No symmetric matrix over any field can have* `NSA` *in its epr-sequence. Further, no symmetric matrix over any field can have the epr-sequence* $\cdots \mathtt{ASN} \cdots \mathtt{A} \cdots$.

Given a matrix B with a nonsingular principal submatrix $B[\alpha]$, we denote by $B/B[\alpha]$ the Schur complement of $B[\alpha]$ in B [12]. The next fact is a generalization of [4, Proposition 2.13] to any field; the proof is exactly the same, and is omitted here (we note that the proof was also omitted in [4]).

Theorem 3.1.8. (Schur Complement Theorem.) *Suppose B is an $n \times n$ symmetric matrix over a field $\mathbb{F}$, with $\operatorname{rank} B = r$. Let $B[\alpha]$ be a nonsingular principal submatrix of B with $|\alpha| = k \leq r$, and let $C = B/B[\alpha]$. Then the following results hold.*

(i) *C is an $(n-k) \times (n-k)$ symmetric matrix.*

(ii) *Assuming the indexing of C is inherited from B, any principal minor of C is given by*

$$\det C[\gamma] = \det B[\gamma \cup \alpha] / \det B[\alpha].$$

(iii) $\operatorname{rank} C = r - k$.

The next result, which is immediate from the Schur Complement Theorem, has been used implicitly in [4] and [10], but we state it here in the interest of clarity (it should be noted that this result appeared in [5] for Hermitian matrices).

Corollary 3.1.9. (Schur Complement Corollary.) *Let B be a symmetric matrix over a field $\mathbb{F}$, $\operatorname{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$, and let $B[\alpha]$ be a nonsingular principal submatrix of B, with $|\alpha| = k \leq \operatorname{rank} B$. Let $C = B/B[\alpha]$ and $\operatorname{epr}(C) = \ell'_1 \ell'_2 \cdots \ell'_{n-k}$. Then, for $j = 1, \ldots, n-k$, $\ell'_j = \ell_{j+k}$ if $\ell_{j+k} \in \{\mathtt{A}, \mathtt{N}\}$.*

Observation 3.1.10. [4, Observation 2.19] *Let B be a symmetric matrix over a field $\mathbb{F}$, with epr-sequence $\ell_1\ell_2\cdots\ell_n$.*

1. *Form a matrix B' from B by copying the last row down and then the last column across. Then the epr-sequence of B' is $\ell_1\ell_2'\cdots\ell_n'\mathtt{N}$ with $\ell_i' = \mathtt{N}$ if $\ell_i = \mathtt{N}$ and $\ell_i' = \mathtt{S}$ otherwise for $2 \le i \le n$.*
2. *Form a matrix B'' from B by taking the direct sum with $[0]$. Then the epr-sequence of B'' is $\ell_1''\ell_2''\cdots\ell_n''\mathtt{N}$ with $\ell_i'' = \mathtt{N}$ if $\ell_i = \mathtt{N}$ and $\ell_i'' = \mathtt{S}$ otherwise for $1 \le i \le n$.*

3.2 Restrictions on attainable epr-sequences over a field of characteristic 2

Before stating our main results in Section 3.3, we devote this section towards establishing restrictions for the attainability of epr-sequences over a field of characteristic 2.

Observation 3.2.1. ($\mathtt{NA}$-$\mathtt{NS}$ Observation.) *Let B be a symmetric matrix over a field of characteristic 2, with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. If $\ell_k\ell_{k+1} = \mathtt{NA}$ or $\ell_k\ell_{k+1} = \mathtt{NS}$ for some k, then k is odd and $\ell_j = \mathtt{N}$ when j is odd.*

Proof. Let $\operatorname{pr}(B) = r_0]r_1\cdots r_n$. Suppose $\ell_k\ell_{k+1} = \mathtt{NA}$ or $\ell_k\ell_{k+1} = \mathtt{NS}$. Then $r_kr_{k+1} = 01$. Since $k \ge 1$, Theorem 3.1.1 implies that $\operatorname{pr}(B) = 1]01\,\overline{01}\,\overline{0}$, and therefore that k is odd, and that $\ell_j = \mathtt{N}$ when j is odd. □

Over a field of characteristic 2, the $\mathtt{NN}$ Theorem admits a generalization when the first $\mathtt{N}$ occurs in an even position of the epr-sequence, which is immediate from the $\mathtt{NA}$-$\mathtt{NS}$ Observation and the $\mathtt{NN}$ Theorem:

Observation 3.2.2. ($\mathtt{N}$-Even Observation.) *Let B be a symmetric matrix over a field of characteristic 2, with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_k = \mathtt{N}$ with k even. Then $\ell_j = \mathtt{N}$ for all $j \ge k$.*

The next observation establishes another generalization of the `NN` Theorem for epr-sequences beginning with `S` or `A`, and it is immediate from Theorem 3.1.1.

Observation 3.2.3. *Let B be a symmetric matrix over a field of characteristic 2, with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Suppose $\ell_1 \neq \mathtt{N}$. If $\ell_k = \mathtt{N}$ for some k, then $\ell_j = \mathtt{N}$ for all $j \geq k$.*

In the interest of brevity, adopting the notation in [2], the principal minor $\det(B[I])$ is denoted by B_I (when $I = \emptyset$, $B_\emptyset$ is defined to have the value 1). Moreover, when $I = \{i_1, i_2, \ldots, i_k\}$, B_I is written as $B_{i_1 i_2 \cdots i_k}$.

The next result will be of particular relevance later in this section, and its proof resorts to Muir's law of extensible minors [11]; for a more recent treatment of this law, the reader is referred to [3].

Lemma 3.2.4. *Let $n \geq 2$, and let B be a symmetric matrix over a field of characteristic 2, with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Suppose that $\ell_{n-1}\ell_n = \mathtt{AN}$. Then every minor of B of order $n-1$ is nonzero.*

Proof. Since the desired conclusion is obvious when $n = 2$, we assume that $n \geq 3$. By hypothesis, every principal minor of B of order $n-1$ is nonzero. Let $i, j \subseteq \{1, 2, \ldots, n\}$ be distinct, and let $I = \{1, 2, \ldots, n\} \setminus \{i, j\}$. Consider the $(n-1) \times (n-1)$ *non-principal* submatrix resulting from deleting row i and column j, i.e., the submatrix $B[I \cup \{j\} | I \cup \{i\}]$. Since I does not contain i and j, using Muir's law of extensible minors (see [11] or [3]), one may extend the homogenous polynomial identity

$$B_\emptyset B_{ij} = B_i B_j - \det(B[\{i\}|\{j\}])\det(B[\{j\}|\{i\}]),$$

to obtain the identity

$$B_I B_{I\cup\{i,j\}} = B_{I\cup\{i\}} B_{I\cup\{j\}} - \det(B[I\cup\{i\}|I\cup\{j\}])\det(B[I\cup\{j\}|I\cup\{i\}]).$$

Since $B_{I\cup\{i,j\}} = \det(B)$, and because $\ell_n = \mathtt{N}$, we must have

$$\det(B[I\cup\{i\}|I\cup\{j\}])\det(B[I\cup\{j\}|I\cup\{i\}]) = B_{I\cup\{i\}} B_{I\cup\{j\}}.$$

Then, as $\ell_{n-1} = \mathtt{A}$, $B_{I\cup\{i\}} B_{I\cup\{j\}} \neq 0$, implying that $\det(B[I\cup\{j\}|I\cup\{i\}]) \neq 0$. □

3.2.1 Restrictions on attainable epr-sequences over $\mathbb{Z}_2$

This section focuses on establishing restrictions for epr-sequences over $\mathbb{Z}_2$.

With the purpose of establishing a contrast between the attainable epr-sequences over $\mathbb{Z}_2$ and those over other fields of characteristic 2, the next example exhibits matrices over a particular field of characteristic 2 attaining epr-sequences that are not attainable over $\mathbb{Z}_2$ (their unattainability over $\mathbb{Z}_2$ is established in this section).

Example 3.2.5. Let $\mathbb{F} = \mathbb{Z}_2$. Consider the field $\mathbb{F}[z] = \{0, 1, z, z+1\}$, where $z^2 = z + 1$. For each of the following (symmetric) matrices over the field $\mathbb{F}[z]$, $\text{epr}(M_\sigma) = \sigma$, where σ is an epr-sequence.

$$M_{\texttt{AAN}} = \begin{bmatrix} 1 & z & z+1 \\ z & 1 & 0 \\ z+1 & 0 & 1 \end{bmatrix}, \quad M_{\texttt{ASSAN}} = \begin{bmatrix} z & 1 & z & z+1 & 0 \\ 1 & z & z+1 & 0 & 1 \\ z & z+1 & z & 1 & z \\ z+1 & 0 & 1 & z & z+1 \\ 0 & 1 & z & z+1 & z \end{bmatrix},$$

$$M_{\texttt{NANSNN}} = \begin{bmatrix} 0 & z & z+1 & 1 & 1 & 1 \\ z & 0 & 1 & 1 & 1 & 1 \\ z+1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad M_{\texttt{SAAA}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & z & z \\ 1 & z & 0 & 1 \\ 1 & z & 1 & 0 \end{bmatrix},$$

$$M_{\texttt{SASN}} = \begin{bmatrix} 1 & z & z & 1 \\ z & 1 & 1 & 1 \\ z & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad M_{\texttt{SASSA}} = \begin{bmatrix} 1 & z & z & z & 1 \\ z & 1 & 0 & 1 & 1 \\ z & 0 & 1 & 1 & 1 \\ z & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Remark 3.2.6.

1. If B is an $n \times n$ symmetric matrix over $\mathbb{Z}_2$ having an epr-sequence starting with $\mathtt{AA}$, then $B = I_n$. This is because a symmetric matrix with nonzero diagonal must have each of its off-diagonal entries equal to zero in order to have all of its order-2 principal minors be nonzero.

2. A similar argument shows that if an $n \times n$ symmetric matrix B over $\mathbb{Z}_2$ has an epr-sequence starting with $\mathtt{NA}$, then $B = A(K_n)$.

Given a sequence $t_{i_1}t_{i_2}\cdots t_{i_k}$, the notation $\overline{t_{i_1}t_{i_2}\cdots t_{i_k}}$ indicates that the sequence may be repeated as many times as desired (or it may be omitted entirely).

Proposition 3.2.7. *Let $n \geq 2$. Then, over $\mathbb{Z}_2$,* $\operatorname{epr}(A(K_n)) = \mathtt{NA}\overline{\mathtt{NA}}$ *when n is even, and* $\operatorname{epr}(A(K_n)) = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$ *when n is odd.*

Proof. Let $\operatorname{epr}(A(K_n)) = \ell_1\ell_2\cdots\ell_n$. Obviously, $\ell_1 = \mathtt{N}$. Observe that, for $2 \leq q \leq n$, every $q \times q$ principal submatrix of B is equal to $A(K_q)$. Since $A(K_q) = J_q - I_q$, $\det(A(K_q)) = (-1)^{q-1}(q-1) = q-1$ (in characteristic 2). Hence, $\ell_q = \mathtt{N}$ when q is odd and $\ell_q = \mathtt{A}$ when q is even. $\square$

Lemma 3.2.8. ($\mathtt{NA}$ Lemma.) *Let B be a symmetric matrix over $\mathbb{Z}_2$, with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *If $\ell_k\ell_{k+1} = \mathtt{NA}$, then $\ell_k\cdots\ell_n = \mathtt{NA}\overline{\mathtt{NA}}$ or $\ell_k\cdots\ell_n = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$.*

Proof. Suppose $\ell_k\ell_{k+1} = \mathtt{NA}$. If $k = 1$, then Remark 3.2.6 implies that $B = A(K_n)$, and therefore that $\operatorname{epr}(B) = \mathtt{NA}\overline{\mathtt{NA}}$ or $\operatorname{epr}(B) = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$ (by Proposition 3.2.7). Now, suppose $k \geq 2$, and that $\ell_j \neq \mathtt{A}$ for some even integer $j > k+1$. By the Inheritance Theorem, B contains a singular $j \times j$ principal submatrix, B', whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_j$ has $\ell'_k\ell'_{k+1} = \mathtt{NA}$ and $\ell'_j = \mathtt{N}$. Since $k \geq 2$, the $\mathtt{NN}$ Theorem implies that $\ell'_{k-1} \neq \mathtt{N}$. Let $B'[\alpha]$ be a nonsingular $(k-1) \times (k-1)$ principal submatrix of B'. It follows from the Schur Complement Theorem that $B'/B'[\alpha]$ is a (symmetric) matrix of order $j-k+1$, and from the Schur Complement Corollary that $\operatorname{epr}(B'/B'[\alpha]) = \texttt{NA}\cdots\texttt{N}$. Since $\operatorname{epr}(B'/B'[\alpha])$ begins with `NA`, $B'/B'[\alpha] = A(K_{j-k+1})$ (by Remark 3.2.6). Then, as $\operatorname{epr}(B'/B'[\alpha])$ ends with `N`, Proposition 3.2.7 implies that $\operatorname{epr}(B'/B'[\alpha]) = \texttt{NA}\overline{\texttt{NA}}\texttt{N}$; hence, $j-k+1$ is odd, which is a contradiction, since j is even and k is odd. □

The epr-sequence of the matrix $M_{\texttt{NANSNN}}$ in Example 3.2.5 demonstrates that the `NA` Lemma cannot be generalized to all fields of characteristic 2.

Theorem 3.2.9. (`AA` Theorem.) *If an epr-sequence containing* `AA` *as a non-terminal subsequence is attainable over* $\mathbb{Z}_2$*, then it is the sequence* $\overline{\texttt{A}}\texttt{AAA}\overline{\texttt{A}}$*.*

Proof. Let B be an $n \times n$ symmetric matrix over $\mathbb{Z}_2$, with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose that $\ell_k\ell_{k+1} = \texttt{AA}$, where $k+1 < n$. We now show by contradiction that $\ell_{k+2} = \texttt{A}$; thus, suppose $\ell_{k+2} \neq \texttt{A}$. Hence, by the Inheritance Theorem, B contains a $(k+2)\times(k+2)$ principal submatrix C with $\operatorname{epr}(C) = \ell'_1\ell'_2\cdots\ell'_{k+2}$ having $\ell'_k\ell'_{k+1}\ell'_{k+2} = \texttt{AAN}$. Note that C is singular. By Remark 3.2.6, $k \geq 2$ (otherwise, $C = I_3$, which is nonsingular). Let $I = \{1,2,\ldots,k+2\} \setminus \{1,2,3\}$. By [9, Theorem 2], and because C is over a field of characteristic 2, the following equation holds:

$$C_I^2 C_{I\cup\{1,2,3\}}^2 + C_{I\cup\{1\}}^2 C_{I\cup\{2,3\}}^2 + C_{I\cup\{2\}}^2 C_{I\cup\{1,3\}}^2 + C_{I\cup\{3\}}^2 C_{I\cup\{1,2\}}^2 = 0,$$

which is the hyperdeterminantal relation obtained from the relation (2) appearing on [9, p. 635]. Then, as $|I| = k-1$, the fact that $\ell'_k\ell'_{k+1}\ell'_{k+2} = \texttt{AAN}$ leads to a contradiction, since the quantity on the left side of this relation must be nonzero. Hence, it must be the case that $\ell_{k+2} = \texttt{A}$. It now follows inductively that $\ell_k\cdots\ell_n = \texttt{AAA}\overline{\texttt{A}}$.

Now, suppose that $\ell_j \neq \texttt{A}$ for some $j < k$. Then, as $k+1 < n$, the Inverse Theorem implies that $\operatorname{epr}(B^{-1})$ starts with `AA`, and that $\operatorname{epr}(B^{-1}) \neq \texttt{AA}\overline{\texttt{A}}\texttt{A}$. But, by Remark 3.2.6, $B^{-1} = I_n$, implying that $\operatorname{epr}(B^{-1}) = \texttt{AA}\overline{\texttt{A}}\texttt{A}$, a contradiction.

Since $\overline{\texttt{A}}\texttt{AAA}\overline{\texttt{A}}$ is attained by I_n, the desired conclusion follows. □

The epr-sequence of the matrix $M_{\texttt{AAN}}$ in Example 3.2.5 shows that the `AA` Theorem does not hold for all fields of characteristic 2.

Theorem 3.2.10. *Let $n \geq 3$, and let B be a symmetric matrix over $\mathbb{Z}_2$, with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Suppose that $\ell_1 = \mathtt{A}$ and $\ell_{n-1}\ell_n = \mathtt{AN}$. Then n is even.*

Proof. By Lemma 3.2.4, every minor of B of order $n-1$ is nonzero. We claim that each row of B contains an even number of nonzero entries; to see this, let k be the number of nonzero entries of B in row i, and consider a calculation of $\det(B)$ via a Laplace expansion along row i. Because in the field $\mathbb{Z}_2$ every number is equal to its negative, this expansion calculates $\det(B)$ by adding k minors of B of order $n-1$; since each of these k minors is nonzero, and because $\det(B) = 0$, it follows that k must be even, as claimed. Hence, the total number of nonzero entries of B must also be even. Then, as the number of nonzero off-diagonal entries of a symmetric matrix is always even, it is immediate that the number of nonzero diagonal entries of B must also be even. Finally, since the number of nonzero diagonal entries of B is n (because $\ell_1 = \mathtt{A}$), n is even, as desired. $\square$

We note that the sequence $\mathtt{AS}\overline{\mathtt{S}}\mathtt{AN}$ is attainable over $\mathbb{Z}_2$ when its order is even (see Theorem 3.3.8), implying that a sequence of the form $\mathtt{A}\cdots\mathtt{AN}$ is not completely prohibited. Moreover, Theorem 3.2.10 does not hold for all fields of characterisitic 2 (see Example 3.2.5).

In the interest of brevity when proving the next result, define the $n \times n$ matrix $R_{n,k}$ as follows: For $n \geq 2$, let

$$R_{n,k} := \begin{bmatrix} I_k & J_{k,n-k} \\ J_{n-k,k} & A(K_{n-k}) \end{bmatrix},$$

where $0 \leq k \leq n$ (we assume that $R_{n,k} = I_n$ when $k = n$, and that $R_{n,k} = A(K_n)$ when $k = 0$).

Proposition 3.2.11. *An epr-sequence starting with $\mathtt{SA}$ is attainable by a symmetric matrix over $\mathbb{Z}_2$ if and only if it has one of the following forms.*

$$\mathtt{SA}\overline{\mathtt{SA}}, \qquad \mathtt{SA}\overline{\mathtt{SA}}\mathtt{A}, \qquad \mathtt{SA}\overline{\mathtt{SA}}\mathtt{N}.$$

Proof. Let $0 \leq k \leq n$ be integers. We begin by showing that $\det(R_{n,k}) = 0$ only when n is odd and k is even. The desired conclusion is immediate for the case with $k = 0$ (by Proposition 3.2.7), and, for the case with $k = n$, it is obvious (since $R_{n,k} = I_n$). Now, suppose $0 < k < n$, and let $C = R_{n,k}$. Note that $\det(C) = \det(I_k)\det(C/I_k) = \det(C/I_k)$, where C/I_k is the Schur complement of I_k in C. Then, as

$$C/I_k = A(K_{n-k}) - J_{n-k,k} \cdot J_{k,n-k} = (1-k)J_{n-k} - I_{n-k},$$

$\det(C) = ((1-k)(n-k)-1)(-1)^{n-k-1} = (k+1)n + 1$ (in characteristic 2). It follows that $\det(C) = 1$ when n is even, and that $\det(C) = k$ when n is odd. We can now conclude that $\det(R_{n,k}) = 0$ only when n is odd and k is even, as desired.

Let σ be an epr-sequence starting with $\mathtt{SA}$. For the first direction, suppose that $\sigma = \operatorname{epr}(B)$ for some symmetric matrix B over $\mathbb{Z}_2$. Let $\sigma = \ell_1\ell_2 \cdots \ell_n$. By hypothesis, $\ell_1\ell_2 = \mathtt{SA}$. Without loss of generality, suppose that the first k diagonal entries of B are nonzero, and suppose that the remaining $n-k$ diagonal entries are zero. Note that, since $\ell_1 = \mathtt{S}$, $1 \leq k \leq n-1$. It is easy to verify that the condition that $\ell_2 = \mathtt{A}$ implies that $B = R_{n,k}$. It is also easy to see that for any integer m with $3 \leq m \leq n$, any $m \times m$ principal submatrix of $R_{n,k}$ is of the form $R_{m,p}$, where $0 \leq p \leq k$ (and $0 \leq m-p \leq n-k$). The above argument implies that any principal minor of B of order m is nonzero when m is even, implying that $\ell_j = \mathtt{A}$ whenever j is even. Also, observe that for any odd integer m with $3 \leq m < n$, there exists $0 \leq p \leq k$ even, and $0 \leq q \leq k$ odd, such that $R_{m,p}$ and $R_{m,q}$ are principal submatrices of B; then, as $\det(R_{m,p}) = 0$ and $\det(R_{m,q}) \neq 0$, B contains both a zero and a nonzero principal minor of order m, implying that $\ell_j = \mathtt{S}$ whenever $j < n$ is odd. It now follows that B must have one of the desired epr-sequences.

For the other direction, note that the order-n sequence $\mathtt{SA}\overline{\mathtt{SA}}$ is attained by the matrix $R_{n,1}$ when n is even. Similarly, (when n is odd) the order-n sequences $\mathtt{SA}\overline{\mathtt{SA}}\mathtt{A}$ and $\mathtt{SA}\overline{\mathtt{SA}}\mathtt{N}$ are attained by $R_{n,1}$ and $R_{n,2}$, respectively. $\square$

As with the previous results, Proposition 3.2.11 cannot be generalized either (see Example 3.2.5).

An observation following from the $\mathtt{NA}$ Lemma, the $\mathtt{AA}$ Theorem and Proposition 3.2.11 is in order:

Observation 3.2.12. *Let B be a symmetric matrix over $\mathbb{Z}_2$, with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$*. If $\ell_2 = \mathtt{A}$, then $\ell_j = \mathtt{A}$ when j is even.*

The previous and the next result also do not hold for all fields of characteristic 2 (see Example 3.2.5).

Proposition 3.2.13. *For any $\mathtt{X}$, the epr-sequence $\mathtt{SAXN}$ cannot occur as a subsequence of the epr-sequence of a symmetric matrix over $\mathbb{Z}_2$.*

Proof. Let B be an $n \times n$ symmetric matrix over $\mathbb{Z}_2$, with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_k\cdots\ell_{k+3} = \mathtt{SAXN}$ for some $1 \leq k \leq n-3$, where $\mathtt{X} \in \{\mathtt{A},\mathtt{S},\mathtt{N}\}$. By Proposition 3.2.11, $k \geq 2$. By the $\mathtt{NSA}$ Theorem, $\ell_{k-1} \neq \mathtt{N}$. Let $B[\alpha]$ be a $(k-1)\times(k-1)$ nonsingular principal submatrix of B. By the Schur Complement Corollary, $\operatorname{epr}(B/B[\alpha]) = \mathtt{YAZN}\cdots$, where $\mathtt{Y},\mathtt{Z} \in \{\mathtt{A},\mathtt{S},\mathtt{N}\}$, which contradicts Observation 3.2.12. □

In the epr-sequence of a symmetric matrix over a field of characteristic *not* 2, [4, Theorem 2.15] asserts that $\mathtt{ANS}$ can only occur as the initial subsequence. Over $\mathbb{Z}_2$, the same restriction holds for $\mathtt{ASS}$:

Proposition 3.2.14. *In the epr-sequence of a symmetric matrix over $\mathbb{Z}_2$, the subsequence $\mathtt{ASS}$ can only occur as the initial subsequence.*

Proof. Let B be an $n \times n$ symmetric matrix over $\mathbb{Z}_2$, with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose to the contrary that $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{ASS}$ for some $2 \leq k \leq n-3$. By the Inheritance Theorem, B contains a $(k+2)\times(k+2)$ principal submatrix B' with $\operatorname{epr}(B') = \cdots\mathtt{XAYN}$, where $\mathtt{X},\mathtt{Y} \in \{\mathtt{A},\mathtt{S},\mathtt{N}\}$. By the $\mathtt{NA}$ Lemma, $\mathtt{X} \neq \mathtt{N}$, and, by the $\mathtt{AA}$ Theorem, $\mathtt{X} \neq \mathtt{A}$; hence, $\mathtt{X} = \mathtt{S}$, so that $\operatorname{epr}(B') = \cdots\mathtt{SAYN}$, which contradicts Proposition 3.2.13. □

Once again, the previous result also cannot be generalized to all fields of characteristic 2 (see Example 3.2.5).

Lemma 3.2.15. *Let B be a symmetric matrix over $\mathbb{Z}_2$, with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Suppose $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{ASA}$ for some $1 \leq k \leq n-2$. Then $\ell_1 \neq \mathtt{N}$ and the following hold.*

1. *If $\ell_1 = \mathtt{A}$, then k is odd.*

2. *If $\ell_1 = \mathtt{S}$, then k is even.*

Proof. By the `NN` Theorem and the `NA`-`NS` Observation, $\operatorname{epr}(B)$ does not begin with `NN`, `NA`, nor `NS`; hence, $\ell_1 \neq \mathtt{N}$.

(1): Suppose that $\ell_1 = \mathtt{A}$ and that k is even. Then, by the Inheritance Theorem, B contains a $(k+2) \times (k+2)$ principal submatrix B' with $\operatorname{epr}(B') = \mathtt{A}\cdots\mathtt{ASA}$. By the Inverse Theorem, $\operatorname{epr}(B^{-1}) = \mathtt{SA}\cdots\mathtt{AA}$. Since B^{-1} is of order $k+2$, Proposition 3.2.11 implies that $k+2$ is odd, which is a contradiction to k being even.

(2): Suppose that $\ell_1 = \mathtt{S}$ and that k is odd. Then, by the Inheritance Theorem, B contains a $(k+2) \times (k+2)$ principal submatrix B' with $\operatorname{epr}(B') = \mathtt{S}\cdots\mathtt{AXA}$, where $\mathtt{X} \in \{\mathtt{A}, \mathtt{S}, \mathtt{N}\}$. Since `X` occurs in an even position, the `N`-Even Observation implies that $\mathtt{X} \neq \mathtt{N}$; and, by the `AA` Theorem, $\mathtt{X} \neq \mathtt{A}$; hence, $\mathtt{X} = \mathtt{S}$. By the Inverse Theorem, $\operatorname{epr}((B')^{-1}) = \mathtt{SA}\cdots\mathtt{SA}$. Since $(B')^{-1}$ is of order $k+2$, Proposition 3.2.11 implies that $k+2$ is even, a contradiction. □

The inverse of the matrix $M_{\mathtt{SASSA}}$ in Example 3.2.5, whose epr-sequence is `SSASA`, reveals that the previous result also cannot be generalized to all fields of characteristic 2; and, for the same reasons, the following theorem cannot be generalized either.

Theorem 3.2.16. *Let B be a symmetric matrix over $\mathbb{Z}_2$. Suppose* $\operatorname{epr}(B)$ *contains* `ASA` *as a subsequence. Then* $\operatorname{epr}(B)$ *is one of the following sequences.*

1. $\mathtt{ASA}\overline{\mathtt{SA}}$;

2. $\mathtt{ASA}\overline{\mathtt{SA}}\mathtt{A}$;

3. $\mathtt{ASA}\overline{\mathtt{SA}}\mathtt{N}$;

4. $\mathtt{SASA}\overline{\mathtt{SA}}$;

5. $\mathtt{SASA}\overline{\mathtt{SA}}\mathtt{A}$;

6. $\mathtt{SASA}\overline{\mathtt{SA}}\mathtt{N}$.

Proof. Suppose that $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$, and that $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{ASA}$. By Lemma 3.2.15, $\ell_1 \neq \mathtt{N}$. We proceed by examining two cases.

Case 1: $\ell_1 = \mathtt{S}$. Because of Proposition 3.2.11, it suffices to show that $\ell_2 = \mathtt{A}$. By Lemma 3.2.15, k is even. If $k = 2$, then, obviously, $\ell_2 = \mathtt{A}$. Now, suppose $k \geq 4$. By the Inheritance Theorem, B contains a $(k+2)\times(k+2)$ principal submatrix, B', whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_{k+2}$ has $\ell'_2 = \ell_2$ and $\ell'_k\ell'_{k+1}\ell'_{k+2} = \mathtt{A}\ell'_{k+1}\mathtt{A}$. By the Inverse Theorem, $\operatorname{epr}((B')^{-1}) = \ell'_{k+1}\mathtt{A}\cdots\ell_2\ell'_1\mathtt{A}$. It follows from Observation 3.2.12 that $[\operatorname{epr}((B')^{-1})]_j = \mathtt{A}$ when j is even. Then, as k is even, and because $[\operatorname{epr}((B')^{-1})]_k = \ell_2$, we must have $\ell_2 = \mathtt{A}$.

Case 2: $\ell_1 = \mathtt{A}$. By Lemma 3.2.15, k is odd. Let $1 < j < k$ be an odd integer. By the Inheritance Theorem, B contains a $(k+2)\times(k+2)$ principal submatrix, B', whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_{k+2}$ has $\ell'_j = \ell_j$ and $\ell'_k\ell'_{k+1}\ell'_{k+2} = \mathtt{A}\ell'_{k+1}\mathtt{A}$. By the Inverse Theorem, $\operatorname{epr}((B')^{-1}) = \ell'_{k+1}\mathtt{A}\cdots\ell_j\cdots$. It follows from Observation 3.2.12 that $[\operatorname{epr}((B')^{-1})]_i = \mathtt{A}$ when i is even. Since $k+2-j$ is even, $[\operatorname{epr}((B')^{-1})]_{k+2-j} = \mathtt{A}$. Then, as $[\operatorname{epr}((B')^{-1})]_{k+2-j} = \ell'_j = \ell_j$, we have $\ell_j = \mathtt{A}$. We conclude that $\ell_i = \mathtt{A}$ when i is an odd integer with $1 < i < k$. Then, as $\ell_{k+1} = \mathtt{S}$, the $\mathtt{AA}$ Theorem implies that $\ell_i \neq \mathtt{A}$ when i is an even integer with $1 < i < k$; and, since $\ell_k = \mathtt{A}$, the $\mathtt{N}$-Even Observation implies that $\ell_i \neq \mathtt{N}$ when i is an even integer with $1 < i < k$. Hence, $\operatorname{epr}(B) = \mathtt{ASA}\overline{\mathtt{SA}}\ell_{k+3}\cdots\ell_n$.

If $n = k+2$, then we are done; thus, suppose $n \geq k+3$. Suppose to the contrary that $\ell_q \neq \mathtt{A}$ for some odd integer q with $k+3 \leq q \leq n$. By the Inheritance Theorem, B contains a singular $q \times q$ principal submatrix, B', whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_q$ has

$\ell'_i = \ell_i = \mathtt{A}$ when $i \leq k+2$ is odd, and, obviously, $\ell'_q = \mathtt{N}$. Let $B'[\alpha]$ be a (necessarily nonsingular) 1×1 principal submatrix of B'. By the Schur Complement Theorem, $B'/B'[\alpha]$ is a $(q-1) \times (q-1)$ (symmetric) matrix, and, by the Schur Complement Corollary, $[\mathrm{epr}(B'/B'[\alpha])]_2 = \mathtt{A}$, and $[\mathrm{epr}(B'/B'[\alpha])]_{q-1} = \mathtt{N}$. It follows from Observation 3.2.12 that $[\mathrm{epr}(B'/B'[\alpha])]_i = \mathtt{A}$ when i is even. Then, as $[\mathrm{epr}(B'/B'[\alpha])]_{q-1} = \mathtt{N}$, $q-1$ is odd, which is a contradiction to the fact that q is odd. We conclude that $\ell_i = \mathtt{A}$ for all odd i with $k+3 \leq i \leq n$.

Then, as $\ell_{k+1} = \mathtt{S}$, the AA Theorem implies that $\ell_i \neq \mathtt{A}$ when i is an even integer with $k+3 \leq i \leq n-1$; and, since at least one of ℓ_{n-1} and ℓ_n must be A (because one of $n-1$ and n must be even) the N-Even Observation implies that $\ell_i \neq \mathtt{N}$ when i is an even integer with $k+3 \leq i \leq n-1$. It follows that $\mathrm{epr}(B) = \mathtt{ASA}\overline{\mathtt{SA}}$ when n is odd, and that either $\mathrm{epr}(B) = \mathtt{ASA}\overline{\mathtt{SA}}\mathtt{A}$ or $\mathrm{epr}(B) = \mathtt{ASA}\overline{\mathtt{SA}}\mathtt{N}$ when n is even. □

3.3 Main results

In this section, a complete characterization of the epr-sequences that are attainable by a symmetric matrix over $\mathbb{Z}_2$ is established. We start by characterizing those that begin with N.

Lemma 3.3.1. *Let* $M_1 = A(K_2) \oplus A(K_2) \oplus \cdots \oplus A(K_2)$ *and*

$$M_2 = \begin{bmatrix} M_1 & \mathbb{1}_n \\ \mathbb{1}_n^T & O_1 \end{bmatrix}$$

be over $\mathbb{Z}_2$*. Then* $\mathrm{epr}(M_1) = \overline{\mathtt{NS}}\mathtt{NA}$ *and* $\mathrm{epr}(M_2) = \overline{\mathtt{NS}}\mathtt{NAN}$.

Proof. Let $\mathrm{epr}(M_1) = \ell_1\ell_2\cdots\ell_n$. Note that n is even. The desired conclusion is obvious when $n = 2$; hence, suppose $n \geq 4$. It is clear that $\ell_1\ell_2 = \mathtt{NS}$; thus, by the NA-NS Observation, $\mathrm{epr}(M_1)$ has N in every odd position. Clearly, M_1 is nonsingular, implying that $\ell_n = \mathtt{A}$. It remains to show that $\ell_j = \mathtt{S}$ when $j \leq n-1$ is even. Since $\ell_n = \mathtt{A}$, by the

NN Theorem, $\ell_j \neq \mathtt{N}$ when $j \leq n-1$ is even. Now, because of the NA Lemma, to show that $\ell_j \neq \mathtt{A}$ when $j \leq n-1$ is even, it suffices to show that $\ell_{n-2} = \mathtt{S}$. Clearly, $M_1(\{2,4\})$ is singular (since it contains a zero row). Then, as $\ell_{n-2} \neq \mathtt{N}$ (because $n-2$ is even), $\ell_{n-2} = \mathtt{S}$.

Let $\operatorname{epr}(M_2) = \ell'_1\ell'_2 \cdots \ell'_{n+1}$. The assertion is clear when $n = 2$ (note that n is even, and that M_2 is of order $n+1$, not n); hence, suppose $n \geq 4$. Since (clearly) $\ell'_1\ell'_2 = \mathtt{NS}$, the NA-NS Observation implies that $\operatorname{epr}(M_2)$ has N in every odd position. Since M_1 is a principal submatrix of M_2, and because $\operatorname{epr}(M_1) = \mathtt{NS\overline{NS}NA}$, it is immediate that $\operatorname{epr}(M_2) = \mathtt{NS\overline{NS}N}\ell'_n\mathtt{N}$. We now show that $\ell'_n = \mathtt{A}$. Observe that any $n \times n$ principal submatrix of M_2 is either M_1, which is nonsingular, or is one that is permutationally similar to the matrix

$$C = \begin{bmatrix} C(\{n\}) & \mathbb{1}_{n-1} \\ \mathbb{1}_{n-1}^T & O_1 \end{bmatrix},$$

where $C(\{n\}) = O_1 \oplus A(K_2) \oplus A(K_2) \oplus \cdots \oplus A(K_2)$. Let C' be the matrix obtained from C by first subtracting its first row from rows $2, 3, \ldots, n-1$, and then subtracting the first column of the resulting matrix from columns $2, 3, \ldots, n-1$. Now observe that $\det(C') = -\det(C'(\{1,n\}))$, where $C'(\{1,n\}) = A(K_2) \oplus A(K_2) \oplus \cdots \oplus A(K_2)$, which is a nonsingular matrix (of order $(n-2)$). Hence, $\det(C') \neq 0$. Then, as $\det(C) = \det(C')$, C is nonsingular. We conclude that $\ell'_n = \mathtt{A}$. □

Theorem 3.3.2. *An epr-sequence starting with* N *is attainable by a symmetric matrix over* $\mathbb{Z}_2$ *if and only if it has one of the following forms:*

1. $\mathtt{NA\overline{NA}}$;

2. $\mathtt{NA\overline{NA}N}$;

3. $\mathtt{\overline{NS}N\overline{N}}$;

4. $\mathtt{NS\overline{NS}NA}$;

5. $\mathtt{NS}\overline{\mathtt{NS}}\mathtt{NAN}$.

Proof. Let $\sigma = \ell_1\ell_2\cdots\ell_n$ be an epr-sequence with $\ell_1 = \mathtt{N}$. Suppose that $\sigma = \text{epr}(B)$, where B is a symmetric matrix over $\mathbb{Z}_2$. If $n = 1$, then $\sigma = \overline{\mathtt{NS}}\mathtt{N}\overline{\mathtt{N}}$ with $\overline{\mathtt{NS}}$ and $\overline{\mathtt{N}}$ empty. Suppose $n \geq 2$. If $\ell_2 = \mathtt{N}$, then, by the $\mathtt{NN}$ Theorem, $\sigma = \overline{\mathtt{NS}}\mathtt{NN}\overline{\mathtt{N}}$ with $\overline{\mathtt{NS}}$ empty. If $\ell_2 = \mathtt{A}$, then, by the $\mathtt{NA}$ Lemma, $\sigma = \mathtt{NA}\overline{\mathtt{NA}}$ or $\sigma = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$.

Finally, suppose $\ell_2 = \mathtt{S}$. Since an attainable epr-sequence cannot end in $\mathtt{S}$, $n \geq 3$. By the $\mathtt{NA}$-$\mathtt{NS}$ Observation, $\ell_j = \mathtt{N}$ when j is odd. Hence, rank(B) is even. We now show that $\mathtt{SNA}$ cannot occur as a subsequence of $\ell_1\ell_2\cdots\ell_{n-2}$. Suppose to the contrary that $\ell_{k-1}\ell_k\ell_{k+1} = \mathtt{SNA}$, where $3 \leq k \leq n-3$. Clearly, since $\ell_j = \mathtt{N}$ when j is odd, k is odd and $\ell_{k+2} = \mathtt{N}$. By the Inheritance Theorem, B contains a $(k+3) \times (k+3)$ principal submatrix B' with $\text{epr}(B') = \cdots\mathtt{SNANX}$, where $\mathtt{X} \in \{\mathtt{A}, \mathtt{N}\}$. If $\mathtt{X} = \mathtt{A}$, then, by the Inverse Theorem, $\text{epr}((B')^{-1}) = \mathtt{NANS}\cdots$, which contradicts the $\mathtt{NA}$ Lemma. Hence, $\mathtt{X} = \mathtt{N}$, and therefore $\text{epr}(B') = \cdots\mathtt{SNANN}$, which contradicts the $\mathtt{NA}$ Lemma. We conclude that $\mathtt{SNA}$ cannot occur as a subsequence of $\ell_1\ell_2\cdots\ell_{n-2}$. Now, let $r = \text{rank}(B)$; hence, $\ell_r \neq \mathtt{N}$. Then, as r is even, $\ell_{r-1} = \mathtt{N}$ (because $r-1$ is odd). Since $\ell_j = \mathtt{N}$ when j is odd, the $\mathtt{NN}$ Theorem implies that $\ell_i \neq \mathtt{N}$ when $i \leq r-1$ is even. We proceed by considering two cases.

Case 1: $r \geq n-1$. First, suppose $r = n-1$. Since r is even, $r+1 = n$ is odd, implying that $\ell_n = \mathtt{N}$. Hence, $\ell_{n-1}\ell_n = \mathtt{AN}$ or $\ell_{n-1}\ell_n = \mathtt{SN}$. Then, as $\ell_2 = \mathtt{S}$, and because $\mathtt{SNA}$ cannot occur as a subsequence of $\ell_1\ell_2\cdots\ell_{n-2}$, it follows inductively that $\sigma = \mathtt{NS}\overline{\mathtt{NS}}\mathtt{NAN}$ or $\sigma = \overline{\mathtt{NS}}\mathtt{NSN}$. Now, suppose $r = n$; hence, n is even and $\ell_n = \mathtt{A}$. Since $\ell_{r-1} = \mathtt{N}$, $\ell_{n-1}\ell_n = \mathtt{NA}$. Then, as $\ell_2 = \mathtt{S}$, and because $\mathtt{SNA}$ cannot occur as a subsequence of $\ell_1\ell_2\cdots\ell_{n-2}$, it follows inductively that $\sigma = \mathtt{NS}\overline{\mathtt{NS}}\mathtt{NA}$.

Case 2: $r \leq n-2$. Hence, $\ell_{r+1}\cdots\ell_n = \mathtt{NN}\overline{\mathtt{N}}$. Since $\ell_{r-1} = \mathtt{N}$ and $\ell_r \neq \mathtt{N}$, $\ell_{r-1}\cdots\ell_n = \mathtt{NANN}\overline{\mathtt{N}}$ or $\ell_{r-1}\cdots\ell_n = \mathtt{NSNN}\overline{\mathtt{N}}$; but the former case contradicts the $\mathtt{NA}$ Lemma, implying that $\ell_{r-1}\cdots\ell_n = \mathtt{NSNN}\overline{\mathtt{N}}$. Then, as $\ell_2 = \mathtt{S}$, and because $\mathtt{SNA}$ cannot occur as a subsequence of $\ell_1\ell_2\cdots\ell_{n-2}$, it follows inductively that $\sigma = \mathtt{NS}\overline{\mathtt{NS}}\mathtt{NN}\overline{\mathtt{N}}$.

For the other direction, we show that each of the sequences listed above is attainable. Assume that the sequence under consideration has order n. The sequences $\mathtt{NA}\overline{\mathtt{NA}}$ and $\mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$ are attainable by Proposition 3.2.7. When $\overline{\mathtt{NS}}$ is non-empty the sequence $\overline{\mathtt{NS}}\mathtt{N}\overline{\mathtt{N}}$ is attainable by applying Observation 3.1.10(2) to the sequence $\mathtt{NA}\overline{\mathtt{NA}}$; and, when $\overline{\mathtt{NS}}$ is empty, it is attained by O_n. Finally, the sequences $\mathtt{NS}\overline{\mathtt{NS}}\mathtt{NA}$ and $\mathtt{NS}\overline{\mathtt{NS}}\mathtt{NAN}$ are attainable by Lemma 3.3.1. □

Naturally, due to the dependence of Theorem 3.3.2 on the results of Section 3.2.1, this theorem does not hold for other fields.

Some lemmas are necessary before stating the second of our three main results in Theorem 3.3.8.

Lemma 3.3.3. *Let* $n \geq 4$, $m \geq 5$, *and let*

$$M_{\mathtt{ASA}} = \begin{bmatrix} I_2 & \mathbb{1}_2 \\ \mathbb{1}_2^T & J_1 \end{bmatrix}, \quad M_{\mathtt{ASAA}} = \begin{bmatrix} I_2 & J_2 \\ J_2 & I_2 \end{bmatrix}$$

be over $\mathbb{Z}_2$. *Let* $B = I_{n-3} \oplus M_{\mathtt{ASA}}$, $B' = I_{m-4} \oplus M_{\mathtt{ASAA}}$, $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$ *and* $\operatorname{epr}(B') = \ell'_1\ell'_2\cdots\ell'_m$. *Then* $\operatorname{epr}(M_{\mathtt{ASA}}) = \mathtt{ASA}$, $\operatorname{epr}(M_{\mathtt{ASAA}}) = \mathtt{ASAA}$, $\ell_1\ell_2\ell_3 = \ell'_1\ell'_2\ell'_3 = \mathtt{ASS}$, $\ell_{n-1}\ell_n = \mathtt{SA}$ *and* $\ell'_{m-1}\ell'_m = \mathtt{AA}$.

Proof. All of the assertions above are easily verified, except $\ell'_{m-1} = \mathtt{A}$, which we now prove. The case with $m = 5$ is easy to check; thus, suppose $m \geq 6$. Note that, since every 3×3 principal submatrix of the (4×4) matrix $M_{\mathtt{ASAA}}$ is nonsingular, and because every $(m-5) \times (m-5)$ principal submatrix of I_{m-4} is also nonsingular, deleting row i and column i of B' results in a matrix that is a direct sum of two nonsingular matrices; hence, every $(m-1) \times (m-1)$ principal submatrix of B' is nonsingular, implying that $\ell'_{m-1} = \mathtt{A}$. □

A matrix that will play an important role here is defined as follows: For $n \geq 2$, let F_n be the $n \times n$ matrix resulting from replacing the first diagonal entry of $A(K_n)$ with 1.

Lemma 3.3.4. *Let $n \geq 2$, and let F_n be over $\mathbb{Z}_2$. Then F_n is nonsingular.*

Proof. The assertion is obvious when $n = 2$; thus, assume $n \geq 3$. Observe that

$$\det(F_n) = \det(F_n[\{1\}])\det(F_n/F_n[\{1\}]) = \det(J_1)\det(F_n/J_1) = \det(F_n/J_1),$$

where

$$F_n/J_1 = F_n[\{2,\ldots,n\}] - \mathbb{1}_{n-1} \cdot (J_1)^{-1} \cdot \mathbb{1}_{n-1}^T = A(K_{n-1}) - J_{n-1} = -I_{n-1}.$$

Hence, $\det(F_n) = \det(-I_{n-1}) \neq 0$. □

Lemma 3.3.5. *Let $n = 4k + 2$, where $k \geq 1$ is an integer. Let $m = \frac{n}{2}$, let*

$$B = \begin{bmatrix} J_m & I_m \\ I_m & I_m \end{bmatrix}$$

be over $\mathbb{Z}_2$, and let $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Then $\ell_1\ell_2\ell_3 = \mathtt{ASS}$ and $\ell_{n-1}\ell_n = \mathtt{AN}$.

Proof. It is easily seen that $\ell_1\ell_2\ell_3 = \mathtt{ASS}$. Next, we show that $\ell_n = \mathtt{N}$. Observe that $\det(B) = \det(I_m)\det(B/I_m)$, where

$$B/I_m = J_m - I_m \cdot (I_m)^{-1} \cdot I_m = A(K_m).$$

Since $m = \frac{n}{2} = 2k + 1$ is odd, Proposition 3.2.7 implies that $A(K_m)$ is singular, implying that B/I_m is singular, and therefore that $\det(B) = 0$; hence, $\ell_n = \mathtt{N}$.

Now, to see that $\ell_{n-1} = \mathtt{A}$, note that the $(n-1) \times (n-1)$ principal submatrix resulting from the deletion of the ith row and ith column of B must be one of the following two matrices:

$$C_1 = \begin{bmatrix} J_{m-1} & X^T \\ X & I_m \end{bmatrix}, \; C_2 = \begin{bmatrix} J_m & X \\ X^T & I_{m-1} \end{bmatrix},$$

where $X = I_m(\emptyset, \{q\})$ and $q \in \{1, 2, \ldots, m\}$ is the unique integer such that $i = q$ or $i = m + q$ (that is, $q = i$ if $1 \leq i \leq m$, and $q = i - m$ if $m + 1 \leq i \leq n$). Observe that

$\det(C_1) = \det(I_m)\det(C_1/I_m)$ and $\det(C_2) = \det(I_{m-1})\det(C_2/I_{m-1})$, where $C_1/I_m = J_{m-1} - X^TX$ and $C_2/I_{m-1} = J_m - XX^T$ are the Schur complements of I_m and I_{m-1} in C_1 and C_2, respectively. Since $X^TX = I_{m-1}$, $C_1/I_m = A(K_{m-1})$. Then, as $m-1 = 2k$ is even, Proposition 3.2.7 implies that C_1/I_m is nonsingular; hence, $\det(C_1) \neq 0$.

Finally, observe that XX^T is the $m \times m$ matrix resulting from replacing the qth diagonal entry of I_m with 0. Hence, C_2/I_{m-1} is the matrix resulting from replacing the qth diagonal entry of $A(K_m)$ with 1. Then, as C_2/I_{m-1} is permutationally similar to the nonsingular matrix F_m (see Lemma 3.3.4), C_2/I_{m-1} is nonsingular, implying that $\det(C_2) \neq 0$. □

A worthwhile observation is that the condition that n is equal to 2 modulo 4 in Lemma 3.3.5 was of relevance when showing that $\det(B) = 0$, as it is consistent with the proof of Theorem 3.2.10, from which it can be deduced that, in order to have $\det(B) = 0$, it is necessary for B to contain an even number of nonzero entries in each row (observe that B contains $\frac{n}{2}+1 = 2(k+1)$ nonzero entries in each of the first $\frac{n}{2}$ rows, and 2 nonzero entries in each of the remaining rows). For the same reasons, the congruence modulo 4 of n in the following lemma will once again be of relevance.

Lemma 3.3.6. *Let $n = 4k$, where $k \geq 2$ is an integer. Let $m = \frac{n}{2}$, let*

$$B = \begin{bmatrix} J_{m-1} & W \\ W^T & I_{m+1} \end{bmatrix}$$

be over $\mathbb{Z}_2$, where $W = [I_{m-1}, J_{m-1,2}]$, and let $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Then $\ell_1\ell_2\ell_3 =$ `ASS` and $\ell_{n-1}\ell_n =$ `AN`.*

Proof. It is easily verified that $\ell_1\ell_2\ell_3 =$ `ASS`. Now we verify that $\ell_n =$ `N`. Observe that $\det(B) = \det(I_{m+1})\det(B/I_{m+1})$, where B/I_{m+1} is the Schur complement of $B[\{m, m+1, \ldots, n\}] = I_{m+1}$ in B. Note that

$$B/I_{m+1} = J_{m-1} - WW^T = J_{m-1} - I_{m-1} - 2J_{m-1} = A(K_{m-1}) - 2J_{m-1}.$$

Hence, $B/I_{m+1} = A(K_{m-1})$ (in characteristic 2). Then, as $m-1 = 2k-1$ is odd, Proposition 3.2.7 implies that B/I_{m+1} is singular. It follows that $\det(B) = 0$, and therefore that $\ell_n = \mathtt{N}$.

Now we show that $\ell_{n-1} = \mathtt{A}$. Let $\alpha_1 = \{1, 2, \ldots, m-1\}$, $\alpha_2 = \{m, m+1, \ldots, n-2\}$ and $\alpha_3 = \{n-1, n\}$.

Let B' be the matrix obtained from B by deleting its ith row and ith column. Let $q = i - (m-1)$. Suppose that $M_1 = B'$ if $i \in \alpha_1$, that $M_2 = B'$ if $i \in \alpha_2$, and that $M_3 = B'$ if $i \in \alpha_3$. It is easy to see that

$$C_1 = \begin{bmatrix} J_{m-2} & X \\ X^T & I_{m+1} \end{bmatrix}, \; C_2 = \begin{bmatrix} J_{m-1} & Y \\ Y^T & I_m \end{bmatrix}, \; C_3 = \begin{bmatrix} J_{m-1} & Z \\ Z^T & I_m \end{bmatrix},$$

where

$$X = [I_{m-1}(\{i\}, \emptyset), J_{m-2,2}], \;\; Y = [I_{m-1}(\emptyset, \{q\}), J_{m-1,2}], \;\; Z = [I_{m-1}, \mathbb{1}_{m-1}].$$

We proceed to show that B' is nonsingular by considering the three cases outlined above.

Case 1: $B' = C_1$. Note that $\det(C_1) = \det(I_{m+1})\det(C_1/I_{m+1})$, where C_1/I_{m+1} is the Schur Complement of I_{m+1} in C_1, and that

$$C_1/I_{m+1} = J_{m-2} - XX^T = J_{m-2} - I_{m-2} - 2J_{m-2} = A(K_{m-2}) - 2J_{m-2}.$$

Hence, $C_1/I_{m+1} = A(K_{m-2})$ (in characteristic 2). Then, as $m-2 = 2k-2$ is even, Proposition 3.2.7 implies that C_1/I_{m+1} is nonsingular, implying that $\det(C_1) \neq 0$.

Case 2: $B' = C_2$. Then $\det(C_2) = \det(I_m)\det(C_2/I_m)$, where C_2/I_m is the Schur complement of I_m in C_2, and

$$C_2/I_m = J_{m-1} - YY^T = J_{m-1} - I_{m-1}(\emptyset, \{q\}) \cdot I_{m-1}(\{q\}, \emptyset) - 2J_{m-1}.$$

Note that $I_{m-1}(\emptyset, \{q\}) \cdot I_{m-1}(\{q\}, \emptyset)$ is the matrix obtained from I_{m-1} by replacing its qth diagonal entry with 0. Then, as $2J_{m-1} = O_{m-1}$ (in characteristic 2), C_2/I_m is the matrix obtained from $A(K_{m-1})$ by replacing its qth diagonal entry with 1. Hence, C_2/I_m

is permutationally similar to the nonsingular matrix F_{m-1} (see Lemma 3.3.4). It follows that C_2/I_m is nonsingular, and therefore that $\det(C_2) \neq 0$.

Case 3: $B' = C_3$. Then $\det(C_3) = \det(I_m)\det(C_3/I_m)$, where C_3/I_m is the Schur complement of I_m in C_3, and

$$C_3/I_m = J_{m-1} - ZZ^T = J_{m-1} - I_{m-1} - J_{m-1} = -I_{m-1}.$$

It follows that C_3/I_m is nonsingular, and therefore that $\det(C_3) \neq 0$. $\square$

Lemma 3.3.7. *The following epr-sequences are attainable by a symmetric matrix over* $\mathbb{Z}_2$.

$$\mathtt{ASA}\overline{\mathtt{SA}}, \qquad \mathtt{ASA}\overline{\mathtt{SA}}\mathtt{A}, \qquad \mathtt{ASA}\overline{\mathtt{SA}}\mathtt{N}.$$

Proof. The attainability of $\mathtt{ASA}\overline{\mathtt{SA}}$ follows by observing that, by the Inverse Theorem, the inverse of any (symmetric) matrix attaining the sequence $\mathtt{SA}\overline{\mathtt{SA}}\mathtt{A}$, which is attainable by Proposition 3.2.11, has epr-sequence $\mathtt{ASA}\overline{\mathtt{SA}}$.

Now, for $n \geq 4$ even, we show that the matrix

$$B = \begin{bmatrix} I_{n-2} & J_{n-2,2} \\ J_{2,n-2} & I_2 \end{bmatrix}$$

has epr-sequence $\mathtt{ASA}\overline{\mathtt{SA}}\mathtt{A}$. Because of Theorem 3.2.16, it suffices to show that $\operatorname{epr}(B)$ begins with $\mathtt{ASA}$, and that it ends with $\mathtt{A}$. Observe that $\det(B) = \det(I_{n-2})\det(B/I_{n-2})$, where

$$B/I_{n-2} = I_2 - J_{2,n-2} \cdot (I_{n-2})^{-1} \cdot J_{n-2,2} = I_2 - (n-2)J_2.$$

Since n is even, $B/I_{n-2} = I_2$ (in characteristic 2); hence, B is nonsingular. It is clear that $\operatorname{epr}(B)$ begins with $\mathtt{AS}$. Finally, note that each 3×3 principal submatrix of B must be I_3 or one of the following:

$$\begin{bmatrix} I_2 & \mathbb{1}_2 \\ \mathbb{1}_2^T & J_1 \end{bmatrix}, \qquad \begin{bmatrix} J_1 & \mathbb{1}_2^T \\ \mathbb{1}_2 & I_2 \end{bmatrix}.$$

Then, as each of these 3×3 matrices is nonsingular, $\operatorname{epr}(B)$ begins with $\mathtt{ASA}$, as desired.

With $n \geq 5$, let B be an $n \times n$ symmetric matrix with epr-sequence $\mathtt{SASA\overline{SA}N}$, which is attainable by Proposition 3.2.11. Note that n is odd. Let $\alpha \subseteq \{1, 2, \ldots, n\}$ with $|\alpha| = 1$ be such that $B[\alpha]$ is nonsingular. Let $\operatorname{epr}(B/B[\alpha]) = \ell'_1 \ell'_2 \cdots \ell'_{n-1}$. We now show that $\operatorname{epr}(B/B[\alpha]) = \mathtt{ASA\overline{SA}N}$. By the Schur Complement Corollary, $\ell'_j = \mathtt{A}$ when j is odd, and $\ell'_{n-1} = \mathtt{N}$. Since $n-2$ is odd, $\ell'_{n-2} = \mathtt{A}$. Since $\ell'_{n-1} = \mathtt{N}$, the $\mathtt{AA}$ Theorem implies that $\ell'_j \neq \mathtt{A}$ when $j \leq n-3$ is even. Finally, as $\ell'_{n-2} = \mathtt{A}$, the $\mathtt{N}$-Even Observation implies that $\ell'_j = \mathtt{S}$ when $j \leq n-3$ is even. It follows that $\operatorname{epr}(B/B[\alpha]) = \mathtt{ASA\overline{SA}N}$, as desired. ☐

Before stating our characterization of the epr-sequences that begin with $\mathtt{A}$ in the next theorem, something needs to be clarified: [4, Corollary 2.22] claims that the sequence $\mathtt{AS\overline{S}AAA\overline{A}}$ is attainable over $\mathbb{Z}_2$; this claim is false: Observe that it contradicts the $\mathtt{AA}$ Theorem. But it should be noted that [4, Corollary 2.22] becomes true once the field is restricted to be of characteristic 0, since it relies on [4, Proposition 2.18].

Theorem 3.3.8. *An epr-sequence of order n, and starting with $\mathtt{A}$, is attainable by a symmetric matrix over $\mathbb{Z}_2$ if and only if it has one of the following forms:*

1. $\mathtt{A\overline{A}}$;
2. $\mathtt{A\overline{S}N\overline{N}}$;
3. $\mathtt{ASS\overline{S}A}$;
4. $\mathtt{ASS\overline{S}AA}$;
5. $\mathtt{ASSS\overline{S}AN}$ *with n even;*
6. $\mathtt{ASA\overline{SA}}$;
7. $\mathtt{ASA\overline{SA}A}$;
8. $\mathtt{ASA\overline{SA}N}$.

Proof. Let $\sigma = \ell_1\ell_2\cdots\ell_n$ be an epr-sequence with $\ell_1 = \mathtt{A}$. Suppose that $\sigma = \operatorname{epr}(B)$, where B is a symmetric matrix over $\mathbb{Z}_2$. If $n = 1$ or $n = 2$, then σ is A, AA, or AN, all of which are listed above. Suppose $n \geq 3$. If $\ell_2 = \mathtt{A}$ or $\ell_2 = \mathtt{N}$, then the AA Theorem and the N-Even Observation imply that σ is either $\mathtt{AAA\overline{A}}$ or $\mathtt{ANN\overline{N}}$. Now, suppose $\ell_2 = \mathtt{S}$. If σ contains the subsequence ASA, then, by Theorem 3.2.16, σ is either $\mathtt{ASA\overline{SA}}$, $\mathtt{ASA\overline{SA}A}$, or $\mathtt{ASA\overline{SA}N}$. Now, suppose σ does not contain ASA. Hence, $\ell_3 = \mathtt{N}$ or $\ell_3 = \mathtt{S}$, and $n \geq 4$. If $\ell_3 = \mathtt{N}$, then Observation 3.2.3 implies that $\sigma = \mathtt{ASNN\overline{N}}$. Now, assume that $\ell_3 = \mathtt{S}$. Let k be a minimal integer with $3 \leq k \leq n-1$ such that $\ell_k\ell_{k+1} = \mathtt{SN}$ or $\ell_k\ell_{k+1} = \mathtt{SA}$. Hence, $\ell_1\ell_2\cdots\ell_k = \mathtt{ASS\overline{S}}$. If $\ell_{k+1} = \mathtt{N}$, then Observation 3.2.3 implies that $\sigma = \mathtt{ASS\overline{S}N\overline{N}}$. Now, assume that $\ell_{k+1} = \mathtt{A}$. If $n = k+1$, then $\sigma = \mathtt{ASS\overline{S}A}$. Thus, suppose $n \geq k+2$.

We now show that $n = k+2$. Suppose to the contrary that $n \geq k+3$. By the AA Theorem, $\ell_{k+2} \neq \mathtt{A}$. If $\ell_{k+2} = \mathtt{N}$, then Observation 3.2.3 implies that σ contains SANN, which is prohibited by Proposition 3.2.13; hence, $\ell_{k+2} = \mathtt{S}$, so that $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{SAS}$. Then, as σ does not contain ASA, and because SASN is prohibited by Proposition 3.2.13, $\ell_{k+3} = \mathtt{S}$, implying that σ contains ASS as a non-initial subsequence, which contradicts Proposition 3.2.14. It follows that $n = k+2$, and therefore that σ is either $\mathtt{ASS\overline{S}AA}$ or $\mathtt{ASS\overline{S}AN}$; in the case with $\sigma = \mathtt{ASS\overline{S}AN}$, Theorem 3.2.10 implies that n is even, and therefore that $\sigma = \mathtt{ASSS\overline{S}AN}$.

Now, we establish the other direction. As before, we assume that the sequence under consideration has order n. First, the sequence $\mathtt{A\overline{A}}$ is attained by I_n. The sequence $\mathtt{A\overline{S}N\overline{N}}$ is attainable by applying Observation 3.1.10(1) to the sequence $\mathtt{A\overline{A}}$. To see that $\mathtt{ASS\overline{S}A}$ and $\mathtt{ASS\overline{S}AA}$ are attainable, observe that the matrices B and B' in Lemma 3.3.3 must attain these sequences, respectively, since the epr-sequence of these matrices must be one of those listed above. Similarly, when n is even, one of the two matrices in the statements of Lemma 3.3.5 and Lemma 3.3.6 is required to attain the sequence $\mathtt{ASSS\overline{S}AN}$. Finally, the attainability of $\mathtt{ASA\overline{SA}}$, $\mathtt{ASA\overline{SA}A}$, and $\mathtt{ASA\overline{SA}N}$ follows from Lemma 3.3.7. $\square$

The reader is once again referred to Example 3.2.5 to see why Theorem 3.3.8 cannot be generalized to other fields.

As before, we need more lemmas in order to prove the last of our three main results.

For an integer $n \geq 2$ and $k \in \{1, 2, \ldots, n\}$, we let e_k^n denote the column vector of length n with the kth entry equal to 1 and every other entry equal to zero; moreover, let

$$G_n := \begin{bmatrix} J_1 & (e_1^{n-1})^T \\ e_1^{n-1} & F_{n-1} \end{bmatrix}.$$

Lemma 3.3.9. *Let $n \geq 4$ be an even integer, let G_n be over $\mathbb{Z}_2$, and let $\operatorname{epr}(G_n) = \ell_1\ell_2 \cdots \ell_n$. Then $\ell_1\ell_2 = \texttt{SS}$ and $\ell_{n-1}\ell_n = \texttt{AN}$.*

Proof. It is easily verified that $\ell_1\ell_2 = \texttt{SS}$. The final assertion is easy to check when $n = 4$; thus, suppose $n \geq 5$. Observe that any $(n-1) \times (n-1)$ principal submatrix of G_n has one of the following forms: G_{n-1}, F_{n-1} or $J_1 \oplus A(K_{n-2})$. Hence, to show that $\ell_{n-1}\ell_n = \texttt{AN}$, it suffices to show that G_{n-1}, F_{n-1} and $J_1 \oplus A(K_{n-2})$ are nonsingular, and that G_n is singular. By Lemma 3.3.4, F_{n-1} is nonsingular. By Proposition 3.2.7, and because $n-2$ is even, $J_1 \oplus A(K_{n-2})$ is also nonsingular.

Finally, we show that $\det(G_{n-1}) \neq 0$ and $\det(G_n) = 0$. Let $q \in \{n-1, n\}$. Observe that $\det(G_q) = \det(F_{q-1}) - \det(A(K_{q-2}))$. Since $\det(F_{q-1}) \neq 0$, $\det(G_q) = 1 - \det(A(K_{q-2}))$ (in characteristic 2). Hence, $\det(G_q) = 0$ if and only if $\det(A(K_{q-2})) \neq 0$. It follows from Proposition 3.2.7 that $\det(G_q) = 0$ if and only if q is even. Then, as n is even, $\det(G_{n-1}) \neq 0$ and $\det(G_n) = 0$. □

Lemma 3.3.10. *Let $n \geq 5$ be an odd integer. Then there exists a symmetric matrix over $\mathbb{Z}_2$ whose epr-sequence $\ell_1\ell_2 \cdots \ell_n$ has $\ell_1\ell_2 = \texttt{SS}$ and $\ell_{n-1}\ell_n = \texttt{AN}$.*

Proof. Clearly, $n+1$ is even and $n+1 \geq 6$. Let $m = \frac{n+1}{2}$, and let

$$B' = \begin{bmatrix} J_m & I_m \\ I_m & I_m \end{bmatrix}, \quad B'' = \begin{bmatrix} J_{m-1} & W \\ W^T & I_{m+1} \end{bmatrix},$$

where $W = [I_{m-1}, J_{m-1,2}]$. Observe that B' and B'' are $(n+1) \times (n+1)$ symmetric matrices. Let $\operatorname{epr}(B') = \ell'_1\ell'_2 \cdots \ell'_{n+1}$ and $\operatorname{epr}(B'') = \ell''_1\ell''_2 \cdots \ell''_{n+1}$. We consider two cases:

Case 1: $n+1 = 4k+2$ for some integer $k \geq 1$. Observe that, by Lemma 3.3.5, $\ell'_n\ell'_{n+1} = \mathtt{AN}$. Let $\alpha = \{n+1\}$, let $C = B'/B'[\alpha]$, and let $\operatorname{epr}(C) = \ell_1\ell_2 \cdots \ell_n$. We now show that C is a matrix with the desired properties. By the Schur Complement Corollary, $\ell_{n-1}\ell_n = \mathtt{AN}$. To show that $\ell_1\ell_2 = \mathtt{SS}$, first, observe that, by the Schur Complement Theorem, and because $\det(B'[\alpha]) = 1$ (in characteristic 2),

$$\det(C[\{n\}]) = \det(B'[\{n\} \cup \alpha]), \quad \det(C[\{m\}]) = \det(B'[\{m\} \cup \alpha]),$$

$$\det(C[\{n-1, n\}]) = \det(B'[\{n-1, n\} \cup \alpha]), \quad \det(C[\{1, 2\}]) = \det(B'[\{1, 2\} \cup \alpha]).$$

Then, by observing that $B'[\{n\} \cup \alpha] = I_2$, that $B'[\{m\} \cup \alpha] = J_2$, that $B'[\{n-1, n\} \cup \alpha] = I_3$ and that $B'[\{1, 2\} \cup \alpha] = J_2 \oplus J_1$, we conclude that $\det(C[\{n\}])$ and $\det(C[\{n-1, n\}])$ are nonzero, and that $\det(C[\{m\}])$ and $\det(C[\{1, 2\}])$ are zero. Hence, $\ell_1\ell_2 = \mathtt{SS}$.

Case 2: $n+1 = 4k$ for some integer $k \geq 2$. Observe that, by Lemma 3.3.6, $\ell''_n\ell''_{n+1} = \mathtt{AN}$. Let $\alpha = \{n+1\}$, let $C = B''/B''[\alpha]$, and let $\operatorname{epr}(C) = \ell_1\ell_2 \cdots \ell_n$. As in Case 1, we show that C is a matrix with the desired properties. By the Schur Complement Corollary, $\ell_{n-1}\ell_n = \mathtt{AN}$. To show that $\ell_1\ell_2 = \mathtt{SS}$, first, observe that, by the Schur Complement Theorem, and because $\det(B''[\alpha]) = 1$ (in characteristic 2),

$$\det(C[\{n\}]) = \det(B''[\{n\} \cup \alpha]), \quad \det(C[\{1\}]) = \det(B''[\{1\} \cup \alpha]),$$

$$\det(C[\{n-1, n\}]) = \det(B''[\{n-1, n\} \cup \alpha]), \quad \det(C[\{1, 2\}]) = \det(B''[\{1, 2\} \cup \alpha]).$$

Then, by observing that $B''[\{n\} \cup \alpha] = I_2$, $B''[\{1\} \cup \alpha] = J_2$, $B''[\{n-1, n\} \cup \alpha] = I_3$ and $B''[\{1, 2\} \cup \alpha] = J_3$, we conclude that $\det(C[\{n\}])$ and $\det(C[\{n-1, n\}])$ are nonzero, and that $\det(C[\{1\}])$ and $\det(C[\{1, 2\}])$ are zero. Hence, $\ell_1\ell_2 = \mathtt{SS}$. □

Together with Theorems 3.3.2 and 3.3.8, the next result completes the characterization of the attainable epr-sequences over $\mathbb{Z}_2$.

Theorem 3.3.11. *An epr-sequence starting with* S *is attainable by a symmetric matrix over* $\mathbb{Z}_2$ *if and only if it has one of the following forms:*

1. $\mathtt{S\overline{S}N\overline{N}}$;
2. $\mathtt{S\overline{S}A}$;
3. $\mathtt{S\overline{S}AA}$;
4. $\mathtt{SS\overline{S}AN}$;
5. $\mathtt{SASA\overline{SA}}$;
6. $\mathtt{SASA\overline{SA}A}$;
7. $\mathtt{SA\overline{SA}N}$.

Proof. Let $\sigma = \ell_1\ell_2\cdots\ell_n$ be an epr-sequence with $\ell_1 = \mathtt{S}$. Suppose that $\sigma = \operatorname{epr}(B)$, where B is a symmetric matrix over $\mathbb{Z}_2$. Since an attainable epr-sequence cannot end with S, $n \geq 2$. If $n = 2$, then σ is SA or SN. Suppose $n \geq 3$. If $\ell_2 = \mathtt{A}$ or $\ell_2 = \mathtt{N}$, then Proposition 3.2.11 and the N-Even Observation imply that σ is either $\mathtt{SA\overline{SA}}$, $\mathtt{SA\overline{SA}A}$, $\mathtt{SA\overline{SA}N}$, or $\mathtt{SNN\overline{N}}$. Thus, suppose $\ell_2 = \mathtt{S}$. Hence, by Theorem 3.2.16, σ does not contain ASA. Let k be a minimal integer with $2 \leq k \leq n-1$ such that $\ell_k\ell_{k+1} = \mathtt{SN}$ or $\ell_k\ell_{k+1} = \mathtt{SA}$; in the former case, Observation 3.2.3 implies that $\sigma = \mathtt{SS\overline{S}N\overline{N}}$. Now consider the latter case, namely $\ell_k\ell_{k+1} = \mathtt{SA}$. If $n = k+1$, then $\sigma = \mathtt{SS\overline{S}A}$. Thus, suppose $n \geq k+2$.

We now show that $n = k+2$. Suppose to the contrary that $n \geq k+3$. By the AA Theorem, $\ell_{k+2} \neq \mathtt{A}$. If $\ell_{k+2} = \mathtt{N}$, then Observation 3.2.3 implies that σ contains SANN, which is prohibited by Proposition 3.2.13; hence, $\ell_{k+2} = \mathtt{S}$, so that $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{SAS}$. Then, as σ does not contain ASA, and because SASN is prohibited by Proposition 3.2.13, $\ell_{k+3} = \mathtt{S}$, implying that σ contains ASS as a non-initial subsequence, a contradiction to Proposition 3.2.14. It follows that $n = k+2$, and therefore that σ is either $\mathtt{SS\overline{S}AA}$ or $\mathtt{SS\overline{S}AN}$.

Now, we establish the other direction. We assume that the sequence under consideration has order n. The sequence $\mathtt{S\overline{S}N\overline{N}}$ is attainable by applying Observation 3.1.10(2) to the attainable sequence $\mathtt{A\overline{A}}$. The sequence $\mathtt{S\overline{S}A}$ is attainable by [4, Observation 2.16]. The attainability of $\mathtt{S\overline{S}AA}$ follows by observing that, by the Inverse Theorem, the inverse of any symmetric matrix attaining the sequence $\mathtt{A\overline{S}SA}$, which is attainable by Theorem 3.3.8, has epr-sequence $\mathtt{S\overline{S}AA}$. To see that the sequence $\mathtt{SS\overline{S}AN}$ is attainable, observe that the argument above forces the matrix G_n in Lemma 3.3.9 to attain this sequence when n is even, and that it forces the matrix whose existence was established in Lemma 3.3.10 to attain this sequence when n is odd. Finally, the sequences $\mathtt{SASA\overline{SA}}$, $\mathtt{SASA\overline{SA}A}$ and $\mathtt{SA\overline{SA}N}$ are attainable by Proposition 3.2.11. □

To conclude, we note that there is no known characterization of the epr-sequences that are attainable by symmetric matrices over the real field or any other field besides $\mathbb{Z}_2$. However, the results of Theorems 3.3.2, 3.3.8 and 3.3.11 provide such a characterization for symmetric matrices over $\mathbb{Z}_2$.

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CHAPTER 4. THE SIGNED ENHANCED PRINCIPAL RANK CHARACTERISTIC SEQUENCE

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Abstract

The signed enhanced principal rank characteristic sequence (sepr-sequence) of an $n \times n$ Hermitian matrix is the sequence $t_1 t_2 \cdots t_n$, where t_k is either $\mathtt{A}^*$, $\mathtt{A}^+$, $\mathtt{A}^-$, $\mathtt{N}$, $\mathtt{S}^*$, $\mathtt{S}^+$, or $\mathtt{S}^-$, based on the following criteria: $t_k = \mathtt{A}^*$ if B has both a positive and a negative order-k principal minor, and each order-k principal minor is nonzero. $t_k = \mathtt{A}^+$ (respectively, $t_k = \mathtt{A}^-$) if each order-k principal minor is positive (respectively, negative). $t_k = \mathtt{N}$ if each order-k principal minor is zero. $t_k = \mathtt{S}^*$ if B has each a positive, a negative, and a zero order-k principal minor. $t_k = \mathtt{S}^+$ (respectively, $t_k = \mathtt{S}^-$) if B has both a zero and a nonzero order-k principal minor, and each nonzero order-k principal minor is positive (respectively, negative). Such sequences provide more information than the $(\mathtt{A}, \mathtt{N}, \mathtt{S})$ epr-sequence in the literature, where the kth term is either $\mathtt{A}$, $\mathtt{N}$, or $\mathtt{S}$ based on whether all, none, or some (but not all) of the order-k principal minors of the matrix are nonzero. Various sepr-sequences are shown to be unattainable by Hermitian matrices. In particular, by applying Muir's law of extensible minors, it is shown that subsequences such as $\mathtt{A}^*\mathtt{N}$ and $\mathtt{N}\mathtt{A}^*$ are prohibited in the sepr-sequence of a Hermitian matrix. The notion of a nonnegative and nonpositive subsequence is introduced, which leads to a connection with positive semidefinite matrices. For Hermitian matrices of orders $n = 1, 2, 3$, all attainable sepr-sequences are classified. For real symmetric matrices, a complete characterization of the attainable sepr-sequences whose underlying epr-sequence contains `ANA` as a non-terminal subsequence is established.

Keywords. Signed enhanced principal rank characteristic sequence; enhanced principal rank characteristic sequence; minor; rank; Hermitian matrix.

AMS Subject Classifications. 15A15, 15A03, 15B57.

4.1 Introduction

The *principal minor assignment problem*, introduced in [1], asks the following question: Can we find an $n \times n$ matrix with prescribed principal minors? As a simplification of the principal minor assignment problem, Brualdi et al. [2] associated a sequence with a symmetric matrix, which they defined as follows: Given an $n \times n$ symmetric matrix B over a field F, *the principal rank characteristic sequence* (abbreviated pr-sequence) of B is defined as $\text{pr}(B) = r_0]r_1 \cdots r_n$, where, for $k \geq 1$,

$$r_k = \begin{cases} 1 & \text{if } B \text{ has a nonzero principal minor of order } k \text{, and} \\ 0 & \text{otherwise,} \end{cases}$$

while $r_0 = 1$ if and only if B has a 0 diagonal entry [2]. (The *order* of a minor is k if it is the determinant of a $k \times k$ submatrix.) We note that the original definition of the pr-sequence was for real symmetric, complex symmetric and Hermitian matrices only; but Barrett et al. [3] later extended it to symmetric matrices over any field. In the context of the principal minor assignment problem, the pr-sequence is somewhat limited, as it only records the presence or absence of a full-rank principal submatrix of each possible order; thus, in order to provide more insight, the pr-sequence was "enhanced" by Butler et al. [4] with the introduction of another sequence: Given an $n \times n$ symmetric matrix B over a field F, the *enhanced principal rank characteristic sequence* (abbreviated epr-sequence) of B is defined as $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$, where

$$\ell_k = \begin{cases} \texttt{A} & \text{if all the principal minors of order } k \text{ are nonzero;} \\ \texttt{S} & \text{if some but not all the principal minors of order } k \text{ are nonzero;} \\ \texttt{N} & \text{if none of the principal minors of order } k \text{ are nonzero, i.e., all are zero.} \end{cases}$$

There has been substantial work done on pr- and epr-sequences (see [2, 3, 4, 5, 6, 7, 8], for example). Here, we introduce a sequence that extends the pr- and epr-sequence, which we think remains tractable, while providing further help for working on the principal minor assignment problem for Hermitian matrices:

Definition 4.1.1. Let B be a complex Hermitian matrix with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. The *signed enhanced principal rank characteristic sequence* (abbreviated sepr-sequence) of B is the sequence $\operatorname{sepr}(B) = t_1t_2\cdots t_n$, where

$$t_k = \begin{cases} \texttt{A}^* & \text{if } \ell_k = \texttt{A} \text{ and } B \text{ has both a positive and a negative order-}k \text{ principal minor;} \\ \texttt{A}^+ & \text{if each order-}k \text{ principal minor of } B \text{ is positive;} \\ \texttt{A}^- & \text{if each order-}k \text{ principal minor of } B \text{ is negative;} \\ \texttt{N} & \text{if each order-}k \text{ principal minor of } B \text{ is zero;} \\ \texttt{S}^* & \text{if } \ell_k = \texttt{S} \text{ and } B \text{ has both a positive and a negative order-}k \text{ principal minor;} \\ \texttt{S}^+ & \text{if } \ell_k = \texttt{S} \text{ and each order-}k \text{ principal minor of } B \text{ is nonnegative;} \\ \texttt{S}^- & \text{if } \ell_k = \texttt{S} \text{ and each order-}k \text{ principal minor of } B \text{ is nonpositive.} \end{cases}$$

Further motivation for studying pr-, epr- and sepr-sequences are the instances where the principal minors of a matrix are of interest; as stated in [9], these instances include the detection of P-matrices in the study of the complementarity problem, Cartan matrices in Lie algebras, univalent differentiable mappings, self-validating algorithms, interval matrix analysis, counting of spanning trees of a graph using the Laplacian, D-nilpotent automorphisms, and in the solvability of the inverse multiplicative eigenvalue problem (see [9] and the references therein).

Section 4.2 is devoted to developing some of the tools used to establish results in subsequent sections. In Section 4.3, various sepr-sequences are shown to be unattainable by Hermitian matrices, and, at the end, the notion of a nonnegative and nonpositive subsequence is introduced, which leads to a connection with positive semidefinite matrices. Section 4.4 is devoted to providing a classification of the sepr-sequences of orders $n = 1, 2, 3$ that can be attained by an $n \times n$ Hermitian matrix. Finally, Section 4.5 focuses on the sepr-sequences of real symmetric matrices, where a complete characterization of the sepr-sequences whose underlying epr-sequence contains `ANA` as a non-terminal subsequence is established.

For the rest of the paper, all matrices are Hermitian. For any sepr-sequence σ, the epr-sequence resulting from removing the superscripts of each term in σ is called the *underlying* epr-sequence of σ. A (pr-, epr- or sepr-) sequence is said to be *attainable* by a class of matrices provided that there exists a matrix B in the class that attains it; otherwise, we say that it is *unattainable* (by the given class). A subsequence that does not appear in an attainable sequence is *prohibited*. A sequence is said to have *order* n if it consists of n terms. Given a sequence $t_{i_1}t_{i_2}\cdots t_{i_k}$, the notation $\overline{t_{i_1}t_{i_2}\cdots t_{i_k}}$ indicates that the sequence may be repeated as many times as desired (or it may be omitted entirely). Let $B = [b_{ij}]$ and let $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$; then the submatrix lying in rows indexed by α, and columns indexed by β, is denoted by $B[\alpha, \beta]$; if $\alpha = \beta$, then the principal submatrix $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$. The matrices O_n, I_n and J_n are the matrices of order n denoting the zero matrix, the identity matrix and the all-1s matrix, respectively. The block diagonal matrix with the matrices B and C on the diagonal is denoted by $B \oplus C$.

4.1.1 Results cited

This section lists results that will be cited frequently, which will be referenced by the assigned nomenclature (if any). Each instance of $\cdots$ below is permitted to be empty.

Proposition 4.1.2. [4, Proposition 2.5] *No Hermitian matrix can have the epr-sequence* $\mathtt{SN}\cdots\mathtt{A}\cdots$.

Corollary 4.1.3. [4, Corollary 2.7] ($\mathtt{NSA}$ Theorem.) *No Hermitian matrix can have* $\mathtt{NSA}$ *in its epr-sequence. Further, no Hermitian matrix can have the epr-sequence* $\cdots\mathtt{ASN}\cdots\mathtt{A}\cdots$.

For an $n \times n$ matrix B with a nonsingular principal submatrix $B[\alpha]$, recall that the Schur complement of $B[\alpha]$ in B is the matrix $B/B[\alpha] := B[\alpha^c] - B[\alpha^c, \alpha](B[\alpha])^{-1}B[\alpha, \alpha^c]$, where $\alpha^c = \{1, 2, \ldots, n\} \setminus \alpha$.

Theorem 4.1.4. [7, Theorem 1.10] (Schur Complement Theorem.) *Suppose B is an $n \times n$ Hermitian matrix with $\operatorname{rank} B = r$. Let $B[\alpha]$ be a nonsingular principal submatrix of B with $|\alpha| = k \leq r$, and let $C = B/B[\alpha]$. Then the following results hold.*

- *(i)* *C is an $(n-k) \times (n-k)$ Hermitian matrix.*
- *(ii)* *Assuming the indexing of C is inherited from B, any principal minor of C is given by*

$$\det C[\gamma] = \det B[\gamma \cup \alpha] / \det B[\alpha].$$

- *(iii)* $\operatorname{rank} C = r - k$.

Corollary 4.1.5. [7, Corollary 1.11] (Schur Complement Corollary.) *Let B be a Hermitian matrix, let $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$, and let $B[\alpha]$ be a nonsingular principal submatrix of B, with $|\alpha| = k \leq \operatorname{rank} B$. Let $C = B/B[\alpha]$ and $\operatorname{epr}(C) = \ell'_1\ell'_2\cdots\ell'_{n-k}$. Then, for $j = 1, \ldots, n-k$, $\ell'_j = \ell_{j+k}$ if $\ell_{j+k} \in \{\mathtt{A}, \mathtt{N}\}$.*

In the interest of brevity, the notation B_I for $\det(B[I])$ in [2] and [8] is adopted here (when $I = \emptyset$, $B_\emptyset$ is defined to have the value 1). Moreover, when $I = \{i_1, i_2, \ldots, i_k\}$, B_I is written as $B_{i_1 i_2 \cdots i_k}$.

Given a matrix B, the determinant of the 2×2 principal submatrix $B[\{i,j\}]$ can be stated as an homogenous polynomial identity as follows:

$$B_\emptyset B_{ij} = B_i B_j - \det(B[\{i\}|\{j\}]) \det(B[\{j\}|\{i\}]).$$

The identity in the next result is already known, and can be obtained by applying Muir's law of extensible minors [10] to the above identity (for a more recent treatment of this law, the reader is referred to [11]).

Remark 4.1.6. Let $n \geq 2$, let B be an $n \times n$ Hermitian matrix, let $i, j \in \{1, 2, \ldots, n\}$ be distinct, and let $I \subseteq \{1, 2, \ldots, n\} \setminus \{i, j\}$. Then

$$B_I B_{I \cup \{i,j\}} = B_{I \cup \{i\}} B_{I \cup \{j\}} - |\det(B[I \cup \{i\}|I \cup \{j\}])|^2.$$

Remark 4.1.6 will be invoked as "Muir's law of extensible minors."

4.2 The signed enhanced principal rank characteristic sequence

We begin this section with simple observations, and conclude with results that will serve as tools in establishing the results of subsequent sections.

Observation 4.2.1. *The sepr-sequence of a Hermitian matrix must end in* $\mathtt{A}^+$*,* $\mathtt{A}^-$ *or* $\mathtt{N}$*.*

Given an sepr-sequence $t_1 t_2 \cdots t_n$, the *negative* of this sequence, denoted $\operatorname{neg}(t_1 t_2 \cdots t_n)$, is the sequence resulting from replacing "+" superscripts with "-" superscripts in $t_1 t_2 \cdots t_n$, and viceversa. For example, the negative of the sequence $\mathtt{N}\mathtt{S}^-\mathtt{S}^*\mathtt{A}^*\mathtt{A}^+$ is $\mathtt{N}\mathtt{S}^+\mathtt{S}^*\mathtt{A}^*\mathtt{A}^-$. Given a matrix B, the ith term in its sepr-sequence (respectively, epr-sequence) is $[\operatorname{sepr}(B)]_i$ (respectively, $[\operatorname{epr}(B)]_i$).

Observation 4.2.2. *Let B be an $n \times n$ Hermitian matrix, and let i be an integer with $1 \leq i \leq n$.*

1. *If i is even, then $[\mathrm{sepr}(-B)]_i = [\mathrm{sepr}(B)]_i$.*

2. *If i is odd, then $[\mathrm{sepr}(-B)]_i = \mathrm{neg}([\mathrm{sepr}(B)]_i)$.*

The following is immediate from [4, Theorem 2.3].

Theorem 4.2.3. (`NN` Theorem.) *Suppose B is a Hermitian matrix,* $\mathrm{sepr}(B) = t_1 t_2 \cdots t_n$, *and $t_k = t_{k+1} = \mathtt{N}$ for some k. Then $t_j = \mathtt{N}$ for all $j \geq k$.*

Theorem 4.2.4. (Inverse Theorem.) *Suppose B is a nonsingular Hermitian matrix.*

(i) If $\mathrm{sepr}(B) = t_1 t_2 \cdots t_{n-1}\mathtt{A}^+$, *then* $\mathrm{sepr}(B^{-1}) = t_{n-1} t_{n-2} \cdots t_1 \mathtt{A}^+$.

(ii) If $\mathrm{sepr}(B) = t_1 t_2 \cdots t_{n-1}\mathtt{A}^-$, *then* $\mathrm{sepr}(B^{-1}) = \mathrm{neg}(t_{n-1} t_{n-2} \cdots t_1)\mathtt{A}^-$.

Proof. Let $\alpha \subseteq \{1, 2, \ldots, n\}$ be nonempty. By Jacobi's determinantal identity, $\det B^{-1}[\alpha] = \det B(\alpha)/\det B$. The desired conclusions are now immediate. □

The next lemma is proven by replicating part of the proof of [4, Theorem 2.6].

Lemma 4.2.5. *Let k and n be integers with $1 \leq k < n$. Suppose that each k-element subset of $\{1, 2, \ldots, n\}$ is associated with exactly one of two given properties, and that not every pair of k-element subsets is associated with the same property. Then there exist distinct integers $i, j \in \{1, 2, \ldots, n\}$, and a $(k-1)$-element subset $I \subseteq \{1, 2, \ldots, n\} \setminus \{i, j\}$, such that $I \cup \{i\}$ and $I \cup \{j\}$ are not associated with the same property.*

Proof. By hypothesis, there exists two lists of indices, say, $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$, with property 1 and property 2, respectively. Without loss of generality, we may assume that these lists are ordered so that any common indices occur in the same position in each list. Consider the following lists of indices.

$$p_1, p_2, p_3, \ldots, p_k;$$

$$q_1, p_2, p_3, \ldots, p_k;$$

$$q_1, q_2, p_3, \ldots, p_k;$$

$$q_1, q_2, q_3, \ldots, p_k;$$

$$\ldots$$

$$q_1, q_2, q_3, \ldots, q_k.$$

Since the first list corresponds with property 1, and because the last list corresponds with property 2, somewhere in between these two lists there are two consecutive lists with one list corresponding with property 1, and the other list corresponding with property 2. Hence, as two consecutive lists differ in at most one position, there exists a $(k-1)$-element subset $I \subseteq \{1, 2, \ldots, n\}$, and $i, j \in \{1, 2, \ldots, n\} \setminus I$, such that $I \cup \{i\}$ is associated with property 1, and $I \cup \{j\}$ is associated with property 2. □

Lemma 4.2.6. *Let B be an $n \times n$ Hermitian matrix with $[\mathrm{sepr}(B)]_k = \mathtt{A}^*$. Then there exists a $(k-1)$-element subset $I \subseteq \{1, 2, \ldots, n\}$, and $i, j \in \{1, 2, \ldots, n\} \setminus I$, such that $B_{I \cup \{i\}} > 0$ and $B_{I \cup \{j\}} < 0$.*

Proof. Since $\mathrm{sepr}(B)$ cannot end in $\mathtt{A}^*$, $k < n$. Then, as every k-element subset of $\{1, 2, \ldots, n\}$ is associated with either a positive or a negative order-k principal minor, but not both, the conclusion follows from Lemma 4.2.5. □

Theorem 4.2.7. (Inheritance Theorem.) *Let B be an $n \times n$ Hermitian matrix, $m \leq n$, and $1 \leq i \leq m$.*

1. *If $[\mathrm{sepr}(B)]_i = \mathtt{N}$, then $[\mathrm{sepr}(C)]_i = \mathtt{N}$ for all $m \times m$ principal submatrices C.*
2. *If $[\mathrm{sepr}(B)]_i = \mathtt{A}^+$, then $[\mathrm{sepr}(C)]_i = \mathtt{A}^+$ for all $m \times m$ principal submatrices C.*

3. *If* $[\mathrm{sepr}(B)]_i = \mathtt{A}^-$*, then* $[\mathrm{sepr}(C)]_i = \mathtt{A}^-$ *for all* $m \times m$ *principal submatrices* C*.*

4. *If* $[\mathrm{sepr}(B)]_m = \mathtt{A}^*$*, then there exist* $m \times m$ *principal submatrices* C_{A^+} *and* C_{A^-} *of* B *such that* $[\mathrm{epr}(C_{A^+})]_m = \mathtt{A}^+$ *and* $[\mathrm{sepr}(C_{A^-})]_m = \mathtt{A}^-$*.*

5. *If* $[\mathrm{sepr}(B)]_m = \mathtt{S}^+$*, then there exist* $m \times m$ *principal submatrices* C_{A^+} *and* C_N *of* B *such that* $[\mathrm{epr}(C_{A^+})]_m = \mathtt{A}^+$ *and* $[\mathrm{sepr}(C_N)]_m = \mathtt{N}$*.*

6. *If* $[\mathrm{sepr}(B)]_m = \mathtt{S}^-$*, then there exist* $m \times m$ *principal submatrices* C_{A^-} *and* C_N *of* B *such that* $[\mathrm{sepr}(C_{A^-})]_m = \mathtt{A}^-$ *and* $[\mathrm{sepr}(C_N)]_m = \mathtt{N}$*.*

7. *If* $[\mathrm{sepr}(B)]_m = \mathtt{S}^*$*, then there exist* $m \times m$ *principal submatrices* C_{A^+}*,* C_{A^-} *and* C_N *of* B *such that* $[\mathrm{sepr}(C_{A^+})]_m = \mathtt{A}^+$*,* $[\mathrm{sepr}(C_{A^-})]_m = \mathtt{A}^-$ *and* $[\mathrm{sepr}(C_N)]_m = \mathtt{N}$*.*

8. *If* $i < m$ *and* $[\mathrm{sepr}(B)]_i = \mathtt{A}^*$*, then there exists an* $m \times m$ *principal submatrix* C_{A^*} *such that* $[\mathrm{sepr}(C_{A^*})]_i = \mathtt{A}^*$*.*

9. *If* $i < m$ *and* $[\mathrm{sepr}(B)]_i = \mathtt{S}^+$*, then there exists an* $m \times m$ *principal submatrix* C_{S^+} *such that* $[\mathrm{sepr}(C_{S^+})]_i = \mathtt{S}^+$*.*

10. *If* $i < m$ *and* $[\mathrm{sepr}(B)]_i = \mathtt{S}^-$*, then there exists an* $m \times m$ *principal submatrix* C_{S^-} *such that* $[\mathrm{sepr}(C_{S^-})]_i = \mathtt{S}^-$*.*

11. *If* $i < m$ *and* $[\mathrm{sepr}(B)]_i = \mathtt{S}^*$*, then there exists an* $m \times m$ *principal submatrix* C_S *such that* $[\mathrm{sepr}(C_S)]_i \in \{\mathtt{S}^*, \mathtt{S}^+, \mathtt{S}^-\}$*.*

12. *If* $i < m$ *and* $[\mathrm{sepr}(B)]_i = \mathtt{S}^*$*, then there exists an* $m \times m$ *principal submatrix* C_+ *such that* $[\mathrm{sepr}(C_+)]_i \in \{\mathtt{A}^*, \mathtt{S}^*, \mathtt{S}^+\}$*.*

13. *If* $i < m$ *and* $[\mathrm{sepr}(B)]_i = \mathtt{S}^*$*, then there exists an* $m \times m$ *principal submatrix* C_- *such that* $[\mathrm{sepr}(C_-)]_i \in \{\mathtt{A}^*, \mathtt{S}^*, \mathtt{S}^-\}$*.*

Proof. (1)–(3): Statements (1)–(3) simply follow by noting that a principal submatrix of a principal submatrix, is also principal submatrix.

(4)–(7): If $[\mathrm{sepr}(B)]_m = \mathtt{A}^*$, then B contains an $m \times m$ principal submatrix with positive determinant, say, C_{A^+}, as well as one with negative determinant, say, C_{A^-}; these two matrices each have the desired sepr-sequence, which establishes Statement (4). Statements (5)–(7) are established in the same manner as Statement (4).

(8): By Lemma 4.2.6, there exists an $(i-1)$-element subset $I \subseteq \{1, 2, \ldots, m\}$, and $p, q \in \{1, 2, \ldots, m\} \setminus I$, such that $B_{I \cup \{p\}} > 0$ and $B_{I \cup \{q\}} < 0$. Then, by arbitrarily adding $m - i - 1$ indices to $I \cup \{p, q\}$, to obtain an m-element subset α, one obtains the principal submatrix $B[\alpha]$, for which $[\mathrm{sepr}(B[\alpha])]_i = \mathtt{A}^*$.

(9)–(11): These three statements are immediate from [4, Theorem 2.6].

(12) and (13): By hypothesis, there exists two lists of indices, say, $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$, such that $B_{p_1, p_2, \ldots, p_i} > 0$ and $B[q_1, q_2, \ldots, q_i] < 0$. Without loss of generality, we may assume that these lists are ordered so that any common indices occur in the same position in each list. Consider the following lists of indices.

$$\begin{gathered} p_1, p_2, p_3, \ldots, p_k; \\ q_1, p_2, p_3, \ldots, p_k; \\ q_1, q_2, p_3, \ldots, p_k; \\ q_1, q_2, q_3, \ldots, p_k; \\ \cdots \\ q_1, q_2, q_3, \ldots, q_k. \end{gathered}$$

As one moves down these lists, one must eventually encounter two consecutive lists satisfying one of the following: (i) One list corresponds with a positive principal minor, and the other corresponds with a negative principal minor; (ii) one list corresponds with a positive principal minor, and the other corresponds with a zero principal minor. If every pair of lists does not satisfy (i) or (ii), then each list corresponds with a positive principal minor, which is a contradiction, since the last list corresponds with a zero minor. Hence, as two consecutive lists differ in at most one position, the union of these two (distinct) lists generates an index set of cardinality $i + 1$; then, by arbitrarily adding

$m-i-1$ indices to this index set, to obtain an m-element subset α, one obtains the principal submatrix $B[\alpha]$, for which $[\operatorname{sepr}(B[\alpha])]_i \in \{\mathtt{S}^*, \mathtt{S}^+\}$ if the two lists used to generate α satisfy (ii), while $[\operatorname{sepr}(B[\alpha])]_i \in \{\mathtt{A}^*, \mathtt{S}^*\}$ if the two lists satisfy (i). Hence, with $C_+ = B[\alpha]$, $[\operatorname{sepr}(C_+)]_i \in \{\mathtt{A}^*, \mathtt{S}^*, \mathtt{S}^+\}$.

Statement (13) is established in the same manner as (12). $\square$

Given an $n \times n$ Hermitian matrix B whose sepr-sequence contains $\mathtt{S}^+$ (respectively, $\mathtt{S}^-$) in position i, by the Inheritance Theorem, for all m with $i < m < n$, this matrix must contain at least one $m \times m$ principal submatrix whose sepr-sequence inherits $\mathtt{S}^+$ (respectively, $\mathtt{S}^-$) in position i. However, the next example reveals that $\mathtt{S}^*$ is not necessarily inherited.

Example 4.2.8. The (Hermitian) matrix

$$B = \begin{bmatrix} -1 & 2 & i & 4 & 0 \\ 2 & 0 & 6 & 1 & 8 \\ -i & 6 & 1 & i & 1+i \\ 4 & 1 & -i & -1 & 1+i \\ 0 & 8 & 1-i & 1-i & 0 \end{bmatrix}$$

has sepr-sequence $\mathtt{S}^*\mathtt{S}^-\mathtt{S}^*\mathtt{A}^+\mathtt{A}^+$. It is easily verified that none of the sepr-sequences of the five 4×4 principal submatrices of B inherit the $\mathtt{S}^*$ appearing in the third position.

With the next result, we add an additional tool to our arsenal for studying epr- and sepr-sequences, which is analogous to that of the inheritance of an $\mathtt{S}^+$, $\mathtt{S}^-$ or $\mathtt{A}^*$ by a principal submatrix (see the Inheritance Theorem).

Proposition 4.2.9. *Let B be a Hermitian matrix with* $\operatorname{sepr}(B) = t_1t_2\cdots t_n$. *Suppose* $t_p \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$ *and* $t_q = \mathtt{A}^*$, *where* $1 \le p < q < n$. *Then there exists a (nonsingular)* $p \times p$ *principal submatrix* $B[\alpha]$ *such that the* $(n-p) \times (n-p)$ *(Hermitian) matrix* $C = B/B[\alpha]$ *with* $\operatorname{sepr}(C) = t_1't_2'\cdots t_{n-p}'$ *has* $t_{q-p}' = t_q = \mathtt{A}^*$.

Proof. By Lemma 4.2.5, there exist distinct integers $i, j \in \{1, 2, \ldots, n\}$, and a $(q-1)$-element subset $I \subseteq \{1, 2, \ldots, n\} \setminus \{i, j\}$, such that $\det B[I \cup \{i\}] > 0$ and $\det B[I \cup \{j\}] < 0$. Let $\alpha \subseteq I$ be a p-element subset. By hypothesis, $B[\alpha]$ is nonsingular. Let $C = B/B[\alpha]$, $\operatorname{sepr}(C) = t'_1 t'_2 \cdots t'_{n-p}$ and $\beta = I \backslash \alpha$. By the Schur Complement Theorem, $\det(C[\beta \cup \{i\}])$ and $\det(C[\beta \cup \{j\}])$ have opposite signs. Then, as $|\beta \cup \{i\}| = |\beta \cup \{j\}| = q - p$, $t'_{q-p} \in \{\mathtt{A}^*, \mathtt{S}^*\}$. But, by the Schur Complement Corollary, we must have $t'_{q-p} = \mathtt{A}^*$, as desired. ☐

4.3 Sepr-sequences of Hermitian matrices

With our attention confined to Hermitian matrices, in this section we establish restrictions for the attainability of sepr-sequences.

Proposition 4.3.1. (Basic Proposition.) *No Hermitian matrix can have any of the following sepr-sequences.*

1. $\mathtt{A}^*\mathtt{A}^+ \cdots$;

2. $\mathtt{A}^*\mathtt{S}^+ \cdots$;

3. $\mathtt{A}^*\mathtt{N} \cdots$;

4. $\mathtt{S}^*\mathtt{A}^+ \cdots$;

5. $\mathtt{S}^*\mathtt{S}^+ \cdots$;

6. $\mathtt{S}^*\mathtt{N} \cdots$;

7. $\mathtt{S}^+\mathtt{A}^+ \cdots$;

8. $\mathtt{S}^-\mathtt{A}^+ \cdots$;

9. $\mathtt{N}\mathtt{A}^* \cdots$;

10. `NA`$^+ \cdots$;

11. `NS`$^* \cdots$;

12. `NS`$^+ \cdots$.

Proof. To see that the sequences 1, 2, 3, 4, 5 and 6 are prohibited, note that a Hermitian matrix containing both a positive and a negative diagonal entry, must contain a negative principal minor of order 2.

The sequences 7 and 8 are prohibited because a Hermitian matrix with both a zero and a nonzero diagonal entry, must contain a nonpositive principal minor of order 2.

Finally, the fact that the sequences 9, 10, 11, and 12 are prohibited follows from the fact that the principal minors of order 2 of a Hermitian matrix with zero diagonal are nonpositive. □

Although the following result is surely known, we offer a brief proof.

Lemma 4.3.2. *Let B be a Hermitian matrix with* $\operatorname{rank}(B) = r$. *Then all the nonzero principal minors of B of order r have the same sign.*

Proof. The conclusion is immediate if B has full-rank; thus, assume that B does not have full-rank. Let B' be a nonsingular $r \times r$ principal submatrix of B (which must exist, since B is Hermitian), and, by use of a permutation similarity, suppose B' is the leading principal submatrix of order r. Since B is Hermitian, it must have exactly r nonzero eigenvalues, which we denote by $\lambda_1, \ldots, \lambda_r$; moreover, there exists a unitary matrix U such that $B = U^*DU$, where $D = \Lambda \oplus O_{n-r}$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$. Let U_r be the $r \times r$ leading principal submatrix of U. It then follows that $B' = U_r^* \Lambda U_r$. Hence, $\det(B') = |\det(U_r)|^2 \prod_{j=1}^{r} \lambda_j$. Since B' was arbitrary, it follows that every nonzero order-r principal minor of B has the same sign as $\prod_{j=1}^{r} \lambda_j$. That completes the proof. □

Corollary 4.3.3. *Neither the sepr-sequences* `A`*`NN`, *nor* `S`*`NN`, *can occur as a subsequence of the sepr-sequence of a Hermitian matrix.*

Proof. Let B be a Hermitian matrix with $\operatorname{sepr}(B)$ containing `A`*`NN` or `S`*`NN`, where the `A`* or `S`* of this subsequence occurs in position k. Then, by the `NN` Theorem, $\operatorname{rank}(B) = k$. Hence, by Lemma 4.3.2, every nonzero principal minor of order k has the same sign, which is a contradiction. □

In order to generalize one of the assertions of Corollary 4.3.3, we will now apply Muir's law of extensible minors.

Theorem 4.3.4. *Neither the sepr-sequence* `A`*`N`*, nor* `NA`**, can occur as a subsequence of the sepr-sequence of a Hermitian matrix.*

Proof. Let B be a Hermitian matrix with $\operatorname{sepr}(B) = t_1t_2\cdots t_n$. Suppose to the contrary that $t_kt_{k+1} =$ `A`*`N` for some k. By Lemma 4.2.6, there exists a $(k-1)$-element subset $I \subseteq \{1, 2, \dots, n\}$, and $i, j \in \{1, 2, \dots, n\} \setminus I$, such that $B_{I\cup\{i\}} > 0$ and $B_{I\cup\{j\}} < 0$; hence, $B_{I\cup\{i\}}B_{I\cup\{j\}} < 0$. Now, since I does not contain i and j, and because B is Hermitian, Muir's law of extensible minors implies that,

$$B_I B_{I\cup\{i,j\}} = B_{I\cup\{i\}}B_{I\cup\{j\}} - |\det(B[I\cup\{i\}|I\cup\{j\}])|^2.$$

Then, as $B_{I\cup\{i\}}B_{I\cup\{j\}} < 0$, $B_I B_{I\cup\{i,j\}} < 0$, implying that $B_{I\cup\{i,j\}} \neq 0$, a contradiction to $t_{k+1} =$ `N`. It follows that `A`*`N` is prohibited.

To establish the final assertion, we again proceed by contradiction. Suppose $t_kt_{k+1} =$ `NA`* for some k. By the Basic Proposition, $k \geq 2$. Since $t_{k+1} =$ `A`*, By Lemma 4.2.6, there exists a k-element subset $I \subseteq \{1, 2, \dots, n\}$, and $i, j \in \{1, 2, \dots, n\} \setminus I$, such that $B_{I\cup\{i\}} > 0$ and $B_{I\cup\{j\}} < 0$; hence, $B_{I\cup\{i\}}B_{I\cup\{j\}} < 0$. Once again, we use the identity

$$B_I B_{I\cup\{i,j\}} = B_{I\cup\{i\}}B_{I\cup\{j\}} - |\det(B[I\cup\{i\}|I\cup\{j\}])|^2.$$

Since $t_k =$ `N`, we have $B_I = 0$, implying that

$$0 = B_{I\cup\{i\}}B_{I\cup\{j\}} - |\det(B[I\cup\{i\}|I\cup\{j\}])|^2 < 0,$$

a contradiction. □

For the rest of the paper, we invoke Theorem 4.3.4 by just stating that $\mathtt{A}^*\mathtt{N}$ or $\mathtt{N}\mathtt{A}^*$ is prohibited.

Theorem 4.3.5. *For any* $\mathtt{X}$, *if any of the sepr-sequences* $\mathtt{A}^+\mathtt{X}\mathtt{A}^+$ *or* $\mathtt{A}^-\mathtt{X}\mathtt{A}^-$ *occurs in the sepr-sequence of a Hermitian matrix, then* $\mathtt{X} \in \{\mathtt{A}^+, \mathtt{A}^-\}$.

Proof. Let B be a Hermitian matrix with $\operatorname{sepr}(B) = t_1t_2\cdots t_n$. Suppose $t_k = t_{k+2} = \mathtt{A}^+$ or $t_k = t_{k+2} = \mathtt{A}^-$, for some k with $1 \leq k \leq n-2$. Suppose to the contrary that $t_{k+1} \neq \mathtt{A}^+$ and $t_{k+1} \neq \mathtt{A}^-$. Let $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Let $i, j \in \{1, 2, \ldots, n\}$, with $i \neq j$, and $I \subseteq \{1, 2, \ldots, n\} \setminus \{i, j\}$, where $|I| = k$. Since B is Hermitian, Muir's law of extensible minors implies that

$$B_I B_{I\cup\{i,j\}} = B_{I\cup\{i\}} B_{I\cup\{j\}} - |\det(B[I \cup \{i\}|I \cup \{j\}])|^2.$$

By hypothesis, $B_I B_{I\cup\{i,j\}} > 0$. Then, as i, j and I were arbitrary, we must have $B_{I\cup\{i\}} B_{I\cup\{j\}} > 0$ whenever $I \subseteq \{1, 2, \ldots, n\} \setminus \{i, j\}$ and $i, j \in \{1, 2, \ldots, n\}$ are distinct (otherwise, the expression on the right side of the above identity would be nonpositive). It follows that $\ell_{k+1} = \mathtt{A}$. By hypothesis, $t_{k+1} = \mathtt{A}^*$. By Lemma 4.2.6, there exists a k-element subset $I \subseteq \{1, 2, \ldots, n\}$, and $i, j \in \{1, 2, \ldots, n\} \setminus I$, such that $B_{I\cup\{i\}} > 0$ and $B_{I\cup\{j\}} < 0$; hence, $B_{I\cup\{i\}} B_{I\cup\{j\}} < 0$, which is a contradiction to the above argument. $\square$

Theorem 4.3.5 raises the following question: Can the subsequences $\mathtt{A}^+\mathtt{A}^+\mathtt{A}^+$, $\mathtt{A}^+\mathtt{A}^-\mathtt{A}^+$, $\mathtt{A}^-\mathtt{A}^-\mathtt{A}^-$, $\mathtt{A}^-\mathtt{A}^+\mathtt{A}^-$ occur in the sepr-sequence of a Hermitian matrix? In Section 4.4, we demonstrate that the answer is affirmative.

Proposition 4.3.6. *For* $\mathtt{X} \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$, *the sepr-sequences* $\mathtt{S}^+\mathtt{S}^+\cdots\mathtt{X}\cdots$ *and* $\mathtt{S}^-\mathtt{S}^+\cdots\mathtt{X}\cdots$ *are prohibited for Hermitian matrices.*

Proof. Let $B = [b_{ij}]$ be an $n \times n$ Hermitian matrix with sepr-sequence $\mathtt{S}^+\mathtt{S}^+\cdots$ or $\mathtt{S}^-\mathtt{S}^+\cdots$. Without loss of generality, we may assume that $b_{11} = 0$. Let $j \in \{2, 3, \ldots, n\}$. By hypothesis, the order-2 principal minor $B_{1j} = b_{11}b_{jj} - |b_{1j}|^2 = -|b_{1j}|^2$ is nonnegative, implying that $b_{ij} = 0$. Since j was arbitrary, it follows that the first row of B is zero, implying that B is singular. We conclude that a Hermitian matrix with sepr-sequence $\mathtt{S}^+\mathtt{S}^+\cdots$, or $\mathtt{S}^-\mathtt{S}^+\cdots$, is singular.

Now, suppose to the contrary that B has sepr-sequence $\mathtt{S}^+\mathtt{S}^+\cdots\mathtt{X}\cdots$ or $\mathtt{S}^-\mathtt{S}^+\cdots\mathtt{X}\cdots$, where $\mathtt{X} \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$ occurs in position k. By the Inheritance Theorem, B has a nonsingular $k \times k$ principal submatrix with sepr-sequence $\mathtt{S}^+\mathtt{Y}\cdots$ or $\mathtt{S}^-\mathtt{Y}\cdots$, where $\mathtt{Y} \in \{\mathtt{A}^+, \mathtt{S}^+, \mathtt{N}\}$. It follows from Proposition 4.1.2 and the Basic Proposition that $\mathtt{Y} = \mathtt{S}^+$, a contradiction to the assertion in the previous paragraph. □

Corollary 4.3.7. *None of the following sepr-sequences can occur as a subsequence of the sepr-sequence of a Hermitian matrix.*

1. $\mathtt{S}^+\mathtt{S}^*\mathtt{A}^+$;
2. $\mathtt{S}^-\mathtt{S}^*\mathtt{A}^-$;
3. $\mathtt{S}^+\mathtt{S}^+\mathtt{A}^+$;
4. $\mathtt{S}^-\mathtt{S}^-\mathtt{A}^-$;
5. $\mathtt{S}^+\mathtt{S}^-\mathtt{A}^+$;
6. $\mathtt{S}^-\mathtt{S}^+\mathtt{A}^-$.

Proof. Let B be a Hermitian matrix with $\operatorname{sepr}(B) = t_1t_2\cdots t_n$, where $n \geq 3$. Let $k \in \{1, 2, \ldots, n-2\}$. We proceed by contradiction.

(1): Suppose that $t_kt_{k+1}t_{k+2} = \mathtt{S}^+\mathtt{S}^*\mathtt{A}^+$. By the Inheritance Theorem, B contains a $(k+2) \times (k+2)$ principal submatrix B' whose sepr-sequence ends with $\mathtt{XYA}^+$, where $\mathtt{X} \in \{\mathtt{A}^+, \mathtt{S}^+, \mathtt{N}\}$ and $\mathtt{Y} \in \{\mathtt{S}^*, \mathtt{S}^+, \mathtt{S}^-\}$. By the Inverse Theorem, $\operatorname{sepr}((B')^{-1}) = \mathtt{YX}\cdots\mathtt{A}^+$, which contradicts the Basic Proposition or Proposition 4.1.2 or Proposition 4.3.6. It follows that $\mathtt{S}^+\mathtt{S}^*\mathtt{A}^+$ is prohibited.

(2): Suppose that $t_k t_{k+1} t_{k+2} = \mathtt{S}^-\mathtt{S}^*\mathtt{A}^-$. By the Inheritance Theorem, B contains a $(k+2)\times(k+2)$ principal submatrix B' whose sepr-sequence ends with $\mathtt{XYA}^-$, where $\mathtt{X} \in \{\mathtt{A}^-, \mathtt{S}^-, \mathtt{N}\}$ and $\mathtt{Y} \in \{\mathtt{S}^*, \mathtt{S}^+, \mathtt{S}^-\}$. By the Inverse Theorem, $\text{sepr}((B')^{-1}) = \text{neg}(\mathtt{YX})\cdots\mathtt{A}^-$, which contradicts the Basic Proposition or Proposition 4.1.2 or Proposition 4.3.6. Hence, $\mathtt{S}^-\mathtt{S}^*\mathtt{A}^-$ is prohibited.

(3): Suppose that $t_k t_{k+1} t_{k+2} = \mathtt{S}^+\mathtt{S}^+\mathtt{A}^+$. By the Inheritance Theorem, B contains a $(k+2)\times(k+2)$ principal submatrix whose sepr-sequence ends with $\mathtt{XS}^+\mathtt{A}^+$, where $\mathtt{X} \in \{\mathtt{A}^+, \mathtt{S}^+, \mathtt{N}\}$. By the Inverse Theorem, $\text{sepr}(B^{-1}) = \mathtt{S}^+\mathtt{X}\cdots\mathtt{A}^+$. Then, as $\mathtt{X} \in \{\mathtt{A}^+, \mathtt{S}^+, \mathtt{N}\}$, we have a contradiction to the Basic Proposition or Proposition 4.1.2 or Proposition 4.3.6. It follows that $\mathtt{S}^+\mathtt{S}^+\mathtt{A}^+$ is prohibited.

(4): Suppose that $t_k t_{k+1} t_{k+2} = \mathtt{S}^-\mathtt{S}^-\mathtt{A}^-$. By the Inheritance Theorem, B contains a $(k+2)\times(k+2)$ principal submatrix whose sepr-sequence ends with $\mathtt{XS}^-\mathtt{A}^-$, where $\mathtt{X} \in \{\mathtt{A}^-, \mathtt{S}^-, \mathtt{N}\}$. By the Inverse Theorem, $\text{sepr}(B^{-1}) = \text{neg}(\mathtt{S}^-\mathtt{X})\cdots\mathtt{A}^- = \mathtt{S}^+\,\text{neg}(\mathtt{X})\cdots\mathtt{A}^-$, which contradicts the Basic Proposition or Proposition 4.1.2 or Proposition 4.3.6.

(5) and (6): If any of $\mathtt{S}^+\mathtt{S}^-\mathtt{A}^+$ or $\mathtt{S}^-\mathtt{S}^+\mathtt{A}^-$ was a subsequence of $\text{sepr}(B)$, then applying Observation 4.2.2 to $\text{sepr}(B)$ would contradict items (3) or (4) above. □

Proposition 4.3.8. *None of the following sepr-sequences can occur as a subsequence of the sepr-sequence of a Hermitian matrix.*

1. $\mathtt{A}^+\mathtt{A}^*\mathtt{S}^+$;

2. $\mathtt{A}^-\mathtt{A}^*\mathtt{S}^-$;

3. $\mathtt{S}^+\mathtt{A}^*\mathtt{A}^+$;

4. $\mathtt{S}^-\mathtt{A}^*\mathtt{A}^-$;

5. $\mathtt{S}^+\mathtt{A}^*\mathtt{S}^+$;

6. $\mathtt{S}^-\mathtt{A}^*\mathtt{S}^-$.

Proof. Let B be a Hermitian matrix containing one of the sequences (1)–(6) in positions $k-1, k, k+1$. By Lemma 4.2.6, there exists a $(k-1)$-element subset $I \subseteq \{1, 2, \dots, n\}$, and $i, j \in \{1, 2, \dots, n\} \setminus I$, such that $B_{I\cup\{i\}} > 0$ and $B_{I\cup\{j\}} < 0$; hence, $B_{I\cup\{i\}}B_{I\cup\{j\}} < 0$. But, since $B_I B_{I\cup\{i,j\}} \geq 0$ by hypothesis, the identity

$$B_I B_{I\cup\{i,j\}} = B_{I\cup\{i\}}B_{I\cup\{j\}} - |\det(B[I\cup\{i\}|I\cup\{j\}])|^2$$

leads to a contradiction. □

Proposition 4.3.9. *Let B be a Hermitian matrix with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$ *and* $\operatorname{sepr}(B) = t_1t_2\cdots t_n$. *Suppose* $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{SNA}$. *Then* $t_kt_{k+1}t_{k+2} = \mathtt{S}^+\mathtt{NA}^-$ *or* $t_kt_{k+1}t_{k+2} = \mathtt{S}^-\mathtt{NA}^+$.

Proof. Suppose to the contrary that $t_kt_{k+1}t_{k+2} \neq \mathtt{S}^+\mathtt{NA}^-$ or $t_kt_{k+1}t_{k+2} \neq \mathtt{S}^-\mathtt{NA}^+$. Since $\mathtt{NA}^*$ is prohibited, $t_kt_{k+1}t_{k+2} \in \{\mathtt{S}^*\mathtt{NA}^+, \mathtt{S}^+\mathtt{NA}^+, \mathtt{S}^*\mathtt{NA}^-, \mathtt{S}^-\mathtt{NA}^-\}$. First, suppose $t_kt_{k+1}t_{k+2} \in \{\mathtt{S}^*\mathtt{NA}^+, \mathtt{S}^+\mathtt{NA}^+\}$. By the Inheritance Theorem, B has a $(k+2)\times(k+2)$ principal submatrix C_+ with $\operatorname{sepr}(C_+) \in \{\cdots\mathtt{A}^*\mathtt{NA}^+, \cdots\mathtt{S}^*\mathtt{NA}^+, \cdots\mathtt{S}^+\mathtt{NA}^+\}$. It now follows from the Inverse Theorem that $\operatorname{sepr}(C_+^{-1}) \in \{\mathtt{NA}^*\cdots, \mathtt{NS}^*\cdots, \mathtt{NS}^+\cdots\}$, which contradicts the Basic Proposition.

Finally, suppose $t_kt_{k+1}t_{k+2} \in \{\mathtt{S}^*\mathtt{NA}^-, \mathtt{S}^-\mathtt{NA}^-\}$. By the Inheritance Theorem, B has a $(k+2)\times(k+2)$ principal submatrix C_- with $\operatorname{sepr}(C_-) \in \{\cdots\mathtt{A}^*\mathtt{NA}^-, \cdots\mathtt{S}^*\mathtt{NA}^-, \cdots\mathtt{S}^-\mathtt{NA}^-\}$. Hence, by the Inverse Theorem, $\operatorname{sepr}(C_-^{-1}) \in \{\operatorname{neg}(\mathtt{NA}^*)\cdots, \operatorname{neg}(\mathtt{NS}^*)\cdots, \operatorname{neg}(\mathtt{NS}^-)\cdots\}$, which contradicts the Basic Proposition. □

A result analogous to Theorem 4.3.5 follows from Corollary 4.3.7, and Propositions 4.3.8 and 4.3.9.

Theorem 4.3.10. *For any* $\mathtt{X}$, *if any of the sepr-sequences* $\mathtt{S}^+\mathtt{XA}^+$ *or* $\mathtt{S}^-\mathtt{XA}^-$ *occurs in the sepr-sequence of a Hermitian matrix, then* $\mathtt{X} \in \{\mathtt{A}^+, \mathtt{A}^-\}$.

Proposition 4.3.11. *For* $\mathtt{X} \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$, *the following sepr-sequences are prohibited for Hermitian matrices.*

1. $\cdots \mathtt{A}^+\mathtt{S}^*\mathtt{S}^+ \cdots \mathtt{X} \cdots$;

2. $\cdots \mathtt{A}^+\mathtt{S}^+\mathtt{S}^+ \cdots \mathtt{X} \cdots$;

3. $\cdots \mathtt{A}^+\mathtt{S}^-\mathtt{S}^+ \cdots \mathtt{X} \cdots$;

4. $\cdots \mathtt{A}^-\mathtt{S}^*\mathtt{S}^- \cdots \mathtt{X} \cdots$;

5. $\cdots \mathtt{A}^-\mathtt{S}^-\mathtt{S}^- \cdots \mathtt{X} \cdots$;

6. $\cdots \mathtt{A}^-\mathtt{S}^+\mathtt{S}^- \cdots \mathtt{X} \cdots$.

Proof. Let $\mathtt{X} \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$, and let B be a Hermitian matrix. We first discard the sequences (1)–(3) simultaneously, and then the sequences (4)–(6).

(1)–(3): Suppose to the contrary that $\text{sepr}(B) = \cdots \mathtt{A}^+\mathtt{Y}\mathtt{S}^+ \cdots \mathtt{X} \cdots$, where $\mathtt{Y} \in \{\mathtt{S}^*, \mathtt{S}^+, \mathtt{S}^-\}$ and $\mathtt{X}$ occurs in position k. By the Inheritance Theorem, B has a nonsingular $k \times k$ principal submatrix B' whose sepr-sequence contains $\mathtt{A}^+\mathtt{W}\mathtt{Z}$, where $\mathtt{W} \in \{\mathtt{S}^*, \mathtt{S}^+, \mathtt{S}^-\}$ and $\mathtt{Z} \in \{\mathtt{A}^+, \mathtt{S}^+, \mathtt{N}\}$. By Theorem 4.3.5, $\mathtt{Z} \neq \mathtt{A}^+$. Since B' is nonsingular, the $\mathtt{NSA}$ Theorem implies that $\mathtt{Z} \neq \mathtt{N}$. It follows that $\mathtt{Z} = \mathtt{S}^+$. Hence, B' contains $\mathtt{A}^+\mathtt{W}\mathtt{S}^+$. Then, as B' is nonsingular, the Inverse Theorem implies that $\text{sepr}((B')^{-1})$ contains one of the prohibited sequences $\mathtt{S}^+\mathtt{W}\mathtt{A}^+$ and $\text{neg}(\mathtt{S}^+\mathtt{W}\mathtt{A}^+) = \mathtt{S}^- \,\text{neg}(\mathtt{W})\mathtt{A}^-$, which contradicts Corollary 4.3.7.

(4)–(6): Suppose to the contrary that $\text{sepr}(B) = \cdots \mathtt{A}^-\mathtt{Y}\mathtt{S}^- \cdots \mathtt{X} \cdots$, where $\mathtt{Y} \in \{\mathtt{S}^*, \mathtt{S}^+, \mathtt{S}^-\}$ and $\mathtt{X}$ occurs in position of k. By the Inheritance Theorem, B has a nonsingular $k \times k$ principal submatrix B' whose sepr-sequence contains $\mathtt{A}^-\mathtt{W}\mathtt{Z}$, where $\mathtt{W} \in \{\mathtt{S}^*, \mathtt{S}^+, \mathtt{S}^-\}$ and $\mathtt{Z} \in \{\mathtt{A}^-, \mathtt{S}^-, \mathtt{N}\}$. By Theorem 4.3.5, $\mathtt{Z} \neq \mathtt{A}^-$. Since B' is nonsingular, the $\mathtt{NSA}$ Theorem implies that $\mathtt{Z} \neq \mathtt{N}$. It follows that $\mathtt{Z} = \mathtt{S}^-$. Hence, B' contains $\mathtt{A}^-\mathtt{W}\mathtt{S}^-$. Then, as B' is nonsingular, the Inverse Theorem implies that $\text{sepr}((B')^{-1})$ contains one of the prohibited sequences $\mathtt{S}^-\mathtt{W}\mathtt{A}^-$ and $\text{neg}(\mathtt{S}^-\mathtt{W}\mathtt{A}^-) = \mathtt{S}^+ \,\text{neg}(\mathtt{W})\mathtt{A}^+$, which again contradicts Corollary 4.3.7. $\square$

Proposition 4.3.12. *Let B be a Hermitian matrix with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$ *and* $\operatorname{sepr}(B) = t_1t_2\cdots t_n$*. Suppose* $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{ANS}$*. Then* $t_kt_{k+1}t_{k+2} = \mathtt{A}^+\mathtt{N}\mathtt{S}^-$ *or* $t_kt_{k+1}t_{k+2} = \mathtt{A}^-\mathtt{N}\mathtt{S}^+$*.*

Proof. Suppose to the contrary that $t_kt_{k+1}t_{k+2} \neq \mathtt{A}^+\mathtt{N}\mathtt{S}^-$ and $t_kt_{k+1}t_{k+2} \neq \mathtt{A}^-\mathtt{N}\mathtt{S}^+$. Clearly, $t_{k+1} = \mathtt{N}$. Since $\mathtt{A}^*\mathtt{N}$ is prohibited, $t_k \neq \mathtt{A}^*$. Hence, $t_kt_{k+1}t_{k+2} \in \{\mathtt{A}^+\mathtt{N}\mathtt{S}^*, \mathtt{A}^+\mathtt{N}\mathtt{S}^+, \mathtt{A}^-\mathtt{N}\mathtt{S}^*, \mathtt{A}^-\mathtt{N}\mathtt{S}^-\}$. In each case, by the Inheritance Theorem, B has a $(k+2)\times(k+2)$ principal submatrix B' with $\operatorname{sepr}(B') = \mathtt{A}^+\mathtt{N}\mathtt{A}^+$ or $\operatorname{sepr}(B') = \mathtt{A}^-\mathtt{N}\mathtt{A}^-$, a contradiction to Theorem 4.3.5. □

A natural question to answer is, does a result analogous to Theorems 4.3.5 and 4.3.10 hold for subsequences of the form $\mathtt{A}^+\mathtt{X}\mathtt{S}^+$ and $\mathtt{A}^-\mathtt{X}\mathtt{S}^-$? In other words, are the sequences $\mathtt{A}^+\mathtt{X}\mathtt{S}^+$ and $\mathtt{A}^-\mathtt{X}\mathtt{S}^-$ prohibited in the sepr-sequence of a Hermitian matrix when $\mathtt{X} \notin \{\mathtt{A}^+, \mathtt{A}^-\}$? The answer is negative: An $n\times n$ positive semidefinite matrix with nonzero diagonal and rank $n-1$, and containing principal minors of order 2 and 3 that are equal to zero, serves as a counterexample (by Theorem 4.3.16 below, the sepr-sequence of such a matrix begins with $\mathtt{A}^+\mathtt{S}^+\mathtt{S}^+$). A simple example is $B = I_3 \oplus J_2$. Also, observe that $-B$ begins with $\mathtt{A}^-\mathtt{S}^+\mathtt{S}^-$ (see Observation 4.2.2). With that being said, a relatively similar result to Theorems 4.3.5 and 4.3.10 can still be obtained, which is an immediate consequence of Propositions 4.3.8, 4.3.11 and 4.3.12:

Theorem 4.3.13. *For any* $\mathtt{X}$ *and for* $\mathtt{Y} \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$*, if any of the sepr-sequences* $\cdots\mathtt{A}^+\mathtt{X}\mathtt{S}^+\cdots\mathtt{Y}\cdots$ *or* $\cdots\mathtt{A}^-\mathtt{X}\mathtt{S}^-\cdots\mathtt{Y}\cdots$ *is attainable by a Hermitian matrix, then* $\mathtt{X} \in \{\mathtt{A}^+, \mathtt{A}^-\}$*.*

Corollary 4.3.14. *Any Hermitian matrix with an sepr-sequence containing any of the following subsequences is singular.*

1. $\mathtt{A}^+\mathtt{S}^*\mathtt{S}^+$;
2. $\mathtt{A}^+\mathtt{S}^+\mathtt{S}^+$;
3. $\mathtt{A}^+\mathtt{S}^-\mathtt{S}^+$;

4. $\mathtt{A}^-\mathtt{S}^*\mathtt{S}^-$*;*

5. $\mathtt{A}^-\mathtt{S}^+\mathtt{S}^-$*;*

6. $\mathtt{A}^-\mathtt{S}^-\mathtt{S}^-$*.*

Proposition 4.3.15. *Let B be a Hermitian matrix with* $\mathrm{epr}(B) = \ell_1\ell_2\cdots\ell_n$ *and* $\mathrm{sepr}(B) = t_1t_2\cdots t_n$*. Suppose $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{SNS}$ for some k with $1 \leq k \leq n-2$. Then $t_kt_{k+1}t_{k+2} = \mathtt{S}^*\mathtt{NS}^*$, or $t_kt_{k+1}t_{k+2} = \mathtt{S}^+\mathtt{NS}^-$, or $t_kt_{k+1}t_{k+2} = \mathtt{S}^-\mathtt{NS}^+$.*

Proof. We proceed by contradiction. Suppose that $t_kt_{k+1}t_{k+2} \notin \{\mathtt{S}^*\mathtt{NS}^*, \mathtt{S}^+\mathtt{NS}^-, \mathtt{S}^-\mathtt{NS}^+\}$. Hence, $t_kt_{k+1}t_{k+2}$ is one of the sequences in the set

$$\{\mathtt{S}^*\mathtt{NS}^+, \mathtt{S}^*\mathtt{NS}^-, \mathtt{S}^+\mathtt{NS}^*, \mathtt{S}^+\mathtt{NS}^+, \mathtt{S}^-\mathtt{NS}^*, \mathtt{S}^-\mathtt{NS}^-\}.$$

We examine four cases:

Case 1: $t_kt_{k+1}t_{k+2} = \mathtt{S}^*\mathtt{NS}^+$. Let $B[\alpha]$ be a $k \times k$ nonsingular principal submatrix with $\det(B[\alpha]) > 0$. By the Schur Complement Theorem, $B/B[\alpha]$ is an $(n-k) \times (n-k)$ Hermitian matrix with every diagonal entry equal to zero; moreover, $\mathrm{rank}(B/B[\alpha]) = \mathrm{rank}(B) - k \geq (k+2) - k = 2$, implying that $B/B[\alpha]$ has a nonzero principal minor of order 2, say, $\det((B/B[\alpha])[\{i,j\}])$. Since $B/B[\alpha]$ has zero diagonal, $\det((B/B[\alpha])[\{i,j\}]) < 0$. By the Schur Complement Theorem,

$$\det((B/B[\alpha])[\{i,j\}]) = \det B[\{i,j\} \cup \alpha]/\det B[\alpha].$$

Then, as $\det B[\alpha] > 0$, $\det B[\{i,j\} \cup \alpha] < 0$, a contradiction to $t_{k+2} = \mathtt{S}^+$.

Case 2: $t_kt_{k+1}t_{k+2} = \mathtt{S}^*\mathtt{NS}^-$. Let $B[\alpha]$ be a $k \times k$ nonsingular principal submatrix with $\det(B[\alpha]) < 0$. Just as in Case 1, $B/B[\alpha]$ is an $(n-k) \times (n-k)$ Hermitian matrix with zero diagonal, and with a nonzero principal minor of order 2. Let $\det((B/B[\alpha])[\{i,j\}])$ be a nonzero principal minor of order 2. Since $B/B[\alpha]$ has zero diagonal, $\det((B/B[\alpha])[\{i,j\}]) < 0$. As in Case 1, by the Schur Complement Theorem,

$$\det((B/B[\alpha])[\{i,j\}]) = \det B[\{i,j\} \cup \alpha]/\det B[\alpha].$$

Then, as $\det B[\alpha] < 0$, $\det B[\{i,j\} \cup \alpha] > 0$, a contradiction to $t_{k+2} = \mathtt{S}^-$.

Case 3: $t_k t_{k+1} t_{k+2} \in \{\mathtt{S}^+\mathtt{NS}^*, \mathtt{S}^+\mathtt{NS}^+\}$. By the Inheritance Theorem, B has a $(k+2)\times(k+2)$ principal submatrix B' with $\operatorname{sepr}(B') = t'_1 t'_2 \cdots t'_{k+2}$ having $t'_{k+1}t'_{k+2} = \mathtt{NA}^+$ and $t'_k \in \{\mathtt{A}^+, \mathtt{S}^+, \mathtt{N}\}$. By the $\mathtt{NN}$ Theorem, $t'_k \neq \mathtt{N}$. It follows that $t'_k t'_{k+1} t'_{k+2} \in \{\mathtt{A}^+\mathtt{NA}^+, \mathtt{S}^+\mathtt{NA}^+\}$, which contradicts Theorems 4.3.5 and 4.3.10.

Case 4: $t_k t_{k+1} t_{k+2} \in \{\mathtt{S}^-\mathtt{NS}^*, \mathtt{S}^-\mathtt{NS}^-\}$. By the Inheritance Theorem, B has a $(k+2)\times(k+2)$ principal submatrix B' with $\operatorname{sepr}(B') = t'_1 t'_2 \cdots t'_{k+2}$ having $t'_{k+1}t'_{k+2} = \mathtt{NA}^-$ and $t'_k \in \{\mathtt{A}^-, \mathtt{S}^-, \mathtt{N}\}$. By the $\mathtt{NN}$ Theorem, $t'_k \neq \mathtt{N}$. It follows that $t'_k t'_{k+1} t'_{k+2} \in \{\mathtt{A}^-\mathtt{NA}^-, \mathtt{S}^-\mathtt{NA}^-\}$, a contradiction to Theorems 4.3.5 and 4.3.10. ☐

4.3.1 Nonnegative and nonpositive sepr-sequences

We call a subsequence of an sepr-sequence *nonnegative* (respectively, *nonpositive*) if each of its terms is in $\{\mathtt{A}^+, \mathtt{S}^+, \mathtt{N}\}$ (respectively, $\{\mathtt{A}^-, \mathtt{S}^-, \mathtt{N}\}$).

Theorem 4.3.16. *Let B be an $n \times n$ Hermitian matrix, and let $\sigma = x_1x_2\cdots x_k$ be a nonnegative subsequence of* $\operatorname{sepr}(B)$*, where $2 \le k \le n$. Then $x_2x_3\cdots x_k = \overline{\mathtt{A}^+}\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$.*

Proof. We first show that if $x_q = \mathtt{N}$ for some $q > 1$, then $x_j = \mathtt{N}$ for all $j \ge q$. To see this, suppose $x_q = \mathtt{N}$ for some $q > 1$. If $x_{q-1} = \mathtt{N}$, then our claim follows from the $\mathtt{NN}$ Theorem. Now, suppose $x_{q-1} \neq \mathtt{N}$. Since the subsequences $\mathtt{A}^+\mathtt{NA}^+$, $\mathtt{S}^+\mathtt{NA}^+$, $\mathtt{A}^+\mathtt{NS}^+$, and $\mathtt{S}^+\mathtt{NS}^+$ are prohibited by Theorems 4.3.5 and 4.3.10, and by Propositions 4.3.12 and 4.3.15, $x_{q-1}x_qx_{q+1} = \mathtt{S}^+\mathtt{NN}$ or $x_{q-1}x_qx_{q+1} = \mathtt{A}^+\mathtt{NN}$; hence, our claim now follows from the $\mathtt{NN}$ Theorem. Now we examine three cases based on the value of x_1.

Case 1: $x_1 = \mathtt{A}^+$. If $x_2 = \mathtt{N}$, then, by the above assertion, we must have $x_j = \mathtt{N}$ for $j \ge 2$, so that $x_2x_3\cdots x_k = \mathtt{N}\overline{\mathtt{N}}$. Now, suppose $x_2 = \mathtt{S}^+$. If $k = 2$, then $x_2x_3\cdots x_k = \mathtt{S}^+$, and therefore we are done; thus, suppose $k > 2$. Then, as $\mathtt{A}^+\mathtt{S}^+\mathtt{A}^+$ is prohibited by Theorem 4.3.5, $\sigma = \mathtt{A}^+\mathtt{S}^+\mathtt{N}\cdots$ or $\sigma = \mathtt{A}^+\mathtt{S}^+\mathtt{S}^+\cdots$. If $\sigma = \mathtt{A}^+\mathtt{S}^+\mathtt{N}\cdots$, then the assertion in the previous paragraph implies that $\sigma = \mathtt{A}^+\mathtt{S}^+\mathtt{N}\overline{\mathtt{N}}$, so that $x_2x_3\cdots x_k = \mathtt{S}^+\mathtt{N}\overline{\mathtt{N}}$. Now, suppose $\sigma = \mathtt{A}^+\mathtt{S}^+\mathtt{S}^+\cdots$. Let p be a minimal integer with $3 \le p \le k-1$ such that $x_px_{p+1} = \mathtt{S}^+\mathtt{A}^+$ or $x_px_{p+1} = \mathtt{S}^+\mathtt{N}$ (if no such p exists, then $x_2x_3\cdots x_k = \mathtt{S}^+\mathtt{S}^+\overline{\mathtt{S}^+}$). Note that $x_2x_3\cdots x_p = \mathtt{S}^+\mathtt{S}^+\overline{\mathtt{S}^+}$. Since $\mathtt{S}^+\mathtt{S}^+\mathtt{A}^+$ is prohibited by Theorem 4.3.10, $x_px_{p+1} = \mathtt{S}^+\mathtt{N}$. Hence, by the assertion in the first paragraph, $x_2x_3\cdots x_k = \mathtt{S}^+\mathtt{S}^+\overline{\mathtt{S}^+}\mathtt{N}\overline{\mathtt{N}}$. Observe that we have shown that any nonnegative subsequence of $\operatorname{sepr}(B)$ that starts with $\mathtt{A}^+$ is of the form $\sigma = \mathtt{A}^+\ \overline{\mathtt{A}^+}\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$.

Case 2: $x_1 = \mathtt{S}^+$. If $x_2 = \mathtt{A}^+$, then $x_2x_3\cdots x_k$ is a nonnegative subsequence starting with $\mathtt{A}^+$, and therefore, by applying Case 1 to this subsequence, we have $x_2x_3\cdots x_k = \mathtt{A}^+\ \overline{\mathtt{A}^+}\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$. If $x_2 = \mathtt{N}$, then, by the assertion in the first paragraph, $x_2x_3\cdots x_k = \mathtt{N}\overline{\mathtt{N}}$. Now, suppose $x_2 = \mathtt{S}^+$. If $k = 2$, then $x_2x_3\cdots x_k = \mathtt{S}^+$, and therefore we are done; thus, suppose $k > 2$. Let p be a minimal integer with $2 \le p \le k-1$ such that $x_px_{p+1} = \mathtt{S}^+\mathtt{A}^+$ or $x_px_{p+1} = \mathtt{S}^+\mathtt{N}$ (if no such p exists, then $x_2x_3\cdots x_k = \mathtt{S}^+\overline{\mathtt{S}^+}$). Observe that $x_2x_3\cdots x_p = \mathtt{S}^+\overline{\mathtt{S}^+}$. Since $x_1 = \mathtt{S}^+$, and because $\mathtt{S}^+\mathtt{S}^+\mathtt{A}^+$ is prohibited by Theorem 4.3.10, $x_px_{p+1} = \mathtt{S}^+\mathtt{N}$. Hence, by the assertion in the first paragraph, $x_2x_3\cdots x_k = \mathtt{S}^+\overline{\mathtt{S}^+}\mathtt{N}\overline{\mathtt{N}}$.

Case 3: $x_1 = \mathtt{N}$. If $x_2 = \mathtt{N}$, then the $\mathtt{NN}$ Theorem implies that $x_2x_3\cdots x_k = \mathtt{N}\overline{\mathtt{N}}$. If $x_2 = \mathtt{A}^+$, then, by applying Case 1 to $x_2x_3\cdots x_k$, we have $x_2x_3\cdots x_k = \mathtt{A}^+\ \overline{\mathtt{A}^+}\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$. Now, suppose $x_2 = \mathtt{S}^+$. Then, by applying Case 2 to $x_2x_3\cdots x_k$, we obtain that $\sigma = \mathtt{N}\ \mathtt{S}^+\ \overline{\mathtt{A}^+}\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$; but, as $\mathtt{NSA}$ is prohibited by the $\mathtt{NSA}$ Theorem, it follows that $\overline{\mathtt{A}^+}$ is empty, and therefore that $\sigma = \mathtt{N}\ \mathtt{S}^+\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$, implying that $x_2x_3\cdots x_k = \mathtt{S}^+\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$. $\square$

By simply replacing "+" superscripts with "−" superscripts, and "nonnegative" with "nonpositive," in the proof of Theorem 4.3.16, one obtains a proof for a result analogous to Theorem 4.3.16:

Theorem 4.3.17. *Let B be an $n \times n$ Hermitian matrix, and let $\sigma = x_1x_2\cdots x_k$ be a nonpositive subsequence of* $\operatorname{sepr}(B)$*, where $2 \le k \le n$. Then $x_2x_3\cdots x_k = \overline{\mathtt{A}^-}\ \overline{\mathtt{S}^-}\ \overline{\mathtt{N}}$.*

A corollary to Theorem 4.3.16 relates nonnegative sepr-sequences to positive semidefinite matrices:

Corollary 4.3.18. *Let B be a (Hermitian) positive semidefinite matrix. Then* $\mathrm{sepr}(B) = \overline{\mathtt{A}^+}\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$, *where* $\overline{\mathtt{N}}$ *is nonempty if* $\overline{\mathtt{S}^+}$ *is nonempty.*

Proof. Let $\mathrm{sepr}(B) = t_1 t_2 \cdots t_n$. Since the principal minors of a (Hermitian) positive semidefinite matrix must be nonnegative, $\mathrm{sepr}(B)$ must be nonnegative. It is easy to see that if $t_1 = \mathtt{N}$, $B = O_n$, implying that $\mathrm{sepr}(B) = \mathtt{N}\overline{\mathtt{N}}$. Now, suppose $t_1 = \mathtt{S}^+$. Then, as B contains at least one zero diagonal entry, B contains at least one zero principal minor of order 2; hence, $t_2 \neq \mathtt{A}^+$, and thus, by Theorem 4.3.16, $t_2 t_3 \cdots t_n = \overline{\mathtt{S}^+}\mathtt{N}\overline{\mathtt{N}}$, implying that $\mathrm{sepr}(B) = \mathtt{S}^+\overline{\mathtt{S}^+}\mathtt{N}\overline{\mathtt{N}}$. Finally, if $t_1 = \mathtt{A}^+$, the desired conclusion is immediate from Theorem 4.3.16. ☐

4.4 Sepr-sequences of order $n \leq 3$

This section is devoted towards classifying all the sepr-sequences of orders $n = 1, 2, 3$ that can be attained by Hermitian matrices.

For $n = 1$, it is obvious that the only attainable sepr-sequences are $\mathtt{A}^+, \mathtt{A}^-$ and $\mathtt{N}$.

For $n = 2$, there are a total of 21 sepr-sequences ending in $\mathtt{A}^+$, $\mathtt{A}^-$ or $\mathtt{N}$; of these, the 3 sequences that start with $\mathtt{S}^*$ are not attainable, since a matrix of order 2 only contains two diagonal entries. Of the remaining 18 sequences, $\mathtt{A}^*\mathtt{A}^+$, $\mathtt{A}^*\mathtt{N}$, $\mathtt{S}^+\mathtt{A}^+$, $\mathtt{S}^-\mathtt{A}^+$ and $\mathtt{N}\mathtt{A}^+$ are not attainable by the Basic Proposition. That leaves 13 sequences, which constitute the sepr-sequences of order $n = 2$ that are attainable by Hermitian matrices. These 13 sequences are listed in Table 4.1, where a Hermitian matrix attaining each sequence is provided; in the case where the matrix provided is expressed as the negative of another matrix, its sepr-sequence can be verified by applying Observation 4.2.2 to the sepr-sequence of the corresponding matrix.

Example 4.4.1. Matrices for Table 4.1:

$$M_{\mathtt{A}^*\mathtt{A}^-} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad M_{\mathtt{S}^+\mathtt{A}^-} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Table 4.1: All sepr-sequences of order $n = 2$ that are attainable by Hermitian matrices.

Sepr-sequence	Hermitian matrix	Result
$\mathtt{A}^*\mathtt{A}^-$	$M_{\mathtt{A}^*\mathtt{A}^-}$	Example 4.4.1
$\mathtt{A}^+\mathtt{A}^+$	I_2	
$\mathtt{A}^+\mathtt{A}^-$	$2J_2 - I_2$	
$\mathtt{A}^+\mathtt{N}$	J_2	
$\mathtt{A}^-\mathtt{A}^+$	$-I_2$	
$\mathtt{A}^-\mathtt{A}^-$	$-(2J_2 - I_2)$	
$\mathtt{A}^-\mathtt{N}$	$-J_2$	
$\mathtt{N}\mathtt{A}^-$	$J_2 - I_2$	
$\mathtt{NN}$	O_2	
$\mathtt{S}^+\mathtt{A}^-$	$M_{\mathtt{S}^+\mathtt{A}^-}$	Example 4.4.1
$\mathtt{S}^+\mathtt{N}$	$\operatorname{diag}(1, 0)$	
$\mathtt{S}^-\mathtt{A}^-$	$-M_{\mathtt{S}^+\mathtt{A}^-}$	Example 4.4.1
$\mathtt{S}^-\mathtt{N}$	$\operatorname{diag}(-1, 0)$	

As just shown, the results developed before this section sufficed to decide the attainability of all the sepr-sequences of order $n = 2$. However, for $n = 3$, there remain sequences unaccounted for.

Proposition 4.4.2. (Order-3 Proposition) *For any* $\mathtt{X}$*, the sepr-sequences* $\mathtt{S}^*\mathtt{S}^*\mathtt{X}$*,* $\mathtt{S}^*\mathtt{A}^*\mathtt{X}$*,* $\mathtt{A}^*\mathtt{S}^*\mathtt{X}$ *and* $\mathtt{X}\mathtt{S}^*\mathtt{N}$ *are prohibited as the sepr-sequence of a* 3×3 *Hermitian matrix.*

Proof. To see why $\mathtt{S}^*\mathtt{S}^*\mathtt{X}$ and $\mathtt{S}^*\mathtt{A}^*\mathtt{X}$ are prohibited, observe that any 3×3 Hermitian matrix whose sepr-sequence starts with $\mathtt{S}^*$ cannot contain a positive principal minor of order 2, since such a matrix does not contain two nonzero diagonal entries having the same sign.

To discard $\mathtt{A}^*\mathtt{S}^*\mathtt{X}$, note that a 3×3 Hermitian matrix with an sepr-sequence starting with $\mathtt{A}^*$ must contain at least two negative principal minors of order 2. Then, as a 3×3 matrix contains only 3 principal minors of order 2, a 3×3 Hermitian matrix cannot have an sepr-sequence starting with $\mathtt{A}^*\mathtt{S}^*$, since it cannot have both a zero and a positive principal minor of order 2.

Finally, the fact that $\mathtt{X}\mathtt{S}^*\mathtt{N}$ is prohibited follows from Lemma 4.3.2, since a Hermitian matrix attaining this sequence would have rank 2. □

Since the underlying epr-sequence of an attainable sepr-sequence must also be attainable, to decide the attainability of the sepr-sequences of order 3, we will take advantage of what is known about the epr-sequences of 3 × 3 Hermitian matrices. We proceed by first determining the sepr-sequences that are attainable by Hermitian matrices but not by real symmetric matrices, and then we determine the remaining ones, namely those that can be attained by real symmetric matrices.

It was established in [7] that $\mathtt{NAN}$ is the only epr-sequence of order 3 that is attainable by a Hermitian matrix but not by a real symmetric matrix. Since $\mathtt{NA^{*}N}$ and $\mathtt{NA^{+}N}$ are not attainable by a Hermitian matrix (because of the Basic Proposition), the only sepr-sequence of order 3 that is attainable by a Hermitian matrix but not by a real symmetric matrix is $\mathtt{NA^{-}N}$.

It now remains to determine the sepr-sequences that are attainable by real symmetric matrices. The epr-sequences of order 3 that are attainable by real symmetric matrices are listed in [4, Table 3], which are $\mathtt{AAA}$, $\mathtt{AAN}$, $\mathtt{ANA}$, $\mathtt{ANN}$, $\mathtt{ASA}$, $\mathtt{ASN}$, $\mathtt{NAA}$, $\mathtt{NNN}$, $\mathtt{NSN}$, $\mathtt{SAA}$, $\mathtt{SAN}$, $\mathtt{SNN}$, $\mathtt{SSA}$ and $\mathtt{SSN}$. Then, as an attainable sepr-sequence must end in $\mathtt{A^{+}}$, $\mathtt{A^{-}}$ or $\mathtt{N}$, by counting the sepr-sequences whose underlying epr-sequence is one of those just listed, we find that only 130 sepr-sequences are potentially attainable (note that we are *not* counting the sequence $\mathtt{NA^{-}N}$ among these 130 sequences, since we are now only counting those that are attainable by real symmetric matrices). We now discard certain sequences from these 130 sequences, and show that the remaining ones are all attainable. The 3 sepr-sequences starting with $\mathtt{A^{*}A^{+}}$ are not attainable by the Basic Proposition; that leaves 127 sequences. The 10 sequences having one of the forms $\mathtt{A^{+}XA^{+}}$ or $\mathtt{A^{-}XA^{-}}$, with $\mathtt{X} \notin \{\mathtt{A^{+}}, \mathtt{A^{-}}\}$, are not attainable by Theorem 4.3.5; that leaves 117 sequences. The 11 sequences containing the prohibited subsequences $\mathtt{A^{*}N}$ and $\mathtt{NA^{*}}$ are discarded; that leaves 106 sequences. Of the remaining sequences (which do not include the already-discarded sequence $\mathtt{S^{*}A^{*}N}$), 13 are discarded by the Order-3 Proposition; that leaves 93 sequences. The 3 sequences of the form $\mathtt{A^{*}S^{+}X}$, as well as $\mathtt{NA^{+}A^{+}}$, $\mathtt{NA^{+}A^{-}}$ and $\mathtt{NS^{+}N}$, are discarded by the Basic Proposition; that leaves sequences 87 sequences. The 9 sequences starting with $\mathtt{XA^{+}}$, where $\mathtt{X} \in \{\mathtt{S^{*}}, \mathtt{S^{+}}, \mathtt{S^{-}}\}$, are discarded by the Basic Proposition; that leaves 78 sequences. The 8 sequences of the form $\mathtt{S}^+\mathtt{X}\mathtt{A}^+$ and $\mathtt{S}^-\mathtt{X}\mathtt{A}^-$, with $\mathtt{X} \in \{\mathtt{A}^*, \mathtt{S}^*, \mathtt{S}^+, \mathtt{S}^-\}$, are discarded by Theorem 4.3.10; that leaves 70 sequences. The sequence $\mathtt{S}^*\mathtt{NN}$ is discarded by the Basic Proposition; that leaves 69 sequences. The sequences $\mathtt{S}^+\mathtt{S}^+\mathtt{A}^-$ and $\mathtt{S}^-\mathtt{S}^+\mathtt{A}^+$ are discarded by Proposition 4.3.6; that leaves 67 sequences. Finally, the 3 sequences of the form $\mathtt{S}^*\mathtt{S}^+\mathtt{X}$ are discarded by the Basic Proposition; that leaves 64 sequences, which constitute the sepr-sequences of order $n = 3$ that are attainable by real symmetric matrices. By adding the sequence $\mathtt{N}\mathtt{A}^-\mathtt{N}$ to these 64 sequences, we obtain all the sepr-sequences that are attainable by Hermitian matrices; these 65 sequences are listed in Table 4.2, where a Hermitian matrix attaining each sequence is provided.

Example 4.4.3. Matrices for Table 4.2:

$$M_{\mathtt{A}^*\mathtt{A}^-\mathtt{A}^+} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \quad M_{\mathtt{A}^+\mathtt{A}^+\mathtt{A}^-} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \quad M_{\mathtt{A}^+\mathtt{A}^-\mathtt{A}^+} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

$$M_{\mathtt{A}^+\mathtt{A}^-\mathtt{A}^-} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}, \quad M_{\mathtt{A}^*\mathtt{A}^-\mathtt{N}} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & -1 \end{bmatrix}, \quad M_{\mathtt{A}^+\mathtt{A}^+\mathtt{N}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix},$$

$$M_{\mathtt{A}^+\mathtt{A}^-\mathtt{N}} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 7 \\ 2 & 7 & 1 \end{bmatrix}, \quad M_{\mathtt{A}^*\mathtt{S}^-\mathtt{A}^+} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad M_{\mathtt{A}^+\mathtt{S}^*\mathtt{A}^-} = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 2 \\ -4 & 2 & 4 \end{bmatrix},$$

$$M_{\mathtt{A}^+\mathtt{S}^+\mathtt{A}^-} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad M_{\mathtt{A}^+\mathtt{S}^-\mathtt{A}^-} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \quad M_{\mathtt{A}^*\mathtt{S}^-\mathtt{N}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$M_{\mathtt{A}^+\mathtt{S}^-\mathtt{N}} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \quad M_{\mathtt{N}\mathtt{A}^-\mathtt{N}} = \begin{bmatrix} 0 & i & 1 \\ -i & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad M_{\mathtt{N}\mathtt{S}^-\mathtt{N}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M_{\mathtt{S^+A^*A^-}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix},\ M_{\mathtt{S^+A^-A^+}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},\ M_{\mathtt{S^+A^-A^-}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$M_{\mathtt{S^*A^-N}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix},\ M_{\mathtt{S^+A^-N}} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix},\ M_{\mathtt{S^*S^-A^+}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$M_{\mathtt{S^+S^-A^-}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ M_{\mathtt{S^+S^+N}} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ M_{\mathtt{S^+S^-N}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Table 4.2: All sepr-sequences of order $n = 3$ that are attainable by Hermitian matrices.

Sepr-sequence	Hermitian matrix	Result
$\mathtt{A^*A^*A^+}$	$\operatorname{diag}(1, -1, -1)$	
$\mathtt{A^*A^*A^-}$	$\operatorname{diag}(-1, 1, 1)$	
$\mathtt{A^*A^-A^+}$	$M_{\mathtt{A^*A^-A^+}}$	Example 4.4.3
$\mathtt{A^*A^-A^-}$	$-M_{\mathtt{A^*A^-A^+}}$	Example 4.4.3
$\mathtt{A^+A^*A^-}$	$(-M_{\mathtt{A^*A^-A^+}})^{-1}$	Example 4.4.3
$\mathtt{A^+A^+A^+}$	I_3	
$\mathtt{A^+A^+A^-}$	$M_{\mathtt{A^+A^+A^-}}$	Example 4.4.3
$\mathtt{A^+A^-A^+}$	$M_{\mathtt{A^+A^-A^+}}$	Example 4.4.3
$\mathtt{A^+A^-A^-}$	$M_{\mathtt{A^+A^-A^-}}$	Example 4.4.3
$\mathtt{A^-A^*A^+}$	$-(-M_{\mathtt{A^*A^-A^+}})^{-1}$	Example 4.4.3
$\mathtt{A^-A^+A^+}$	$-M_{\mathtt{A^+A^+A^-}}$	Example 4.4.3
$\mathtt{A^-A^+A^-}$	$-I_3$	
$\mathtt{A^-A^-A^+}$	$-M_{\mathtt{A^+A^-A^-}}$	Example 4.4.3
$\mathtt{A^-A^-A^-}$	$-M_{\mathtt{A^+A^-A^+}}$	Example 4.4.3
$\mathtt{A^*A^-N}$	$M_{\mathtt{A^*A^-N}}$	Example 4.4.3
$\mathtt{A^+A^+N}$	$M_{\mathtt{A^+A^+N}}$	Example 4.4.3
$\mathtt{A^+A^-N}$	$M_{\mathtt{A^+A^-N}}$	Example 4.4.3
$\mathtt{A^-A^+N}$	$-M_{\mathtt{A^+A^+N}}$	Example 4.4.3
$\mathtt{A^-A^-N}$	$-M_{\mathtt{A^+A^-N}}$	Example 4.4.3
$\mathtt{A^+NA^-}$	$-(J_3 - 2I_3)$	
$\mathtt{A^-NA^+}$	$J_3 - 2I_3$	
$\mathtt{A^+NN}$	J_3	
$\mathtt{A^-NN}$	$-J_3$	
$\mathtt{A^*S^-A^+}$	$M_{\mathtt{A^*S^-A^+}}$	Example 4.4.3
$\mathtt{A^*S^-A^-}$	$-M_{\mathtt{A^*S^-A^+}}$	Example 4.4.3
$\mathtt{A^+S^*A^-}$	$M_{\mathtt{A^+S^*A^-}}$	Example 4.4.3
$\mathtt{A^+S^+A^-}$	$M_{\mathtt{A^+S^+A^-}}$	Example 4.4.3
$\mathtt{A^+S^-A^-}$	$M_{\mathtt{A^+S^-A^-}}$	Example 4.4.3
$\mathtt{A^-S^*A^+}$	$-M_{\mathtt{A^+S^*A^-}}$	Example 4.4.3
$\mathtt{A^-S^+A^+}$	$-M_{\mathtt{A^+S^+A^-}}$	Example 4.4.3
$\mathtt{A^-S^-A^+}$	$-M_{\mathtt{A^+S^-A^-}}$	Example 4.4.3

Table 4.2 (continued): All sepr-sequences of order $n = 3$ that are attainable by Hermitian matrices.

Sepr-sequence	Hermitian matrix	Result
$\mathtt{A^*S^-N}$	$M_{\mathtt{A^*S^-N}}$	Example 4.4.3
$\mathtt{A^+S^+N}$	$J_1 \oplus J_2$	
$\mathtt{A^+S^-N}$	$M_{\mathtt{A^+S^-N}}$	Example 4.4.3
$\mathtt{A^-S^+N}$	$-(J_1 \oplus J_2)$	
$\mathtt{A^-S^-N}$	$-M_{\mathtt{A^+S^-N}}$	Example 4.4.3
$\mathtt{NA^-A^+}$	$J_3 - I_3$	
$\mathtt{NA^-A^-}$	$-(J_3 - I_3)$	
$\mathtt{NA^-N}$	$M_{\mathtt{NA^-N}}$	Example 4.4.3
$\mathtt{NNN}$	O_3	
$\mathtt{NS^-N}$	$M_{\mathtt{NS^-N}}$	Example 4.4.3
$\mathtt{S^*A^-A^+}$	$(-M_{\mathtt{A^+S^*A^-}})^{-1}$	Example 4.4.3
$\mathtt{S^*A^-A^-}$	$-(-M_{\mathtt{A^+S^*A^-}})^{-1}$	Example 4.4.3
$\mathtt{S^+A^*A^-}$	$M_{\mathtt{S^+A^*A^-}}$	Example 4.4.3
$\mathtt{S^+A^-A^+}$	$M_{\mathtt{S^+A^-A^+}}$	Example 4.4.3
$\mathtt{S^+A^-A^-}$	$M_{\mathtt{S^+A^-A^-}}$	Example 4.4.3
$\mathtt{S^-A^*A^+}$	$-M_{\mathtt{S^+A^*A^-}}$	Example 4.4.3
$\mathtt{S^-A^-A^+}$	$-M_{\mathtt{S^+A^-A^-}}$	Example 4.4.3
$\mathtt{S^-A^-A^-}$	$-M_{\mathtt{S^+A^-A^+}}$	Example 4.4.3
$\mathtt{S^*A^-N}$	$M_{\mathtt{S^*A^-N}}$	Example 4.4.3
$\mathtt{S^+A^-N}$	$M_{\mathtt{S^+A^-N}}$	Example 4.4.3
$\mathtt{S^-A^-N}$	$-M_{\mathtt{S^+A^-N}}$	Example 4.4.3
$\mathtt{S^+NN}$	$J_1 \oplus O_2$	
$\mathtt{S^-NN}$	$-(J_1 \oplus O_2)$	
$\mathtt{S^*S^-A^+}$	$M_{\mathtt{S^*S^-A^+}}$	Example 4.4.3
$\mathtt{S^*S^-A^-}$	$-M_{\mathtt{S^*S^-A^+}}$	Example 4.4.3
$\mathtt{S^+S^*A^-}$	$(-M_{\mathtt{S^*S^-A^+}})^{-1}$	Example 4.4.3
$\mathtt{S^+S^-A^-}$	$M_{\mathtt{S^+S^-A^-}}$	Example 4.4.3
$\mathtt{S^-S^*A^+}$	$-(-M_{\mathtt{S^*S^-A^+}})^{-1}$	Example 4.4.3
$\mathtt{S^-S^-A^+}$	$-M_{\mathtt{S^+S^-A^-}}$	Example 4.4.3
$\mathtt{S^*S^-N}$	$\operatorname{diag}(1, -1, 0)$	
$\mathtt{S^+S^+N}$	$M_{\mathtt{S^+S^+N}}$	Example 4.4.3
$\mathtt{S^+S^-N}$	$M_{\mathtt{S^+S^-N}}$	Example 4.4.3
$\mathtt{S^-S^+N}$	$-M_{\mathtt{S^+S^+N}}$	Example 4.4.3
$\mathtt{S^-S^-N}$	$-M_{\mathtt{S^+S^-N}}$	Example 4.4.3

4.5 Sepr-sequences of real symmetric matrices

This section focuses on real symmetric matrices, and its main result is a complete characterization of the sepr-sequences whose underlying epr-sequence contains `ANA` as a non-terminal subsequence (see Theorem 4.5.6).

Proposition 4.5.1. *For any* `X`*, the sepr-sequence* $\mathtt{NXS^*N}$ *cannot occur in the sepr-sequence of a real symmetric matrix.*

Proof. Let B be a real symmetric matrix with $\operatorname{sepr}(B)$ containing $\mathtt{NXS^*N}$, where the penultimate term of this subsequence occurs in position k. By [6, Proposition 2.4], $\operatorname{rank}(B) = k$. It follows from Lemma 4.3.2 that the nonzero principal minors of order k of B have the same sign, which contradicts our hypothesis. □

Proposition 4.5.2. *Let B be a real symmetric matrix with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$ *and* $\operatorname{sepr}(B) = t_1t_2\cdots t_n$*. Suppose* $\ell_1\ell_2\ell_3 = \mathtt{ANA}$*. Then* $t_1t_2t_3 = \mathtt{A^+NA^-}$ *or* $t_1t_2t_3 = \mathtt{A^-NA^+}$*. Furthermore, the following hold.*

1. *If* $t_1t_2t_3 = \mathtt{A^+NA^-}$*, then* $t_i = \mathtt{A^-}$ *for* $i \geq 4$*.*
2. *If* $t_1t_2t_3 = \mathtt{A^-NA^+}$*, then* $t_i = \mathtt{A^+}$ *for odd* $i \geq 4$*, and* $t_j = \mathtt{A^-}$ *for even* $j \geq 4$*.*

Proof. By [6, Proposition 2.5], B is conjugate by a nonsingular diagonal matrix to one of the matrices $\pm(J_n - 2I_n)$. Since $\operatorname{sepr}(B)$ remains invariant under this type of conjugation, we may assume that $B = \pm(J_n - 2I_n)$. It is now easy to check that $t_1t_2t_3 = \mathtt{A^+NA^-}$ or $t_1t_2t_3 = \mathtt{A^-NA^+}$. We examine each case separately.

Case 1: $t_1t_2t_3 = \mathtt{A^+NA^-}$. Hence, $B = -(J_n - 2I_n)$. Let k be an integer with $4 \leq k \leq n$. Observe that any order-k principal submatrix is of the form $-(J_k - 2I_k)$; hence, each order-k principal minor is $2^{k-1}(2-k) < 0$ (the eigenvalues of $-(J_k - 2I_k)$ are $2-k$ and 2, with multiplicity 1 and $k-1$, respectively). It follows that $t_k = \mathtt{A^-}$.

Case 2: $t_1t_2t_3 = \mathtt{A}^-\mathtt{N}\mathtt{A}^+$. Hence, $B = J_n - 2I_n$. The desired conclusion now follows by applying Observation 4.2.2 to the matrix $-B$, which, by Case 1, has $\operatorname{sepr}(B) = \mathtt{A}^+\mathtt{N}\mathtt{A}^-\overline{\mathtt{A}^-}$. □

Corollary 4.5.3. *The sepr-sequence* $\mathtt{A}^-\mathtt{N}\mathtt{A}^+\mathtt{A}^+$ *cannot occur as a subsequence of the sepr-sequence of a real symmetric matrix.*

Proof. If a real symmetric matrix existed with an sepr-sequence containing $\mathtt{A}^-\mathtt{N}\mathtt{A}^+\mathtt{A}^+$, then, by the Inheritance Theorem, it would contain a principal submatrix whose sepr-sequence ends with $\mathtt{A}^-\mathtt{N}\mathtt{A}^+\mathtt{A}^+$, and whose inverse has sepr-sequence $\mathtt{A}^+\mathtt{N}\mathtt{A}^- \cdots \mathtt{A}^+$ (see the Inverse Theorem), which would contradict Proposition 4.5.2. □

Corollary 4.5.4. *Let B be a real symmetric matrix with* $\operatorname{sepr}(B) = t_1t_2\cdots t_n$, *and let k be an integer with $k \leq n-2$. Then the following hold.*

1. *If $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^+\mathtt{N}\mathtt{A}^-\mathtt{A}^+$, then k is odd.*

2. *If $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^-\mathtt{N}\mathtt{A}^+\mathtt{A}^-$, then k is even.*

Proof. Suppose $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^+\mathtt{N}\mathtt{A}^-\mathtt{A}^+$. If k were even, then, by Observation 4.2.2, $\operatorname{sepr}(-B)$ would contain $\mathtt{A}^-\mathtt{N}\mathtt{A}^+\mathtt{A}^+$, which would contradict Proposition 4.5.3. That establishes Statement (1). Statement (2) is proven similarly. □

Lemma 4.5.5. *Let B be a real symmetric matrix with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$ *and* $\operatorname{sepr}(B) = t_1t_2\cdots t_n$. *Suppose $\ell_{k-1}\ell_k\ell_{k+1} = \mathtt{ANA}$, where $k \leq n-2$. Then $t_i \in \{\mathtt{A}^+, \mathtt{A}^-\}$ for all $i \neq k$ and* $t_{k+1} = \operatorname{neg}(t_{k-1})$.

Proof. If $k = 2$, then all the conclusions are immediate from Proposition 4.5.2; thus, we assume that $k \geq 3$. By [6, Theorem 2.6], $t_i \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$ for all $i \neq k$. By Theorem 4.3.5, and because $\mathtt{A}^*\mathtt{N}$ and $\mathtt{N}\mathtt{A}^*$ are prohibited, $t_{k-1}t_kt_{k+1} = \mathtt{A}^+\mathtt{N}\mathtt{A}^-$ or $t_{k-1}t_kt_{k+1} = \mathtt{A}^-\mathtt{N}\mathtt{A}^+$; hence, $t_{k+1} = \operatorname{neg}(t_{k-1})$. We now show by contradiction that $t_i \neq \mathtt{A}^*$ for all $i \neq k$; thus, suppose $t_j = \mathtt{A}^*$ for some $j \neq k$. We proceed by examining two cases.

Case 1: $j < k$. Since $t_{k+2} \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$, the Inheritance Theorem implies that B has a (necessarily nonsingular) $(k+2) \times (k+2)$ principal submatrix B' with $\operatorname{sepr}(B') = \cdots \mathtt{A}^* \cdots \mathtt{XN}\operatorname{neg}(\mathtt{X})\mathtt{Y}$, where $\mathtt{X}, \mathtt{Y} \in \{\mathtt{A}^+, \mathtt{A}^-\}$. By the Inverse Theorem, $\operatorname{sepr}((B')^{-1}) = \mathtt{ZN}\operatorname{neg}(\mathtt{Z}) \cdots \mathtt{A}^* \cdots$, where $\mathtt{Z} \in \{\mathtt{A}^+, \mathtt{A}^-\}$; now observe that this contradicts Proposition 4.5.2.

Case 2: $j > k$. Since $t_{k-2} \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$, Proposition 4.2.9 implies that there exists a (necessarily nonsingular) $(k-2) \times (k-2)$ principal submatrix $B[\alpha]$ such that the sepr-sequence of $C = B/B[\alpha]$ has $\mathtt{A}^*$ in the $(j-(k-2))$-th position. By the Schur Complement Corollary, $\operatorname{epr}(C) = \mathtt{ANA}\cdots$; hence, by Proposition 4.5.2, $\operatorname{sepr}(C)$ does not contain $\mathtt{A}^*$, which leads to a contradiction. □

We are now in position to completely characterize all the sepr-sequences that are attainable by real symmetric matrices and whose underlying epr-sequence contains $\mathtt{ANA}$ as a non-terminal subsequence.

Theorem 4.5.6. *Let $\sigma = t_1 t_2 \cdots t_n$ be an sepr-sequence whose underlying epr-sequence is $\ell_1 \ell_2 \cdots \ell_n$. Suppose $\ell_{k-1}\ell_k\ell_{k+1} = \mathtt{ANA}$, where $2 \le k \le n-2$. Let $\alpha = \{1, \ldots, n-1\} \setminus \{k-1, k\}$. Then σ is attainable by a real symmetric matrix if and only if one of the following holds.*

1. *$\sigma = \overline{\mathtt{A}^+}\mathtt{A}^+\mathtt{N}\mathtt{A}^-\mathtt{A}^-\overline{\mathtt{A}^-}$;*
2. *k is odd, $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^+\mathtt{N}\mathtt{A}^-\mathtt{A}^+$ and $t_{i+1} = \operatorname{neg}(t_i)$ for all $i \in \alpha$;*
3. *k is even, $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^-\mathtt{N}\mathtt{A}^+\mathtt{A}^-$ and $t_{i+1} = \operatorname{neg}(t_i)$ for all $i \in \alpha$.*

Proof. First, we show that if any of Statements (1)–(3) holds, then σ is attainable. Suppose (1) holds. Let $B = -(J_n - kI_n)$. We claim that $\operatorname{sepr}(B) = \sigma$. Obviously, $[\operatorname{sepr}(B)]_1 = \mathtt{A}^+ = t_1$. Since every principal submatrix of order $q \ge 2$ is of the form $-(J_q - kI_q)$, each principal minor of order q is $k^{q-1}(k-q)$. Hence, $[\operatorname{sepr}(B)]_q = \mathtt{A}^+ = t_q$ for $2 \le q \le k-1$, $[\operatorname{sepr}(B)]_k = \mathtt{N} = t_k$, and $[\operatorname{sepr}(B)]_q = \mathtt{A}^- = t_q$ for $k+1 \le q \le n$.

It follows that $\operatorname{sepr}(B) = \sigma$. To show that σ is attainable if Statements (2) or (3) hold, let $C = J_n - kI_n$. Note that each principal minor of order $q \geq 2$ is $(-k)^{q-1}(q-k) = (-k)^q(k-q)$. It is now easy to check that $\operatorname{sepr}(C) = \sigma$, with σ as in Statement (2) or (3), depending on the parity of k.

For the other direction, suppose σ is attainable by a real symmetric matrix, say, B, so that $\operatorname{sepr}(B) = \sigma = t_1t_2\cdots t_n$. By Lemma 4.5.5, $t_i \in \{\mathtt{A}^+, \mathtt{A}^-\}$ for all $i \neq k$ and $t_{k+1} = \operatorname{neg}(t_{k-1})$. It follows that $t_{k-1}t_kt_{k+1} = \mathtt{A}^+\mathtt{NA}^-$ or $t_{k-1}t_kt_{k+1} = \mathtt{A}^-\mathtt{NA}^+$. Since $\mathtt{A}^-\mathtt{NA}^+\mathtt{A}^+$ is prohibited by Corollary 4.5.3, $t_{k-1}t_kt_{k+1}t_{k+2}$ must be either $\mathtt{A}^+\mathtt{NA}^-\mathtt{A}^-$, $\mathtt{A}^+\mathtt{NA}^-\mathtt{A}^+$ or $\mathtt{A}^-\mathtt{NA}^+\mathtt{A}^-$. We now examine these three possibilities in two cases.

Case i: $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^+\mathtt{NA}^-\mathtt{A}^-$. We now show that $\operatorname{sepr}(B) = \overline{\mathtt{A}^+}\mathtt{A}^+\mathtt{NA}^-\mathtt{A}^-\overline{\mathtt{A}^-}$. We start by showing that $t_i = \mathtt{A}^+$ for all $i \leq k-2$. Suppose to the contrary that there exists $j \leq k-2$ such that $t_j = \mathtt{A}^-$. By the Inheritance and Inverse Theorems, the sepr-sequence of the inverse of any (necessarily nonsingular) $(k+2)\times(k+2)$ principal submatrix of B has the form $\mathtt{A}^+\mathtt{NA}^-\cdots\mathtt{A}^+\cdots$, which contradicts Proposition 4.5.2. We conclude that $t_i = \mathtt{A}^+$ for all $i \leq k-2$. Now we show that $t_i = \mathtt{A}^-$ for all $i \geq k+3$. Suppose to the contrary that there exists $j \geq k+3$ such that $t_j = \mathtt{A}^+$. Then, as every principal minor of order $k-2$ is positive, the Schur Complement Theorem and the Schur Complement Corollary imply that for any (necessarily nonsingular) $(k-2)\times(k-2)$ principal submatrix $B[\alpha]$, $\operatorname{sepr}(B/B[\alpha]) = \mathtt{A}^+\mathtt{NA}^-\cdots\mathtt{A}^+\cdots$, which contradicts Proposition 4.5.2. We conclude that $\operatorname{sepr}(B) = \overline{\mathtt{A}^+}\mathtt{A}^+\mathtt{NA}^-\mathtt{A}^-\overline{\mathtt{A}^-}$. Then, as $\sigma = \operatorname{sepr}(B)$, Statement (1) holds. Note that we have shown that if the sepr-sequence of a real symmetric matrix contains $\mathtt{A}^+\mathtt{NA}^-\mathtt{A}^-$, then its sepr-sequence must be $\overline{\mathtt{A}^+}\mathtt{A}^+\mathtt{NA}^-\mathtt{A}^-\overline{\mathtt{A}^-}$.

Case ii: $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^+\mathtt{NA}^-\mathtt{A}^+$ or $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^-\mathtt{NA}^+\mathtt{A}^-$. By Corollary 4.5.4, k is odd if $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^+\mathtt{NA}^-\mathtt{A}^+$, and k is even if $t_{k-1}t_kt_{k+1}t_{k+2} = \mathtt{A}^-\mathtt{NA}^+\mathtt{A}^-$. Thus, it remains to show that $t_{i+1} = \operatorname{neg}(t_i)$ for all $i \in \alpha$, from which it would follow that either Statement (2) or Statement (3) holds. Suppose to the contrary that $t_{i+1} \neq \operatorname{neg}(t_i)$ for some $i \in \alpha$; hence, $t_{i+1} = t_i$. Let $\operatorname{sepr}(-B) = t'_1t'_2\cdots t'_n$. It follows from Observation

4.2.2 that $t'_{k-1}t'_kt'_{k+1}t'_{k+2} = \mathtt{A}^{+}\mathtt{NA}^{-}\mathtt{A}^{-}$ and that $t'_{i+1} = \operatorname{neg}(t'_i)$. Now, observe that the last sentence at the end of Case i implies that $\operatorname{sepr}(-B) = \overline{\mathtt{A}^{+}}\mathtt{A}^{+}\mathtt{NA}^{-}\mathtt{A}^{-}\overline{\mathtt{A}^{-}}$. Hence, $t'_{i+1} = t'_i$. Then, as $t'_{i+1} = \operatorname{neg}(t'_i)$, we must have $t'_{i+1} = t'_i = \mathtt{N}$, a contradiction. We conclude that it must be the case that $t_{i+1} = \operatorname{neg}(t_i)$ for all $i \in \alpha$, implying that either Statement (2) or Statement (3) holds. □

To see that Theorem 4.5.6 cannot be generalized to Hermitian matrices, the reader is referred to [7, Theorem 3.3]. Moreover, Theorem 4.5.6 cannot be generalized to include the case when `ANA` occurs as a terminal sequence, since the epr-sequence `SAANA` is attainable by a real symmetric matrix (see [4, Table 5]).

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CHAPTER 5. GENERAL CONCLUSIONS

In Chapter 2, new restrictions for the attainability of epr-sequences by the class of real symmetric matrices were established, which allowed for a classification of two families of sequences that are attainable by real symmetric matrices: the family of pr-sequences not containing three consecutive 1s and the family of epr-sequences containing an `N` in every subsequence of length 3. In the context of real symmetric matrices, the classification of the latter family served as an attempt to understand those epr-sequences that are not covered by Theorem 1.3.5, namely those that are allowed to contain the subsequences `NA` and `NS`, which, as was argued above, remain difficult to understand.

In Chapter 3, the question of whether there were fields over which the existing tools could allow one to obtain a complete characterization of the epr-sequences that are attainable by symmetric matrices was considered, where it was shown that such a characterization was indeed possible for the field $\mathbb{Z}_2$. Thanks to this characterization, the principal minor assignment problem for symmetric matrices over $\mathbb{Z}_2$ can be reduced as follows: For each *attainable* epr-sequence $\ell_1\ell_2\cdots\ell_n$ containing one or more `S`s, determine what integers can be assigned to each `S` in order to guarantee the existence of a matrix, B, for which the following two conditions hold: (i) $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$; (ii) if $\ell_k = \texttt{S}$ and the number assigned to ℓ_k is s_k, then the number of nonzero order-k principal minors of B is s_k.

In Chapter 4, a sequence that "enhances" the epr-sequence, the sepr-sequence, was introduced for the class of Hermitian matrices, which provides further aid towards studying the principal minors of Hermitian matrices. The level of "enhancement" was evidenced in part by the fact that the presence of some subsequences of length 2 (namely $\mathtt{A}^*\mathtt{N}$ and $\mathtt{NA}^*$) in an sepr-sequence was shown to imply that the sequence is unattainable. Hence, for some lists of $2^n - 1$ real numbers, there exists some $k \in \{1, 2, \ldots, n-1\}$ such that the non-realizability of these numbers as the principal minors of a Hermitian matrix can be deduced based on just $\binom{n}{k} + \binom{n}{k+1}$ of their numbers. Epr-sequences are not capable of being this efficient, since any sequence of length 2 can appear in an attainable epr-sequence (in the case of symmetric or Hermitian matrices).

In the context of Hermitian matrices, there are only three sequences of length 3 whose presence in an epr-sequence always implies that the epr-sequence is not attainable: `NNA`, `NNS` and `NSA` [1]. However, as we saw in Chapter 4, in the case of sepr-sequences, there are numerous sequences of length 3 that lead to the same conclusion (that is true even if we do not consider those sequences whose underlying epr-sequence is `NNA`, `NNS` or `NSA`). This is obviously further aid for the study of principal minors of Hermitian matrices.

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