ACTA ARITHMETICA
214 (2024)

Character sums and the Riemann Hypothesis

by

Brian Conrey (Pasadena, CA)

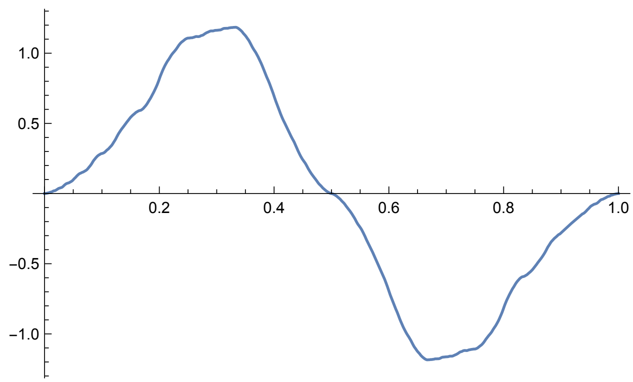
Dedicated to Henryk on his semisesquicentennial

Abstract. We prove that an innocent looking inequality implies the Riemann Hypothesis and show a way to approach this inequality through sums of Legendre symbols.

Introduction. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{\lambda(n)\sin 2\pi nx}{n^2}$$

where λ is the Liouville lambda-function (1). Since $|\lambda(n)| = 1$, this series is absolutely convergent for real x, so that f is continuous, odd and periodic with period 1 on $\mathbb{R}$. Here is a plot of $f(x)$ for $0 \leqslant x \leqslant 1$ using 1000 terms of the series defining f:



2020 *Mathematics Subject Classification*: Primary 11L40; Secondary 11M26.
Key words and phrases: Riemann Hypothesis, Dirichlet L-functions, Dirichlet characters.
Received 30 May 2023; revised 13 November 2023.
Published online 11 April 2024.

(1) λ is completely multiplicative and takes the value -1 on primes so that $\lambda(p_1^{e_1} \dots p_r^{e_r}) = (-1)^{e_1+\dots+e_r}$.

DOI: 10.4064/aa230530-13-11

Theorem 1. *If $f(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/4$, then the Riemann Hypothesis is true.*

Theorem 1 is deceptive in that it looks like it should be a simple matter to prove that $f(x)$ is non-negative. A problem is that it is not clear whether $f(x)$ is differentiable or not, and even if it is, it would be difficult to estimate the derivative. So, proving that $f(x) > 0$ at some point does not immediately tell us about $f(x)$ at nearby points.

The "1/4" in Theorem 1 can be replaced by any positive constant. So the real issue is trying to prove that $f(x) > 0$ for small positive x.

Note that

$$\left|\sum_{n=N+1}^{\infty} \frac{\lambda(n)\sin 2\pi nx}{n^2}\right| < \int_N^{\infty} u^{-2}\,du = \frac{1}{N}$$

so that if for some x there is an N such that

$$\sum_{n=1}^{N} \frac{\lambda(n)\sin 2\pi nx}{n^2} \geqslant \frac{1}{N} \tag{1}$$

then it must be the case that $f(x) > 0$. We will use this idea a little later.

We can give an "explicit formula" for f in terms of the zeros $\rho = \beta + i\gamma$ of ζ:

Theorem 2. *Assuming the Riemann Hypothesis,*

$$f(x) = -\frac{4\pi^2 x^{3/2}}{3\zeta(1/2)} - \frac{8\pi^2}{3} x^{3/2} \sum_{n\leqslant 4x} \frac{\ell(n)}{\sqrt{n}}\left(1 - \frac{n}{4x}\right)^{3/2}$$
$$+ \pi \lim_{T\to\infty} \sum_{\substack{\rho=1/2+i\gamma\\ |\gamma|\leqslant T}} \operatorname*{Res}_{z=\rho-1} \frac{X(1-z)\zeta(2z+2)x^{1-z}}{(1-z)\zeta(z+1)}.$$

Here $\ell(n)$ is defined through its generating function

$$\sum_{n=1}^{\infty} \ell(n) n^{-s} = \frac{\zeta(2s-1)}{\zeta(s)}$$

for $\Re s > 1$. Also, $X(s)$ is the factor from the functional equation for $\zeta(s)$ which can be defined by

$$X(s)^{-1} = X(1-s) = \frac{\zeta(1-s)}{\zeta(s)} = 2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}.$$

Note that if the zeros of $\zeta(s)$ are simple, then the term with the sum over the zeros of ζ becomes

$$\pi \sum_{\rho} \frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)}.$$

Theorem 2 is nearly a converse to Theorem 1 in the sense that if RH is true and all the zeros are simple and

$$\sum_{\rho}\left|\frac{X(2-\rho)\zeta(2\rho)}{(2-\rho)\zeta'(\rho)}\right| \leqslant -\frac{4\pi}{3\zeta(1/2)} \tag{2}$$

then $f(x)\geqslant 0$ for $0\leqslant x\leqslant 1/4$. Note that

$$-\frac{4\pi}{3\zeta(1/2)} = 2.86834\ldots \quad \text{and} \quad \sum_{|\gamma|\leqslant 1000}\left|\frac{X(2-\rho)\zeta(2\rho)}{(2-\rho)\zeta'(\rho)}\right| = 0.264954\ldots$$

so that the inequality (2) seems plausible.

Finally, we remark that the formula of Theorem 2 for $f(x)$ hides very well the fact that $f(x)$ is periodic with period 1!

1. Prior results. There has been quite a lot of work connecting partial weighted sums of the Liouville lambda-function and the Riemann Hypothesis. We refer to [BFM] for a nice description of past work. In that paper the authors prove that the smallest value of x for which

$$\sum_{n\leqslant x}\frac{\lambda(n)}{n} < 0$$

is $x = 72185376951205$.

2. Character sums. A possible approach to proving that $f(x) > 0$ for small $x > 0$ lies in the fact that λ is completely multiplicative and takes the values ± 1. This scenario resembles quadratic Dirichlet characters (for simplicity think Legendre symbols) except that Dirichlet characters can also take the value 0. By the Chinese Remainder Theorem, for any N there is a prime number q such that $\lambda(n) = \left(\frac{n}{q}\right)$ for all $n\leqslant N$, where $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol ([2]) modulo q. As an example,

$$\lambda(n) = \left(\frac{n}{163}\right)$$

for all $n\leqslant 40$, but they differ at $n = 41$.

Let

$$f_q(x) = \sum_{n=1}^{\infty}\frac{\left(\frac{n}{q}\right)\sin 2\pi n x}{n^2}$$

be the Fourier sine series with $\lambda(n)$ replaced by $\left(\frac{n}{q}\right)$. If $f_q(x)\geqslant 0$ for $0\leqslant x\leqslant 1/4$ for a sufficiently large set of q, then it must also be the case that $f(x)\geqslant 0$ for $0\leqslant x\leqslant 1/4$. (The proof is that if $f(x_0) < 0$ for some $0 < x_0 < 1/4$,

([2]) $\left(\frac{n}{q}\right) = 0$ if $(n,q) > 1$; $\left(\frac{n}{q}\right) = +1$ if n is a square modulo q; and $\left(\frac{n}{q}\right) = -1$ if n is not a square modulo q.

then we can find a q such that $\left(\frac{n}{q}\right) = \lambda(n)$ for all $n \leqslant N$ where N is chosen so large that $|f(x_0)| > 1/N$; then it must be the case by the analogue of (1) for f_q that $f_q(x_0) < 0$.) The same assertion but with q restricted to primes congruent to 3 modulo 8 is also valid, since the Legendre symbols for these q can also imitate $\lambda(n)$ for arbitrarily long stretches $1 \leqslant n \leqslant N$. We can express this as follows:

Theorem 3. *If*

$$f_q(x) \geqslant 0$$

for all $0 \leqslant x \leqslant 1/4$ *and all primes* q *congruent to* 3 *modulo* 8, *then the Riemann Hypothesis is true.*

Remark 1. We could just as well have stated this theorem for $q \equiv 3 \bmod 4$. However, the intention is that we are interested in q for which χ_q imitates λ. Insisting that $\chi_q(2) = -1$ leads to the condition that $q \equiv 3 \bmod 8$.

The sums $f_q(x)$ still have the same problem in that it is tricky to prove for sure that they are positive for small positive x. However, the analogue of Theorem 2 above is much simpler, is unconditional, and leads to a straightforward way to check, for any given fixed q, that $f_q(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/4$.

Theorem 4. *Let* $x \geqslant 0$. *Let* $q \equiv 3 \bmod 8$ *be squarefree. Then*

$$f_q(x) = 2\pi x L_q(1) - \frac{2\pi^2 x}{\sqrt{q}} \sum_{n \leqslant xq} \left(\frac{n}{q}\right)\left(1 - \frac{n}{xq}\right)$$

where

$$L_q(1) = \sum_{n=1}^{\infty} \frac{\left(\frac{n}{q}\right)}{n}.$$

Now Dirichlet's class number formula enters the picture. Let $K = \mathbb{Q}(\sqrt{-q})$ be the imaginary quadratic field obtained by adjoining $\sqrt{-q}$ to the rationals $\mathbb{Q}$. Let $h(q)$ be the class number ([3]) of K. Then Dirichlet's formula reads

$$h(q) = \frac{\sqrt{q}}{\pi} L_q(1)$$

for squarefree $q \equiv 3 \bmod 4$ and $q > 3$ (see [D] or [IK]). Thus, the theorem above can be rephrased in terms of $h(q)$. Moreover, we can express $L_q(1)$ as a finite character sum:

$$L_q(1) = -\frac{\pi}{q^{3/2}} \sum_{n=1}^{q} n \left(\frac{n}{q}\right).$$

([3]) The class number is a measure of how close to unique factorization the integers of K are; $h(q) = 1$ means the integers of K can be factored into primes in only one way.

Since $\left(\frac{n}{q}\right)$ is an odd function of q, we also have

$$L_q(1) = -\frac{2\pi}{q^{3/2}} \sum_{n=1}^{(q-1)/2} n \left(\frac{n}{q}\right)$$

and

$$h(q) = S_q\left(\frac{q}{2}\right) \quad \text{where} \quad S_q(N) := \sum_{n \leqslant N} \left(\frac{n}{q}\right)\left(1 - \frac{n}{N}\right).$$

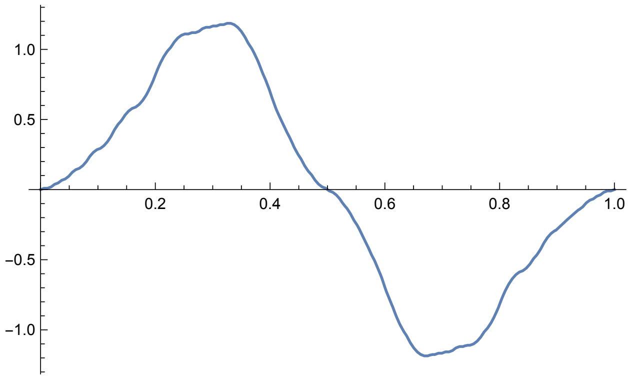
Corollary 1. *Let* $q > 3$ *be squarefree with* $q \equiv 3 \bmod 8$. *Then*

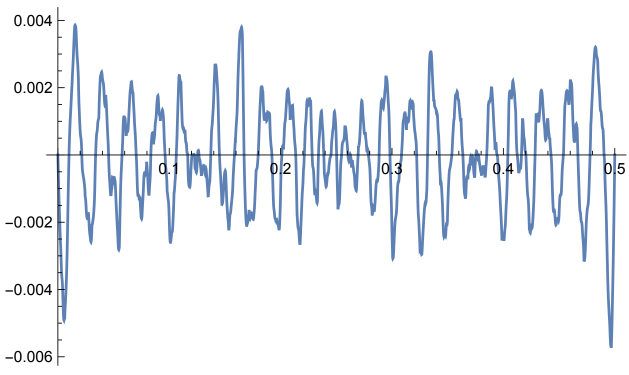
$$f_q(x) = \frac{2\pi^2 x}{\sqrt{q}} \left(S_q\left(\frac{q}{2}\right) - S_q(qx)\right).$$

Here is a plot of

$$f_{163}(x) = \frac{2\pi^2 x}{\sqrt{163}} \left(S_{163}\left(\frac{163}{2}\right) - S_{163}(163x)\right)$$

for $0 \leqslant x \leqslant 1$ and a plot of the difference $f(x) - f_{163}(x)$:





We can use the corollary to prove that $f_{163}(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/2$ and consequently that $f(x) \geqslant 0$ for $1/4 > x \geqslant 0.043$ as follows:

$$f(x)=\sum_{n=1}^{40}\frac{\lambda(n)\sin 2\pi nx}{n^2}+\frac{\Theta}{40}=\sum_{n=1}^{40}\frac{\chi_{163}(n)\sin 2\pi nx}{n^2}+\frac{\Theta}{40}$$
$$=f_{163}(x)+\frac{\Theta}{20}=\frac{2\pi^2x}{\sqrt{163}}\left(S_{163}\left(\frac{163}{2}\right)-S_{163}(163x)\right)+\frac{\Theta}{20}$$

where Θ denotes a number with absolute value at most 1, not necessarily the same at each occurrence. Now for a an integer, $S_{163}(163x)$ is constant for x in the interval $\left[\frac{a}{163},\frac{a+1}{163}\right)$. Therefore, $f_{163}(x)\geqslant\min\left\{f_{163}\left(\frac{a}{163}\right),f_{163}\left(\frac{a+1}{163}\right)\right\}$ for x in this interval. We can tabulate these values:

a	1	2	3	4	5	6	7	8	9	10
$f_{163}\left(\frac{a}{163}\right)$	0.0095	0.0095	0.019	0.038	0.047	0.066	0.076	0.095	0.12	0.14

Since $\frac{\Theta}{20}\leqslant 0.05$, it follows from (1) that $f(x)\geqslant 0$ for $0.25\geqslant x\geqslant\frac{7}{163}=0.043$.

Corollary 2. *$f(x)\geqslant 0$ for $0.043\leqslant x\leqslant 0.25$.*

It seems clear that for any given $\epsilon>0$ we could replace 0.043 by ϵ in this inequality with enough computation time. Also, if we use Euler products instead of Dirichlet series, we can show that $f(x)\geqslant 0$ for $1/4\geqslant x\geqslant 0.011$.

The following conjecture seems surprising.

Conjecture 1. *If $q\equiv 3\bmod 8$ is squarefree, then $f_q(x)\geqslant 0$ for $0\leqslant x\leqslant 1/2$.*

Remark 2. J. Bober has checked that this inequality is true for all primes $q\equiv 3\bmod 8$ up to 10^9.

Now we turn to the proofs.

3. Useful lemmas

Lemma 1. *For $y>0$ we have*

$$\frac{1}{2\pi i}\int_{(c)}\frac{X(1-s)y^{1-s}}{1-s}\,ds=\frac{\sin 2\pi y}{\pi}$$

for any c satisfying $0<c<1$ where (c) denotes the path from $c-i\infty$ to $c+i\infty$.

The integrand has simple poles at $s=0,-2,-4,\dots$ with the residue at $s=-2n$ equal to

$$\frac{1}{\pi}\,\frac{(-1)^n(2\pi y)^{2n+1}}{(2n+1)!}.$$

Summing these leads to the desired formula. See also [T1]; the above is the integral of formula (7.9.5) in [T1].

LEMMA 2. *If $c > 0$ and $\Re a > 0$, then*

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s)\Gamma(a)}{\Gamma(s+a)} x^{-s}\, ds = \begin{cases} (1-x)^{a-1} & \text{if } 0 < x < 1, \\ 0 & \text{if } x \geqslant 1. \end{cases}$$

This is formula (7.7.14) of [T1].

LEMMA 3. *If $c > 0$, then*

$$\frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s(s+1)}\, ds = \begin{cases} 1 - \frac{1}{x} & \text{if } x > 1, \\ 0 & \text{if } 0 < x \leqslant 1. \end{cases}$$

This lemma is well-known and is easy to verify.

4. Proofs of theorems

Proof of Theorem 1. This assertion is a consequence of Landau's Theorem: "If $g(n) \geqslant 0$ then the rightmost singularity of $\sum_{n=1}^{\infty} g(n)n^{-s}$ is real." This is Theorem 10 of [HR] and Theorem 1.7 of [MV2]. What we actually need is an integral version of this theorem: "If $g(x) \geqslant 0$ then the rightmost singularity of $\int_1^{\infty} g(x)x^{-s}\, dx$ is real." The proof of this version is essentially the same as that of the first version (see [MV2, Lemma 15.1]). The application to our situation is slightly subtle. We argue as follows. Since

$$\sum_{n=1}^{\infty} \lambda(n)n^{-s} = \frac{\zeta(2s)}{\zeta(s)},$$

it follows from Lemma 1 that

$$\frac{f(x)}{\pi} = \frac{1}{2\pi i}\int_{(c)} \frac{X(1-s)}{1-s}\,\frac{\zeta(2s+2)}{\zeta(s+1)} x^{1-s}\, ds$$

where $0 < c < 1$. The integral is absolutely convergent for $0 < c < 1/2$. By Mellin inversion we have

$$\frac{\pi X(1-s)}{1-s}\,\frac{\zeta(2s+2)}{\zeta(s+1)} = \int_0^{\infty} f(x)x^{s-2}\, dx.$$

We split the integral into two integrals at $x = 4$ so that

$$\frac{\pi X(1-s)}{1-s}\,\frac{\zeta(2s+2)}{\zeta(s+1)} = \int_0^4 f(x)x^{s-2}\, dx + \int_4^{\infty} f(x)x^{s-2}\, dx = I_1(s) + I_2(s),$$

say. The integral defining $I_1(s)$ is absolutely convergent for $\sigma > 1$ and the second integral is absolutely convergent for $\sigma < 1$. Using the periodicity of f we can show that the second integral converges for $\sigma < 2$. Indeed, let

$$F(x) = \int_0^x f(t)\, dt.$$

Then $F(n) = 0$ for all integers n and F is bounded. Therefore,

$$\begin{aligned} I_2(s) &= \sum_{n=4}^{\infty} \int_n^{n+1} f(x)x^{s-2}\,dx \\ &= \sum_{n=4}^{\infty} \Big(F(x)x^{s-2}\big|_{x=n}^{x=n+1} - (s-2)\int_n^{n+1} F(x)x^{s-3}\,dx \Big) \\ &= -(s-2)\int_4^{\infty} F(x)x^{s-3}\,dx. \end{aligned}$$

This integral converges for $\Re s < 2$. So, we now have I_2 analytic for $\Re s < 2$. Clearly, $I_1 + I_2$ is analytic for $\Re s > \max\{-1/2, \rho - 1\}$, i.e. for $\Re s > 0$. (The pole of $X(1-s)$ at $s = 0$ is canceled by the zero of $1/\zeta(s+1)$ at $s = 0$.) It follows that $I_1(s) = (I_1(s) + I_2(s)) - I_2(s)$ is analytic for $\Re s > 0$. Hence $I_2(s)$ is also analytic for $\Re s > 0$, and since we already knew it was analytic for $\Re s < 2$, it follows that $I_2(s)$ is entire. Now, we can write I_1 as

$$I_1(s) = \int_{1/4}^{\infty} f(1/x)x^{-s}\,dx.$$

Recall we have assumed that $f(1/x) \geqslant 0$ for $x \geqslant 4$. Therefore, by Landau's Theorem, the rightmost singularity of $I_1(s)$ is real. Since I_2 is entire, it follows that the rightmost pole of $I_1(s) + I_2(s)$ must also be real. But the rightmost real pole of

$$I_1(s) + I_2(s) = \frac{\pi X(1-s)}{1-s}\,\frac{\zeta(2s+2)}{\zeta(s+1)}$$

is at $s = -1/2$. This must be the rightmost pole. Therefore the poles at $\rho - 1$ must all have their real parts less than or equal to $-1/2$. In particular, $\Re\rho \leqslant 1/2$, which is RH.

Proof of Theorem 2. We start again from

$$\frac{f(x)}{\pi} = \frac{1}{2\pi i}\int_{(c)} \frac{X(1-s)\zeta(2s+2)x^{1-s}}{(1-s)\zeta(s+1)}\,ds$$

where $0 < c < 1/2$. The integrand has poles only at $s = -1/2$ and at $s = \rho - 1$ where ρ is a complex zero of $\zeta(s)$, and nowhere else in the s-plane. The residue at $s = -1/2$ is

$$\frac{X\left(\frac{3}{2}\right)}{\frac{3}{2}\zeta\left(\frac{1}{2}\right)}x^{3/2} = -\frac{4\pi}{3\zeta\left(\frac{1}{2}\right)}x^{3/2}.$$

Assuming that the zeros are simple, the residue at $s = \rho - 1$ is

$$\frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)}.$$

We (carefully) move the path of integration to (c) where $-2 < c < -1$. To do this we have to cross through a field of poles arising from the zeros of the zeta-function. We use Theorem 14.16 of [T1] (see also [R]) to find a path on which $1/\zeta(s+1) \ll T^{\epsilon}$ where we can safely cross. Using the bounds $|X(1-s)| \ll T^{\sigma-1/2}$ and $\zeta(2s+2) \ll T^{-1/2-\sigma}$ we can get the sum of the residues arising from the zeros up to height T together with an error term that tends to 0 as $T \to \infty$. Thus, assuming the zeros are simple,

$$\begin{aligned}\frac{f(x)}{\pi} &= -\frac{4\pi x^{3/2}}{3\zeta(1/2)} + \sum_{\rho} \frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)} \\ &\quad + \frac{1}{2\pi i}\int_{(c)} \frac{X(1-s)\zeta(2s+2)x^{1-s}}{(1-s)\zeta(s+1)}\,ds.\end{aligned}$$

If the zeros are not simple, we modify the sum over zeros appropriately. We make the change of variable $s \mapsto -s$ in the integral. Using the functional equation for the ζ-function and functional relations for the Γ-function, we see that the new integrand is

$$\frac{X(1+s)\zeta(2-2s)x^{1+s}}{(1+s)\zeta(1-s)} = -\pi^{3/2}2^{2s}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+2)}\,\frac{\zeta(2s-1)}{\zeta(s)}x^{1+s}.$$

By Lemma 2,

$$\frac{1}{2\pi i}\int_{(c)} \pi^{3/2}2^{2s}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+2)}\,\frac{\zeta(2s-1)}{\zeta(s)}x^{1+s} = \frac{8\pi}{3}x^{3/2}\sum_{n\leqslant 4x}\frac{\ell(n)}{\sqrt{n}}\left(1-\frac{n}{4x}\right)^{3/2}.$$

Then Theorem 2 follows.

Proof of Theorem 4. We denote $\chi_q(n) = \left(\frac{n}{q}\right)$. By Lemma 1,

$$f_q(x) = \frac{\pi}{2\pi i}\int_{(c)} L(s+1,\chi_q)X(1-s)x^{1-s}\,\frac{ds}{1-s} \tag{3}$$

where $0 < c < 1$. Since χ_q is odd, we find that the integrand has a pole at $s = 0$ and nowhere else in the complex plane. We move the path of integration to (c) where $c < -1$ to see that

$$f_q(x) = 2\pi x L(1,\chi_q) + \frac{\pi}{2\pi i}\int_{(c)} L(s+1,\chi_q)X(1-s)x^{1-s}\,\frac{ds}{1-s}.$$

Now let $s \mapsto -s$ in the integral and use the functional equation (see [D], [IK] or [MV2])

$$L(1-s,\chi_q) = 2q^{s-1/2}(2\pi)^{-s}\Gamma(s)\sin\frac{\pi s}{2}L(s,\chi_q).$$

After simplification, the integral above is

$$\frac{-2\pi^2}{2\pi i}\int_{(c)} q^{s-1/2}x^{1+s}L(s,\chi_q)\,\frac{ds}{s(s+1)}.$$

By Lemma 3, this integral is

$$\frac{-2\pi^2 x}{\sqrt{q}}\sum_{n\leqslant xq}\chi_q(n)\Big(1-\frac{n}{xq}\Big).$$

The proof of Theorem 4 is complete.

REMARK 3. Note that the non-negativity, for $0<x<1/4$, of the right-hand side of (3) implies the Riemann Hypothesis. This condition only involves Dirichlet L-functions with quadratic characters. Thus, information solely about Dirichlet L-functions potentially gives the Riemann Hypothesis. This example shows that different L-functions somehow know about each other.

5. Further remarks. Since

$$h(q)\gg_\epsilon q^{1/2-\epsilon},$$

we see that

$$f_q(x)\geqslant 0 \quad \text{for } a\ll x\ll q^{-1/2-\epsilon}.$$

In particular,

$$f_q(a/q)\geqslant 0 \quad \text{for } a\ll q^{1/2-\epsilon}.$$

But this does not give information about $f(x)$.

Also, the Pólya–Vinogradov inequality tells us that

$$\max_N\Big|\sum_{n=1}^{N}\chi_q(n)\Big|\ll q^{1/2}\log q$$

and the work of Montgomery and Vaughan [MV1] shows that the Riemann Hypothesis for $L(s,\chi)$ implies that

$$\max_N\Big|\sum_{n=1}^{N}\chi_q(n)\Big|\ll q^{1/2}\log\log q.$$

Moreover, it is known that the right-hand side here cannot be replaced by any function that goes to infinity slower. It is also known, assuming the Riemann Hypothesis for $L(s,\chi)$, that

$$L(1,\chi)\ll\log\log q.$$

Our desired inequality can be expressed in terms of $L(1,\chi)$ as

$$\max_{N\leqslant q/4}\sum_{n=1}^{N}\chi(n)\Big(1-\frac{n}{N}\Big)\leqslant\frac{\sqrt{q}}{\pi}L(1,\chi). \tag{4}$$

It appears that both sides of this inequality can be as big as $\sqrt{q}\log\log q$.

A question is whether the converse of Theorem 1 is true. It might be possible to approach this by showing that the "3/2" derivative of $f(x)$ is positive at $x=0$ so that there is a small interval to the right of 0 for which $f(x)\geqslant 0$. This method, or trying to prove (2) directly, would involve explicit estimates (assuming RH) for $1/\zeta(s)$ in the critical strip; see [MV2, Section 13.2] for a good approach to such explicit estimates.

Finally, we mention that $f(x)$ can be evaluated at a rational number $x=a/q$ as an average involving Dirichlet L-functions $L(s,\chi)$ where χ is a character modulo q.

6. Evaluation of $f_q(a/p)$. Let $p<q$ and $(a,p)=1$. We explicitly evaluate $f_q(a/p)$ as a sum over characters modulo p as follows. We have

$$\begin{aligned} f_q(a/p) &= \sum_{n=1}^{\infty}\frac{\chi_q(n)\sin\frac{2\pi an}{p}}{n^2}=\sum_{n=1}^{\infty}\frac{\chi_q(n)}{n^2}\,\frac{1}{\phi(p)}\Im\Big\{\sum_{\psi \bmod p}\tau(\psi)\overline{\psi}(an)\Big\}\\ &=\frac{1}{\phi(p)}\Im\Big\{\sum_{\psi \bmod p}\tau(\psi)\overline{\psi}(a)\sum_{n=1}^{\infty}\frac{\chi_q(n)\psi(n)}{n^2}\Big\}\\ &=\frac{1}{\phi(p)}\Im\Big\{\sum_{\psi \bmod p}\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\Big\}.\end{aligned}$$

Now, if ψ is even then

$$\overline{\tau(\psi)}=\sum_{n=1}^{p}\overline{\psi(n)}e(-an/p)=\sum_{n=1}^{p}\overline{\psi}(-n)e(an/p)=\sum_{n=1}^{p}\overline{\psi}(n)e(an/p)=\tau(\overline{\psi}),$$

while if ψ is odd then

$$\overline{\tau(\psi)}=-\tau(\overline{\psi}).$$

Thus, for even ψ,

$$\Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})+\tau(\overline{\psi})\psi(a)L(2,\chi_q\psi)\}=0,$$

and for odd ψ,

$$\Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})+\tau(\overline{\psi})\psi(a)L(2,\chi_q\psi)\}=2\Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\}.$$

Therefore, using the fact that $\tau(\chi_p)=i\sqrt{p}$ when $p\equiv 3 \bmod 4$, we have

$$f_q(a/p)=\frac{1}{\phi(p)}\sum_{\substack{\psi \bmod p\\ \psi(-1)=-1\\ \psi^2\neq\psi_0}}\Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\}$$
$$+\,\delta(p\equiv 3 \bmod 4)\frac{\sqrt{p}}{\phi(p)}\Re\{\overline{\psi}(a)L(2,\chi_q\overline{\psi})\}.$$

We use this to prove that

$$f_q(1/3)>0 \quad\text{and}\quad f_q(1/5)>0$$

for any q. By the formula above we have

$$f_q(1/3)=\frac{\sqrt{3}}{2}L(2,\chi_q\chi_3)>0$$

and

$$f_q(1/5)=\frac{2}{\phi(5)}\Im\{(-1.17557+1.90211i)L(2,\chi_q\psi_1)\}=1.9\alpha-1.17\beta$$

where $\psi_1=\{1,i,-i,-1,0\}$ with $\tau(\psi_1)=-1.17557+1.90211i$ and

$$\alpha+i\beta=L(2,\chi_q\psi_1)=1+\frac{\chi_q(2)i}{2^2}-\frac{\chi_q(3)i}{3^2}-\frac{\chi_q(4)}{4^2}+\cdots.$$

Now

$$\alpha\geqslant 1-\frac{1}{4^2}-\frac{1}{5^2}-\cdots=0.716\ldots \quad\text{and}\quad |\beta|<\frac{1}{2^2}+\frac{1}{3^2}+\cdots=0.64\ldots.$$

Thus,

$$f_q(1/5)>0.6.$$

A couple of formulas may help us move forward here. One is that if θ_1 and θ_2 are characters with coprime moduli m_1 and m_2 respectively, then (see [IK, (3.16)])

$$\tau(\theta_1\theta_2)=\theta_1(m_2)\theta_2(m_1)\tau(\theta_1)\tau(\theta_2).$$

The other is that

$$L(1-r,\theta)=-\frac{m^{r-1}}{r}\sum_{b=1}^{m}\theta(b)B_r(b/m)$$

for a character θ modulo m and a positive integer r where B_r is the rth Bernoulli polynomial (see [Wa, Theorem 4.2]). Recall the functional equation (see [D]) for a primitive character θ modulo m:

$$L(1-s,\theta)=\left(\frac{m}{2\pi}\right)^s\Gamma(s)(e^{\pi is/2}+\theta(-1)e^{-\pi is/2})L(s,\overline{\theta})/\tau(\overline{\theta}).$$

It follows that for an even $\theta = \chi_q\psi$, with $q \equiv 3 \bmod 4$ and ψ an odd character modulo p, we have

$$\begin{aligned} L(2, \chi\overline{\psi}) &= -\pi\left(\frac{pq}{2\pi}\right)^{-1} L(-1, \theta)/\tau(\theta) \\ &= -\pi\left(\frac{pq}{2\pi}\right)^{-1} L(-1, \chi\psi)/(\chi(p)\psi(q)\tau(\psi)i\sqrt{q}). \end{aligned}$$

Therefore,

$$\begin{aligned} \Im\{\tau(\psi)\overline{\psi}(a)L(2, \chi_q\overline{\psi})\} &= \Re\left\{\frac{2\pi^2\chi_q(p)\overline{\psi}(aq)}{pq^{3/2}} L(-1, \chi_q\psi)\right\} \\ &= -\Re\left\{\frac{\pi^2\chi_q(p)\overline{\psi}(aq)}{\sqrt{q}} \sum_{b=1}^{pq} \chi_q(b)\psi(b)B_2(b/(pq))\right\}. \end{aligned}$$

We sum this equation over the odd characters modulo p using

$$\begin{aligned} \sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \psi\left(\frac{b}{aq}\right) &= \frac{1}{2} \sum_{\psi \bmod p} \left(\psi\left(\frac{b}{aq}\right) - \psi\left(-\frac{b}{aq}\right)\right) \\ &= \frac{\phi(p)}{2} \begin{cases} 1 & \text{if } b \equiv aq \bmod p, \\ -1 & \text{if } b \equiv -aq \bmod p. \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} &\sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \Im\{\tau(\psi)\overline{\psi}(a)L(2, \chi_q\overline{\psi})\} \\ &= -\frac{\phi(p)}{2} \frac{\pi^2\chi_q(p)}{\sqrt{q}} \Bigg(\sum_{\substack{b \leqslant pq \\ b \equiv aq \bmod p}} \chi_q(a)B_2(b/(pq)) - \sum_{\substack{b \leqslant pq \\ b \equiv -aq \bmod p}} \chi_q(a)B_2(b/(pq)) \Bigg). \end{aligned}$$

Note that

$$B_2(x) = x^2 - x + 1/6.$$

Also,

$$\begin{aligned} \sum_{\substack{b \leqslant pq \\ b \equiv aq \bmod p}} \chi_q(b) - \sum_{\substack{b \leqslant pq \\ b \equiv -aq \bmod p}} \chi_q(b) = 0, \\ \sum_{\substack{b \leqslant pq \\ b \equiv aq \bmod p}} b\chi_q(b) - \sum_{\substack{b \leqslant pq \\ b \equiv -aq \bmod p}} b\chi_q(b) = 0. \end{aligned}$$

Thus,

$$\sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\}$$
$$= -\frac{\pi^2\chi_q(p)}{2p^2q^{5/2}}\Big(\sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}} b^2\chi_q(b) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}} b^2\chi_q(b)\Big).$$

Hence, we have

THEOREM 5. *For primes p and q both congruent to 3 modulo 4 and for $1 \leqslant a < p/2$ we have*

$$f_q(a/p) = -\frac{\pi^2\chi_q(p)}{2p^2q^{5/2}}\Big(\sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}} b^2\chi_q(b) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}} b^2\chi_q(b)\Big).$$

As a consequence we also have

COROLLARY 3. *If*

$$\text{test}_a(p,q) := -\chi_q(p)\Big(\sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}} b^2\chi_q(b) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}} b^2\chi_q(b)\Big) > 0 \tag{5}$$

for all primes $p < q$ congruent to 3 modulo 8 and all $0 < a < p/2$, then the Riemann Hypothesis follows.

We note that by these techniques one can show

THEOREM 6.

$$f_q(a/q) = \frac{\pi^2}{2\sqrt{q}}\bigg(\chi_q(a) - \frac{1}{q^2}\sum_{c=1}^{q-1} c^2(\chi_q(c-a) - \chi_q(c+a))\bigg).$$

When this formula is compared with our earlier formula

$$f_q\left(\frac{a}{q}\right) = \frac{2\pi^2}{q^{3/2}}\bigg(\frac{a}{3}\sum_{n\leqslant \frac{q-1}{2}} \chi_q(n) - \sum_{n=1}^{a}(a-n)\chi_q(n)\bigg),$$

we deduce the identity

$$\frac{a}{3}\sum_{n\leqslant (q-1)/2} \chi_q(n) - \sum_{n=1}^{a}(a-n)\chi_q(n)$$
$$= \frac{q}{4}\bigg(\chi_q(a) - \frac{1}{q^2}\sum_{c=1}^{q-1} c^2(\chi_q(c-a) - \chi_q(c+a))\bigg)$$

for $q \equiv 3 \bmod 4$.

Now we indicate another possible direction.

Proposition 1. *If*

$$f_q(x) = 0$$

then x is a rational number.

Proof. By Corollary 1, $f_q(x) = 0$ implies that $S_q(q/2) - S_q(qx) = 0$. But $S_q(q/2) = h(q)$ is an integer. So $f_q(x) = 0$ implies that $S_q(qx)$ is a rational number. Now

$$S_q(qx) = \sum_{n \leqslant [qx]} \chi_q(n)\left(1 - \frac{n}{qx}\right) = \sum_{n \leqslant [qx]} \chi_q(n) - \frac{\sum_{n \leqslant [qx]} n\chi_q(n)}{qx}.$$

This has the shape integer $-$ $\frac{\text{integer}}{qx}$, which can only be rational if x is a rational number. ∎

So, it suffices to show that $f_q(x)$ has no rational zeros; perhaps a congruence argument could work. However, Theorem 5 is not of much use here because the hypothetical x for which $f_q(x) = 0$ would likely have a denominator that is divisible by q, so the conditions of Theorem 5 do not hold.

We remark that there are rational values of x for which the numerator of $f_q(x)$ is congruent to 0 modulo q; for example

$$f_{19}\left(\tfrac{25}{76}\right) = \tfrac{19}{25}, \quad f_{19}\left(\tfrac{29}{190}\right) = \tfrac{19}{29}, \quad f_{19}\left(\tfrac{30}{209}\right) = \tfrac{19}{30}.$$

These examples, which all seem to have an x with denominator divisible by q, might be worth studying further.

Here is one final formula that may or may not be useful. Suppose that $f_q(x) = 0$. Let $y = xq$. Then either

$$\sum_{n \leqslant y} \chi_q(n) = h(q) \quad \text{and} \quad \sum_{n \leqslant y} n\chi_q(n) = 0$$

or else

$$y = \frac{\sum_{n \leqslant [y]} n\chi_q(n)}{\sum_{n \leqslant [y]} \chi_q(n) - h(q)}.$$

The first alternative seems unlikely as in that case there would be an interval on which $f_q(x)$ would be identically 0.

7. Conclusion. Conjecture 1 has been checked for primes up to 10^9 and it holds for those primes. However, probabilistic grounds call into question its truth for all primes $q \equiv 3 \bmod 8$. Of course, one only needs its truth for a set of characters χ_q for which $\chi_q(n) = \lambda(n)$ for all $n \leqslant N_q$ where $N_q \to \infty$ with q. Presumably something like this is correct (and should be equivalent to RH), but it is not clear how to proceed. But the results of Section 6 suggest a slightly alternative way forward which may have a more arithmetic flavor.

Acknowledgements. Research supported by an FRG grant from NSF.

References

[BFM] P. Borwein, R. Ferguson, and M. J. Messinghoff, *Sign changes in sums of the Liouville function*, Math. Comp. 77 (2008), no. 263, 1681–1694; also available at https://www.wstein.org/simuw/misc/borwein-parity.pdf.

[D] H. Davenport, *Multiplicative Number Theory*, 3rd ed., Grad. Texts in Math. 74, Springer, New York, 2000.

[HR] G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge Tracts in Math. Math. Phys. 18, Cambridge Univ. Press, Cambridge, 1915.

[IK] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ. 53, Amer. Math. Soc., Providence, RI, 2004.

[MV1] H. L. Montgomery and R. C. Vaughan, *Exponential sums with multiplicative coefficients*, Invent. Math. 43 (1977), 69–82.

[MV2] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge Stud. Adv. Math. 97, Cambridge Univ. Press, Cambridge, 2007.

[R] K. Ramachandra, *On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$, J. London Math. Soc. (2) 8 (1974), 683–690.

[T1] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed., Clarendon Press, Oxford, 1948.

[T2] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Clarendon Press, Oxford Univ. Press, New York, 1986.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Grad. Texts in Math. 83, Springer, New York, 1997.

Brian Conrey
American Institute of Mathematics
Caltech 8-32
Pasadena, CA 91125, USA
E-mail: conrey@aimath.org