An approximation formula for the shifted cubic moment of automorphic *L*-functions in the weight aspect

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Abstract. Consider the family of automorphic *L*-functions associated with primitive cusp forms of level one, ordered by weight k. Assuming that k tends to infinity, we prove a new approximation formula for the cubic moment of shifted *L*-values over this family which relates it to the fourth moment of the Riemann zeta function. More precisely, the formula includes a conjectural main term, the fourth moment of the Riemann zeta function and error terms of size smaller than that predicted by the recipe conjectures.

1 Introduction

This paper is a continuation and generalization of [6], where the cubic moment of central values $L(f,1/2)$ of automorphic *L*-functions associated with primitive cusp forms of level one and large weight has been considered. In the present work, we study the shifted cubic moment, namely,

$$(1.1) \qquad \mathcal{M}(\alpha_1, \alpha_2, \alpha_3) := \sum_{f \in H_{2k}} \omega(f) L(f, 1/2 + \alpha_1) L(f, 1/2 + \alpha_3) L(f, 1/2 + \alpha_3),$$

where H_{2k} is the normalized Hecke basis of the space of holomorphic cusp forms of weight $2k \geq 2$ and level one. The harmonic weight $\omega(f)$ is defined in a standard way (see [6, (1.6)]).

One of the reasons why we are interested in investigating the shifted cubic moment is that its asymptotic behavior can be conjectured using the "recipe" by Conrey, Farmer, Keating, Rubinstein, and Snaith [2]. These conjectures predict that the main term of (1.1) is given by

$$\mathcal{MT}_3(\alpha_1, \alpha_2, \alpha_3) = \sum_{\varepsilon_1=\pm 1} \sum_{\varepsilon_2=\pm 1} \sum_{\varepsilon_3=\pm 1} \mathcal{C}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$
$$(1.2) \qquad \times \zeta(1 + \varepsilon_1\alpha_1 + \varepsilon_2\alpha_2)\zeta(1 + \varepsilon_1\alpha_1 + \varepsilon_3\alpha_3)\zeta(1 + \varepsilon_2\alpha_2 + \varepsilon_3\alpha_3),$$

where

$$(1.3) \qquad \mathcal{C}(1,1,1) = 1, \quad \mathcal{C}(1,1,-1) = (-1)^k X_k(\alpha_3),$$

$$(1.4) \qquad \mathcal{C}(1,-1,1) = (-1)^k X_k(\alpha_2), \quad \mathcal{C}(-1,1,1) = (-1)^k X_k(\alpha_1),$$

$$(1.5) \qquad \mathcal{C}(1,-1,-1) = X_k(\alpha_2) X_k(\alpha_3), \quad \mathcal{C}(-1,1,-1) = X_k(\alpha_1) X_k(\alpha_3),$$

$$(1.6) \quad \mathcal{C}(-1,-1,1) = X_k(\alpha_1) X_k(\alpha_2), \quad \mathcal{C}(-1,-1,-1) = (-1)^k X_k(\alpha_1) X_k(\alpha_2) X_k(\alpha_3),$$

and

$$(1.7) \qquad X_k(\alpha) := (2\pi)^{2\alpha} \frac{\Gamma(k-\alpha)}{\Gamma(k+\alpha)}.$$

Unfortunately, an unconditional proof of an asymptotic formula for (1.1) is out of current technology. Nevertheless, we are able to investigate the structure and different properties of (1.1). In general, shifts reveal more clearly the combinatorial structure of moments, while the case of central values is just the limit case when all of the shifts are zero. For this reason, it is quite useful to introduce shifts. Accordingly, in [6], the majority of computations was performed for the shifted cubic moment, and only at the last step shifts are taken to be zero. Unfortunately, the method used in [6] has a disadvantage of not being fully symmetric in terms of shifts. Without any doubts, (1.1) is symmetric in $\alpha_1, \alpha_2, \alpha_3$, and therefore, the asymptotic formula for this moment should have the same property. The method of studying the cubic moment in [6] is based on the decomposition $3 = 2 + 1$. This means that we first use the series representation for one of the L-functions and reduce the problem to the study of the second moment. This process surely shuffles a bit the structure of the cubic moment; and in this paper, we try to overcome this issue.

We introduce some notation in order to formulate our main result. First, let $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and

$$\zeta_4(\bar{\alpha}; w) = \zeta\left(\frac{1+\alpha_1+\alpha_2-\alpha_3}{2} - w\right) \zeta\left(\frac{1+\alpha_1-\alpha_2+\alpha_3}{2} - w\right)$$

$$(1.8) \qquad \times \zeta\left(\frac{1-\alpha_1+\alpha_2+\alpha_3}{2} - w\right) \zeta\left(\frac{1+\alpha_1+\alpha_2+\alpha_3}{2} + w\right).$$

Note that the product (1.8) is symmetric in $\alpha_1, \alpha_2, \alpha_3$. To shorten formulas, we also introduce the following notation $_pI_q$ for the generalized hypergeometric function multiplied by Gamma factors

$$_pI_q\begin{pmatrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{pmatrix}; z) := \frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)} {}_pF_q\begin{pmatrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{pmatrix}; z) =$$

$$(1.9) \qquad = \sum_{j=0}^{\infty} \frac{\Gamma(a_1+j)\ldots\Gamma(a_p+j)}{\Gamma(b_1+j)\ldots\Gamma(b_q+j)} \frac{z^j}{j!}.$$

Finally, let

$$(1.10) \qquad \mathcal{H}_1(\bar{\alpha}, k; w) = {}_3I_2\begin{pmatrix} \frac{1+\alpha_1+\alpha_2-\alpha_3}{2} - w, \frac{1+\alpha_1-\alpha_2+\alpha_3}{2} - w, \frac{1-\alpha_1+\alpha_2+\alpha_3}{2} - w \\ k + \frac{1+\alpha_1+\alpha_2+\alpha_3}{2} - w, \frac{3+\alpha_1+\alpha_2+\alpha_3}{2} - w - k \end{pmatrix}; 1)$$

and

$$(1.11) \quad l_1(\tilde{\alpha}, k; w) = (2\pi)^{\frac{1+\alpha_1+\alpha_2+\alpha_3}{2}+w} \exp\left(\frac{\pi|\Im w|}{2} - \pi i \operatorname{sgn}(\Im w)\frac{1+\alpha_1+\alpha_2+\alpha_3}{4}\right).$$

Note that the expressions (1.10) and (1.11) are also symmetric in $\alpha_1, \alpha_2, \alpha_3$. The main result of the paper is contained in the following theorem.

Theorem 1.1 *For any $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that for $|\alpha_j| \ll \delta(\varepsilon)$ the following formula holds:*

$$\mathcal{M}(\alpha_1, \alpha_2, \alpha_3) = \mathcal{MT}_3(\alpha_1, \alpha_2, \alpha_3) +$$

$$(1.12) \qquad + \frac{(-1)^k}{2\pi i} \int_{(0)} \zeta_4(\tilde{\alpha}; w) \mathcal{H}_1(\tilde{\alpha}, k; w) l_1(\tilde{\alpha}, k; w) dw + O\left(\frac{k^\varepsilon}{k}\right).$$

The requirement of $\alpha_1, \alpha_2, \alpha_3$ being small is more or less of a technical nature and allows us to simplify the proof of several estimates on special functions.

Note that the error term in (1.12) is smaller than $O(k^{-1/2})$ predicted by the recipe conjectures.

The structure of the paper is as follows: First, in Section 2, we provide a statement of the formula for the shifted cubic moment proved in [6]. Then, in Section 3, we prove various results on special functions that will be used later to obtain a symmetric version of the formula for the cubic moment. Finally, in Section 4, we prove Theorem 1.1.

2 Reciprocity type formula for the cubic moment

In this section, we state the formula proved in [6], which relates the considered cubic moment to the fourth moment of the Riemann zeta function. Let

$$(2.1) \qquad \rho = \alpha_1, \quad u = \frac{\alpha_2 + \alpha_3}{2}, \quad v = \frac{\alpha_2 - \alpha_3}{2}.$$

For such u, v, ρ, the cubic moment in [6, (1.9)] coincides with (1.1). Let

$$\mathcal{MT}_D(\tilde{\alpha}) = \zeta(1+\alpha_1+\alpha_2)\zeta(1+\alpha_1+\alpha_3)\zeta(1+\alpha_2+\alpha_3)\mathcal{C}(1,1,1) +$$
$$+ \zeta(1+\alpha_1+\alpha_2)\zeta(1+\alpha_1-\alpha_3)\zeta(1+\alpha_2-\alpha_3)\mathcal{C}(1,1,-1) +$$
$$+ \zeta(1+\alpha_1-\alpha_2)\zeta(1+\alpha_1+\alpha_3)\zeta(1-\alpha_2+\alpha_3)\mathcal{C}(1,-1,1) +$$
$$(2.2) \qquad + \zeta(1+\alpha_1-\alpha_2)\zeta(1+\alpha_1-\alpha_3)\zeta(1-\alpha_2-\alpha_3)\mathcal{C}(1,-1,-1),$$

where $\mathcal{C}(\pm 1, \pm 1, \pm 1)$ are defined in (1.3)–(1.6). We also introduce the functions

$$Z_1(u, v, \rho, k; w) = \zeta\left(\frac{1+\rho}{2} + v + w\right)\zeta\left(\frac{1+\rho}{2} - v + w\right)$$

$$(2.3) \qquad \times \zeta\left(\frac{1+\rho}{2} + u - w\right)\zeta\left(\frac{1+\rho}{2} - u - w\right)\hat{g}_1(u, v, \rho, k; w),$$

$$\mathcal{ET}_1(u, v, \rho) = (-1)^k (R_{1,1} + R_{1,2} - R_{1,3} - R_{1,4})(u, v, \rho, k)$$

$$(2.4) \qquad + \frac{(-1)^k}{2\pi i} \int_{(0)} Z_1(u, v, \rho, k; w) dw,$$

where $\hat{g}_1(u, v, \rho, k; w)$ is given in [6, Lemma 4.7] and

$$(2.5) \qquad R_{1,1}(u, v, \rho, k) = \zeta(1 - 2v)\zeta(\rho + u + v)\zeta(\rho - u + v)\hat{g}_1(u, v, \rho, k; 1/2 - \rho/2 - v),$$

$$(2.6) \qquad R_{1,2}(u, v, \rho, k) = \zeta(1 + 2v)\zeta(\rho + u - v)\zeta(\rho - u - v)\hat{g}_1(u, v, \rho, k; 1/2 - \rho/2 + v),$$

$$(2.7) \qquad R_{1,3}(u, v, \rho, k) = -\zeta(1 - 2u)\zeta(\rho + u + v)\zeta(\rho + u - v)\hat{g}_1(u, v, \rho, k; -1/2 + \rho/2 + u),$$

$$(2.8) \qquad R_{1,4}(u, v, \rho, k) = -\zeta(1 + 2u)\zeta(\rho - u + v)\zeta(\rho - u - v)\hat{g}_1(u, v, \rho, k; -1/2 + \rho/2 - u).$$

Let us also define

$$
Z_2(u, v, \rho, k; w) = \zeta\left(\frac{1 + \rho}{2} + v - w\right)\zeta\left(\frac{1 + \rho}{2} - v - w\right)
$$
$$(2.9) \qquad \qquad \times \zeta\left(\frac{1 + \rho}{2} + u + w\right)\zeta\left(\frac{1 + \rho}{2} - u + w\right)\hat{g}_2(u, v, \rho, k; w),$$

$$
\mathcal{ET}_2(u, v, \rho) = \left(R_{2,1} + R_{2,2} - R_{2,3} - R_{2,4}\right)(u, v, \rho, k)
$$
$$(2.10) \qquad \qquad + \frac{1}{2\pi i}\int_{(0)} Z_2(u, v, \rho, k; w)\, dw,$$

where $\hat{g}_2(u, v, \rho, k; w)$ is given in [6, Lemma 4.3] and

$$(2.11) \qquad R_{2,1}(u, v, \rho, k) = \zeta(1 - 2u)\zeta(\rho + u + v)\zeta(\rho + u - v)\hat{g}_2(u, v, \rho, k; 1/2 - \rho/2 - u),$$

$$(2.12) \qquad R_{2,2}(u, v, \rho, k) = \zeta(1 + 2u)\zeta(\rho - u + v)\zeta(\rho - u - v)\hat{g}_2(u, v, \rho, k; 1/2 - \rho/2 + u),$$

$$(2.13) \qquad R_{2,3}(u, v, \rho, k) = -\zeta(1 + 2v)\zeta(\rho + u - v)\zeta(\rho - u - v)\hat{g}_2(u, v, \rho, k; -1/2 + \rho/2 - v),$$

$$(2.14) \qquad R_{2,4}(u, v, \rho, k) = -\zeta(1 - 2v)\zeta(\rho + u + v)\zeta(\rho - u + v)\hat{g}_2(u, v, \rho, k; -1/2 + \rho/2 + v).$$

Finally, let

$$(2.15) \qquad \mathcal{ET}_3(u, v, \rho) = (-1)^k \frac{\Gamma(k - u + v)}{\Gamma(k + u - v)}(2\pi)^{2u - 2v}\mathcal{ET}_2(v, u, \rho).$$

Theorem 2.1 *The following formula holds:*

$$(2.16) \qquad \mathcal{M}(\alpha_1, \alpha_2, \alpha_3) = \mathcal{MT}_D(\bar{\alpha}) + \mathcal{ET}_1(u, v, \rho) + \mathcal{ET}_2(u, v, \rho) + \mathcal{ET}_3(u, v, \rho).$$

Proof Formula (2.16) is a modification of [6, (5.1)]. The term $\mathcal{MT}_D(\bar{\alpha})$ corresponds to $\mathcal{MT}_1(u, v, \rho)$ defined in [6, (5.2)]. Formula (2.2) follows immediately from [6, (3.8) and (5.2)] together with (2.1). We are left to prove (2.15). It follows from [6, (3.5)] and [8, 15.8.1] that

$$\psi_k(x; u, v) = 2(2\pi)^{2u} \frac{\Gamma(k - u + v)\Gamma(k - u - v)}{\Gamma(2k)}$$

(2.17)
$$\times \sin\left(\pi(1/2 + v)\right) x^k (1 + x)^{-k+v} {}_2F_1\left(\begin{matrix} k + u - v, k - u - v \\ 2k \end{matrix}; \frac{x}{1 + x}\right).$$

Comparing (2.17) with [6, (3.4)], we conclude

(2.18)
$$\psi_k(x; u, v) = \frac{\Gamma(k - u + v)}{\Gamma(k + u - v)} (2\pi)^{2u - 2v} \Phi_k\left(\frac{x}{1 + x}; v, u\right).$$

Now, (2.15) follows from [6, (5.24) and (5.5)] and (2.18). Note that there are several typos in [6, (4.18) and (4.19)]. According to (2.18) and [6, (4.14) and (4.17)], we have

(2.19)
$$\hat{g}_3(u, v, \rho, k; w) = \frac{\Gamma(k - u + v)}{\Gamma(k + u - v)} (2\pi)^{2u - 2v} \hat{g}_2(v, u, \rho, k; w).$$

Therefore, the correct versions of [6, (4.18) and (4.19)] could be obtained from [6, (4.15) and (4.16)] with the use of (2.19). Typos in [6, (4.18) and (4.19)] do not affect other formulas or results in [6] since in [6], we deal with the case $u = v = 0$. In that case [6, (4.18) and (4.19)] are correct. ∎

We are going to deduce Theorem 1.1 from Theorem 2.1. As one can see from (2.2) and (1.2), the term $\mathcal{MT}_D(\bar{\alpha})$ in (2.16) contains four of eight summands in (1.2). The remaining four terms are hidden in the expressions $R_{i,j}(u, v, \rho, k)$. To evaluate $R_{1,1}$, $R_{1,2}$ and to estimate $R_{2,1}$, $R_{2,2}$, we will prove some new results on the behavior of functions $\hat{g}_j(v, u, \rho, k; w)$, $j = 1, 2$. This is done in the next section.

3 Special functions

In this section, we first recall some definitions and properties of special functions appearing in [6, Section 4]. Then, we will obtain some new results concerning these functions required to prove Theorem 1.1. First, it follows from (2.1) and the statement of Theorem 1.1 that

(3.1)
$$|v| + |u| + |\rho| < \delta$$

for some small $\delta > 0$. Also throughout the section, we will frequently use the Stirling bound [8, 5.11.9] on Gamma factors

(3.2)
$$|\Gamma(x + iy)| \ll |y|^{x - 1/2} e^{-\pi|y|/2}.$$

3.1 Properties of $g_2(u, v, \rho, k; x)$

Let (see [6, (3.4)])

$$\Phi_k(x; u, v) := 2(2\pi)^{2u} \frac{\Gamma(k - u + v)\Gamma(k - u - v)}{\Gamma(2k)}$$

(3.3)
$$\times \sin\left(\pi(1/2 + u)\right) x^k (1 - x)^{-u} {}_2F_1(k - u + v, k - u - v, 2k; x),$$

and for $\Re w > -1/2 - \Re \rho/2 + \Re u$ (see [6, (4.14)]),

$$(3.4) \qquad \hat{g}_2(u, v, \rho, k; w) = \int_0^1 (1-x)^{w+\rho/2-1/2} x^{-1-\rho} \Phi_k(x; u, v)\, dx.$$

Lemma 3.1 *For $|v| + |u| + |\rho| < \delta$ and*

$$\max(-1/2 + \Re(\rho/2) + \Re v, -1/2 + |\Re u| - \Re(\rho/2)) < \Re w,$$

we have

$$\hat{g}_2(u, v, \rho, k; w) = 2(2\pi)^{2u} \cos(\pi u) \frac{\Gamma(k - u - v)}{\Gamma(k + u + v)} \frac{1}{2\pi i} \int_{(\Delta)} \frac{\Gamma(k - z)}{\Gamma(k + z)}$$

$$(3.5) \qquad \times \frac{\Gamma(1/2 + \rho/2 - v + w - z)}{\Gamma(1/2 - \rho/2 - v + w)} \Gamma(z - u + v)\, \Gamma(z + u + v)\, \Gamma(z - \rho)\, dz,$$

where

$$\max(\Re(\rho), |\Re u| - \Re(v)) < \Delta < \min(1/2 + \Re(\rho/2 - v + w), k).$$

Proof Substituting (3.3) to (3.4), applying [8, 15.8.1] and making the change of variable $x = (1+t)^{-1}$, we obtain

$$\frac{\hat{g}_2(u, v, \rho, k; w)}{2(2\pi)^{2u} \cos(\pi u)} = \int_0^\infty t^{w-v+\rho/2-1/2-k}(1+t)^{v-w+\rho/2-1/2}$$

$$(3.6) \qquad \times {}_2F_1\left(\begin{matrix} k - u + v, k + u + v; \\ 2k \end{matrix} ; -t^{-1} \right) dt.$$

Using the Mellin integral representation [8, 15.6.6] of the hypergeometric function in (3.6), changing the orders of integration and evaluating the integral over t with the help of [8, 5.12.3], we find

$$\frac{\hat{g}_2(u, v, \rho, k; w)}{2(2\pi)^{2u} \cos(\pi u)} = \frac{\Gamma(k - u - v)}{\Gamma(k + u + v)} \frac{1}{2\pi i} \int_{(\Delta)} \frac{\Gamma(k - u + v + s)\Gamma(k + u + v + s)}{\Gamma(2k + s)}$$

$$(3.7) \qquad \times \frac{\Gamma(1/2 + \rho/2 - v + w - k - s)}{\Gamma(1/2 - \rho/2 - v + w)} \Gamma(-s)\, \Gamma(k - \rho + s)\, ds,$$

where

$$\max(\Re(\rho) - k, |\Re u| - \Re(v) - k) < \Delta < \min(1/2 + \Re(\rho/2 - v + w) - k, 0).$$

Finally, making the change of variable $s = z - k$, we obtain (3.5). $\blacksquare$

Lemma 3.2 *For $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that for $|u|, |v|, |\rho| \ll \delta(\varepsilon)$ and any fixed A such that $1 < A < K$ we have*

$$(3.8) \qquad \hat{g}_2(u, v, \rho, k; ir) \ll \frac{(k(|r|+1))^\varepsilon}{k(1+|r|)^A}.$$

Proof Moving the line of integration in (3.5) to the right on $\Re z = \sigma < k$, we cross poles at the points

$$(3.9) \qquad z_p(j) = 1/2 + \rho/2 - v + ir + j \quad \text{for} \quad j = 0, 1, 2, \ldots.$$

The residues are estimated in the following way:

$$\frac{\Gamma(k-u-v)}{\Gamma(k+u+v)}\frac{\Gamma(k-1/2-\rho/2+v-ir-j)}{\Gamma(k+1/2+\rho/2-v+ir+j)}\frac{\Gamma(1/2-\rho/2-v+ir+j)}{\Gamma(1/2-\rho/2-v+ir)}$$
$$\times\,\Gamma(1/2+\rho/2+ir+j-u)\Gamma(1/2+\rho/2+ir+j+u)\ll$$
$$(3.10)\qquad\ll\frac{k^{4\delta}(1+|r|)^{3j+\delta}e^{-\pi|r|}}{(k+|r|)^{1+2j-3\delta}}\ll\frac{(k(|r|+1))^{\varepsilon}}{k(1+|r|)^A}.$$

Therefore,

$$\hat{g}_2(u,v,\rho,k;ir)\ll\frac{(k(|r|+1))^{\varepsilon}}{k(1+|r|)^A}+\left|\frac{k^{4\delta}}{2\pi i}\int_{(\sigma)}\frac{\Gamma(k-z)}{\Gamma(k+z)}\right.$$
$$(3.11)\qquad\left.\times\frac{\Gamma(1/2+\rho/2-v+ir-z)}{\Gamma(1/2-\rho/2-v+ir)}\Gamma(z-u+v)\,\Gamma(z+u+v)\,\Gamma(z-\rho)\,dz\right|.$$

Let $z=\sigma+iy$ and $y_0=C\log^2(k(1+|r|))$, where the constant C is chosen in such way that $|r-y_0|>r/2$. To estimate the integral (3.11), we apply (3.2) and split it in two parts, the first one over $|y|>y_0$ and the second over $|y|\le y_0$. The first integral is smaller than

$$(3.12)\qquad\int_{|y|>y_0}\frac{|y|^{3\sigma+3\delta-3/2}e^{-\pi(3|y|+|y-r|-|r|)/2}}{(|y-r|+1)^{\sigma-3\delta/2}(|r|+1)^{-3\delta/2}(k+|y|)^{2\sigma}}dy\ll\frac{1}{(k(1+|r|))^A}.$$

The second integral is smaller than

$$\int_{|y|\le y_0}\frac{|y|^{3\sigma+3\delta-3/2}(|y-r|+1)^{-\sigma+3\delta/2}}{(|r|+1)^{-3\delta/2}(k+|y|)^{2\sigma}}dy\ll$$
$$(3.13)\qquad\ll\int_{|y|\le y_0}\frac{(k(1+|r|))^{\varepsilon}|y|^{3\sigma-3/2}dy}{(k+|y|)^{2\sigma}(|y-r|+1)^{\sigma}}\ll\frac{(k(1+|r|))^{\varepsilon}}{(k^2(1+|r|))^{\sigma}}.$$

Substituting (3.12) and (3.13) to (3.11), we prove (3.8). ∎

Lemma 3.3 *For any $\varepsilon>0$, there exists $\delta(\varepsilon)$ such that for $|u|,|v|,|\rho|\ll\delta(\varepsilon)$, we have*

$$(3.14)\qquad\hat{g}_2(u,v,\rho,k;1/2-\rho/2\pm u)\ll\frac{k^{\varepsilon}}{k^2}.$$

Proof Using (3.5) with $\Delta=1-\varepsilon_0$ (ε_0 is chosen in such way that all poles of $\Gamma(1-v\pm u-z)$ are located to the right of the line $\Re z=\Delta$), writing $z=\Delta+iy$ and applying (3.2), we obtain

$$(3.15)\qquad\hat{g}_2(u,v,\rho,k;1/2-\rho/2\pm u)\ll k^{4\delta}\int_{-\infty}^{\infty}\frac{(1+|y|)^{2\Delta+3\delta-1}e^{-2\pi|y|}}{(k+|y|)^{2\Delta}}dy\ll\frac{k^{\varepsilon}}{k^2}.$$

Note that estimating the integral representation [6, (4.15)] of $\hat{g}_2(\dots)$ by absolute value with the use of the Stirling bound (3.2) on Gamma factors, we are able to obtain only a weaker estimate $k^{-1+\varepsilon}$. This estimate is sufficient for our further computations. ∎

3.2 Properties of $g_1(u, v, \rho, k; x)$

Let

$$F(v, u; y) := (1-y)^v y^u {}_2F_1 \left(\begin{matrix} k+u+v, 1-k+u+v \\ 1+2u \end{matrix} ; y \right).$$

For $0 < y < 1$, let (see [1, Lemma 5.1])

(3.16) $$\phi_k(1-y; u, v) = \tilde{\phi}_k(1-y; u, v) + \tilde{\phi}_k(1-y; u, -v),$$

where

$$\tilde{\phi}_k(1-y; u, v) = (-1)^k \frac{(2\pi)^{2u} \pi}{\sin(\pi v)} \left(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)} \right.$$

(3.17) $$\left. \times \frac{\sin(\pi(v+u))}{\Gamma(1-2u)\sin(2\pi u)} F(v, -u; y) + \frac{\sin(\pi(v-u))}{\Gamma(1+2u)\sin(-2\pi u)} F(v, u; y) \right).$$

For $-1/2 - \mathfrak{R}\rho/2 + |\mathfrak{R}v| < \mathfrak{R}w < 1/2 + \mathfrak{R}\rho/2 - |\mathfrak{R}u|$, let (see [6, (4.27)])

(3.18) $$\hat{g}_1(u, v, \rho, k; w) = \int_0^1 (y(1-y))^{\rho/2-1/2} \frac{(1-y)^w}{y^w} \phi_k(1-y; u, v)\, dy.$$

We are going to prove an analog of [6, Lemma 4.7] which will be used later to deduce a version of [6, Lemma 4.8] for $\hat{g}_1(u, v, \rho, k; 1/2 - \rho/2 \pm v)$.

Lemma 3.4 *For $|v| + |u| + |\rho| < \delta$ and*

$$-1/2 - \mathfrak{R}\rho/2 + |\mathfrak{R}v| < \mathfrak{R}w < 2 + \mathfrak{R}\rho/2 - |\mathfrak{R}u|,$$

we have

$$\hat{g}_1(u, v, \rho, k; w) = \sum_{\pm} \frac{(-1)^k (2\pi)^{2u}}{2\sin(\pm\pi v)} \frac{\Gamma(k-u\mp v)}{\Gamma(k+u\pm v)} \Gamma\left(\frac{1+\rho}{2} \pm v + w \right)$$

$$\times \frac{1}{2\pi i} \int_{(\Delta_\pm)} \frac{\Gamma(k-1/2+s/2)}{\Gamma(k+1/2-s/2)} \Gamma\left(\frac{1-s}{2} + u \pm v \right) \Gamma\left(\frac{1-s}{2} - u \pm v \right)$$

(3.19) $$\times \cos(\pi(\pm 2v - s/2)) \frac{\Gamma(\rho/2 \mp v - w + s/2)}{\Gamma(1/2 + \rho + s/2)}\, ds,$$

where $\max(1 - 2k, \mathfrak{R}(-\rho + 2w) \pm 2\mathfrak{R}v) < \Delta_\pm < 1 - |2\mathfrak{R}u| \pm 2\mathfrak{R}v.$

Proof It follows from [7, 6.574.1] that for $-\mathfrak{R}(k+u) < \mathfrak{R}(v) < 1/2$, we have

(3.20)
$$F(v, u; y) = (1-y)^v 2^{-2v} \frac{\Gamma(k-u-v)\Gamma(1+2u)}{\Gamma(k+u+v)} \int_0^\infty J_{2u}(t\sqrt{y}) J_{2k-1}(t) t^{2v}\, dt.$$

Substituting (3.20) to (3.17), we obtain, for $-\Re(k - |u|) < \Re(v) < 1/2$,

$$\tilde{\phi}_k(1 - y; u, v) = (-1)^k \frac{(2\pi)^{2u}\pi(1-y)^v 2^{-2v}}{\sin(\pi v)\sin(2\pi u)} \frac{\Gamma(k - v - u)}{\Gamma(k + v + u)}$$

(3.21)
$$\times \sum_{\pm} \pm \sin\pi(v \pm u) \int_0^\infty J_{\pm 2u}(t\sqrt{y}) J_{2k-1}(t) t^{2v} dt.$$

Applying [8, 1.14.36] together with [3, p. 326 f.(1)], we prove that

(3.22)
$$\int_0^\infty J_{2u}(t\sqrt{y}) J_{2k-1}(t) t^{2v} dt = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(k + v - s/2)\Gamma(u + s/2)}{\Gamma(k - v + s/2)\Gamma(1 + u - s/2)} \frac{2^{2v-1}ds}{y^{s/2}}.$$

Next, we change the variable s to $1 + 2v - s$, and then apply [8, 5.5.3] to transform $\Gamma(1/2 + u - v + s/2)$. As a result,

$$\int_0^\infty J_{2u}(t\sqrt{y}) J_{2k-1}(t) t^{2v} dt = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(k - 1/2 + s/2)}{\Gamma(k + 1/2 - s/2)}$$

(3.23)
$$\times \Gamma(1/2 + u + v - s/2)\Gamma(1/2 - u + v - s/2)\cos\pi(u - v + s/2) \frac{2^{2v-1}ds}{\pi y^{1/2 + v - s/2}}.$$

Substituting (3.23) to (3.21), we have

$$\tilde{\phi}_k(1 - y; u, v) = (-1)^k \frac{(2\pi)^{2u}(1-y)^v}{2\sin(\pi v) y^{1/2+v}} \frac{\Gamma(k - v - u)}{\Gamma(k + v + u)}$$

$$\times \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(k - 1/2 + s/2)}{\Gamma(k + 1/2 - s/2)} \Gamma(1/2 + u + v - s/2)$$

(3.24)
$$\times \Gamma(1/2 - u + v - s/2)\cos\pi(2v - s/2) y^{s/2} ds.$$

According to [8, 5.12.1] for $\Re(-\rho/2 - 1/2 - v) < \Re w < \Re(\rho/2 + s/2 - v)$, we have

(3.25)
$$\int_0^1 (y(1-y))^{\rho/2 - 1/2}(1 - y)^{w+v} y^{s/2 - 1/2 - v - w} dy = \frac{\Gamma(\rho/2 + s/2 - v - w)\Gamma(\rho/2 + 1/2 + v + w)}{\Gamma(\rho + s/2 + 1/2)}.$$

Using (3.16), (3.18), (3.24) and evaluating the resulting integral over y using (3.25), we prove (3.19). ∎

Lemma 3.5 *For any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $|v| + |u| + |\rho| < \delta(\varepsilon)$ the following formula holds:*

$$\hat{g}_1(u, v, \rho, k; 1/2 - \rho/2 \pm v) = \frac{(-1)^k (2\pi)^{2u}}{\sin(\pm\pi v)} \Gamma(\rho + u \mp v)\Gamma(\rho - u \mp v)$$

$$\times \left(\sin(\pi\rho) \frac{\Gamma(k - u \mp v)}{\Gamma(k + u \pm v)} \frac{\Gamma(k - \rho \pm 2v)}{\Gamma(k + \rho \mp 2v)} - \sin\pi(\rho \mp 2v) \frac{\Gamma(k - u \pm v)}{\Gamma(k + u \mp v)} \frac{\Gamma(k - \rho)}{\Gamma(k + \rho)} \right)$$

$$+ J_1(u, \pm v, \rho, k) + J_2(u, \pm v, \rho, k),$$

(3.26)

where

$$J_1(u, \pm v, \rho, k) = \frac{(-1)^k (2\pi)^{2u}}{2 \sin(\pm\pi v)} \frac{\Gamma(k - u \mp v)}{\Gamma(k + u \pm v)} \Gamma(1 \pm 2v)$$

$$\times \frac{1}{2\pi i} \int_{(\Delta_1)} \frac{\Gamma(k - 1/2 + s/2)}{\Gamma(k + 1/2 - s/2)} \cos\left(\pi(\pm 2v - s/2)\right)$$

$$(3.27) \qquad \times \Gamma\left(\frac{1-s}{2} + u \pm v\right) \Gamma\left(\frac{1-s}{2} - u \pm v\right) \frac{\Gamma(-1/2 + \rho \mp 2v + s/2)}{\Gamma(1/2 + \rho + s/2)} ds,$$

$$J_2(u, \pm v, \rho, k) = \frac{(-1)^k (2\pi)^{2u}}{2 \sin(\mp\pi v)} \frac{\Gamma(k - u \pm v)}{\Gamma(k + u \mp v)}$$

$$\times \frac{1}{2\pi i} \int_{(\Delta_2)} \frac{\Gamma(k - 1/2 + s/2)}{\Gamma(k + 1/2 - s/2)} \cos\left(\pi(\mp 2v - s/2)\right)$$

$$(3.28) \qquad \times \Gamma\left(\frac{1-s}{2} + u \mp v\right) \Gamma\left(\frac{1-s}{2} - u \mp v\right) \frac{\Gamma(-1/2 + \rho + s/2)}{\Gamma(1/2 + \rho + s/2)} ds,$$

and

$$\max(1 - 2k, -1 - 2\Re(\rho \pm 4v)) < \Delta_1 < 1 - 2|\Re u| - 2|\Re v|,$$

$$\max(1 - 2k, -1 - 2\Re(\rho)) < \Delta_2 < 1 - 2|\Re u| - 2|\Re v|.$$

Moreover, the following estimates hold:

$$(3.29) \qquad J_1(u, \pm v, \rho, k) \ll \frac{k^\varepsilon}{k}, \quad J_2(u, \pm v, \rho, k) \ll \frac{k^\varepsilon}{k}.$$

Proof The proof is similar to the one of [6, Lemma 4.8]. We move the line of integration in (3.19) to the left, crossing the pole at $s = \pm 2v + 2w - \rho$. Evaluating the residues and letting $w = 1/2 - \rho/2 \pm v$, we obtain (3.26). To prove (3.29), we estimate Gamma factors in (3.27) and (3.28) by means of the Stirling formula (3.2). Taking into account that $|v| + |u| + |\rho| < \varepsilon$ and choosing $\Delta_1 = \Delta_2 = 0$, we have

$$J_1(u, \pm v, \rho, k) \ll k^\varepsilon \int_{-\infty}^{\infty} \frac{(k + |y|)^{\Delta_1 - 1}}{(1 + |y|)^{\Delta_1 + 1}} dy \ll \frac{k^\varepsilon}{k},$$

$$J_2(u, \pm v, \rho, k) \ll k^\varepsilon \int_{-\infty}^{\infty} \frac{(k + |y|)^{\Delta_2 - 1}}{(1 + |y|)^{\Delta_2 + 1 \pm 2\Re v}} dy \ll \frac{k^\varepsilon}{k},$$

thus proving (3.29). ∎

The function $\hat{g}_1(\dots)$ can be expressed in terms of $\mathcal{H}_1(\bar{\alpha}, k; w)$ (see (1.10)) and $\hat{g}_2(\dots)$ (see [6, (4.16)]).

Lemma 3.6 *The following equality holds:*

$$\hat{g}_1(u, v, \rho, k; -w) = 2(-1)^k (2\pi)^{\alpha_2 + \alpha_3} \Gamma\left(\frac{1 + \alpha_1 - \alpha_2 - \alpha_3}{2} + w\right)$$

$$\times \frac{\sin \pi(\alpha_1/2 - w) \cos \pi \left(\frac{\alpha_1 - \alpha_2 - \alpha_3}{2} + w\right)}{\cos \pi \left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} - w\right)} \mathcal{H}_1(\bar{\alpha}, k; w) -$$

(3.30)
$$- \frac{(-1)^k \sin(\pi \alpha_1)}{\cos \pi \left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} - w\right)} \hat{g}_2(u, v, \rho, k; w).$$

Proof Moving the line of integration in [6, (4.28)] to the left and evaluating residues at the points

$$s_1(j) = 1 - 2k - 2j, \quad s_2(j) = -\rho - 2u - 2w - 2j \quad j = 0, 1, 2 \dots,$$

we obtain (see also [6, Lemma 4.11])

$$\hat{g}_1(u, v, \rho, k; w) = -2(-1)^k (2\pi)^{2u} \Gamma \left(\frac{1+\rho}{2} - u - w\right) \frac{\sin(\pi\rho) \cos(\pi u)}{\cos \pi(\rho/2 + u + w)}$$

$$\times {}_3I_2 \begin{pmatrix} k - u + v, k - u - v, k - \rho \\ 2k, k + \frac{1-\rho}{2} - u - w \end{pmatrix} + 2(-1)^k (2\pi)^{2u} \Gamma \left(\frac{1+\rho}{2} - u - w\right)$$

$$\times \frac{\sin \pi(\rho/2 + w) \cos \pi(\rho/2 - u - w)}{\cos \pi(\rho/2 + u + w)} {}_3I_2 \begin{pmatrix} \frac{1+\rho}{2} + v + w, \frac{1+\rho}{2} - v + w, \frac{1-\rho}{2} + u + w, \\ k + \frac{1+\rho}{2} + u + w, \frac{3+\rho}{2} + u + w - k \end{pmatrix}.$$

(3.31)

Rewriting (3.31) in terms of $\alpha_1, \alpha_2, \alpha_3$ using (2.1) and applying (1.10), [6, (4.16)], we prove (3.30). ∎

3.3 Properties of $\mathcal{H}_1(\bar{\alpha}, k; w)$

The following statement is the core part of the paper.

Proposition 3.7 *For any $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that for $|\alpha_j| \ll \delta(\varepsilon)$, we have*

(3.32)
$$\mathcal{H}_1(\bar{\alpha}, k; ir) \ll e^{-\pi|r|/2} (1 + |r|)^\varepsilon \frac{k^\varepsilon}{\sqrt{k}},$$

and for $|r| \ll \log^2 k$, we have

(3.33)
$$\mathcal{H}_1(\bar{\alpha}, k; ir) \ll \frac{k^\varepsilon}{k}.$$

The proof of the estimates (3.32) and (3.33) is based on the use of the following integral representation for $\mathcal{H}_1(\bar{\alpha}, k; w)$.

Lemma 3.8 *We have*

$$\mathcal{H}_1(\bar{\alpha}, k; w) = \frac{\Gamma \left(\frac{1-\alpha_1+\alpha_2-\alpha_3}{2} - w\right)}{\Gamma (1 + \alpha_2 + \alpha_3)} \int_0^1 (1 - x)^{\frac{-1+\alpha_1+\alpha_2-\alpha_3}{2} - w}$$

(3.34)
$$\times x^{\frac{-1+\alpha_1+\alpha_2+\alpha_3}{2} + w} {}_2F_1 \left(1 + \alpha_2 - k, k + \alpha_2; 1 + \alpha_2 + \alpha_3\right) x \, dx.$$

Proof Applying [9, (7.4.4.2)] with

$$a = \frac{1 + \alpha_1 - \alpha_2 + \alpha_3}{2} - w, \, b = \frac{1 + \alpha_1 + \alpha_2 - \alpha_3}{2} - w, \, c = \frac{1 - \alpha_1 + \alpha_2 + \alpha_3}{2} - w,$$

$$d = \frac{3 + \alpha_1 + \alpha_2 + \alpha_3}{2} - w - k, \quad e = k + \frac{1 + \alpha_1 + \alpha_2 + \alpha_3}{2} - w, \quad s = \frac{1 + \alpha_1 + \alpha_2 + \alpha_3}{2} + w,$$

we deduce

$$\mathcal{H}_1(\bar{\alpha}, k; w) = \frac{\Gamma\left(\frac{1 + \alpha_1 + \alpha_2 - \alpha_3}{2} - w\right) \Gamma\left(\frac{1 - \alpha_1 + \alpha_2 + \alpha_3}{2} - w\right)}{\Gamma(1 + \alpha_1 + \alpha_2) \Gamma(1 + \alpha_2 + \alpha_3)}$$

$$(3.35) \qquad \times \Gamma\left(\frac{1 + \alpha_1 + \alpha_2 + \alpha_3}{2} + w\right) {}_3F_2\left(\begin{matrix} 1 + \alpha_2 - k, k + \alpha_2, \frac{1 + \alpha_1 + \alpha_2 + \alpha_3}{2} + w \\ 1 + \alpha_1 + \alpha_2, 1 + \alpha_2 + \alpha_3 \end{matrix}; 1\right).$$

This formula can be viewed as an analog of [6, (4.41)]. For the hypergeometric function on the right-hand side of (3.35), we use the integral representation [8, 16.5.2] with

$$a_0 = \frac{1 + \alpha_1 + \alpha_2 + \alpha_3}{2} + w, \quad b_0 = 1 + \alpha_1 + \alpha_2,$$

thus proving (3.34). ■

   Therefore, we have reduced the problem of estimating $\mathcal{H}_1(\bar{\alpha}, k; w)$ to the problem of finding an asymptotic formula uniform in x for ${}_2F_1(; x)$ on the right-hand side of (3.34). We will independently consider the case of x being small, x being close to 1 and the intermediate case $\varepsilon < x < 1 - \varepsilon$. In the last case, one can apply the following result of Farid Khwaja and Olde Daalhuis [4, Theorem 3.2].

Lemma 3.9 *For some small fixed $\varepsilon > 0$ such that $\varepsilon < y < \pi/2 - \varepsilon$ and $k \to \infty$, we have*

$${}_2F_1\left(\begin{matrix} 1 + \alpha_2 - k, k + \alpha_2 \\ 1 + \alpha_2 + \alpha_3 \end{matrix}; \cos^2 y\right) = (-1)^k \frac{\Gamma(1 + \alpha_2 + \alpha_3)\Gamma(k - \alpha_3)}{\Gamma(k + \alpha_2)}$$

$$\times \Bigg(O(\Phi_n(k, y)) + y^{\alpha_3 - \alpha_2} \left(J_{\alpha_2 - \alpha_3}(2y\lambda)\cos(\pi\alpha_2) - Y_{\alpha_2 - \alpha_3}(2y\lambda)\sin(\pi\alpha_2)\right) \sum_{j=0}^{n-1} \frac{c_j(y)}{k^j}$$

$$- iy^{\alpha_3 - \alpha_2 - 1} \left(J_{1 + \alpha_2 - \alpha_3}(2y\lambda)\sin(\pi\alpha_2) + Y_{1 + \alpha_2 - \alpha_3}(2y\lambda)\cos(\pi\alpha_2)\right) \sum_{j=1}^{n-1} \frac{d_j(y)}{k^j} \Bigg),$$

(3.36)

where $\lambda = k - \frac{1}{2}$, $|c_j(y)| \ll_\varepsilon 1$, $|d_j(y)| \ll_\varepsilon 1$, and

$$(3.37) \qquad c_0(y) = -\frac{y^{1/2 + \alpha_2 - \alpha_3}}{\sin^{1/2 + \alpha_2 - \alpha_3} y \cos^{1/2 + \alpha_2 + \alpha_3} y}, \quad d_0(y) = 0,$$

$$\Phi_n(k, y) = \frac{y^{\Re(\alpha_3 - \alpha_2)}}{k^n} \left(|J_{\alpha_2 - \alpha_3}(2y\lambda)| + |Y_{\alpha_2 - \alpha_3}(2y\lambda)| + \right.$$

$$(3.38) \qquad \left. + y^{-1}|Y_{1 + \alpha_2 - \alpha_3}(2y\lambda)| + y^{-1}|J_{1 + \alpha_2 - \alpha_3}(2y\lambda)| \right).$$

Proof We apply [4, Theorem 3.2] with

$$z = -\cos(2y), \quad \lambda = k - \frac{1}{2}, \quad a = \frac{1}{2} + \alpha_2, \quad c = 1 + \alpha_2 + \alpha_3.$$

In this case, [4, (3.9)] becomes

$$\xi = \log\left(-z - i\sqrt{1 - z^2}\right) = -2iy,$$

and [4, (3.10)] proves (3.37). Finally, the K-Bessel functions of purely imaginary argument in [4, (3.8)] can be transformed into combination of Y and J-Bessel functions by means of [8, 10.27.8 and 10.4.3]

$$K_\nu(2iy\lambda) = \frac{-\pi i}{2} e^{-\nu\pi i/2} H_\nu^{(2)}(2y\lambda) = \frac{-\pi i}{2} e^{-\nu\pi i/2} \left(J_\nu(2y\lambda) - i Y_\nu(2y\lambda) \right),$$

$$K_\nu(-2iy\lambda) = \frac{\pi i}{2} e^{\nu\pi i/2} H_\nu^{(1)}(2y\lambda) = \frac{\pi i}{2} e^{\nu\pi i/2} \left(J_\nu(2y\lambda) + i Y_\nu(2y\lambda) \right).$$

After some straightforward computations, we obtain (3.36). ∎

It is explained in [5, Section 4], why [4, Theorem 3.2] does not provide an adequate asymptotic formula either for y close to 0 or to $\pi/2$. The case y close to $\pi/2$ is studied in [5, Section 4.1], and the case of y close to 0 is considered in [5, Section 4.4]. Both asymptotic formulas are given not in terms of Bessel functions, but in terms of Kummer functions. See [8, Section13] for the definition and properties of these functions.

Lemma 3.10 *Assume that $|\alpha_j| \ll \delta(\varepsilon)$. For some small fixed $\varepsilon > 0$ such that $0 < x < \varepsilon$ and $k \to \infty$, we have*

$$
{}_2F_1 \begin{pmatrix} 1 + \alpha_2 - k, k + \alpha_2 \\ 1 + \alpha_2 + \alpha_3 \end{pmatrix}; x = \frac{\Gamma(1 + \alpha_2 + \alpha_3)\Gamma(k - \alpha_3)}{\Gamma(k + \alpha_2)} e^{-k\zeta_0}
$$

$$
\times \left(O(\Phi_n^M(k, x)) + \frac{k^{\alpha_2 + \alpha_3}}{\Gamma(1 + \alpha_2 + \alpha_3)} M(1 + \alpha_2 - k, 1 + \alpha_2 + \alpha_3, 2k\zeta_0) \sum_{j=0}^{n-1} \frac{a_{j,0}}{k^j} \right.
$$

$$
(3.39) \qquad + \frac{k^{\alpha_2 + \alpha_3 - 1}}{\Gamma(\alpha_2 + \alpha_3)} M(1 + \alpha_2 - k, \alpha_2 + \alpha_3, 2k\zeta_0) \sum_{j=0}^{n-1} \frac{a_{j,1}}{k^j} \Bigg),
$$

where $|a_{j,0}|, |a_{j,1}| \ll 1$ and

$$(3.40) \qquad\qquad \cos\theta_0 = 1 - 2x, \quad \sigma_0 + \sin\sigma_0 = \theta_0, \quad \zeta_0 = 1 - \cos\sigma_0,$$

$$\Phi_n^M(k, x) = k^{\varepsilon - n} |M(1 + \alpha_2 - k, 1 + \alpha_2 + \alpha_3, 2k\zeta_0)|$$

$$(3.41) \qquad\qquad + k^{\varepsilon - 1 - n} |M(1 + \alpha_2 - k, \alpha_2 + \alpha_3, 2k\zeta_0)|.$$

Proof This is [5, (48)]. The estimate (3.41) of the error term can be obtained in the same way as in [4, Theorem 3.2]. ∎

Lemma 3.11 *Assume that $|\alpha_j| \ll \delta(\varepsilon)$. For some small fixed $\varepsilon > 0$ such that $1 - \varepsilon < x < 1$ and $k \to \infty$, we have*

$$
{}_2F_1 \begin{pmatrix} 1 + \alpha_2 - k, k + \alpha_2 \\ 1 + \alpha_2 + \alpha_3 \end{pmatrix}; x = \frac{\Gamma(1 + \alpha_2 + \alpha_3)}{\Gamma(k + \alpha_2)} e^{-k\zeta_1}
$$

$$
\times \left(O(\Phi_n^U(k, x)) + U(1 + \alpha_2 - k, 1 + \alpha_2 - \alpha_3, 2k\zeta_1) \sum_{j=0}^{n-1} \frac{b_{j,0}}{k^{j + \alpha_3 - \alpha_2}} \right.
$$

$$
(3.42) \qquad - (k - 1 - \alpha_3) U(1 + \alpha_2 - k, \alpha_2 - \alpha_3, 2k\zeta_1) \sum_{j=0}^{n-1} \frac{b_{j,1}}{k^{j + 1 + \alpha_3 - \alpha_2}} \Bigg),
$$

where $|b_{j,0}|, |b_{j,1}| \ll 1$ and

$$(3.43) \qquad \cos\theta_1 = 2x - 1, \quad \sigma_1 + \sin\sigma_1 = \theta_1, \quad \zeta_1 = 1 - \cos\sigma_1,$$

$$
\begin{aligned}
\Phi_n^U(k, x) &= k^{\varepsilon - n}|U(1 + \alpha_2 - k, 1 + \alpha_2 - \alpha_3, 2k\zeta)| \\
(3.44) \qquad &\quad + k^{\varepsilon - n}|U(1 + \alpha_2 - k, \alpha_2 - \alpha_3, 2k\zeta)|.
\end{aligned}
$$

Proof This is [5, (74)]. The estimate (3.44) of the error term can be obtained in the same way as in [4, Theorem 3.2]. ∎

We are left to obtain asymptotic formulas for the Kummer functions of the form

$$(3.45) \qquad M(\alpha - a, c, az), \quad U(\alpha - a, c, az) \text{ as } a \to +\infty,$$

which appears on the right-hand sides of (3.39) and (3.42). To do this, we generalize the results of Temme [10, Sections 27.4.4 and 27.4.5]. Note that the M-Kummer function is in fact $_1F_1$-hypergeometric function, i.e., by [8, 13.2.2 and 16.2.1], we have

$$(3.46) \qquad M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{\Gamma(a + j)}{\Gamma(b + j)} \frac{z^j}{j!} = {}_1F_1\begin{pmatrix} a \\ b \end{pmatrix}; z\end{pmatrix}.$$

So, we are left to obtain an asymptotic formula for

$$(3.47) \qquad {}_1F_1\begin{pmatrix} \alpha - a \\ c \end{pmatrix}; az\end{pmatrix}.$$

The next Lemma is a generalization of [10, (27.4.71)].

Lemma 3.12 *For $0 \le z < 4$, fixed values of α and c, and $a \to \infty$, we have*

$$
\begin{aligned}
(3.48) \qquad {}_1F_1\begin{pmatrix} \alpha - a \\ c \end{pmatrix}; az\end{pmatrix} &= \frac{\Gamma(1 - \alpha + a)\Gamma(c)}{\Gamma(c + a - \alpha)} e^{az/2} \gamma^{1-c} \\
&\quad \times \left(O(\Phi_n^J(a, z)) + J_{c-1}(2\gamma a) \sum_{j=0}^{n-1} \frac{\tilde{A}_j(a, z)}{a^j} + \gamma J_{c-2}(2\gamma a) \sum_{j=0}^{n-1} \frac{\tilde{B}_j(a, z)}{a^j} \right),
\end{aligned}
$$

where $|\tilde{A}_j(a, z)|, |\tilde{B}_j(a, z)| \ll 1$ and

$$(3.49) \qquad \theta = \arcsin\frac{\sqrt{z}}{2}, \quad \gamma = \theta + \frac{1}{2}\sin(2\theta),$$

$$(3.50) \qquad \Phi_n^J(a, z) = a^{-n}|J_{c-1}(2\gamma a)| + \gamma a^{-n}|J_{c-2}(2\gamma a)|.$$

Proof Using [10, (10.1.9) and (10.3.46)], we first show that

$$(3.51) \qquad {}_1F_1\begin{pmatrix} \alpha - a \\ c \end{pmatrix}; az\end{pmatrix} = \frac{\Gamma(1 - \alpha + a)\Gamma(c)}{\Gamma(c + a - \alpha)} \frac{e^{az/2}}{2\pi i} \int_L e^{(c+a-\alpha)s - az/s} f(-az, -s) \frac{ds}{s^c},$$

where L is the Hankel contour (see [10, Figure 2.2]), and (see [10, (10.3.28)])

$$(3.52) \qquad f(z, s) = e^{zg_0(s)}\left(\frac{s}{1 - e^{-s}}\right)^c, \quad g_0(s) = \frac{1}{s} - \frac{1}{e^s - 1} - \frac{1}{2}.$$

Note that

$$(3.53) \qquad e^{cs} f(-z, -s) = f(az, s),$$

and therefore,

$$(3.54) \qquad e^{az/2} e^{(c+a)s - az/s} f(-az, -s) = e^{as - az/(e^s - 1)} \left(\frac{s}{1 - e^{-s}} \right)^c.$$

Substituting (3.54) to (3.51), we obtain an analog of [10, (27.4.65) and (27.4.66)]:

$$(3.55) \qquad {}_1F_1 \left(\begin{matrix} \alpha - a \\ c \end{matrix}; az \right) = \frac{\Gamma(1 - \alpha + a)\Gamma(c)}{\Gamma(c + a - \alpha)} \frac{1}{2\pi i} \int_L e^{a\psi(w)} \tilde{g}(w) \frac{dw}{w^c},$$

where

$$(3.56) \qquad \psi(w) = w - \frac{z}{e^w - 1}, \quad \tilde{g}(w) = e^{-\alpha w} \left(\frac{w}{1 - e^{-w}} \right)^c.$$

The only difference with [10, (27.4.66)] is the presence of the multiple $e^{-\alpha w}$ in (3.56). Proceeding as in [10], we obtain [10, (27.4.69)] with $p(t)$ being replaced by

$$(3.57) \qquad \tilde{p}(t) = \left(\frac{w}{t} \right)^{-c} \tilde{g}(w) \frac{dw}{dt} = e^{-\alpha w} \left(\frac{1 - e^{-w}}{t} \right)^{-c+2} \frac{t^2 + \gamma^2}{(1 - e^{-w})^2 + z e^{-w}},$$

and [10, (27.4.71)] with A_k, B_k being replaced by

$$(3.58) \qquad \tilde{p}_j(t) = -t^c \frac{d}{dt}(t^{1-c} \tilde{q}_{j-1}(t)) = \tilde{A}_j(a, z) + \tilde{B}_j(a, z)t + (t + \gamma^2/t)\tilde{q}_j(t),$$

$$(3.59) \qquad \tilde{A}_j(a, z) = \frac{\tilde{p}_j(i\gamma) + \tilde{p}_j(i\gamma)}{2}, \quad \tilde{B}_j(a, z) = \frac{\tilde{p}_j(i\gamma) - \tilde{p}_j(i\gamma)}{2i\gamma},$$

where $\tilde{p}_0(t) = \tilde{p}(t)$. The closed formulas for the first coefficients are (see [10, (27.4.74)]):

$$(3.60) \qquad \tilde{A}_0(a, z) = \left(\frac{\gamma}{2 \sin \theta} \right)^c \sqrt{\frac{2 \tan \theta}{\gamma}} \cos(c - 2\alpha)\theta,$$

$$(3.61) \qquad \tilde{B}_0(a, z) = \left(\frac{\gamma}{2 \sin \theta} \right)^c \sqrt{\frac{2 \tan \theta}{\gamma}} \frac{\sin(c - 2\alpha)\theta}{\gamma}.$$

Finally, the estimate (3.50) of the error term can be obtained in the same way as in [4, Theorem 3.2]. ∎

Now, we consider the case of the U-Kummer function [10, Section 27.4.5].

Lemma 3.13 *For $0 \le z < 4$, fixed values of α and c, and $a \to \infty$, we have*

$$U(\alpha - a, c, az) = \Gamma(1 - \alpha + a)e^{az/2}\gamma^{1-c}$$

$$(3.62) \qquad \times \left(O(\Phi_n^C(a, z)) + C_{c-1}(2\gamma a) \sum_{j=0}^{n-1} \frac{\tilde{A}_j(a, z)}{a^j} + \gamma C_{c-2}(2\gamma a) \sum_{j=0}^{n-1} \frac{\tilde{B}_j(a, z)}{a^j} \right),$$

where $\tilde{A}_j(a,z), \tilde{B}_j(a,z)$ are the same as in (3.48),

$$(3.63) \qquad C_\nu(z) = \cos\pi(a-\alpha)J_\nu(z) + \sin\pi(a-\alpha)Y_\nu(z),$$

and

$$(3.64) \qquad \Phi_n^C(a,z) = a^{-n}\left(|J_{c-1}(2\gamma a)| + |J_{c-2}(2\gamma a)| + |Y_{c-1}(2\gamma a)| + |Y_{c-2}(2\gamma a)|\right).$$

Proof   Using [10, (10.1.11) and (10.3.61)], we prove an analog of [10, (27.4.80)]:

$$(3.65) \qquad \begin{aligned} \frac{U(\alpha-a,c,az)}{\Gamma(c+a-\alpha)} &= \frac{e^{\mp\pi i(a-\alpha)}}{\Gamma(c)}\,{}_1F_1\left(\begin{matrix}\alpha-a\\c\end{matrix}; az\right) \\ &\quad - \frac{e^{az}e^{\pm\pi ic}}{\Gamma(\alpha-a)}U(c+a-\alpha,c,aze^{\pm\pi i}). \end{aligned}$$

For the ${}_1F_1$ function in (3.65), we can apply (3.48). Therefore, we are left to consider $U(c+a-\alpha,c,az)$ (at the end, we will replace az by $aze^{\pm\pi i}$). Applying [10, (10.3.27)] and (3.53), we prove an analog of [10, (10.3.62)]:

$$(3.66) \qquad U(c+a-\alpha,c,az) = \frac{e^{az/2}}{\Gamma(c-\alpha+a)}\int_0^\infty e^{-as-az/s}e^{\alpha s}f(-az,-s)\frac{ds}{s^c},$$

where the function $f(\cdot,\cdot)$ is defined by (3.52). Simplifying the function under the integral in (3.66), we have the following analog of [10, (27.4.81)]:

$$(3.67) \qquad U(c+a-\alpha,c,az) = \frac{1}{\Gamma(c-\alpha+a)}\int_0^\infty e^{-a\varphi(w)}\tilde{g}(-w)\frac{dw}{w^c},$$

where

$$(3.68) \qquad \varphi(w) = w + \frac{z}{e^w-1}, \quad \tilde{g}(-w) = e^{\alpha w}\left(\frac{e^w-1}{w}\right)^{-c}.$$

The only difference with [10, (27.4.81)] is the presence of the factor $e^{\alpha w}$. Arguing in the same way as in [10, Section 27.4.1], making the change of variable [10, (27.4.32)], we obtain an analog of [10, (27.4.38) and (27.4.82)], namely,

$$(3.69) \qquad U(c+a-\alpha,c,az) = \frac{e^{az/2}}{\Gamma(c-\alpha+a)}\int_0^\infty e^{-a(t+\beta^2/t)}\tilde{f}(-t)\frac{dt}{t^c},$$

where

$$(3.70) \qquad \beta = \frac{w_0+\sinh w_0}{2}, \quad w_0 = 2\operatorname{arcsinh}\frac{\sqrt{z}}{2}$$

and

$$(3.71) \qquad \tilde{f}(-t) = e^{\alpha w(t)}\left(\frac{e^{w(t)}-1}{t}\right)^{-c+2}\frac{t^2-\beta^2}{(e^{w(t)}-1)^2 - ze^{w(t)}}$$

is an analog of [10, (27.4.41)]. Note that $\tilde{f}(t)$ coincides with $\tilde{p}(t)$ defined in (3.57) after changing z to $-z$. Therefore, we show the following analog of [10, (27.4.84)]:

$$U(\alpha - a, c, aze^{\pm\pi i}) = 2\gamma^{1-c}e^{\pm\pi i(1-c)/2}\frac{e^{-az/2}}{\Gamma(c-\alpha+a)}$$

$$(3.72) \qquad \times \left(K_{1-c}(\pm 2i\gamma a)\sum_{j=0}^{\infty}\frac{\tilde{A}_j(a,z)}{a^j} \mp i\gamma K_{c-2}(\pm 2i\gamma a)\sum_{j=0}^{\infty}\frac{\tilde{B}_j(a,z)}{a^j}\right).$$

Using the relation [8, (10.27.3)], i.e., $K_{-\nu}(z) = K_\nu(z)$, and [10, (10.3.64)], we infer

$$(3.73) \qquad e^{\pm\pi i(c+1)/2}K_{c-1}(\pm 2i\gamma a) = \frac{\pi}{2}\left(Y_{c-1}(2\gamma a) \pm iJ_{c-1}(2\gamma a)\right)$$

$$(3.74) \qquad e^{\pm\pi i(c+1)/2}K_{c-2}(\pm 2i\gamma a) = \frac{\pi}{2}e^{\pm\pi i/2}\left(Y_{c-2}(2\gamma a) \pm iJ_{c-2}(2\gamma a)\right).$$

Truncating the series (3.72) at the point $j = n$ with the error (3.64) (see the proof of [4, Theorem 3.2]), we obtain

$$U(\alpha - a, c, aze^{\pm\pi i}) = \pi\gamma^{1-c}e^{\mp\pi ic}\frac{e^{-az/2}}{\Gamma(c-\alpha+a)}$$

$$\times \left(O(\Phi_n^C(a,z)) + (Y_{c-1}(2\gamma a) \pm iJ_{c-1}(2\gamma a))\sum_{j=0}^{n-1}\frac{\tilde{A}_j(a,z)}{a^j}\right.$$

$$(3.75) \qquad \left. + \gamma\left(Y_{c-2}(2\gamma a) \pm iJ_{c-2}(2\gamma a)\right)\sum_{j=0}^{n-1}\frac{\tilde{B}_j(a,z)}{a^j}\right).$$

Substituting (3.48) and (3.75) into (3.65), we have

$$\frac{U(\alpha - a, c, az)}{\Gamma(c+a-\alpha)} = \frac{\Gamma(1-\alpha+a)}{\Gamma(c+a-\alpha)}e^{az/2}\gamma^{1-c}$$

$$(3.76) \qquad \times \left(O(\Phi_n^C(a,z)) + C_{c-1}(2\gamma a)\sum_{j=0}^{n-1}\frac{\tilde{A}_j(a,z)}{a^j} + \gamma C_{c-2}(2\gamma a)\sum_{j=0}^{n-1}\frac{\tilde{B}_j(a,z)}{a^j}\right),$$

where

$$(3.77) \qquad C_\nu(2\gamma a) = e^{\mp\pi i(a-\alpha)}J_\nu(2\gamma a) - \sin\pi(\alpha - a)\left(Y_\nu(2\gamma a) \pm iJ_\nu(2\gamma a)\right).$$

Opening the brackets in (3.77), we complete the proof of (3.63). ∎

Proof of Proposition 3.7 First, we decompose the integral in (3.34) smoothly (say by inserting functions $\chi_0(x)$, $\chi_{1/2}(x)$, $\chi_1(x)$) into three ranges:

$$(3.78) \qquad 0 < x < \delta_0, \quad \delta_0 < x < 1 - \delta_1, \quad 1 - \delta_1 < x < 1,$$

where δ_0, δ_1 are some small constants.

Consider first the part over $0 < x < \delta_0$. In this case, we apply Lemma 3.10 followed by Lemma 3.12. Taking n in (3.39) and (3.48) sufficiently large, we obtain that the contribution of the error terms is negligible, and thus, it is enough to investigate only

the contribution of the main term (all other terms will have the same structure and will be smaller) which is bounded by the sum of the integrals of the following type:

$$e^{-\pi|\Im w|/2}(1+|w|)^\varepsilon k^\varepsilon \sum_{j=0}^{2} \int_0^1 \chi_0(x)(1-x)^{\frac{-1+\alpha_1+\alpha_2-\alpha_3}{2}-w}$$

$$(3.79) \qquad\qquad \times x^{\frac{-1+\alpha_1+\alpha_2+\alpha_3}{2}+w} A_j(x)\gamma(x)^{-\nu_j} J_{\nu_j}(2\gamma(x)k)dx,$$

where $\nu = -j + \alpha_2 + \alpha_3$ for $j = 0, 1, 2$ and $A_j(x)$ are products of coefficients in (3.39) and (3.48). Note that

$$(3.80) \qquad\qquad \gamma(x) = \frac{\arccos(1-2x)}{2} = \sqrt{x} + O(x^{3/2}),$$

since we first perform transformations (3.40) and then (3.49) with $z = 2\zeta_0$. Now, we decompose the integral (3.79) smoothly at the point $x = k^{-2+\varepsilon}$. For $0 < x < k^{-2+\varepsilon}$ using [8, (10.7.3)] and (3.80), we show that $\gamma(x)^{-\nu_j} J_{\nu_j}(2\gamma(x)k) \ll k^{\nu_j}$, and therefore,

$$(3.81)$$
$$\int_0^{k^{-2+\varepsilon}} (1-x)^{\frac{-1+\alpha_1+\alpha_2-\alpha_3}{2}-w} x^{\frac{-1+\alpha_1+\alpha_2+\alpha_3}{2}+w} A_j(x)\gamma(x)^{-\nu_j} J_{\nu_j}(2\gamma(x)k)dx \ll k^{-1+\varepsilon}.$$

In the case of $k^{-2+\varepsilon} < x < \delta_0$, we apply the asymptotic formula [7, 8.451.1] for the Bessel function. The error term is again negligible and we are left to estimate the integral of the following type (here, we set $w = ir$):

$$(3.82) \qquad\qquad \int_{k^{-2+\varepsilon}}^1 \chi_0(x)W(x)x^{\frac{-1+\alpha_1+\alpha_2+\alpha_3}{2}}\gamma(x)^{\alpha_2+\alpha_3}e^{ig_\pm(x)}\frac{dx}{\sqrt{\gamma(x)k}},$$

where $W(x) = (1-x)^{\frac{-1+\alpha_1+\alpha_2-\alpha_3}{2}} A_j(x)$ and the phase function is equal to

$$(3.83) \qquad\qquad g_\pm(x) = \pm 2\gamma(x)k + r\log x - r\log(1-x).$$

Estimating it by absolute value using (3.79), we prove (3.32). For $|r| \ll \log^2 k$, we have $g'_\pm(x) \gg k/\sqrt{x}$, and therefore, using the first derivative test, we estimate (3.82) as

$$(3.84) \qquad\qquad \max_{k^{-2+\varepsilon}<x<\delta_0} \frac{x^{\frac{\alpha_1+\alpha_2+\alpha_3}{2}}\gamma(x)^{\alpha_2+\alpha_3}}{k\sqrt{\gamma(x)k}} \ll \frac{k^\varepsilon}{k}.$$

As a result, in view of (3.79), we prove (3.33).

Consider the third part of the integral (3.34) over $1 - \delta_1 < x < 1$. In this case, we apply Lemma 3.11 followed by Lemma 3.13. Again the contribution of the error terms is negligible, and thus, it is enough to investigate only the contribution of the main term (all other terms will have the same structure and will be smaller), which is the sum of the integrals of the following type:

$$e^{-\pi|\Im w|/2}(1+|w|)^\varepsilon k^\varepsilon \sum_{j=0}^{2} \int_0^1 \chi_1(x)(1-x)^{\frac{-1+\alpha_1+\alpha_2-\alpha_3}{2}-w}$$

$$(3.85) \qquad\qquad \times x^{\frac{-1+\alpha_1+\alpha_2+\alpha_3}{2}+w} D_j(x)\gamma(x)^{-\nu_j} C_{\nu_j}(2\gamma_1(x)k)dx,$$

where $v_j = -j + \alpha_2 - \alpha_3$ for $j = 0, 1, 2$, $D_j(x)$ are products of coefficients in (3.42) and (3.62), and by (3.63), we have

(3.86)
$$C_{v_j}(2\gamma_1(x)k) = (-1)^{k-1}\cos(\pi\alpha_2)J_{v_j}(2\gamma_1(x)k) + (-1)^k\sin(\pi\alpha_2)Y_{v_j}(2\gamma_1(x)k).$$

Note that now since we first perform transformations (3.43) and then (3.49) with $z = 2\zeta_1$, we have

(3.87)
$$\gamma_1(x) = \frac{\arccos(2x-1)}{2} = \sqrt{1-x} + O((1-x)^{3/2}).$$

Now, we decompose the integral in (3.85) smoothly $x = 1 - k^{-2+\varepsilon}$. For $1 - k^{-2+\varepsilon} < x < 1$ using [8, (10.7.3)–(10.7.5)] and (3.87), we have

(3.88)
$$\gamma_1(x)^{-v_j}C_{v_j}(2\gamma_1(x)k) \ll \gamma_1(x)^{-v_j}(\gamma_1(x)k)^{-|v_j|} \ll (k(1-x))^\varepsilon,$$

and therefore,

(3.89)
$$\int_{1-k^{-2+\varepsilon}}^1 (1-x)^{\frac{-1+\alpha_1+\alpha_2-\alpha_3}{2}-w} x^{\frac{-1+\alpha_1+\alpha_2+\alpha_3}{2}+w} D_j(x)\gamma_1(x)^{-v_j}C_{v_j}(2\gamma_1(x)k)dx \ll k^{-1+\varepsilon}.$$

To handle the case of $1 - \delta_1 < x < 1 - k^{-2+\varepsilon}$, we apply an asymptotic formulas [7, 8.451.1 and 8.451.2] for the Bessel functions. Using them, one can write $C_{v_j}(2\gamma_1(x)k)$ as

(3.90)
$$C_{v_j}(y) = \sum_\pm \frac{e^{\pm iy}}{\sqrt{y}} V_\pm(y),$$

where $V_\pm(y)$ smoothed non-oscillating functions uniformly bounded by a constant. So we are left to estimate integral of the following type (here, we set $w = ir$):

(3.91)
$$\int_0^{1-k^{-2+\varepsilon}} \chi_1(x)W(x)(1-x)^{\frac{-1+\alpha_1+\alpha_2-\alpha_3}{2}}\gamma(x)^{-\alpha_2+\alpha_3}e^{ig_\pm(x)}\frac{dx}{\sqrt{\gamma_1(x)k}},$$

where $W(x) = x^{\frac{-1+\alpha_1+\alpha_2+\alpha_3}{2}}V_\pm(2\gamma(x)k)D_j(x)$ and the phase function

(3.92)
$$g_\pm(x) = \pm 2\gamma_1(x)k + r\log x - r\log(1-x).$$

Estimating the integral by absolute value and in view of (3.85), we obtain the estimate (3.32). For $|r| \ll \log^2 k$, we have $g'_\pm(x) \gg k/\sqrt{1-x}$ and thus using the first derivative test, we estimate (3.91) as

(3.93)
$$\max_{1-\delta_1 < x < 1-k^{-2+\varepsilon}} \frac{(1-x)^{\frac{\alpha_1+\alpha_2-\alpha_3}{2}}\gamma_1(x)^{-\alpha_2+\alpha_3}}{k\sqrt{\gamma_1(x)k}} \ll \frac{k^\varepsilon}{k}.$$

Finally, in view of (3.85), we prove the estimate (3.33).

Consider the second part of the integral (3.34) over $\delta_0 < x < 1 - \delta_1$. In this case, we make the change of variable $x = \cos^2 y$ and apply Lemma 3.9. Again the contribution of the error terms is negligible and thus it is enough to investigate only the contribution of the main term (all other terms will have the same structure and will be smaller). Note that now y is bounded away from 0 and $\pi/2$. So the arguments of Y and J-Bessel

functions in (3.36) are comparable with k and we can apply for their combination an asymptotic formula similar to (3.90). Finally, we obtain the integrals

$$(3.94) \qquad \frac{e^{-\pi|\Im w|/2}(1+|w|)^{\varepsilon}}{k^{1/2+\alpha_2+\alpha_3}} \int_0^1 \chi_{1/2}(\cos^2 y)\, W_{\pm}(y)\, e^{if_{\pm}(y)}\, dy,$$

where $W_{\pm}(y)$ are smoothed non-oscillating functions uniformly bounded by a constant and

$$(3.95) \qquad f_{\pm}(y) = \pm(2k-1)y - 2r\log(\tan y).$$

Estimating the integral (3.94) by absolute value, we obtain the estimate (3.32). For $|r| \ll \log^2 k$, we have $f_{\pm}'(y) \gg k$, and thus using the first derivative test, we estimate (3.94) as

$$(3.96) \qquad \frac{e^{-\pi|\Im w|/2}(1+|w|)^{\varepsilon}}{k^{1/2+\alpha_2+\alpha_3}} \max_{0<y<\pi/2} \frac{\chi_{1/2}(\cos^2 y)\, W_{\pm}(y)}{k} \ll \frac{e^{-\pi|\Im w|/2}(1+|w|)^{\varepsilon} k^{\varepsilon}}{k},$$

completing the proof of (3.33). ∎

4   The proof of Theorem 1.1

As one can see from (2.2) and (1.2), the term $\mathcal{MT}_D(\tilde{\alpha})$ in (2.16) contains four of eight summands appearing in (1.2). The remaining four terms are hidden in the expressions $R_{i,j}(u,v,\rho,k)$. Using (2.11), (2.12), and (3.14), we obtain that $R_{2,1}$ and $R_{2,2}$ are bounded by $k^{-2+\varepsilon}$. It follows from [6, (5.18)] and [6, (5.19)] that

$$-R_{2,3}(u,v,\rho,k) = \zeta(1-\alpha_1+\alpha_2)\zeta(1-\alpha_1-\alpha_3)\zeta(1+\alpha_2-\alpha_3)$$

$$(4.1) \qquad \times \mathcal{C}(-1,1,-1)\frac{\cos\frac{\pi(\alpha_2+\alpha_3)}{2}}{2\cos\frac{\pi(\alpha_1+\alpha_3)}{2}\cos\frac{\pi(\alpha_1-\alpha_2)}{2}},$$

$$-R_{2,4}(u,v,\rho,k) = \zeta(1-\alpha_1-\alpha_2)\zeta(1-\alpha_1+\alpha_3)\zeta(1-\alpha_2+\alpha_3)$$

$$(4.2) \qquad \times \mathcal{C}(-1,-1,1)\frac{\cos\frac{\pi(\alpha_2+\alpha_3)}{2}}{2\cos\frac{\pi(\alpha_1+\alpha_2)}{2}\cos\frac{\pi(\alpha_1-\alpha_3)}{2}}.$$

Now using (2.15), (4.1), and (4.2), the term $\mathcal{ET}_3(u,v,\rho)$ produces the following parts of the main term $\mathcal{MT}_3(\alpha_1,\alpha_2,\alpha_3)$:

$$-R_{3,3}(u,v,\rho,k) = \zeta(1-\alpha_1+\alpha_2)\zeta(1-\alpha_1+\alpha_3)\zeta(1+\alpha_2+\alpha_3)$$

$$(4.3) \qquad \times \mathcal{C}(-1,1,1)\frac{\cos\frac{\pi(\alpha_2-\alpha_3)}{2}}{2\cos\frac{\pi(\alpha_1-\alpha_3)}{2}\cos\frac{\pi(\alpha_1-\alpha_2)}{2}},$$

and

$$-R_{3,4}(u,v,\rho,k) = \zeta(1-\alpha_1-\alpha_2)\zeta(1-\alpha_1-\alpha_3)\zeta(1-\alpha_2-\alpha_3)$$

$$(4.4) \qquad \times \mathcal{C}(-1,-1,-1)\frac{\cos\frac{\pi(\alpha_2-\alpha_3)}{2}}{2\cos\frac{\pi(\alpha_1+\alpha_2)}{2}\cos\frac{\pi(\alpha_1+\alpha_3)}{2}}.$$

Substituting [6, (4.29)] into [6, (5.37)] and [6, (5.38)], we obtain (the estimate [6, (4.31)] could be easily generalized in case of small values of u, v, ρ)

$$R_{1,3}(u, v, \rho, k) = -\zeta(1 - \alpha_1 - \alpha_2)\zeta(1 - \alpha_1 - \alpha_3)\zeta(1 - \alpha_2 - \alpha_3)$$

$$(4.5) \qquad \times \frac{\Gamma(k - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(k + \alpha_1 + \alpha_2 + \alpha_3)} \frac{(2\pi)^{2\alpha_1 + 2\alpha_2 + 2\alpha_3} \cos \frac{\pi(2\alpha_1 + \alpha_2 + \alpha_3)}{2}}{2 \cos \frac{\pi(\alpha_1 + \alpha_2)}{2} \cos \frac{\pi(\alpha_1 + \alpha_3)}{2}} + O(k^{-1+\varepsilon}),$$

$$R_{1,4}(u, v, \rho, k) = -\zeta(1 - \alpha_1 + \alpha_2)\zeta(1 - \alpha_1 + \alpha_3)\zeta(1 + \alpha_2 + \alpha_3)$$

$$(4.6) \qquad \times (-1)^k \mathcal{C}(-1, 1, 1) \frac{\cos \frac{\pi(2\alpha_1 - \alpha_2 - \alpha_3)}{2}}{2 \cos \frac{\pi(\alpha_1 - \alpha_2)}{2} \cos \frac{\pi(\alpha_1 - \alpha_3)}{2}} + O(k^{-1+\varepsilon}).$$

Substituting (3.26) into (2.5), (2.6) and using the functional equation for the Riemann zeta functions:

$$\zeta(\rho + u \pm v)\zeta(\rho - u \pm v)\Gamma(\rho + u \pm v)\Gamma(\rho - u \pm v)$$

$$= \frac{(2\pi)^{2\rho \pm 2v}}{4 \cos \frac{\pi(\rho + u \pm v)}{2} \cos \frac{\pi(\rho - u \pm v)}{2}} \zeta(1 - \rho - u \mp v)\zeta(1 - \rho + u \mp v),$$

we obtain

$$R_{1,1}(u, v, \rho, k) = \zeta(1 - \alpha_1 - \alpha_2)\zeta(1 - \alpha_1 + \alpha_3)\zeta(1 - \alpha_2 + \alpha_3)$$

$$\times \frac{(-1)^k (2\pi)^{2\alpha_1 + 2\alpha_2}}{4 \cos \frac{\pi(\alpha_1 + \alpha_2)}{2} \cos \frac{\pi(\alpha_1 - \alpha_3)}{2}} \left(\frac{\Gamma(k - \alpha_3)\Gamma(k - \alpha_1 - \alpha_2 + \alpha_3)}{\Gamma(k + \alpha_3)\Gamma(k + \alpha_1 + \alpha_2 - \alpha_3)} \frac{\sin(\pi\alpha_1)}{\sin \frac{\pi(\alpha_3 - \alpha_2)}{2}} \right.$$

$$(4.7) \qquad \left. + \frac{\Gamma(k - \alpha_1)\Gamma(k - \alpha_2)}{\Gamma(k + \alpha_1)\Gamma(k + \alpha_2)} \frac{\sin \pi(\alpha_1 + \alpha_2 - \alpha_3)}{\sin \frac{\pi(\alpha_2 - \alpha_3)}{2}} \right) + O(k^{-1+\varepsilon}),$$

$$R_{1,2}(u, v, \rho, k) = \zeta(1 - \alpha_1 + \alpha_2)\zeta(1 - \alpha_1 - \alpha_3)\zeta(1 + \alpha_2 - \alpha_3)$$

$$\times \frac{(-1)^k (2\pi)^{2\alpha_1 + 2\alpha_3}}{4 \cos \frac{\pi(\alpha_1 + \alpha_3)}{2} \cos \frac{\pi(\alpha_1 - \alpha_2)}{2}} \left(\frac{\Gamma(k - \alpha_2)\Gamma(k - \alpha_1 + \alpha_2 - \alpha_3)}{\Gamma(k + \alpha_2)\Gamma(k + \alpha_1 - \alpha_2 + \alpha_3)} \frac{\sin(\pi\alpha_1)}{\sin \frac{\pi(\alpha_2 - \alpha_3)}{2}} \right.$$

$$(4.8) \qquad \left. + \frac{\Gamma(k - \alpha_1)\Gamma(k - \alpha_3)}{\Gamma(k + \alpha_1)\Gamma(k + \alpha_3)} \frac{\sin \pi(\alpha_1 - \alpha_2 + \alpha_3)}{\sin \frac{\pi(\alpha_3 - \alpha_2)}{2}} \right) + O(k^{-1+\varepsilon}).$$

Therefore, we have considered all $R_{i,j}(u, v, \rho, k)$. Applying (4.3), (4.6) (note that there is a factor $(-1)^{k+1}$ in (2.4)), and

$$\cos \frac{\pi(\alpha_2 - \alpha_3)}{2} + \cos \frac{\pi(2\alpha_1 - \alpha_2 - \alpha_3)}{2} = 2 \cos \frac{\pi(\alpha_1 - \alpha_3)}{2} \cos \frac{\pi(\alpha_1 - \alpha_2)}{2},$$

we find that

$$-(-1)^k R_{1,4}(u, v, \rho, k) - R_{3,3}(u, v, \rho, k)$$

$$(4.9) \qquad = \zeta(1 - \alpha_1 + \alpha_2)\zeta(1 - \alpha_1 + \alpha_3)\zeta(1 + \alpha_2 + \alpha_3)\mathcal{C}(-1, 1, 1) + O(k^{-1+\varepsilon}).$$

To transform the remaining $R_{i,j}(\cdot)$ functions, we will apply the following asymptotic formula for (1.7):

$$(4.10) \qquad X_k(\alpha) = \frac{(2\pi)^{2\alpha}}{k^{2\alpha}}\left(1 + O\left(\frac{1}{k}\right)\right),$$

which is a consequence of the Stirling formula [8, 5.11.3 and 5.11.13].

Applying (4.1), (4.8) (note that there is a factor $(-1)^k$ in (2.4)), and the relation (see (4.10))

$$(2\pi)^{2\alpha_1+2\alpha_3}\frac{\Gamma(k-\alpha_2)\Gamma(k-\alpha_1+\alpha_2-\alpha_3)}{\Gamma(k+\alpha_2)\Gamma(k+\alpha_1-\alpha_2+\alpha_3)} = X_k(\alpha_2)X_k(\alpha_1-\alpha_2+\alpha_3) + O(k^{-1+\varepsilon})$$

$$= \mathcal{C}(-1,1,-1) + O(k^{-1+\varepsilon}),$$

we find that

$$(-1)^k R_{1,2}(u,v,\rho,k) - R_{2,3}(u,v,\rho,k) = \zeta(1-\alpha_1+\alpha_2)\zeta(1-\alpha_1-\alpha_3)\zeta(1+\alpha_2-\alpha_3)$$

$$\times \frac{\mathcal{C}(-1,1,-1)}{4\cos\frac{\pi(\alpha_1+\alpha_3)}{2}\cos\frac{\pi(\alpha_1-\alpha_2)}{2}\sin\frac{\pi(\alpha_2-\alpha_3)}{2}}$$

$$\times \left(2\cos\frac{\pi(\alpha_2+\alpha_3)}{2}\sin\frac{\pi(\alpha_2-\alpha_3)}{2} + \sin(\pi\alpha_1) - \sin\pi(\alpha_1-\alpha_2+\alpha_3)\right) + O(k^{-1+\varepsilon})$$

$$= \zeta(1-\alpha_1+\alpha_2)\zeta(1-\alpha_1-\alpha_3)\zeta(1+\alpha_2-\alpha_3)\mathcal{C}(-1,1,-1) + O(k^{-1+\varepsilon}).$$

(4.11)

Using (4.2), (4.7), and the relation (see (4.10)):

$$(2\pi)^{2\alpha_1+2\alpha_2}\frac{\Gamma(k-\alpha_3)\Gamma(k-\alpha_1-\alpha_2+\alpha_3)}{\Gamma(k+\alpha_3)\Gamma(k+\alpha_1+\alpha_2-\alpha_3)} = X_k(\alpha_3)X_k(\alpha_1+\alpha_2-\alpha_3) + O(k^{-1+\varepsilon})$$

$$= \mathcal{C}(-1,-1,1) + O(k^{-1+\varepsilon}),$$

we prove that

$$(-1)^k R_{1,1}(u,v,\rho,k) - R_{2,4}(u,v,\rho,k)$$

$$= \frac{\zeta(1-\alpha_1-\alpha_2)\zeta(1-\alpha_1+\alpha_3)\zeta(1-\alpha_2+\alpha_3)\mathcal{C}(-1,-1,1)}{4\cos\frac{\pi(\alpha_1+\alpha_2)}{2}\cos\frac{\pi(\alpha_1-\alpha_3)}{2}\sin\frac{\pi(\alpha_3-\alpha_2)}{2}}$$

$$\times \left(2\cos\frac{\pi(\alpha_2+\alpha_3)}{2}\sin\frac{\pi(\alpha_3-\alpha_2)}{2} + \sin(\pi\alpha_1) - \sin\pi(\alpha_1+\alpha_2-\alpha_3)\right)$$

$$+ O(k^{-1+\varepsilon}) = \zeta(1-\alpha_1-\alpha_2)\zeta(1-\alpha_1+\alpha_3)\zeta(1-\alpha_2+\alpha_3)\mathcal{C}(-1,-1,1) + O(k^{-1+\varepsilon}).$$

(4.12)

Combining (4.4), (4.5), and the relation (see (4.10)):

$$(2\pi)^{2\alpha_1+2\alpha_2+2\alpha_3}\frac{\Gamma(k-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(k+\alpha_1+\alpha_2+\alpha_3)} = X_k(\alpha_1+\alpha_2+\alpha_3) + O(k^{-1+\varepsilon})$$

$$= (-1)^k\mathcal{C}(-1,-1,-1) + O(k^{-1+\varepsilon}),$$

we infer that

$$
\begin{aligned}
-(-1)^k R_{1,3}(u,v,\rho,k) &- R_{3,4}(u,v,\rho,k) = \zeta(1-\alpha_1-\alpha_2)\zeta(1-\alpha_1-\alpha_3)\zeta(1-\alpha_2-\alpha_3) \\
&\times \frac{\mathcal{C}(-1,-1,-1)}{2\cos\frac{\pi(\alpha_1+\alpha_2)}{2}\cos\frac{\pi(\alpha_1+\alpha_3)}{2}}\left(\cos\frac{\pi(\alpha_2-\alpha_3)}{2}+\cos\frac{\pi(2\alpha_1+\alpha_2+\alpha_3)}{2}\right)+O(k^{-1+\varepsilon}) \\
&= \zeta(1-\alpha_1-\alpha_2)\zeta(1-\alpha_1-\alpha_3)\zeta(1-\alpha_2-\alpha_3)\mathcal{C}(-1,-1,-1)+O(k^{-1+\varepsilon}).
\end{aligned}
$$
(4.13)

Substituting (2.4), (2.10), (2.15) into (2.16) and applying (2.2), (4.9), (4.11), (4.12), (4.13), we obtain

$$
\begin{aligned}
\mathcal{M}(\alpha_1,\alpha_2,\alpha_3) &= \mathcal{MT}_3(\alpha_1,\alpha_2,\alpha_3) + \frac{(-1)^k}{2\pi i}\int_{(0)} Z_1(u,v,\rho,k;w)\,dw \\
&+ \frac{1}{2\pi i}\int_{(0)} Z_2(u,v,\rho,k;w) + (-1)^k\frac{\Gamma(k-u+v)}{\Gamma(k+u-v)}(2\pi)^{2u-2v}Z_2(v,u,\rho,k;w)\,dw \\
&+ O(k^{-1+\varepsilon}).
\end{aligned}
$$
(4.14)

Using (2.9) and (3.8), we show that

(4.15)
$$
\mathcal{M}(\alpha_1,\alpha_2,\alpha_3) = \mathcal{MT}_3(\alpha_1,\alpha_2,\alpha_3) + \frac{(-1)^k}{2\pi i}\int_{(0)} Z_1(u,v,\rho,k;-w)\,dw + O(k^{-1+\varepsilon}).
$$

Applying the functional equation [8, 25.4.1] to the $\zeta\left(\frac{1+\rho}{2}-u-w\right)$ in (2.3), rewriting the obtained expression in terms of $\tilde{\alpha}$ and using (1.8), we have

(4.16)
$$
Z_1(u,v,\rho,k;-w) = \frac{\zeta_4(\tilde{\alpha};w)(2\pi)^{\frac{1+\alpha_1-\alpha_2-\alpha_3}{2}+w}\hat{g}_1(u,v,\rho,k;-w)}{2\cos\frac{\pi}{2}\left(\frac{1+\alpha_1-\alpha_2-\alpha_3}{2}+w\right)\Gamma\left(\frac{1+\alpha_1-\alpha_2-\alpha_3}{2}+w\right)}.
$$

Substituting (3.30) into (4.16) and using (3.8), we find

$$
\begin{aligned}
&\frac{(-1)^k}{2\pi i}\int_{(0)} Z_1(u,v,\rho,k;-w)\,dw \\
&= \frac{1}{2\pi i}\int_{(0)}\frac{\sin\pi(\alpha_1/2-w)\cos\pi\left(\frac{\alpha_1-\alpha_2-\alpha_3}{2}+w\right)}{\cos\frac{\pi}{2}\left(\frac{1+\alpha_1-\alpha_2-\alpha_3}{2}+w\right)\cos\pi\left(\frac{\alpha_1+\alpha_2+\alpha_3}{2}-w\right)} \\
&\qquad\times \mathcal{H}_1(\tilde{\alpha},k;w)\zeta_4(\tilde{\alpha};w)(2\pi)^{\frac{1+\alpha_1+\alpha_2+\alpha_3}{2}+w}\,dw + O(k^{-1+\varepsilon}).
\end{aligned}
$$
(4.17)

Let $w = ir$, then

$$
\begin{aligned}
&\frac{\sin\pi(\alpha_1/2-w)\cos\pi\left(\frac{\alpha_1-\alpha_2-\alpha_3}{2}+w\right)}{\cos\frac{\pi}{2}\left(\frac{1+\alpha_1-\alpha_2-\alpha_3}{2}+w\right)\cos\pi\left(\frac{\alpha_1+\alpha_2+\alpha_3}{2}-w\right)} \\
&\qquad = \exp\left(\frac{\pi|r|}{2}-\pi i\,\mathrm{sgn}(r)\frac{1+\alpha_1+\alpha_2+\alpha_3}{4}\right)+O(\exp(-\pi|r|/2)).
\end{aligned}
$$
(4.18)

Substituting (4.18) into (4.17) and using (3.33), we obtain

$$\frac{(-1)^k}{2\pi i} \int_{(0)} Z_1(u, v, \rho, k; -w)\, dw$$

$$(4.19) \qquad = \frac{1}{2\pi i} \int_{(0)} l_1(\tilde{\alpha}, k; w)\mathcal{H}_1(\tilde{\alpha}, k; w)\zeta_4(\tilde{\alpha}; w)\, dw + O(k^{-1+\varepsilon}).$$

Finally, (1.12) follows from (4.15) and (4.19).

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