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Simple zeros of the zeta function of a quadratic number field. I

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1. Introduction

Let $\zeta_K(s)$ be the Dedekind zeta function of a quadratic extension K of the rationals. It is known that all the non-real zeros of ζ_K are in the strip $0<\mathrm{Re}.\ s<1$ and that the number of zeros of ζ_K in the rectangle $\mathscr{R}=\{s=\sigma+it: 0<\sigma<1,\ 0<t<T\}$ is asymptotically $\pi^{-1}T\log T$. It is generally believed that all the zeros of ζ_K are simple zeros, but it is not known whether ζ_K has infinitely many non-real simple zeros. In fact, prior to 1970 it was not even known that the Riemann zeta function $\zeta(s)$ has infinitely many complex simple zeros. The only related theorem was Selberg's result of 1942 that a positive proportion of the zeros of $\zeta(s)$ are of odd order and lie on the critical line. In 1972, Montgomery showed that if the Riemann Hypothesis is true then at least 2/3 of the zeros of $\zeta(s)$ are simple and in 1974 Levinson proved that at least 1/3 of the zeros lie on the critical line. It was observed (independently) by Heath-Brown and Selberg that Levinson's work actually implied that 1/3 of the zeros are simple and on the critical line.

Thus, Levinson's method is the only method to date which gives unconditional results on simple zeros of Dirichlet series. It is rather unlikely that his method could be successfully applied to $\zeta_K(s)$. Here we develop a new method and prove

Theorem. *The number of simple zeros of $\zeta_K(s)$ in $\mathscr{R}$ exceeds $T^{\frac{6}{11}}$ if T is sufficiently large.*

The zeros we find are also simple zeros of $\zeta(s)$ and we prove slightly more, namely that the number of simple zeros exceeds $T^{\theta-\varepsilon}$ for any $\varepsilon>0$ and $T>T_0(\varepsilon)$ where

$$\theta=\max\left\{\frac{1}{1+6c},\frac{(1+16c+16c^2)^{\frac{1}{2}}-1-4c}{4c}\right\} \tag{1}$$

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and c is any number for which

$$\zeta(\tfrac{1}{2}+it)\ll_{\varepsilon} t^{c+\varepsilon}$$

holds for all $\varepsilon>0$ and $t>1$. By the recent work of Bombieri and Iwaniec we may take $c=9/56$ which leads to $\theta=0.54959\ldots$. If $c=0$ (the Lindelof hypothesis) then our theorem yields $T^{1-\varepsilon}$ simple zeros of ζ_K in $\mathscr{R}$.

In another paper (to appear in the Proceedings of the Stillwater Number Theory Conference) we prove that the Riemann Hypothesis implies that a positive proportion (at least 1/54) of the zeros of $\zeta_K(s)$ are simple. This result appears to be unattainable by Montgomery's pair-correlation method.

2. Notation, conventions, and standard results

The occurence of ε in an estimate should be interpreted to mean that the estimate is true for all $\varepsilon>0$ and that the implicit constant depends on ε.

Sums with omitted limits are sums over all positive integers (subject to other stated conditions).

Throughout the paper ψ is a fixed real primitive character to the modulus q and χ denotes a character with modulus less than or equal to q. Gauss' sum is

$$\tau(\psi)=\sum_{a=1}^{q}\psi(a)e(a/q)$$

where $e(x)=\exp(2\pi ix)$. Since ψ is real and primitive

$$\sum_{a=1}^{q}\psi(a)e(ab/q)=\psi(b)\tau(\psi) \tag{2}$$

(see Davenport [2], Chap. 9).

The Riemann zeta-function is given by $\zeta(s)=\sum n^{-s}$ for $\sigma>1$, and

$$L(s,\chi)=\sum_{n}\chi(n)n^{-s}\qquad(\sigma>1)$$

is a Dirichlet L-function. Zeros of $\zeta_K(s)$, $\zeta(s)$, and $L(s,\psi)$ are denoted by ρ_K, ρ, and ρ_ψ, respectively, and $\rho=\beta+i\gamma$. Our results are not uniform in K and constants implied by the O- and $\ll$-notation depend in general on K and q. (For example, $\tau(\chi)\ll 1$.)

The functional equations for ζ and L are

$$\zeta(1-s)=X(1-s)\zeta(s) \tag{3}$$

where

$$X(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2} \tag{4}$$

and

$$L(1-s,\psi)=X(1-s,\psi)L(s,\psi) \tag{5}$$

where

$$X(1-s,\psi)=\frac{q^s}{\tau(\psi)}(2\pi)^{-s}\Gamma(s)(e^{\pi i s/2}+\psi(-1)e^{-\pi i s/2}). \tag{6}$$

Some standard estimates we use, which can be found in Titchmarsh [6], are

$$\frac{X'}{X}(s)=-\log\frac{|t|}{2\pi}+O\left(\frac{1}{|t|}\right) \qquad (|t|>1); \tag{7}$$

$$X(1-s,\chi),\ X(1-s)\ll t^{\sigma-\frac{1}{2}} \qquad (t>1,\ |\sigma|<2); \tag{8}$$

$$\zeta(\sigma+it),\ \zeta'(\sigma+it)\ll_\varepsilon t^{2c(1-\sigma)+\varepsilon} \qquad (t>1,\ \sigma\geqq\tfrac{1}{2}), \tag{9}$$

where c is as in the introduction; a similar estimate holds for $L(s,\psi)$;

$$\frac{\zeta'}{\zeta}(s),\ \zeta(s),\ \zeta'(s),\ L(s,\psi)\ll(\log T)^A \tag{10}$$

for some $A>0$ where $\sigma>1-(\log T)^{-1}$, $|t|\leqq T$, and $|s-1|\geqq(\log T)^{-1}$; and for every $T>1$ there is a T' such that

$$|T-T'|\leqq 1,$$

and

$$\frac{\zeta'}{\zeta}(\sigma+iT')\ll\log^2 T \qquad (-1\leqq\sigma\leqq 2). \tag{11}$$

As usual,

$$N(\sigma,T)=\#\{\rho\colon |\gamma|\leqq T,\ \beta\geqq\sigma\}.$$

We use the following density results due to Ingham:

$$N(\sigma,T)\ll_\varepsilon T^{\frac{3}{2-\sigma}(1-\sigma)+\varepsilon} \qquad (\sigma\geqq\tfrac{1}{2}) \tag{12}$$

and

$$N(\sigma,T)\ll_\varepsilon T^{2(1+2c)(1-\sigma)+\varepsilon} \qquad (\sigma\geqq\tfrac{1}{2}), \tag{13}$$

where c is as in (9) (see Titchmarsh [6], Chap. 9).

We use $d_k(n)$ to denote the k-fold divisor function.

3. Lemmas

We quote the following two lemmas which we will use repeatedly.

Lemma 1. *Suppose that* $A(s)=\sum_{n=1}^{\infty}a(n)n^{-s}$ for $\sigma>1$ *where*

$$a(n)\ll d_k(n)(\log n)^l$$

for some non-negative integers k and l. Then

$$R:=\sum_{n\leqq x}a(n)-\frac{1}{2\pi i}\int_{\alpha-iU}^{\alpha+iU}A(s)\frac{x^s}{s}\,ds\ll_\varepsilon x^\varepsilon(1+xU^{-1})$$

where $\alpha=1+(\log x)^{-1}$. *If* $U \ll x^{1-\varepsilon}$, *then*

$$R \ll x U^{-1} (\log x)^{k+l}.$$

This lemma is easy to prove by standard methods. Also, it can be found in Conrey, Ghosh, and Gonek [1], Lemma 4.

Lemma 2. *Let* $A(s)$ *be as above and* $\alpha=1+(\log T)^{-1}$. *If* r *is a (fixed) positive integer, then*

$$\frac{1}{2\pi i}\int_{\alpha+i}^{\alpha+iT} X(1-s) r^s A(s)\,ds - \sum_{n \leq \frac{rT}{2\pi}} a(n) e(-n/r) \ll T^{\frac{1}{2}} (\log T)^{k+l}.$$

We will use this lemma with $r=1$ and with $r=q$. It is proved implicitly in Gonek [4], Lemma 5, and also appears in Conrey, Ghosh, and Gonek [1], Lemma 2. For convenience, we also state

Lemma 3. *Suppose that*

$$A_j(s)=\sum_{j=1}^{\infty} a_j(m) m^{-s} \quad \text{and} \quad A(s)=\sum_{m=1}^{\infty} a(m) m^{-s}=\prod_{j=1}^{J} A_j(s) \qquad \text{for } \sigma>1.$$

Then for any $d>0$ *and any completely multiplicative function* f *we have*

$$\sum_{m=1}^{\infty} a(md) f(m) m^{-s} = \sum_{d_1 \ldots d_J = d} \prod_{j=1}^{J} \sum_{\substack{m=1 \\ \left(m, \prod_{i<j} d_i\right)=1}}^{\infty} a_j(m d_j) f(m) m^{-s},$$

for $\sigma>1$.

This is essentially Lemma 3 of Conrey, Ghosh, and Gonek [1].

4. Beginning of the proof.

Our proof relies heavily on the factorization of ζ_K as

$$\zeta_K(s)=\zeta(s) L(s,\psi), \tag{14}$$

where ψ is a real primitive character to the modulus q which is the absolute value of the discriminant of K. In particular, a zero ρ of ζ is also a zero of ζ_K, so that in looking for simple zeros of ζ_K we will restrict our attention to zeros of ζ. A zero ρ of ζ is a simple zero of ζ_K if and only if $\zeta_K'(\rho) \neq 0$. By (14)

$$\zeta_K'(\rho)=\zeta'(\rho) L(\rho,\psi). \tag{15}$$

The non-trivial zeros of $L(s,\psi)$ are symmetric about the line $\sigma=\frac{1}{2}$ by the functional equation for L, so that $L(\rho,\psi)=0$ if and only if $L(1-\rho,\psi)=0$. Thus, ρ is a simple zero of ζ_K if and only if

$$\zeta'(\rho) L(1-\rho,\psi) \neq 0. \tag{16}$$

To count how often (16) holds in $\mathscr{R}$ we use Cauchy's inequality and find that

$$|\sum_{\rho\in\mathscr{R}} \zeta'(\rho)L(1-\rho,\psi)|^2 \leqq \Big(\sum_{\substack{\rho\in\mathscr{R}\\ \zeta'_K(\rho)\neq 0}} 1\Big) \sum_{\rho\in\mathscr{R}} |\zeta'(\rho)L(1-\rho,\psi)|^2. \tag{17}$$

We will show that

$$\sum_{\rho\in\mathscr{R}} \zeta'(\rho)L(1-\rho,\psi) = L(1,\psi)\frac{T}{4\pi}\log^2 T + O(T\log T) \tag{18}$$

and

$$\sum_{\rho\in\mathscr{R}} |\zeta'(\rho)L(1-\rho,\psi)|^2 \ll_\varepsilon T^{2-\theta+\varepsilon} \tag{19}$$

where θ is defined in (1). The estimates (17), (18), and (19) imply the theorem, since $L(1,\psi)\neq 0$.

5. The mean square

To estimate the second sum on the right hand side of (17) we first reduce it to a sum over zeros in the right half of the critical strip. Thus,

$$S := \sum_{\rho\in\mathscr{R}} |\zeta'(\rho)L(1-\rho,\psi)|^2 = \sum_{\substack{\rho\in\mathscr{R}\\ \beta\geqq\frac12}} |\zeta'(\rho)L(1-\rho,\psi)|^2 + \sum_{\substack{\rho\in\mathscr{R}\\ \beta>\frac12}} |\zeta'(1-\rho)L(\rho,\psi)|^2$$

since L and ζ' are real on the real line and the zeros are symmetric about $\sigma=\frac12$. Now by (5) and (8), for ρ in $\mathscr{R}$,

$$|L(1-\rho,\psi)| \ll T^{\beta-\frac12}|L(\rho,\psi)|.$$

Also by differentiating (3) we obtain

$$\zeta'(1-s) = X(1-s)\left(-\zeta'(s)+\frac{X'}{X}(s)\zeta(s)\right), \tag{20}$$

so that

$$\zeta'(1-\rho) = -X(1-\rho)\zeta'(\rho). \tag{21}$$

Thus,

$$S \ll S' := \sum_{\substack{\rho\in\mathscr{R}\\ \beta\geqq\frac12}} T^{2\beta-1}|\zeta'(\rho)L(\rho,\psi)|^2. \tag{22}$$

To estimate S' we split the sum into two parts: the terms for which $\frac12\leqq\beta\leqq\sigma_1$ and those for which $\sigma_1<\beta<1$. Here σ_1 will turn out to be slightly less than $\frac34$ because of the density estimates. Thus, $S'=S_1+S_2$ where

$$S_1 := \sum_{\substack{\rho\in\mathscr{R}\\ \frac12\leqq\beta\leqq\sigma_1}} T^{2\beta-1}|\zeta'(\rho)L(\rho,\psi)|^2 \ll T^{2\sigma_1-1} \sum_{\substack{\rho\in\mathscr{R}\\ \frac12\leqq\beta}} |\zeta'(\rho)L(\rho,\psi)|^2 \ll_\varepsilon T^{2\sigma_1+\varepsilon} \tag{23}$$

since we shall bound the rightmost sum here by $T^{1+\varepsilon}$. To do this, it suffices by Cauchy's inequality to give the same bound for

$$\sum_{\substack{\rho\in\mathcal{R}\\ \beta\geqq\frac{1}{2}}} |\zeta'(\rho)|^4 \quad \text{and} \quad \sum_{\substack{\rho\in\mathcal{R}\\ \beta\geqq\frac{1}{2}}} |L(\rho,\psi)|^4.$$

Now by Cauchy's theorem,

$$\zeta'(\rho)=\frac{1}{\Delta}\int_{\Delta}^{2\Delta}\frac{1}{2\pi i}\int_{|s-\rho|=\delta}\frac{\zeta(s)}{(s-\rho)^2}\,ds\,d\delta$$

where $\Delta=(\log T)^{-1}$. Hence if $\frac{1}{2}\leqq\beta<1$, then using Cauchy's inequality twice we have

$$|\zeta'(\rho)|^4 \ll \int_{\frac{1}{2}-2\Delta}^{1+2\Delta}\int_{\gamma-1}^{\gamma+1} |\zeta(\sigma+it)|^4\,dt\,d\sigma(\log T)^8.$$

Since the number of ρ in a square of side length 1 is $\ll \log T$, we have

$$\sum_{\substack{\rho\in\mathcal{R}\\ \beta\geqq\frac{1}{2}}} |\zeta'(\rho)|^4 \ll \int_{\frac{1}{2}-2\Delta}^{1+2\Delta}\int_{1}^{T+1} |\zeta(\sigma+it)|^4\,dt\,d\delta(\log T)^9 \ll T(\log T)^{13}$$

(see Montgomery [5], Thm. 10.1 for the integral mean value). The mean fourth power of $L(\rho,\psi)$ can be handled similarly, and this justifies (23).

Next we have

$$S_2:=\sum_{\substack{\rho\in\mathcal{R}\\ \beta>\sigma_1}} T^{2\beta-1}|\zeta'(\rho)L(\rho,\psi)|^2 \ll_\varepsilon \sum_{\substack{\rho\in\mathcal{R}\\ \beta>\sigma_1}} T^{2\beta-1+8c(1-\beta)+\varepsilon}$$

by (9). Using Stieltjes' integration, the sum here is

$$=T^{1+\varepsilon}\int_{\sigma_1}^{1} T^{-2(1-\sigma)(1-4c)}\,dN(\sigma,T) \ll_\varepsilon T^{1+\varepsilon}\max_{\sigma_1\leqq\sigma\leqq 1} T^{-2(1-\sigma)(1-4c)}N(\sigma,T).$$

By (12) and (13) and since $c<\frac{1}{6}$, this is a decreasing function of σ for σ in $[\frac{1}{2},1]$. Therefore,

$$S_2 \ll_\varepsilon T^{1+\varepsilon-2(1-\sigma_1)(1-4c)+\min\{3/(2-\sigma_1),\,2(1+2c)\}(1-\sigma_1)}. \tag{24}$$

If we balance the contributions of S_1 and S_2 by equating the exponents on T in (23) and (24) we find that

$$\sigma_1=\min\left\{1-\frac{1}{12c+2},\,\frac{12c+1-(1+16c+16c^2)^{\frac{1}{2}}}{8c}\right\}. \tag{25}$$

Hence

$$S_1+S_2 \ll_\varepsilon T^{2\sigma_1+\varepsilon}.$$

Note that θ in (1) is equal to $2-2\sigma_1$, so that this estimate establishes (19).

6. Initial treatment of the mean first power

We now consider the sum on the left side of (17). To simplify matters we note that

$$\overline{\sum_{\rho\in\mathcal{R}} \zeta'(\rho)\,L(1-\rho,\psi)} = \sum_{\rho\in\mathcal{R}} \zeta'(1-\rho)\,L(\rho,\psi)$$

$$= -\sum_{\rho\in\mathcal{R}} X(1-\rho)\,\zeta'(\rho)\,L(\rho,\psi); \tag{26}$$

the first equality follows from the fact that ζ' and L are real on the real line and as ρ runs through zeros of ζ with $\gamma>0$, $1-\rho$ runs through their conjugates; the second equality follows from (21).

Now by Cauchy's theorem

$$\mathcal{S} := \sum_{\rho\in\mathcal{R}} X(1-\rho)\zeta'(\rho)L(\rho,\psi) = \frac{1}{2\pi i}\int_{\mathscr{C}} X(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)L(s,\psi)\,ds \tag{27}$$

where $\mathscr{C}$ is the positively oriented rectangle with vertices $1+\Delta+i$, $1+\Delta+iT$, $-\Delta+iT$, $-\Delta+i$, with

$$\Delta=(\log T)^{-1}. \tag{28}$$

We may assume that T is a number for which (11) holds since the addition or deletion of $O(\log T)$ summands of (26) changes our sum by an amount which is

$$\ll_{\varepsilon} T^{\frac{1}{2}+\varepsilon}$$

by (8) and (9). Then, by (8), (9), and (11) the contribution to the integral in (27) of the horizontal sides of the contour is

$$\ll_{\varepsilon} T^{\frac{1}{2}+\varepsilon}.$$

Thus,

$$\mathcal{S} = I + I' + O_{\varepsilon}(T^{\frac{1}{2}+\varepsilon}), \tag{29}$$

where

$$I = \frac{1}{2\pi i}\int_{1+\Delta+i}^{1+\Delta+iT} X(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)L(s,\psi)\,ds \tag{30}$$

and

$$I' = \frac{-1}{2\pi i}\int_{-\Delta+i}^{-\Delta+iT} X(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)L(s,\psi)\,ds. \tag{31}$$

7. Evaluation of I

By Lemma 2,

$$I = \sum_{lmn\le\frac{T}{2\pi}} \Lambda(l)\log m\,\psi(n) + O_{\varepsilon}(T^{\frac{1}{2}+\varepsilon}).$$

Then by Lemma 1 with $U=\exp((\log T)^{\frac{1}{2}})$,

$$I = \frac{1}{2\pi i}\int_{1+\Delta-iU}^{1+\Delta+iU} \frac{\zeta'}{\zeta}(s)\zeta'(s)L(s,\psi)\left(\frac{T}{2\pi}\right)^{s}\frac{ds}{s} + O\left(\frac{T}{U}\log^{A} T\right) \tag{32}$$

for some A. The integrand has a pole of order three at $s=1$ with residue

$$\tfrac{1}{2}L(1,\psi)\frac{T}{2\pi}\log^2 T+O(T\log T) \tag{33}$$

and no other poles with $\sigma>1-b$ and $|t|<U$ where

$$b=(\log U)^{-1}=(\log T)^{-\frac{1}{2}}. \tag{34}$$

Thus by Cauchy's theorem, (32), and (33),

$$I=L(1,\psi)\frac{T}{4\pi}(\log^2 T)+\frac{1}{2\pi i}\int_R \frac{\zeta'}{\zeta}(s)\zeta'(s)L(s,\psi)\left(\frac{T}{2\pi}\right)^s\frac{ds}{s}+O(T\log T) \tag{35}$$

where R is the path consisting of the three line segments $\{\sigma+iU: 1+\Delta\geqq\sigma\geqq 1-b\}$, $\{1-b+it: -U\leqq t\leqq U\}$, and $\{\sigma-iU: 1-b\leqq\sigma\leqq 1+\Delta\}$. On the horizontal segments the integrand is

$$\ll\frac{T}{U}(\log T)^A$$

for some $A>0$, by (10), and the integral along the vertical segment is

$$\ll(\log T)^A T^{1-b}\int_{-U}^{U}\frac{dt}{1-b+|t|}\ll T^{1-b}(\log T)^{A+1}$$

again by (10). Hence

$$I=L(1,\psi)\frac{T}{4\pi}\log^2 T+O(T\log T). \tag{36}$$

8. The main term of I'

By (3)

$$\frac{\zeta'}{\zeta}(s)+\frac{\zeta'}{\zeta}(1-s)=\frac{X'}{X}(s),$$

so that by (31)

$$I'=I_1-I_2, \tag{37}$$

where

$$I_1=\frac{1}{2\pi i}\int_{-\Delta+i}^{-\Delta+iT}\frac{\zeta'}{\zeta}(1-s)X(1-s)\zeta'(s)L(s,\psi)\,ds \tag{38}$$

and

$$I_2=\frac{1}{2\pi i}\int_{-\Delta+i}^{-\Delta+iT}\frac{X'}{X}(s)X(1-s)\zeta'(s)L(s,\psi)\,ds. \tag{39}$$

In this section we evaluate I_2.

The integrand in I_2 is regular for $t>0$ whence by Cauchy's theorem we can replace the path $\{-\Delta+it: 1\leqq t\leqq T\}$ by the path consisting of three line segments: $\{\sigma+i\colon\ -\Delta\leqq\sigma\leqq 1+\Delta\}$, $\{1+\Delta+it\colon\ 1\leqq t\leqq T\}$, and $\{\sigma+iT\colon\ 1+\Delta\geqq\sigma\geqq-\Delta\}$. The integral on the lower horizontal segment is $O(1)$ while the integral along the upper horizontal segment contributes $O_\varepsilon(T^{\frac{1}{2}+\varepsilon})$ by (7), (8), and (9). Thus

$$I_2=\frac{1}{2\pi i}\int_{1+\Delta+i}^{1+\Delta+iT} X(1-s)\frac{X'}{X}(s)\zeta'(s)L(s,\psi)ds+O_\varepsilon(T^{\frac{1}{2}+\varepsilon}). \tag{40}$$

By (7) we have

$$\begin{aligned} I_2&=\frac{-1}{2\pi i}\int_{1+\Delta+i}^{1+\Delta+iT} \log\frac{t}{2\pi}X(1-s)\zeta'(s)L(s,\psi)ds+O_\varepsilon(T^{\frac{1}{2}+\varepsilon})\\ &=-\int_1^T \log\frac{v}{2\pi}dJ(v)+O_\varepsilon(T^{\frac{1}{2}+\varepsilon}) \end{aligned} \tag{41}$$

where

$$J(v)=\frac{1}{2\pi i}\int_{1+\Delta+i}^{1+\Delta+iv} X(1-s)\zeta'(s)L(s,\psi)ds.$$

By Lemma 2

$$J(v)=-\sum_{mn\leqq\frac{v}{2\pi}}\log m\psi(n)+O_\varepsilon(T^{\frac{1}{2}+\varepsilon})\qquad\text{for } v\leqq T.$$

We apply Lemma 1 to this and obtain

$$J(v)=-L(1,\psi)\frac{v}{2\pi}\log v+O(T)\qquad\text{for } v\leqq T.$$

(The argument is similar to that of Sect. 7 so we omit the details.) We insert this estimate into (41) and integrate by parts to obtain

$$I_2=L(1,\psi)\frac{T}{2\pi}\log^2 T+O(T\log T). \tag{42}$$

9. Reduction of I_1

We will eventually show that

$$I_1\ll T\log T$$

which is of smaller order than I_2; this is the most difficult part of the argument. In this section we relate I_2 to a sum with oscillating coefficients.

We let $s\to 1-s$ in (38) and use (5) and (20). This leads to

$$I_1=\bar{J}_1 \tag{43}$$

where

$$J_1=\frac{1}{2\pi i}\int_{1+\Delta+i}^{1+\Delta+iT} X(1-s,\psi)\frac{\zeta'}{\zeta}(s)\left(-\zeta'(s)+\frac{X'}{X}(s)\zeta(s)\right)L(s,\psi)ds.$$

By (4) and (6)

$$X(1-s,\psi)=\frac{\psi(-1)q^s}{\tau(\psi)}X(1-s)+O(t^{\sigma-\frac{1}{2}}e^{-\pi t})$$

for $t>1$ and $|\sigma|\leqq 2$. By this and (7),

$$\begin{aligned}J_1&=\frac{-\psi(-1)}{\tau(\psi)2\pi i}\int_{1+\Delta+i}^{1+\Delta+iT} q^s X(1-s)\frac{\zeta'}{\zeta}(s)\left(\zeta'(s)+\log\frac{t}{2\pi}\zeta(s)\right)L(s,\psi)\,ds\\&\quad+O_\varepsilon(T^{\frac{1}{2}+\varepsilon})\\&=\frac{-\psi(-1)}{\tau(\psi)}\left(J_2(T,\psi)-\int_1^T\log\frac{v}{2\pi}\,dJ_3(v,\psi)\right)+O_\varepsilon(T^{\frac{1}{2}+\varepsilon})\end{aligned}\tag{44}$$

where

$$J_2(T,\psi)=\frac{1}{2\pi i}\int_{1+\Delta+i}^{1+\Delta+iT} X(1-s)q^s\frac{\zeta'}{\zeta}(s)\zeta'(s)L(s,\psi)\,ds\tag{45}$$

and

$$J_3(v,\psi)=\frac{-1}{2\pi i}\int_{1+\Delta+i}^{1+\Delta+iv} X(1-s)q^s\zeta'(s)L(s,\psi)\,ds.\tag{46}$$

By Lemma 2,

$$J_2(v,\psi)=\sum_{lmn\leqq\frac{vq}{2\pi}}\Lambda(l)\log m\,\psi(n)\,e\left(\frac{-lmn}{q}\right)+O_\varepsilon(T^{\frac{1}{2}+\varepsilon})\qquad\text{for } v\leqq T\tag{47}$$

and

$$J_3(v,\psi)=\sum_{mn\leqq\frac{vq}{2\pi}}\log m\,\psi(n)\,e\left(\frac{-mn}{q}\right)+O_\varepsilon(T^{\frac{1}{2}+\varepsilon})\qquad\text{for } v\leqq T.\tag{48}$$

The next step is to use Lemma 1 to evaluate these sums. We first require some information about the generating functions.

10. The poles at $s=1$

Let

$$A_2(s,\psi)=\sum_j\frac{a_2(j,\psi)}{j^s}e(-j/q)=\sum_{l,m,n}\frac{\Lambda(l)\log m\,\psi(n)}{l^s m^s n^s}e\left(\frac{-lmn}{q}\right)\tag{49}$$

and

$$A_3(s,\psi)=\sum_j\frac{a_3(j,\psi)}{j^s}e(-j/q)=\sum_{m,n}\frac{\log m\,\psi(n)}{m^s n^s}e\left(-\frac{mn}{q}\right).\tag{50}$$

In this section we prove that A_2 has at most a double pole at $s=1$ and A_3 has at most a simple pole at $s=1$. First of all, it is easy to see that

$$\sum_{m\equiv a \bmod q} m^{-s}-q^{-s}\zeta(s)\tag{51}$$

is entire, that

$$\sum_m e\left(\frac{am}{q}\right)m^{-s}\tag{52}$$

is entire unless $q|a$ in which case it is equal to $\zeta(s)$, and that

$$\sum_{n\equiv 0 \bmod g} \Lambda(n)n^{-s} \tag{53}$$

is regular for $\sigma>0$ unless $g=1$, in which case it is equal to $-\zeta'/\zeta(s)$. (See Estermann [3] for these facts.) Now

$$\begin{aligned}A_3(s,\psi)&=\sum_{a=1}^{q}\psi(a)\sum_{n\equiv a \bmod q} n^{-s}\sum_m \frac{\log m}{m^s}e\left(-\frac{ma}{q}\right)\\&=-\sum_{a=1}^{q}\psi(a)\Big(\sum_{n\equiv a \bmod q} n^{-s}-q^{-s}\zeta(s)\Big)\frac{d}{ds}\sum_m e\left(-\frac{ma}{q}\right)m^{-s}\\&\quad+q^{-s}\zeta(s)\sum_m\frac{\log m}{m^s}\sum_{a=1}^{q}\psi(a)e(-ma/q).\end{aligned}$$

The first term here is regular at $s=1$, by (51) and (52), since $\psi(a)\neq 0$ implies that $(q,a)=1$. The second term here is, by (2),

$$\begin{aligned}&=q^{-s}\zeta(s)\psi(-1)\tau(\psi)\sum_m\frac{\psi(m)\log m}{m^s}\\&=-q^{-s}\psi(-1)\,\tau(\psi)\,\zeta(s)\,L'(s,\psi).\end{aligned}$$

Thus, $A_3(s,\psi)$ has at most a simple pole at $s=1$.

Next, by (49),

$$\begin{aligned}A_2(s,\psi)&=\sum_{a,b=1}^{q}\psi(b)\sum_{l\equiv a \bmod q}\Lambda(l)l^{-s}\sum_{n\equiv b \bmod q}n^{-s}\sum_m\frac{\log m}{m^s}e\left(-\frac{mab}{q}\right)\\&=-\sum_{g|q}\sum_{a=1}^{q/g}{}'\sum_{l\equiv ag \bmod q}\Lambda(l)l^{-s}\sum_{b=1}^{q}\psi(b)\\&\quad\cdot\Big(\sum_{n\equiv b \bmod q}n^{-s}-q^{-s}\zeta(s)\Big)\frac{d}{ds}\sum_m e\left(\frac{-mab}{q/g}\right)m^{-s}\\&\quad-q^{-s}\zeta(s)\sum_{g|q}\sum_{a=1}^{q/g}{}'\sum_{l\equiv ag \bmod q}\Lambda(l)l^{-s}\frac{d}{ds}\\&\quad\cdot\sum_m m^{-s}\sum_{b=1}^{q}\psi(b)e\left(-\frac{mabg}{q}\right)\end{aligned}\tag{54}$$

where Σ' indicates that the sum is for integers a which are coprime to q/g and the congruence conditions denote congruence modulo q. We write the first term on the right of (54) as

$$-\sum_{g|q}B(s,g,\psi).$$

If $1<g<q$, then by (51)–(53), $B(s,g,\psi)$ is regular at $s=1$; $B(s,1,\psi)$ has at most a simple pole at $s=1$, and

$$B(s,q,\psi)=\frac{\Lambda(lq)}{(lq)^s}\sum_{b=1}^{q}\psi(b)\Big(\sum_{n\equiv b \bmod q}n^{-s}-q^{-s}\zeta(s)\Big)\zeta'(s)$$

has at most a double pole at $s=1$. Hence the first term on the right hand side of (54) has at most a double pole at $s=1$. The second term on the right hand side of (54) is, by (2),

$$= -q^{-s}\zeta(s)\tau(\psi)\psi(-1)\sum_{g|q}\psi(g)\sum_{a=1}^{q/g}{}' \psi(a)\sum_{l\equiv ag \bmod q}\Lambda(l)l^{-s}L'(s,\psi)$$

which has at most a double at $s=1$. Thus, $A_2(s,\psi)$ has at most a double pole at $s=1$.

11. Bounds for the generating functions

We require different expressions for A_2 and A_3 to bound them in the critical strip. We obtain these via the formula

$$e\left(-\frac{j}{q}\right)=\sum_{\substack{d|j\\ d|q}}\frac{1}{\phi\left(\frac{q}{d}\right)}\sum_{\chi \bmod \frac{q}{d}}\tau(\bar{\chi})\chi(-j/d). \tag{55}$$

(This formula follows easily from the orthogonality relationships for characters; see also Conrey, Ghosh, and Gonek [1], Sect. 5.) Using this in (49) and (50) yields

$$A_i(s,\psi)=\sum_{d|q}\frac{d^{-s}}{\phi(q/d)}\sum_{\chi \bmod q/d}\tau(\bar{\chi})\chi(-1)A_i(d,s,\psi,\chi) \qquad (i=2,3) \tag{56}$$

where

$$A_i(d,s,\psi,\chi)=\sum_j a_i(jd,\psi)\chi(j)j^{-s} \qquad \text{for } \sigma>1.$$

By (50) and Lemma 3

$$\begin{aligned} A_3(d,s,\psi,\chi)&=\sum_{d=d_1d_2}(-L'(s,\chi)+L(s,\chi)\log d_1)\psi(d_2)L(s,\chi\psi)\prod_{p|d_1}\left(1-\frac{\chi\psi(p)}{p^s}\right)\\ &\ll(|L'(s,\chi)|+|L(s,\chi)|)|L(s,\chi\psi)|. \end{aligned} \tag{57}$$

Similarly, using (49) we find that

$$A_2(d,s,\psi,\chi)=\sum_{d=d_1d_2d_3}\left(\sum_j\frac{\chi(j)\log d_1 j}{j^s}\right)\left(\sum_{(j,d_1)=1}\frac{\chi(j)\psi(jd_2)}{j^s}\right)\left(\sum_{(j,d_1d_2)=1}\frac{\chi(j)\Lambda(jd_3)}{j^s}\right).$$

Writing $I(1)=1$ and $I(d)=0$ if $d>1$, we find that the last factor in the sum on the right here is

$$=I(d_3)\left(-\frac{L'}{L}(s,\chi)-\sum_{p|d_1d_2}\frac{\chi(p)\log p}{p^s-\chi(p)}\right)+\Lambda(d_3)\left(1+I((d_3,d_1d_2))\sum_{p|d_3}\frac{\chi(p)}{p^s-\chi(p)}\right)$$

and this is

$$\ll 1+\left|\frac{L'}{L}(s,\chi)\right|.$$

Combining this with (57) we obtain

$$A_i(s,\psi) \ll \max_{\substack{\chi \bmod k \\ k \leqq q}} \left(1+\left|\frac{L'}{L}(s,\chi)\right|\right)(|L'(s,\chi)|+|L(s,\chi)|)|L(s,\chi\psi)| \tag{58}$$

for $i=2$ and 3 and $|\sigma| \leqq 2$.

12. Bounds for J_2 and J_3

We can now evaluate J_2 and J_3 using Lemma 1 with $U=\exp(\log v)^{\frac{1}{2}})$. This gives, by (47)-(50),

$$J_i(v,\psi)=\frac{1}{2\pi i}\int_{1+\Delta-iU}^{1+\Delta+iU} A_i(s,\psi)\left(\frac{qv}{2\pi}\right)^s \frac{ds}{s}+O_\varepsilon\left(\frac{v}{U}(\log v)^A+T^{\frac{1}{2}+\varepsilon}\right) \tag{59}$$

for $v \leqq T$, $\Delta=(\log v)^{-1}$ and some constant A. We move the path of integration to the path consisting of the three line segments $\{\sigma-iU\colon 1+\Delta \geqq \sigma \geqq 1-b\}$, $\{1-b+it\colon -U \leqq t \leqq U\}$, and $\{\sigma+iU\colon 1-b \leqq \sigma \leqq 1+\Delta\}$ where $b=(\log U)^{-1}$. The integral on the new path is, by (58) and (10),

$$\ll \left(\frac{v}{U}+v^{1-b}\right)(\log v)^A \tag{60}$$

for some $A>0$ (just as in Sect. 7). In moving the path of integration we cross a pole of $A_i(s,\psi)$ at $s=1$ which is of order at most 2 in the case of A_2 and of order at most 1 in the case of A_3. The residues are bounded by $v \log v$ and v respectively. Thus (59), (60), and the results of Sect. 10 imply that

$$J_2(v,\psi) \ll v\log v; \qquad J_3(v,\psi) \ll v. \tag{61}$$

We combine the estimates (44) and (61) to conclude that

$$J_1 \ll T \log T$$

whence, by (43),

$$I_1 \ll T \log T.$$

Thus, by (37) and (42),

$$I'=-L(1,\psi)\frac{T}{2\pi}\log^2 T+O(T\log T).$$

Combining this with (36) and (29), we have

$$\mathcal{S}=-L(1,\psi)\frac{T}{4\pi}\log^2 T+O(T\log T).$$

The formula in (18) now follows from (26), (27), and the above. Since we have already proven (19) in Sect. 5, the theorem now follows.

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