LARGE GAPS BETWEEN ZEROS OF THE ZETA-FUNCTION

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§1. *Introduction.* Let $0 < \gamma \leqslant \gamma'$ denote the ordinates of consecutive non-trivial zeros of $\zeta(s)$ and set

$$\mu = \liminf_{\gamma} \, (\gamma' - \gamma) \frac{\log \gamma}{2\pi},$$

$$\lambda = \limsup_{\gamma} \, (\gamma' - \gamma) \frac{\log \gamma}{2\pi}.$$

Since the average size of $\gamma' - \gamma$ is $2\pi/\log \gamma$, we see that $\mu \leqslant 1 \leqslant \lambda$. One expects that $\mu = 0$ and $\lambda = \infty$, but at present all we can say unconditionally is that $\mu < 1 < \lambda$ (see A. Fujii [**6**] and A. Selberg [**11**]). However, these bounds may be strengthened if one is willing to assume the Riemann hypothesis (RH). For instance, on RH, H. Montgomery [**8**] has shown that $\mu < 0{\cdot}68$ and J. Mueller [**10**] that $\lambda > 1{\cdot}9$. Later Montgomery and Odlyzko [**9**] proved that $\mu < 0{\cdot}5179$ and $\lambda > 1{\cdot}9799$, whereas the present authors have recently shown that $\mu < 0{\cdot}5172$ and $\lambda > 2{\cdot}337$.

In this paper we shall obtain a better lower bound for λ subject to the truth of the generalized Riemann hypothesis (GRH), that is, the hypothesis that the zeros of each Dirichlet L-function have their real parts not exceeding $\frac{1}{2}$. Our bound is a consequence of a mean value formula involving the zeta-function. Let $s = \sigma + it$ denote a complex variable and for T large write

$$L = \log \frac{T}{2\pi}.$$

The formula in question is given by

THEOREM 1. *Assume* GRH *and let*

$$A(s) = \sum_{k \leqslant y} k^{-s},$$

where

$$y = \left(\frac{T}{2\pi}\right)^{\eta}$$

for some $\eta \in (0, \frac{1}{2})$. *Then if* $\varepsilon > 0$ *and* T *is sufficiently large* (*depending on* ε),

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we have

$$\sum_{0<\gamma\le T} |\zeta A(\tfrac{1}{2}+i(\gamma+\alpha))|^2 = \frac{6}{\pi^2}\frac{T}{2\pi} L^5 \sum_{j=0}^{\infty} \frac{(-1)^{j+1}(\alpha L)^{2j+2}}{(2j+5)!}$$
$$\times\left(\frac{-3\eta^2+(2j+5)\eta^3}{3} - \frac{2j+5}{j+3}\eta^{2j+6} + \eta^{2j+7} + \eta^2(1-\eta)^{2j+5}\right)$$
$$+O_\varepsilon(TL^4)+O_\varepsilon(T^{\frac{1}{2}+\eta+\varepsilon})$$

uniformly for real α *with* $|\alpha| \le 100L^{-1}$.

Assuming only the Riemann hypothesis, S. Gonek [7] has proved an analogous formula with $A(s)$ identically one. In fact Theorem 1 and its consequence, Theorem 2 (below), are both valid on RH. The reason we have assumed GRH here is that doing so allows us to avoid a considerable number of technical details, particularly in Lemma 6. The interested reader may refer to Conrey, Ghosh, and Gonek [3] to see how GRH is avoided in a similar situation. We should also point out that Theorem 1 is true for $|\alpha| < CL^{-1}$, where C is any positive constant; we have taken $C = 100$ as this suffices for our purpose.

Our main result is

THEOREM 2. *Assume* GRH. *Then* $\lambda > 2{\cdot}68$.

We use Theorem 1 to prove Theorem 2 in much the same way that Mueller used Gonek's formula (mentioned above) to prove that $\lambda > 1{\cdot}9$. Write

$$\mathcal{M}_1(T) = \int_1^T |\zeta A(\tfrac{1}{2}+it)|^2 dt, \tag{1.1}$$

$$m_2(\alpha, T) = \sum_{0<\gamma\le T} |\zeta A(\tfrac{1}{2}+i(\gamma+\alpha))|^2, \tag{1.2}$$

and

$$\mathcal{M}_2(c, T) = \int_{-c/L}^{c/L} m_2(\alpha, T)\, d\alpha. \tag{1.3}$$

Then it is not difficult to see that if

$$\mathcal{M}_2(c, 2T) - \mathcal{M}_2(c, T) < \mathcal{M}_1(2T) - \mathcal{M}_1(T) \tag{1.4}$$

for all sufficiently large T, then c/π is a lower bound for λ. We shall show that if η is near $\frac{1}{2}$, (1.4) holds with $c = 2{\cdot}68\pi$; Theorem 2 will then follow. Now $\mathcal{M}_1(T)$ can be estimated by using a result of Balasubramanian, Conrey and Heath-Brown [1] (see Lemma 10 below). Thus our main task is to estimate $\mathcal{M}_2(c, T)$, or what amounts to the same thing, to prove Theorem 1.

§2. *Beginning of the proof of Theorem* 1. Throughout, T is large, $L = \log(T/2\pi)$, and ε denotes an arbitrarily small positive number, though not necessarily the same one at each occurrence. Furthermore, we shall always assume that the real number α satisfies

$$|\alpha| \leq 100L^{-1}.$$

Setting

$$a = 1 + L^{-1},$$

we let R be a positively oriented rectangular contour with vertices $a+i$, $a+i(T+\alpha)$, $1-a+i(T+\alpha)$, $1-a+i$, the top edge of which has a small semicircular indentation centered at $\frac{1}{2}+i(T+\alpha)$ opening downward. By Cauchy's residue theorem, the reflection principle, and RH

$$m_2(\alpha, T) = \frac{1}{2\pi i} \int_R \frac{\zeta'}{\zeta}(s-i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\,ds,$$

where we have written

$$\frac{\zeta'}{\zeta}(s-i\alpha) \quad \text{for} \quad \frac{\zeta'(s-i\alpha)}{\zeta(s-i\alpha)}.$$

Now for s inside or on R we have

$$A(s) \ll_\varepsilon y^{1-\sigma+\varepsilon} \tag{2.1}$$

and

$$\zeta(s) \ll_\varepsilon T^{1/2(1-\sigma)+\varepsilon}$$

(see Titchmarsh [**12**; pp. 81-82]). Thus, for such s

$$\zeta(s)\zeta(1-s)A(s)A(1-s) \ll_\varepsilon yT^{\frac{1}{2}+\varepsilon}. \tag{2.2}$$

Moreover, for each large T, there exists a number T' with $T-2 < T' < T-1$ such that

$$\frac{\zeta'}{\zeta}(\sigma+iT') \ll L^2, \tag{2.3}$$

uniformly for $-1 \leq \sigma \leq 2$ (see Davenport [**4**; p. 108]). Obviously we may also take T' so that $T'+\alpha$ is not the ordinate of any zero of $\zeta(s)$. Then by Cauchy's residue theorem, the contribution of the top edge of R to $m_2(\alpha, T)$ equals

$$\frac{1}{2\pi i}\left(\int_{a+i(T+\alpha)}^{a+iT'} + \int_{a+iT'}^{1-a+iT'} + \int_{1-a+iT'}^{1-a+i(T+\alpha)}\right)\frac{\zeta'}{\zeta}(s-i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\,ds$$
$$+ \sum_{T'-\alpha<\gamma\leq T} |\zeta A(\tfrac{1}{2}+i(\gamma+\alpha))|^2.$$

The sum over γ contains $\ll L$ terms, so by (2.2) and (2.3) the above is

$$\ll_\varepsilon yT^{\frac{1}{2}+\varepsilon},$$

uniformly for $|\alpha| \leq 100L^{-1}$. On the other hand, the bottom edge of the contour contributes

$$\ll_\varepsilon yT^\varepsilon$$

by (2.1), since here $\zeta(s)$ and $\zeta(s-i\alpha)$ are bounded.

To treat the integral along the left edge of R we replace s by $1-s$ and use

$$\frac{\zeta'}{\zeta}(1-s-i\alpha)=\frac{\chi'}{\chi}(1-s-i\alpha)-\frac{\zeta'}{\zeta}(s+i\alpha);$$

this follows from the functional equation

$$\zeta(1-s)=\chi(1-s)\zeta(s), \tag{2.4}$$

where

$$\chi(1-s)=\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)} \tag{2.5}$$

(see Titchmarsh [**12**; p. 16]). We then see that

$$\begin{aligned}\frac{1}{2\pi i}&\int_{1-a+i(T+\alpha)}^{1-a+i}\frac{\zeta'}{\zeta}(s-i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\,ds\\&=\frac{1}{2\pi i}\int_{a-i(T+\alpha)}^{a-i}\left(\frac{\zeta'}{\zeta}(s+i\alpha)-\frac{\chi'}{\chi}(1-s-i\alpha)\right)\zeta(s)\zeta(1-s)A(s)A(1-s)\,ds\\&=\bar{I}_1-\bar{I}_2,\end{aligned}$$

where

$$I_1=\frac{1}{2\pi i}\int_{a+i}^{a+i(T+\alpha)}\frac{\zeta'}{\zeta}(s-i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\,ds \tag{2.6}$$

and

$$I_2=\frac{1}{2\pi i}\int_{a+i}^{a+i(T+\alpha)}\frac{\chi'}{\chi}(1-s+i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\,ds. \tag{2.7}$$

Now I_1 is also the contribution of the right-hand side of R to $m_2(\alpha, T)$. Thus, combining the above results, we obtain

$$m_2(\alpha, T)=2\,\mathrm{Re}\,I_1-\bar{I}_2+O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}). \tag{2.8}$$

The next section contains the necessary lemmas for estimating I_1, I_2, and $\mathcal{M}_1(T)$.

§3. *Lemmas.* We write $e(x)$ for $e^{2\pi ix}$.

LEMMA 1. *Let r be a positive real number and suppose that $\chi(1-s)$ is given by (2.5). Then for $a = 1 + L^{-1}$ and T large, we have*

$$\frac{1}{2\pi i}\int_{a+i}^{a+iT} \chi(1-s)r^{-s}ds = \begin{cases} e(-r) + E(r, T)r^{-a}, & \text{if } r \leq T/2\pi, \\ E(r, T)r^{-a}, & \text{if } r > T/2\pi, \end{cases}$$

where

$$E(r, T) \ll T^{1/2} + \frac{T^{3/2}}{|T - 2\pi r| + T^{1/2}}.$$

Except for minor modifications, this is Lemma 2 of Gonek [7].

The next lemma is a special case of Lemma 2 of Conrey, Ghosh, and Gonek [3]. Alternatively, it can easily be derived from Lemma 1.

LEMMA 2. *Let $\alpha(n)$, $\beta(n)$ be arithmetic functions such that $\alpha(n) \ll 1$ and $\beta(n) \ll d_j(n)\log^l n$, where j and l are non-negative integers with $j \leq 4$, and $d_j(n)$ is the coefficient of n^{-s} in $\zeta^j(s)$. Let $a = 1 + L^{-1}$ and $0 < \delta < 1$. Then if $T^\delta \leq x \leq T$, we have*

$$\frac{1}{2\pi i}\int_{a+i}^{a+iT} \chi(1-s)\left(\sum_{k \leq x} \alpha(k)k^{s-1}\right)\left(\sum_{n=1}^{\infty} \beta(n)n^{-s}\right)ds$$
$$= \sum_{k \leq x} \frac{\alpha(k)}{k} \sum_{m \leq kT/2\pi} \beta(m)e\left(-\frac{m}{k}\right) + O_\delta(xT^{1/2}L^{j+l}).$$

LEMMA 3. *Suppose that*

$$\mathscr{A}_j(s) = \sum_{n=1}^{\infty} \alpha_j(n)n^{-s}$$

is absolutely convergent in $\sigma > 1$, for $j = 1, 2, \ldots, J$, and that

$$\mathscr{A}(s) = \sum_{n=1}^{\infty} \alpha(n)n^{-s} = \prod_{j=1}^{J} \mathscr{A}_j(s).$$

Then for any positive integer d,

$$\sum_{n=1}^{\infty} \alpha(nd)n^{-s} = \sum_{d_1 \ldots d_J = d} \prod_{j=1}^{J} \left(\sum_{\substack{n=1 \\ (n, \prod_{i<j} d_i)=1}}^{\infty} \alpha_j(nd_j)n^{-s} \right) \qquad (\sigma > 1).$$

This is Lemma 3 of Conrey, Ghosh, and Gonek [3].

Our next lemma is a well-known result of Estermann [5].

LEMMA 4. *Let H and K be coprime integers with $K > 0$ and define*

$$D\left(s, \frac{H}{K}\right) = \sum_{n=1}^{\infty} d(n)n^{-s}e\left(\frac{nH}{K}\right) \qquad (\sigma > 1),$$

where $d(n)$ is the number of divisors of n. Then $D(s, H/K)$ is regular in the *entire complex plane except for a double pole at* $s=1$. *At this point it has the same principal part as* $K^{1-2s}\zeta^2(s)$.

LEMMA 5. *Let H and K be integers with $(H, K)=1$ and $K=\Pi p^{\lambda}>0$. For α real and $\sigma>1$ define*

$$Q\left(s, \alpha, \frac{H}{K}\right)=-\sum_{m,n=1}^{\infty}\frac{d(m)\Lambda(n)}{m^s n^{s-i\alpha}}\,e\left(\frac{-mnH}{K}\right),$$

where $\Lambda(n)$ is von Mangoldt's function. Then $Q(s, \alpha, H/K)$ has a meromorphic continuation to the entire complex plane. If $\alpha\neq 0$, $Q(s, \alpha, H/K)$ has

(i) *at most a double pole at $s=1$ with the same principal part as*

$$K^{1-2s}\zeta^2(s)\left(\frac{\zeta'}{\zeta}(s-i\alpha)-G(s, \alpha, K)\right),$$

where

$$G(s, \alpha, K)=\sum_{p|K}\log p\left(\sum_{a=1}^{\lambda-1}p^{a(s-1+i\alpha)}+\frac{p^{\lambda(s-1+i\alpha)}}{1-p^{-s+i\alpha}}-\frac{1}{p^{s-i\alpha}-1}\right); \tag{3.1}$$

(ii) *a simple pole at $s=1+i\alpha$ with residue*

$$-\frac{1}{K^{i\alpha}\varphi(K)}\,\zeta^2(1+i\alpha)D_K(1+i\alpha),$$

where

$$D_K(s)=\prod_{p|K}(1-p^{-1}+\lambda(1-p^{-s})(1-p^{s-1})). \tag{3.2}$$

Moreover, on GRH, *$Q(s, \alpha, H/K)$ is regular in $\sigma>\frac{1}{2}$ except for these two poles.*

Proof. For $\sigma>1$ we have

$$\begin{aligned}Q\left(s, \alpha, \frac{H}{K}\right)&=\sum_{a=1}^{K}D\left(s, \frac{-aH}{K}\right)L(s-i\alpha, a, K)\\&=\sum_{g|K}\sum_{b=1}^{K/g}{}^{*}\,D\left(s, \frac{-bH}{K/g}\right)L(s-i\alpha, bg, K),\end{aligned} \tag{3.3}$$

where $D(s, \cdot)$ is the function defined in Lemma 4,

$$L(s, a, k)=-\sum_{n\equiv a(\operatorname{mod} k)}\Lambda(n)n^{-s}\qquad(\sigma>1),$$

and $\sum^*$ denotes a sum over b coprime to K/g. It is well-known that $L(s, a, k)$ has a meromorphic continuation throughout the plane and is regular on the line $\sigma=1$ except for a simple pole at $s=1$, if, and only if, $(a, k)=1$. Also, by Lemma 4, $D(s, -bHg/K)$ is regular everywhere except for a double pole at $s=1$. Thus (3.3) provides a meromorphic continuation of $Q(s, \alpha, H/K)$ to the whole plane and we see that if $\alpha\neq 0$, Q has at most a double pole at $s=1$ and a simple pole at $s=1+i\alpha$.

By Lemma 4 and (3.3) we see that the principal part of $Q(s, \alpha, H/K)$ at $s=1$ is identical to that of

$$\zeta^2(s)\sum_{g|K}\left(\frac{K}{g}\right)^{1-2s}\sum_{b=1}^{K/g}{}^{*}\,L(s-i\alpha, bg, K). \tag{3.4}$$

Now for $\sigma > 1$,

$$\sum_{b=1}^{K/g}{}^{*} L(s-i\alpha, bg, K) = -\sum_{\substack{n=1\\(n, K/g)=1}}^{\infty} \Lambda(ng)(ng)^{-s+i\alpha}$$

$$= \begin{cases} \dfrac{\zeta'}{\zeta}(s-i\alpha) + \displaystyle\sum_{p|K} \frac{\log p}{p^{s-i\alpha}-1}, & \text{if } g=1, \\ \dfrac{-\log p}{p^{(s-i\alpha)a}}, & \text{if } g=p^a \text{ with } p|K \text{ and } a<\lambda, \\ \dfrac{-\log p}{p^{(s-i\alpha)\lambda}(1-p^{-s+i\alpha})}, & \text{if } g=p^{\lambda} \text{ with } p|K, \\ 0, & \text{otherwise.} \end{cases}$$

Thus (3.4) equals

$$\zeta^2(s)K^{1-2s}\left(\frac{\zeta'}{\zeta}(s-i\alpha) - \sum_{p|K} \log p\left(\sum_{a=1}^{\lambda-1} p^{(s-1+i\alpha)a} + \frac{p^{\lambda(s-1+i\alpha)}}{1-p^{-s+i\alpha}} - \frac{1}{p^{s-i\alpha}-1}\right)\right).$$

This proves (i).

To compute the residue at $s=1+i\alpha$ observe that in (3.3), $L(s-i\alpha, bg, K)$ is regular at $1+i\alpha$ unless $g=1$, in which case its principal part is

$$-\frac{1}{\varphi(K)(s-1-i\alpha)}.$$

Hence the residue of $Q(s, \alpha, H/K)$ is the same as that of

$$-\frac{1}{\varphi(K)(s-1-i\alpha)} \sum_{b=1}^{K}{}^{*} D\left(s, \frac{-bH}{K}\right).$$

For $\sigma > 1$ the sum over b equals

$$\sum_{n=1}^{\infty} d(n)n^{-s} \sum_{b=1}^{K}{}^{*} e\left(\frac{-bnH}{K}\right) = \sum_{n=1}^{\infty} d(n)c_K(n)n^{-s}, \tag{3.5}$$

where

$$c_K(n) = \sum_{a=1}^{K}{}^{*} e\left(\frac{an}{K}\right)$$

is Ramanujan's function. (Note that $c_K(-Hn) = c_K(n)$ when $(H, K) = 1$.) However, by Titchmarsh [**12**; p. 11], (3.5) equals

$$\zeta^2(s)K^{1-s} \prod_{p|K} (1-p^{-1}+\lambda(1-p^{-s})(1-p^{s-1})).$$

Thus (ii) follows.

Finally, on GRH the only pole of $L(s-i\alpha, bg, K)$ in $\sigma > \frac{1}{2}$ occurs when $g=1$ at $s=1+i\alpha$. The last assertion of the lemma follows from this and (3.3).

LEMMA 6. *Assume* GRH. *Let* $y=(T/2\pi)^{\eta}$ *where* $\eta \in (0, 1/2)$, *let k be an integer with* $1 \leq k \leq y$, *and let α be real. Set*

$$Q^*(s, \alpha, k) = \sum_{j=1}^{\infty} b(j)j^{-s}e\left(-\frac{j}{k}\right) \qquad (\sigma > 1),$$

where

$$b(j) = -\sum_{\substack{hmn=j \\ h\leq y}} \Lambda(n) n^{i\alpha} d(m).$$

Then $Q^*(s, \alpha, k)$ has an analytic continuation to $\sigma > \frac{1}{2}$ except for possible poles at $s=1$ and $1+i\alpha$. Furthermore,

$$Q^*(s, \alpha, k) \ll_\varepsilon y^{1/2} T^\varepsilon$$

for $\sigma \geq \frac{1}{2} + L^{-1}$, $|t| \leq T$, and $|s-1|$, $|s-1-i\alpha| > \frac{1}{4}$.

Proof. If χ is a character (mod k), the Gauss sum $\tau(\chi)$ is given by

$$\tau(\chi) = \sum_{h=1}^{k} \chi(h) e\left(\frac{h}{k}\right),$$

and one easily shows that

$$e\left(-\frac{j}{k}\right) = \sum_{\substack{d|j \\ d|k}} \frac{1}{\varphi\left(\frac{k}{d}\right)} \sum_{\chi(\operatorname{mod} k/d)} \tau(\bar{\chi}) \chi\left(-\frac{j}{d}\right).$$

Inserting this into the definition of Q^*, we obtain

$$Q^*(s, \alpha, k) = \sum_{d|k} \frac{1}{\varphi\left(\frac{k}{d}\right) d^s} \sum_{\chi(\operatorname{mod} k/d)} \tau(\bar{\chi}) \bar{\chi}(-d) B(s, d) \qquad (\sigma > 1), \tag{3.6}$$

where

$$B(s, d) = \sum_{j=1}^{\infty} b(jd) \chi(jd) j^{-s} \qquad (\sigma > 1).$$

Now set

$$F(s, r, \chi) = \prod_{p|r} (1 - \chi(p) p^{-s}). \tag{3.7}$$

Then, by Lemma 3,

$$B(s, d) = \sum_{f_1 \cdots f_4 = d} \mathscr{A}_1(s, f_1) \mathscr{A}_2(s, f_2, f_1) \mathscr{A}_2(s, f_3, f_1 f_2) \mathscr{A}_3(s, f_4, f_1 f_2 f_3), \tag{3.8}$$

where

$$\mathscr{A}_1(s, f) = \chi(f) \sum_{n < y/f} \chi(n) n^{-s},$$

$$\mathscr{A}_2(s, f, r) = \sum_{\substack{n=1 \\ (n,r)=1}}^{\infty} \chi(fn) n^{-s} = \chi(f) L(s, \chi) F(s, r, \chi),$$

and

$$\mathscr{A}_3(s, f, r) = -\sum_{\substack{n=1 \\ (n,r)=1}}^{\infty} \chi(fn) \Lambda(fn) (fn)^{i\alpha} n^{-s}$$

$$= -f^{i\alpha}\chi(f) \sum_{\substack{n=1 \\ (n,r)=1}}^{\infty} \chi(n)\Lambda(fn)n^{-s+i\alpha}$$

$$= f^{i\alpha}\chi(f) \begin{cases} \dfrac{L'}{L}(s-i\alpha,\chi)+\dfrac{F'}{F}(s-i\alpha,r,\chi), & \text{if } f=1, \\ \dfrac{-\log p}{1-\chi(p)p^{-s+i\alpha}}, & \text{if } f=p^l \text{ and } (r,p)=1, \\ -\log p, & \text{if } f=p^l \text{ and } p\,|\,r, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\mathscr{A}_1$ and $\mathscr{A}_2$ are regular everywhere except when $\chi=\chi_0$, in which case $\mathscr{A}_2$ may have a pole at $s=1$. Furthermore, on GRH $\mathscr{A}_3$ is regular in $\sigma>\frac{1}{2}$ except perhaps for a pole at $s=1+i\alpha$. Thus, with the possible exception of poles at $s=1$ and $1+i\alpha$, $B(s,d)$ is regular in $\sigma>\frac{1}{2}$; (3.8) therefore provides the required continuation of $Q^*(s,\alpha,k)$.

To bound Q^* we first bound $B(s,d)$. Let r_0 denote the square-free part of r. If $\sigma\geqslant\frac{1}{2}$, then from (3.7) we obtain

$$F(s,r,\chi) \ll \prod_{p|r} 2^r = \sum_{f|r_0} 1 = d(r_0) \ll_\varepsilon r_0^\varepsilon \ll_\varepsilon r^\varepsilon,$$

and

$$\frac{F'}{F}(s,r,\chi) = \sum_{p|r} \frac{\chi(p)\log p}{p^s-\chi(p)} \ll \sum_{p|r} \log p = \log r_0 \ll r^\varepsilon.$$

Moreover, on GRH

$$L(s,\chi), \frac{L'}{L}(s,\chi) \ll_\varepsilon (1+|t|)^\varepsilon \left(\frac{k}{d}\right)^\varepsilon \tag{3.9}$$

provided that $\sigma\geqslant\frac{1}{2}+L^{-1}$ and $|s-1|\gg 1$. Thus, if χ is a character (mod k/d),

$$\mathscr{A}_2(s,f,r) \ll_\varepsilon \left(T\frac{rk}{d}\right)^\varepsilon$$

and

$$\mathscr{A}_3(s,f,r) \ll_\varepsilon \left(T\frac{k}{d}\right)^\varepsilon + r^\varepsilon + f^\varepsilon$$

when $\sigma\geqslant\frac{1}{2}+L^{-1}$, $|t|\leqslant T$, and $|s-1|$, $|s-1-i\alpha|>\frac{1}{4}$. Now, if f and r divide d, they also divide k which is $\ll T^{1/2}$. Hence, for f and r dividing d,

$$\mathscr{A}_2(s,f,r), \qquad \mathscr{A}_3(s,f,r) \ll_\varepsilon T^\varepsilon \tag{3.10}$$

in the region under consideration. To estimate $\mathscr{A}_1$, we distinguish two cases. First, if χ is principal (mod k/d) we use the trivial estimate

$$\mathscr{A}_1(s,f) \ll_\varepsilon \left(\frac{y}{f}\right)^{\frac{1}{2}+\varepsilon} \ll_\varepsilon y^{1/2}T^\varepsilon \qquad (\sigma\geqslant\tfrac{1}{2}). \tag{3.11}$$

If χ is not principal, we apply Perron's formula (*cf.* Titchmarsh [**12**; pp. 53–55]) and obtain

$$\mathcal{A}_1(s,f)=\frac{\chi(f)}{2\pi i}\int_{a-2iT}^{a+2iT} L(w,\chi)\frac{(y/f)^{w-s}}{w-s}\,dw+O_\varepsilon\left(\frac{(y/f)^{a-\sigma}}{T^{1-\varepsilon}}\right)$$

for $\sigma\geqslant\frac{1}{2}+L^{-1}$ and $|t|\leqslant T$ (here $a=1+L^{-1}$). We move the line of integration to $\operatorname{Re} w=\sigma+L^{-1}$ and use (3.9). Then

$$\begin{aligned}\mathcal{A}_1(s,f) &\ll_\varepsilon (2Tk/d)^\varepsilon (y/f)^{L^{-1}}L+T^{\varepsilon-1}(y/f)^{1/2}\\ &\ll_\varepsilon T^\varepsilon,\end{aligned}$$

for $\sigma\geqslant 1/2+L^{-1}$, $|t|\leqslant T$. Combining this with (3.10) and (3.11) in (3.8) we find that

$$B(s,d)\ll_\varepsilon \begin{cases} y^{1/2}T^\varepsilon, & \text{if } \chi \text{ is principal,}\\ T^\varepsilon, & \text{otherwise,}\end{cases}$$

in the region $\sigma\geqslant\frac{1}{2}+L^{-1}$, $|t|\leqslant T$ and $|s-1|$, $|s-1-i\alpha|>\frac{1}{4}$. It follows now from (3.6) that in the same s-region

$$Q^*(s,\alpha,k)\ll_\varepsilon T^\varepsilon \sum_{d|k}\frac{1}{\varphi(k/d)d^{1/2}}\left(y^{1/2}|\tau(\chi_0)|+\sum_{\chi\neq\chi_0(\operatorname{mod} k/d)}|\tau(\chi)|\right).$$

Since

$$\tau(\chi)\ll\begin{cases}(k/d)^{1/2}, & \text{if } \chi\neq\chi_0(\operatorname{mod} k/d),\\ 1, & \text{if } \chi=\chi_0(\operatorname{mod} k/d),\end{cases}$$

this is

$$\begin{aligned}&\ll_\varepsilon T^\varepsilon\sum_{d|k}\left(\frac{(y/d)^{1/2}}{\varphi(k/d)}+\frac{k^{1/2}}{d}\right)\ll_\varepsilon T^\varepsilon(y^{1/2}+k^{1/2}\log\log k)\\ &\ll_\varepsilon y^{1/2}T^\varepsilon.\end{aligned}$$

This completes the proof of the lemma.

LEMMA 7. *Let $G(s,\alpha,K)$ be given by* (3.1). *Then for any real α,*

$$G(1,\alpha,K)=\sum_{p|K}p^{i\alpha}\log p+O(C_1(K)),$$

and

$$G'(1,\alpha,K)=\sum_{p|K}p^{i\alpha}\log^2 p+O(C_2(K)),$$

where the derivative is with respect to s and

$$C_j(K)=\sum_{p^a|K}\frac{\log^j p}{p}+\sum_{\substack{p|K\\ a\geqslant 2}}a\log^j p\qquad(j=1,2).$$

Moreover, for $x\geqslant 1$

$$\sum_{h,k\leqslant x}\frac{(h,k)}{hk}\left(1+C_j\left(\frac{k}{(h,k)}\right)\right)\ll\log^3 2x\qquad(j=1,2).$$

Proof. We prove the formula for G' but not for G as it is similar. Set $K=\Pi p^{\lambda}$. By (3.1) we have

$$G'(1,\alpha,K)=\sum_{p|K}\log^2 p\left\{\sum_{a=1}^{\lambda-1} ap^{ai\alpha}+\frac{\lambda p^{i\lambda\alpha}}{(1-p^{-1+i\alpha})}-\frac{p^{-1+i\alpha(\lambda+1)}}{(1-p^{-1+i\alpha})^2}+\frac{p^{1-i\alpha}}{(p^{1-i\alpha}-1)^2}\right\}$$
$$=\sum_{p|K}\log^2 p\left\{\sum_{a=1}^{\lambda} ap^{ia\alpha}+O\left(\frac{\lambda}{p}\right)\right\}$$
$$=\sum_{p|K}p^{i\alpha}\log^2 p+O\left(\sum_{\substack{p^a|K\\ a\geq 2}} a\log^2 p\right)+O\left(\sum_{p^a|K}\frac{\log^2 p}{p}\right),$$

as desired.

Next,

$$\sum_{h,k\leq x}\frac{(h,k)}{hk}\left(C_j\left(\frac{k}{(h,k)}\right)+1\right)\leq\sum_{h,k\leq x}\frac{C_j(k)+1}{hk}\sum_{\substack{d|h\\ d|k}}\varphi(d)$$
$$=\sum_{d\leq x}\frac{\varphi(d)}{d^2}\sum_{h',k'\leq x/d}\frac{C_j(dk')+1}{h'k'}$$
$$\ll\log^3 2x+\log 2x\sum_{d\leq x}\frac{1}{d}\sum_{k'\leq x/d}\frac{C_j(dk')}{k'}.$$

Since $C_j(dk')\ll C_j(d)+C_j(k')$, the last term is

$$\ll\log^2 2x\sum_{d\leq x}\frac{C_j(d)}{d}.$$

But

$$\sum_{d\leq x}\frac{C_j(d)}{d}=\sum_{d\leq x}\frac{1}{d}\left(\sum_{p^a|d}\frac{\log^j p}{p}+\sum_{\substack{p^a|d\\ a\geq 2}} a\log^j p\right)$$
$$=\sum_{p^a\leq x}\frac{\log^j p}{p^{a+1}}\sum_{r\leq x/p^a}\frac{1}{r}+\sum_{\substack{p^a\leq x\\ a\geq 2}}\frac{a\log^j p}{p^a}\sum_{r\leq x/p^a}\frac{1}{r}$$
$$\ll\log 2x.$$

The last assertion of the lemma now follows.

LEMMA 8. *Let* $|\beta|\leq 100L^{-1}$ *and* $1\leq x\leq T/2\pi$. *Then for* $j,l=0,1,\ldots,$

(i) $\displaystyle\sum_{k\leq x}\frac{\log^j k}{k}=\frac{\log^{j+1}x}{j+1}+O((j+1)L^j);$

(ii) $\displaystyle\sum_{k\leq x}\frac{(\log(T/2\pi k))^j}{k}=\frac{L^{j+1}-(\log(T/2\pi x))^{j+1}}{j+1}+O(jL^j);$

(iii) $\displaystyle\sum_{k\leq x}\frac{1}{k}\sum_{p|k}p^{i\beta}\log p=\log^2 x\int_0^1 x^{i\beta\theta}(1-\theta)\,d\theta+O(L);$

(iv) $$\sum_{k\le x}\frac{\log k}{k}\sum_{p|k}p^{i\beta}\log p=\frac{1}{2}\log^3 x\int_0^1 x^{i\beta\theta}(1-\theta^2)\,d\theta+O(L^2);$$

(v) $$\sum_{k\le x}\frac{1}{k}\sum_{p|k}p^{i\beta}\log^2 p=\log^3 x\int_0^1 x^{i\beta\theta}\theta(1-\theta)\,d\theta+O(L^2);$$

(vi) $$\sum_{k\le x}\frac{\varphi(k)}{k^2}\left(\log\frac{x}{k}\right)^j\left(\frac{x}{k}\right)^{i\beta}=\frac{6}{\pi^2}\log^{j+1}x\int_0^1 x^{i\beta\theta}\theta^j\,d\theta+O((j+1)L^j);$$

(vii) $$\sum_{k\le x}\frac{\varphi(k)}{k^2}\left(\log\frac{x}{k}\right)^j\left(\log\frac{Tk}{2\pi x}\right)^j$$
$$=\frac{6}{\pi^2}L^{j+l+1}\int_0^{\log x/L}\theta^j(1-\theta)^l\,d\theta+O((j+l+1)L^{j+l}).$$

Proof. Parts (i) and (ii) follow easily from Stieltjes' integration. The proofs of (iii), (iv) and (v) are similar so we only prove (iii). We observe that

$$\sum_{k\le x}\frac{1}{k}\sum_{p|k}p^{i\beta}\log p=\sum_{p\le x}\frac{\log p}{p^{1-i\beta}}\sum_{n\le x/p}\frac{1}{n}=\sum_{p\le x}\frac{\log p\log x/p}{p^{1-i\beta}}+O(\log x).$$

By the prime number theorem

$$\theta(u)\overset{\text{def}}{=}\sum_{p\le u}\log p=u+E_1(u),$$

where

$$E_1(u)\ll\frac{u}{\log^2 2u}\qquad(u\ge 1).$$

Thus the last sum above is equal to

$$\int_1^x\frac{\log(x/u)}{u^{1-i\beta}}\,d\theta(u)=\int_1^x\frac{\log(x/u)}{u^{1-i\beta}}\,du+\int_1^x\frac{\log(x/u)}{u^{1-i\beta}}\,dE_1(u).$$

The second integral on the right equals

$$E_1(u)\frac{\log(x/u)}{u^{1-i\beta}}\bigg|_1^x+\int_1^x\frac{E_1(u)}{u^{2-i\beta}}\left(1+(1-i\beta)\log\frac{x}{u}\right)du$$
$$\ll\log x+\log x\int_1^x\frac{du}{u\log^2 2u}\ll\log x.$$

In the first integral, we substitute x^θ for u and obtain

$$\log^2 x\int_0^1 x^{i\beta\theta}(1-\theta)\,d\theta.$$

Combining these results, we obtain (iii).

We now prove (vi). Set

$$S(u)=\sum_{k\leq u}\frac{\varphi(k)}{k^2}.$$

By an elementary argument one sees that

$$S(u)=\frac{6}{\pi^2}\log u+E_2(u)\qquad (u\geq 1),$$

where $E_2(u)\ll 1$. Thus

$$\sum_{k\leq x}\frac{\varphi(k)}{k^2}\left(\log\frac{x}{k}\right)^j\left(\frac{x}{k}\right)^{i\beta}=\frac{6}{\pi^2}\int_1^x\left(\log\frac{x}{u}\right)^j\left(\frac{x}{u}\right)^{i\beta}\frac{du}{u}+\int_{1-}^x\left(\log\frac{x}{u}\right)^j\left(\frac{x}{u}\right)^{i\beta}dE_2(u).$$

We substitute $x^{1-\theta}$ for u in the first integral on the right and obtain

$$\frac{6}{\pi^2}\log^{j+1}x\int_0^1 x^{i\beta\theta}\theta^j d\theta.$$

The second integral equals

$$E_2(u)\left(\log\frac{x}{u}\right)^j\left(\frac{x}{u}\right)^{i\beta}\Big|_{1-}^{x}+\int_1^x E_2(u)\left(\frac{x}{u}\right)^{i\beta}\left(\log\frac{x}{u}\right)^{j-1}\left(j+i\beta\log\frac{x}{u}\right)\frac{du}{u}$$

$$\ll 1+(j+|\beta|\log x)\log^j x\ll (j+1)L^j,$$

since $|\beta|\log x\ll 1$. This gives (vi).

To prove (vii) let $S(u)$ be as above. Then

$$\sum_{k\leq x}\frac{\varphi(k)}{k^2}\left(\log\frac{x}{k}\right)^j\left(\log\frac{Tk}{2\pi x}\right)^l=\int_{1-}^x\left(\log\frac{Tu}{2\pi x}\right)^l dS(u)$$

$$=-\frac{6}{\pi^2}\int_1^x\left(\log\frac{x}{u}\right)^j\left(\log\frac{Tu}{2\pi x}\right)^l d\left(\log\frac{x}{u}\right)+\int_1^x\left(\log\frac{x}{u}\right)^j\left(\log\frac{Tu}{2\pi x}\right)^l dE_2(u).$$

The first integral on the last line is

$$-\frac{6}{\pi^2}L^{j+l+1}\int_1^x\left(\frac{\log(x/u)}{L}\right)^j\left(1-\frac{\log(x/u)}{L}\right)^l d\left(\frac{\log(x/u)}{L}\right)$$

$$=\frac{6}{\pi^2}L^{j+l+1}\int_0^{\log x/L}\theta^j(1-\theta)^l d\theta.$$

The second integral equals

$$E_2(u)\left(\log\frac{x}{u}\right)^j\left(\log\frac{Tu}{2\pi x}\right)^l\Bigg|_{1-}^{x}$$
$$+\int_1^x E_2(u)\left(\log\frac{x}{u}\right)^{j-1}\left(\log\frac{Tu}{2\pi x}\right)^{l-1}\left(j\log\frac{Tu}{2\pi x}-l\log\frac{x}{u}\right)\frac{du}{u}$$
$$\ll L^{j+l}+(j+l)L^{j+l}\ll(j+l+1)L^{j+l}.$$

This completes the proof of (vii).

LEMMA 9. *Assume* RH. *Suppose that* α *is real with* $|\alpha|\leq 100L^{-1}$ *and* $n\leq x\leq T$. *If* $D_K(s)$ *is given by* (3.2), *then for* T *sufficiently large*

$$\sum_{k\leq x}\left(\frac{D_{nk}(1+i\alpha)}{\varphi(nk)}-\frac{(1-i\alpha\log nk)}{nk}\right)\ll_\varepsilon \alpha^2L^2n^{\varepsilon-1}.$$

Proof. To facilitate the proof, we allow α to be a complex variable restricted by $|\alpha|\leq 100L^{-1}$. Let $n=\prod p^\lambda$ and for simplicity set

$$f(k)=D_k(1+i\alpha)/\varphi(k).$$

By (3.2) f is multiplicative and

$$f(p^m)=(1+mA_p)p^{-m}, \tag{3.12}$$

where

$$A_p=(1-p^{i\alpha})\left(\frac{1-p^{-1-i\alpha}}{1-p^{-1}}\right). \tag{3.13}$$

Also, observe that

$$f(p^m)\leq\frac{1+mC_1p^{100L^{-1}}}{p^m}\leq c_2\frac{m+1}{p^{m(1-\varepsilon)}}=c_2d(p^m)p^{-m(1-\varepsilon)}$$

for some positive constants c_1 and c_2. Thus, since $\prod_{p|k}c_2$ and $d(k)$ are $\ll_\varepsilon k^\varepsilon$, we obtain the estimate

$$f(k)\ll_\varepsilon k^{-1+\varepsilon}. \tag{3.14}$$

The generating function for the sum $\sum_{k\leq x}f(nk)$ is given by

$$H_n(s,\alpha)=\sum_{k=1}^\infty f(nk)k^{-s}=\prod_p\left(\sum_{m=0}^\infty f(p^{m+\lambda})p^{-ms}\right)$$
$$=\prod_p\left(\sum_{m=0}^\infty f(p^m)p^{-ms}\right)\prod_{p|n}\left(\sum_{m=0}^\infty f(p^{m+\lambda})p^{-ms}\right)\left(\sum_{m=0}^\infty f(p^m)p^{-ms}\right)^{-1}.$$

Let $G_n(s,\alpha)$ denote the product over p dividing n. By (3.12) and (3.13), the other product is

$$\prod_p\left(\sum_{m=0}^\infty\frac{1+mA_p}{p^{m(s+1)}}\right)=\prod_p\left(\frac{1}{1-p^{-s-1}}+\frac{A_p}{p^{s+1}(1-p^{-s-1})^2}\right)$$

$$=\frac{\zeta^2(s+1)}{\zeta(s+1-i\alpha)}\prod_p\left(1+\frac{(1-p^{i\alpha})(1-p^{-i\alpha})}{p^{s+1}(p-1)(1-p^{-s-1+i\alpha})}\right)$$

$$=\frac{\zeta^2(s+1)}{\zeta(s+1-i\alpha)}F(s,\alpha),$$

say. Hence

$$H_n(s,\alpha)=\frac{\zeta^2(s+1)}{\zeta(s+1-i\alpha)}F(s,\alpha)G_n(s,\alpha). \tag{3.15}$$

The product for $F(s,\alpha)$ is absolutely and uniformly convergent for $\sigma\geqslant-\frac{3}{4}$, $|\alpha|\leqslant 100L^{-1}$, say. It therefore represents a bounded analytic function of s and α in $\sigma\geqslant-\frac{1}{2}$, $|\alpha|\leqslant 100L^{-1}$. Next consider $G_n(s,\alpha)$. By (3.12)

$$G_n(s,\alpha)=\frac{1}{n}\prod_{p|n}\left(\frac{1+\lambda A_p}{1-p^{-s-1}}+\frac{A_p}{p^{s+1}(1-p^{-s-1})^2}\right)\left(\frac{1}{1-p^{-s-1}}+\frac{A_p}{p^{s+1}(1-p^{-s-1})^2}\right)^{-1}$$

$$=\frac{1}{n}\prod_{p|n}\left(1+\frac{\lambda A_p}{\left(1+\frac{A_p}{p^{s+1}-1}\right)}\right).$$

Using (3.13) and considering separately those primes dividing n which are less than 100 and those greater than 100, and recalling that $n\leqslant T$, it is not hard to show that in $\sigma\geqslant-\frac{1}{2}$, $|\alpha|\leqslant 100L^{-1}$, each factor of G_n is less than $c_3(\lambda+1)$ for some positive constant c_3. Thus $G_n(s,\alpha)$ is analytic in s and α in this region and

$$G_n(s,\alpha)<\frac{1}{n}\prod_{p|n}c_3(\lambda+1)=\frac{d(n)}{n}\prod_{p|n}c_3\ll_\varepsilon n^{\varepsilon-1}. \tag{3.16}$$

We now apply Perron's formula to $H_n(s,\alpha)$ (*e.g.*, see Davenport [**4**; p. 105]); we obtain

$$\sum_{k\leqslant x}f(nk)=\frac{1}{2\pi i}\int_{1-iU}^{1+iU}H_n(s,\alpha)\frac{x^s}{s}\,ds$$

$$+O\left(x\sum_{k=1}^{\infty}f(nk)k^{-1}\min\left(1,\frac{1}{U|\log(x/k)|}\right)\right).$$

Splitting the sum on the right-hand side into the ranges $[1,\frac{1}{2}x)$, $[\frac{1}{2}x,\frac{3}{2}x)$, and $[\frac{3}{2}x,\infty)$, and using (3.14), we find that the O-term is

$$O\left(\frac{x}{Un^{1-\varepsilon}}\right).$$

Next we pull the contour left to the line $\sigma=-\frac{1}{4}$. On RH

$$\zeta(s),\frac{1}{\zeta(s)}\ll_\varepsilon(1+|t|)^\varepsilon \qquad (\sigma\geqslant\tfrac{1}{2}+\varepsilon,|s-1|>\tfrac{1}{8}),$$

so by (3.15) and (3.16),

$$H_n(s,\alpha)\ll_\varepsilon U^\varepsilon n^{\varepsilon-1}$$

on the new path of integration. Thus, the top and bottom edges contribute

$$\ll_\varepsilon \frac{x}{Un^{1-\varepsilon}}$$

and the left edge contributes

$$\ll \frac{U^\varepsilon}{x^{1/4}n^{1-\varepsilon}}.$$

Taking $U = x^2$ and combining these results leads to

$$\sum_{k\leq x} f(nk) = \operatorname*{res}_{s=0} H_n(s, \alpha)\frac{x^s}{s} + O_\varepsilon(n^{-1+\varepsilon}). \tag{3.17}$$

To compute the residue, we need the first several terms of the Taylor series for $F(s, \alpha)$, $G_n(s, \alpha)$, $\zeta^2(s+1)$, and $\zeta^{-1}(s+1-i\alpha)$ around $s=0$. We have seen that $F(s, \alpha)$ is analytic and uniformly bounded in $\sigma \geq -\frac{1}{2}$, $|\alpha| \leq 100L^{-1}$, so by Cauchy's estimates

$$\left(\frac{d}{ds}\right)^j_{s=0} F(s, \alpha) \ll 1 \qquad (|\alpha| \leq 100L^{-1}; j = 0, 1, 2).$$

By the analyticity in α, we also have

$$F(0, \alpha) = F(0, 0) + O(|\alpha|) = 1 + O(|\alpha|) \qquad (|\alpha| \leq 100L^{-1}).$$

Similarly, by (3.16)

$$\left(\frac{d}{ds}\right)^j_{s=0} G_n(s, \alpha) \ll n^{\varepsilon-1} \qquad (j = 0, 1, 2),$$

and

$$G_n(0, \alpha) = G_n(0, 0) + O(|\alpha|) = \frac{1}{n} + O(|\alpha|),$$

for $|\alpha| \leq 100L^{-1}$. Finally, since

$$\zeta(s) = \frac{1}{s-1} + \gamma + \ldots$$

for s near 1, we see that

$$\zeta^2(s+1) = \frac{1}{s^2} + \frac{2\gamma}{s} + \ldots,$$

and for $|\alpha| \leq 100L^{-1}$,

$$\frac{1}{\zeta}(1 - i\alpha) = -i\alpha + O(|\alpha|^2),$$

$$\left(\frac{1}{\zeta}\right)'(1 - i\alpha) = 1 + O(|\alpha|^2),$$

$$\left(\frac{1}{\zeta}\right)''(1 - i\alpha) = O(1).$$

Using these estimates and (3.15), we obtain

$$\begin{aligned}\operatorname*{res}_{s=0} H_n(s,\alpha)\frac{x^s}{s} &= \operatorname*{res}_{s=0}\left\{\frac{1}{s}(1+s\log x+\dots)(F(0,\alpha)+F'(0,\alpha)s+\dots)\right.\\ &\qquad\times(G_n(0,\alpha)+G_n'(0,\alpha)s+\dots)\left(\frac{1}{s^2}+\frac{2\gamma}{s}+\dots\right)\\ &\qquad\left.\times\left(\frac{1}{\zeta}(1-i\alpha)+\left(\frac{1}{\zeta}\right)'(1-i\alpha)s+\dots\right)\right\}\\ &= F(0,\alpha)G_n(0,\alpha)\left[\frac{\log^2 x}{2}(-i\alpha+O(|\alpha|^2))+\log x(1+O(|\alpha|^2))\right]\\ &\qquad+O_\varepsilon(n^{\varepsilon-1})\\ &= \frac{-i\alpha\log^2 x}{2n}+\frac{\log x}{n}+O_\varepsilon(n^{\varepsilon-1}).\end{aligned}$$

Combined with (3.17) this leads to

$$\sum_{k\leqslant x} f(nk) = -\frac{i\alpha\log^2 x}{2n}+\frac{\log x}{n}+O_\varepsilon(n^{\varepsilon-1}). \tag{3.18}$$

We next observe that

$$f(nk)=\frac{D_{nk}(1+i\alpha)}{\varphi(nk)}$$

is entire in α and has the expansion

$$f(nk)=\frac{1}{nk}-i\alpha\frac{\log nk}{nk}+\alpha^2 g(\alpha;nk)$$

with g entire for all n and k. Thus

$$\sum_{k\leqslant x}\left(f(nk)-\frac{1-i\alpha\log nk}{nk}\right)=\alpha^2 g^*(\alpha;n,x) \tag{3.19}$$

say, with g^* entire. Now

$$\sum_{k\leqslant x}\frac{1-i\alpha\log nk}{nk}=\frac{\log x}{n}-\frac{i\alpha\log^2 x}{2n}+O\left(\frac{\log n}{n}\right)$$

for $x\leqslant T$, $|\alpha|\leqslant 100L^{-1}$. Hence, by (3.18)

$$\sum_{k\leqslant x}\left(f(nk)-\frac{1-i\alpha\log nk}{nk}\right)\ll_\varepsilon n^{\varepsilon-1}$$

for $|\alpha|\leqslant 100L^{-1}$. From this and (3.19) we see that

$$\max_{|\alpha|\leqslant 100L^{-1}}|\alpha^2 g^*(\alpha;n,x)|\ll_\varepsilon n^{\varepsilon-1},$$

or, by the maximum modulus principle,

$$\max_{|\alpha|\leqslant 100L^{-1}}|g^*(\alpha;n,x)|\ll_\varepsilon n^{\varepsilon-1}L^2.$$

Thus, the right-hand side of (3.19) is

$$\ll_\varepsilon \alpha^2 L^2 n^{\varepsilon-1}.$$

This proves the lemma.

Our last lemma is a special case of a formula of Balasubramanian, Conrey and Heath-Brown [**1**].

LEMMA 10. *Let* $A(s)=\sum_{k\le y} k^{-s}$ *as previously, where* $y=(T/2\pi)^\eta$ *and* $\eta\in(0,\frac{1}{2})$. *Then for* $1\le t\le T$,

$$\int_1^t |\zeta A(\tfrac{1}{2}+iu)|^2 du = t\sum_{h,k\le y}\frac{(h,k)}{hk}\log\frac{t(h,k)^2 e^{2\gamma-1}}{2\pi hk}+O(T);$$

here γ is Euler's constant.

§4. *The estimation of* I_1. From this point on, we shall assume that GRH is true. We recall that

$$y=\left(\frac{T}{2\pi}\right)^\eta$$

for some fixed $\eta\in(0,1/2)$, and that

$$|\alpha|\le 100L^{-1}.$$

We will also assume, until the end of this section, that $\alpha\ne 0$.

By (2.6) and the estimates in (2.2) and (2.3), we see that

$$I_1=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(s-i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\,ds+O_\varepsilon(yT^{1/2+\varepsilon}),$$

where $a=1+L^{-1}$. We next replace $\zeta(1-s)$ by $\zeta(s)\chi(1-s)$ (see (2.4)) and write

$$\sum_{j=1}^{\infty} b(j)j^{-s}=\frac{\zeta'}{\zeta}(s-i\alpha)\zeta^2(s)A(s)\qquad(\sigma>1),$$

so that

$$b(j)=-\sum_{hmn=j}\Lambda(n)n^{i\alpha}d(m).$$

Then by Lemma 2,

$$I_1=\sum_{k\le y}\frac{1}{k}\sum_{j\le kT/2\pi} b(j)e\left(-\frac{j}{k}\right)+O_\varepsilon(yT^{1/2+\varepsilon}). \tag{4.1}$$

Writing

$$Q^*(s,\alpha,k)=\sum_{j=1}^{\infty} b(j)j^{-s}e\left(-\frac{j}{k}\right)$$

as in Lemma 6, we find by Perron's formula (see Titchmarsh [**12**; pp. 53–55]) that

$$\sum_{j\le kT/2\pi} b(j)e\left(-\frac{j}{k}\right)=\frac{1}{2\pi i}\int_{a-iT}^{a+iT} Q^*(s,\alpha,k)\left(\frac{kT}{2\pi}\right)^s\frac{ds}{s}+O_\varepsilon(kT^\varepsilon).$$

Now by Lemma 6, $Q^*(s,\alpha,k)$ has at most two poles in $\sigma>\frac{1}{2}$ and these are at $s=1$ and $s=1+i\alpha$ (we are still assuming that $\alpha\neq 0$). Hence, pulling the path of integration to the left as far as $\sigma=a_0=\frac{1}{2}+L^{-1}$, we find that the integral equals

$$R_1+R_{1+i\alpha}+\frac{1}{2\pi i}\left(\int_{a-iT}^{a_0-iT}+\int_{a_0-iT}^{a_0+iT}+\int_{a_0+iT}^{a+iT}\right)Q^*(s,\alpha,k)\left(\frac{kT}{2\pi}\right)^s\frac{ds}{s};$$

here R_1, $R_{1+i\alpha}$ are the residues of the integrand at $s=1$ and $1+i\alpha$, respectively. By Lemma 6, the left edge of the contour contributes

$$\begin{aligned}&\ll_\varepsilon T^\varepsilon y^{1/2}(kT)^{a_0}\int_{-T}^{T}\frac{dt}{1+|t|}\\&\ll_\varepsilon yT^{(1/2)+\varepsilon},\end{aligned}$$

since $k\le y\ll T^{1/2}$. The bottom and top edges contribute

$$\ll_\varepsilon T^\varepsilon y^{1/2}\frac{(kT)^a}{T}\ll_\varepsilon T^\varepsilon y^{3/2}\ll_\varepsilon yT^{(1/2)+\varepsilon}.$$

Thus

$$\sum_{j\le kT/2\pi} b(j)e\left(\frac{-j}{k}\right)=R_1+R_{1+i\alpha}+O_\varepsilon(yT^{(1/2)+\varepsilon}).\tag{4.2}$$

We now compute the residues R_1 and $R_{1+i\alpha}$. Let $Q(s,\alpha,h/k)$ be as in Lemma 5. Then clearly

$$Q^*(s,\alpha,k)=\sum_{h\le y}Q(s,\alpha,h/k)h^{-s}.\tag{4.3}$$

Thus, by Lemma 4(i)

$$R_1=\operatorname*{res}_{s=1}\left[\frac{\zeta^2(s)}{s}\sum_{h\le y}K\left(\frac{T}{2\pi HK}\right)^s\left(\frac{\zeta'}{\zeta}(s-i\alpha)-G(s,\alpha,k)\right)\right],$$

where

$$H=\frac{h}{(h,k)}\qquad\text{and}\qquad K=\frac{k}{(h,k)}.$$

Since $\zeta(s)=(s-1)^{-1}+\gamma+\ldots$ near $s=1$ (where γ is Euler's constant), we find that

$$\begin{aligned}R_1=\frac{T}{2\pi}\sum_{h\le y}\frac{1}{H}\Bigg(\log\frac{Te^{2\gamma-1}}{2\pi HK}&\left(\frac{\zeta'}{\zeta}(1-i\alpha)-G(1,\alpha,K)\right)\\&+\frac{\zeta'}{\zeta}(1-i\alpha)-G'(1,\alpha,K)\Bigg).\end{aligned}\tag{4.4}$$

To compute $R_{1+i\alpha}$, we again appeal to (4.3) and find that

$$R_{1+i\alpha} = \operatorname*{res}_{s=1+i\alpha}\left[\frac{1}{s}\sum_{h\leq y}\left(\frac{KT}{2\pi H}\right)^{s} Q(s, \alpha, H/K)\right].$$

Thus, by Lemma 5(ii),

$$R_{1+i\alpha} = -\frac{\zeta^2(1+i\alpha)}{1+i\alpha}\left(\frac{T}{2\pi}\right)^{1+i\alpha}\sum_{h\leq y}\frac{KD_K(1+i\alpha)}{H^{1+i\alpha}\varphi(K)}. \tag{4.5}$$

Combining (4.1), (4.2), (4.4) and (4.5), we now obtain

$$I_1 = \frac{T}{2\pi}\sum_{h,k\leq y}\frac{(h,k)}{hk}\left(\log\frac{Te^{2\gamma-1}}{2\pi HK}\left(\frac{\zeta'}{\zeta}(1-i\alpha)-G(1,\alpha,K)\right)\right.$$
$$\left.+\left(\frac{\zeta'}{\zeta}\right)'(1-i\alpha)-G'(1,\alpha,K)-\frac{\zeta^2(1+i\alpha)}{1+i\alpha}\left(\frac{T}{2\pi H}\right)^{i\alpha}\frac{KD_K(1+i\alpha)}{\varphi(K)}\right)$$
$$+O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}).$$

Using Lemma 7 and $0<|\alpha|\leq 100L^{-1}$, we may write this as

$$I_1 = \frac{T}{2\pi}\sum_{h,k\leq y}\frac{(h,k)}{hk}\left((L-\log H-\log K)\left(\frac{1}{i\alpha}-\sum_{p|K}p^{i\alpha}\log p\right)\right.$$
$$\left.+\frac{1}{(i\alpha)^2}-\sum_{p|K}p^{i\alpha}\log^2 p+\frac{1}{(i\alpha)^2}\left(\frac{T}{2\pi H}\right)^{i\alpha}\frac{KD_K(1+i\alpha)}{\varphi(K)}\right)$$
$$+O(TL^4)+O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}).$$

We next apply the Möbius inversion formula

$$f((h,k)) = \sum_{\substack{m|h\\ m|k}}\sum_{n|m}\mu(n)f\left(\frac{m}{n}\right),$$

and obtain

$$I_1 = \frac{T}{2\pi}\sum_{h,k\leq y}\frac{1}{hk}\sum_{\substack{m|h\\ m|k}}\sum_{n|m}\mu(n)\frac{m}{n}\left(\log\frac{Tm^2}{2\pi kn^2}\left(\frac{1}{i\alpha}-\sum_{p|nk/m}p^{i\alpha}\log p\right)\right.$$
$$\left.+\frac{1}{(i\alpha)^2}-\sum_{p|nk/m}p^{i\alpha}\log^2 p+\frac{\left(\frac{T}{2\pi h}\right)^{i\alpha}kn^{1-i\alpha}}{(i\alpha)^2 m^{1-i\alpha}}\frac{D_{nk/m}(1+i\alpha)}{\varphi\left(\frac{nk}{m}\right)}\right)$$
$$+O(TL^4)+O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}).$$

Interchanging summations and replacing h by hm, k by km, we see that

$$I_1 = \frac{T}{2\pi}\sum_{m\leq y}\frac{1}{m}\sum_{n|m}\frac{\mu(n)}{n}\sum_{h,k\leq y/m}\frac{1}{hk}$$
$$\times\left(-\log\frac{T}{2\pi hk}\sum_{p|k}p^{i\alpha}\log p-\sum_{p|k}p^{i\alpha}\log^2 p+O(L\log n)\right)$$

$$+\frac{T}{2\pi}\sum_{m\leq y}\frac{1}{m}\sum_{n|m}\frac{\mu(n)}{n}\sum_{h,k\leq y/m}\frac{1}{hk}$$
$$\times\left(\frac{1+i\alpha\log\frac{T}{2\pi hkn^2}-\left(\frac{T}{2\pi hn}\right)^{i\alpha}kn\frac{D_{nk}(1+i\alpha)}{\varphi(nk)}}{(i\alpha)^2}\right)$$
$$+O(TL^4)+O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}),$$

or

$$I_1=J_1+J_2+O(TL^4)+O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}). \tag{4.6}$$

The term $O(L\log n)$ in J_1 may be omitted because it contributes

$$\ll TL^3\sum_{m\leq y}\frac{1}{m}\sum_{n|m}\frac{\log n}{n}=TL^3\sum_{n\leq y}\frac{\log n}{n^2}\sum_{l\leq y/m}\frac{1}{l}\ll TL^4.$$

Thus, observing that $\sum_{n|m}\mu(n)/n=\varphi(m)/m$ and using Lemma 8 parts (i), (iii), (iv) and (v), we find that

$$J_1=\frac{T}{2\pi}\sum_{m\leq y}\frac{\varphi(m)}{m^2}\left(\frac{\log^4 y/m}{2}\int_0^1\left(\frac{y}{m}\right)^{i\alpha\theta}(1-\theta)(2-\theta)d\theta\right.$$
$$\left.-L\log^3 y/m\int_0^1\left(\frac{y}{m}\right)^{i\alpha\theta}(1-\theta)d\theta\right)+O(TL^4).$$

Applying Lemma 8(vi) to this, we obtain

$$J_1=\frac{6}{\pi^2}\frac{T}{2\pi}\left(\frac{\log^5 y}{2}\int_0^1\int_0^1 y^{i\alpha\theta\varphi}(1-\theta)(2-\theta)\varphi^4 d\varphi d\theta\right.$$
$$\left.-L\log^4 y\int_0^1\int_0^1 y^{i\alpha\theta\varphi}(1-\theta)\varphi^3 d\varphi d\theta\right)+O(TL^4).$$

Since

$$\int_0^1\int_0^1 y^{i\alpha\theta\varphi}\theta^k\varphi^l d\theta d\varphi=\sum_{j=0}^{\infty}\frac{(i\alpha\log y)^j}{j!(j+k+1)(j+l+1)},$$

this may be expressed as

$$J_1=\frac{6}{\pi^2}\frac{T}{2\pi}\left(\frac{\log^5 y}{2}\sum_{j=0}^{\infty}\frac{(i\alpha\log y)^j}{(j+3)!}-L\log^4 y\sum_{j=0}^{\infty}\frac{(i\alpha\log y)^j}{(j+2)!(j+4)}\right)+O(TL^4).$$

Finally, since $\log y=\eta L$,

$$J_1=\frac{6}{\pi^2}\frac{T}{2\pi}L^5\sum_{j=0}^{\infty}\frac{(i\alpha\eta L)^j}{(j+4)!}\left(\frac{\eta^5(j+4)}{2}-\eta^4(j+3)\right)+O(TL^4). \tag{4.7}$$

We now turn to J_2. According to Lemma 9, replacing $D_{nk}(1+i\alpha)/\varphi(nk)$ by $(1-i\alpha\log nk)/nk$ in J_2 introduces an error term of magnitude

$$\ll_\varepsilon TL^3\sum_{m\le y}\frac{1}{m}\sum_{n|m}\frac{1}{n^{1-\varepsilon}}\ll_\varepsilon TL^4.$$

Making this substitution and expanding $(T/(2\pi hn))^{i\alpha}$ in a power series around $\alpha=0$, we find that

$$J_2=\frac{T}{2\pi}\sum_{m\le y}\frac{1}{m}\sum_{n|m}\frac{\mu(n)}{n}\sum_{h,k\le y/m}\frac{1}{hk}\left(\log\frac{T}{2\pi hn}\log nk\right.$$
$$\left.-(1-i\alpha\log nk)\log^2\frac{T}{2\pi hn}\sum_{j=0}^{\infty}\frac{(i\alpha\log(T/(2\pi hn)))^j}{(j+2)!}\right)+O_\varepsilon(TL^4).$$

Now it is not hard to see that if we replace $\log nk$ by $\log k$, $\log(T/(2\pi hn))$ by $\log(T/(2\pi h))$, and use $|\alpha|\le 100L^{-1}$, the expression within the outermost parentheses changes by at most $O(L\log n)$. Hence

$$J_2=\frac{T}{2\pi}\sum_{m\le y}\frac{\varphi(m)}{m^2}\sum_{h,k\le y/m}\frac{1}{hk}\left(\log\frac{T}{2\pi h}\log k\right.$$
$$\left.-(1-i\alpha\log k)\log^2\frac{T}{2\pi h}\sum_{j=0}^{\infty}\frac{(i\alpha\log(T/(2\pi h)))^j}{(j+2)!}\right)+O_\varepsilon(TL^4).$$

Carrying out the sums over h and k by means of Lemma 8(i) and (ii), we obtain

$$J_2=\frac{T}{2\pi}\sum_{m\le y}\frac{\varphi(m)}{m^2}\left(\frac{1}{4}\log^2\frac{y}{m}\left(L^2-\log^2\frac{Tm}{2\pi y}\right)\right.$$
$$-\left(\log\frac{y}{m}-\frac{i\alpha}{2}\log^2\frac{y}{m}\right)\left(L^3\sum_{j=0}^{\infty}\frac{(i\alpha L)^j}{(j+3)!}\right.$$
$$\left.\left.-\log^3\frac{Tm}{2\pi y}\sum_{j=0}^{\infty}\frac{(i\alpha\log(Tm/(2\pi y)))^j}{(j+3)!}\right)\right)+O_\varepsilon(TL^4)$$
$$=\frac{T}{2\pi}\sum_{m\le y}\frac{\varphi(m)}{m^2}\left(\frac{1}{4}\log^2\frac{y}{m}\left(L^2-\log^2\frac{Tm}{2\pi y}\right)+\frac{1}{6}\log\frac{y}{m}\left(\log^3\frac{Tm}{2\pi y}-L^3\right)\right.$$
$$+\sum_{j=0}^{\infty}\frac{(i\alpha)^{j+1}}{(j+3)!}\left(\frac{\log y/m}{j+4}\left(\log^{j+4}\frac{Tm}{2\pi y}-L^{j+4}\right)\right.$$
$$\left.\left.+\frac{1}{2}\log^2\frac{y}{m}\left(L^{j+3}-\log^{j+3}\frac{Tm}{2\pi y}\right)\right)\right)+O_\varepsilon(TL^4).$$

Using Lemma 8(vii) to perform the summation over m, we find that

$$J_2=\frac{6}{\pi^2}\frac{T}{2\pi}L^5\left(\int_0^{(\log y)/L}\left(-\frac{1}{2}\theta^2+\theta^3-\frac{5}{12}\theta^4\right)d\theta\right.$$
$$\left.+\sum_{j=0}^{\infty}\frac{(i\alpha L)^{j+1}}{(j+3)!}\int_0^{(\log y)/L}\frac{1}{j+4}(\theta(1-\theta)^{j+4}-\theta)+\frac{1}{2}(\theta^2-\theta^2(1-\theta)^{j+3})d\theta\right)$$
$$+O_\varepsilon(TL^4).$$

Now $\log y = \eta L$ and

$$\int_0^\eta \theta(1-\theta)^l d\theta = \frac{1}{(l+1)(l+2)} - \frac{(1-\eta)^{l+1}(1+\eta(l+1))}{(l+1)(l+2)},$$

$$\int_0^\eta \theta^2(1-\theta)^l d\theta = \frac{2}{(l+1)(l+2)(l+3)} - \frac{(1-\eta)^{l+1}(2+2\eta(l+1)+\eta^2(l+1)(l+2))}{(l+1)(l+2)(l+3)}.$$

Thus, carrying out the above integrations and simplifying the result, we find that

$$J_2 = \frac{6}{\pi^2}\frac{T}{2\pi}L^5\left(\frac{-2\eta^3+3\eta^4-\eta^5}{12} + \sum_{j=0}^{\infty}\frac{(i\alpha L)^{j+1}}{(j+4)!}\right.$$
$$\left.\times\left(\frac{\eta^2(1-\eta)^{j+4}}{2} + \frac{(j+4)\eta^3-3\eta^2}{6}\right)\right) + O_\varepsilon(TL^4).$$

Combining this result with (4.6) and (4.7), we finally see that for sufficiently large T

$$I_1 = \frac{6}{\pi^2}\frac{T}{2\pi}L^5\left(\frac{-4\eta^3+3\eta^4}{24} + \sum_{j=0}^{\infty}\frac{(i\alpha L)^{j+1}}{(j+4)!}\right.$$
$$\left.\times\left(\frac{-3\eta^2+(j+4)\eta^3}{6} + \frac{\eta^{j+6}}{2} - \frac{j+4}{j+5}\eta^{j+5} + \frac{\eta^2}{2}(1-\eta)^{j+4}\right)\right) \tag{4.8}$$
$$+ O_\varepsilon(TL^4) + O_\varepsilon(T^{\frac{1}{2}+\eta+\varepsilon}).$$

Until now we have assumed that $\alpha \neq 0$. However, the main term in (4.8) is continuous at $\alpha = 0$ and from (2.6) it is clear that I_1 is as well. Thus, (4.8) holds for all $|\alpha| \leqslant 100L^{-1}$.

§5. *The estimation of I_2 and of $\mathcal{M}_1(T)$.* From (2.7) we recall that

$$I_2 = \frac{1}{2\pi i}\int_{a+i}^{a+i(T+\alpha)} \frac{\chi'}{\chi}(1-s+i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\,ds.$$

Taking the logarithmic derivative of (2.5) and using the formula

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right) \qquad (|s|\to\infty,\ |\arg s| < \pi-\delta,\ \delta>0)$$

(see Whittaker and Watson [13; Chaps. 12, 13]), we see that

$$\frac{\chi'}{\chi}(1-s+i\alpha) = -\log\frac{t}{2\pi} + O\left(\frac{1}{t}\right) \qquad \left(t\geqslant 1, \frac{1}{2}\leqslant\sigma\leqslant 2, |\alpha|\leqslant 100L^{-1}\right).$$

Thus, if we move the line of integration above to $\sigma = \frac{1}{2}$ and use the estimate (2.2), we obtain

$$I_2 = -\frac{1}{2\pi}\int_1^T |\zeta A(\tfrac{1}{2}+it)|^2 \log\frac{t}{2\pi}\,dt + O\left(\int_1^T |\zeta A(\tfrac{1}{2}+it)|^2\frac{dt}{t}\right) + O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}).$$

By (2.2) again the second integral on the right is

$$\ll_\varepsilon tT^{\frac{1}{2}+\frac{1}{2}\varepsilon}\int_1^T \frac{dt}{t} \ll_\varepsilon yT^{\frac{1}{2}+\varepsilon},$$

so we have

$$I_2 = -\frac{1}{2\pi}\int_1^T |\zeta A(\tfrac{1}{2}+it)|^2 \log\frac{t}{2\pi}\,dt + O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}). \tag{5.1}$$

Let us set

$$J(t) = \int_1^t |\zeta A(\tfrac{1}{2}+iu)|^2\,du. \tag{5.2}$$

Then by Lemma 10,

$$J(t) = t\sum_{h,k\leqslant y}\frac{(h,k)}{hk}\log\frac{t(h,k)^2e^{2\gamma-1}}{2\pi hk} + O(T)$$

for $1\leqslant t\leqslant T$. To estimate the sum we apply Möbius inversion as in the last section. This leads to

$$J(t) = t\sum_{h,k\leqslant y}\frac{1}{hk}\sum_{\substack{m|h\\ m|k}}\sum_{n|m}\frac{\mu(n)m}{n}\log\frac{te^{2\gamma-1}m^2}{2\pi n^2hk} + O(T).$$

Changing the order of summation and replacing h by hm, k by km, we find that

$$J(t) = t\sum_{m\leqslant y}\frac{1}{m}\sum_{n|m}\frac{\mu(n)}{n}\sum_{h,k\leqslant y/m}\frac{1}{hk}\log\frac{te^{2\gamma-1}}{2\pi n^2hk} + O(T).$$

We next replace the logarithm term by $\log(t/(2\pi hk))$. This changes the sum by an amount which is

$$\ll t\sum_{m\leqslant y}\frac{\log^2(y/m)}{m}\sum_{n|m}\frac{\log n}{n} \ll TL^3,$$

so

$$J(T) = t\sum_{m\leqslant y}\frac{\varphi(m)}{m^2}\sum_{h,k\leqslant y/m}\frac{1}{hk}\left(\log\frac{t}{2\pi} - \log h - \log k\right) + O(TL^3)$$

for $1\leqslant t\leqslant T$. By Lemma 8(i),

$$J(t) = t\sum_{m\leqslant y}\frac{\varphi(m)}{m^2}\left(\log\frac{t}{2\pi}\log^2\frac{y}{m} - \log^3\frac{y}{m}\right) + O(TL^3).$$

Thus, summing over m by means of Lemma 8(vii), we have

$$J(t) = \frac{6}{\pi^2}t\left(\log\frac{t}{2\pi}\frac{\log^3 y}{3} - \frac{\log^4 y}{4}\right) + O(TL^3). \tag{5.3}$$

Using (5.1)-(5.3), we now find that

$$I_2 = -\frac{1}{2\pi}\int_1^T J'(t)\log\frac{t}{2\pi}\,dt + O_\varepsilon(yT^{\frac{1}{2}+\varepsilon})$$

$$= -J(T)\frac{L}{2\pi} + O\left(\int_1^T |J(t)|\frac{dt}{t}\right) + O_\varepsilon(yT^{\frac{1}{2}+\varepsilon})$$

$$= -\frac{6}{\pi^2}\frac{T}{2\pi}L\left(L\frac{\log^3 y}{3} - \frac{\log^4 y}{4}\right) + O(TL^4) + O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}).$$

Therefore, upon substituting ηL for $\log y$, we obtain

$$I_2 = \frac{6}{\pi^2}\frac{T}{2\pi}L^5\left(\frac{3\eta^4 - 4\eta^3}{12}\right) + O(TL^4) + O_\varepsilon(T^{\frac{1}{2}+\eta+\varepsilon}). \tag{5.4}$$

Finally, we observe from the definition of $\mathcal{M}_1(T)$ in (1.2) that

$$\mathcal{M}_1(T) = J(T).$$

Thus, by (5.3),

$$\mathcal{M}_1(T) = \frac{6}{\pi^2}TL^4\left(\frac{4\eta^3 - 3\eta^4}{12}\right) + O(TL^3). \tag{5.5}$$

§6. *Completion of the proofs of Theorems* 1 *and* 2. By (1.2) and (2.8) we have

$$m_2(\alpha, T) = \sum_{0<\gamma\leq T} |\zeta A(\tfrac{1}{2} + i(\gamma+\alpha))|^2 = 2\,\mathrm{Re}\,I_1 - \bar{I}_2 + O_\varepsilon(yT^{\frac{1}{2}+\varepsilon}).$$

Combining this with (4.8) and (5.4), we see that for $0<\eta<\frac{1}{2}$, $|\alpha|\leq 100L^{-1}$, and T large,

$$m_2(\alpha, T) = \frac{6}{\pi^2}\frac{T}{2\pi}L^5\sum_{j=0}^{\infty}\frac{(-1)^{j+1}(\alpha L)^{2j+2}}{(2j+5)!}\left(\frac{-3\eta^2 + (2j+5)\eta^3}{3}\right.$$
$$\left. - \frac{2j+5}{j+3}\eta^{2j+6} + \eta^{2j+7} + \eta^2(1-\eta)^{2j+5}\right) + O_\varepsilon(TL^4)$$
$$+ O_\varepsilon(T^{\frac{1}{2}+\eta+\varepsilon}).$$

This completes the proof of Theorem 1.

To prove Theorem 2, we write

$$\mathcal{M}_2(c, T) = \int_{-c/L}^{c/L} m_2(\alpha, T)\,d\alpha$$

as in (1.3). If $|c|\leq 100$ we may replace $m_2(\alpha, T)$ by the expression above. We then obtain

$$\mathcal{M}_2(c,T)=\frac{12}{\pi^2}\frac{T}{2\pi}L^4\sum_{j=0}^{\infty}\frac{(-1)^{j+1}c^{2j+3}}{(2j+5)!(2j+3)}\left(\frac{-3\eta^2+(2j+5)\eta^3}{3}\right.$$
$$\left.-\frac{2j+5}{j+3}\eta^{2j+6}+\eta^{2j+7}+\eta^2(1-\eta)^{2j+5}\right)+O_\varepsilon(TL^3)+O_\varepsilon(T^{\frac{1}{2}+\eta+\varepsilon}).$$

Next we take $\eta=\frac{1}{2}-\varepsilon$ and find that

$$\mathcal{M}_2(c,T)=\frac{12}{\pi^2}\frac{T}{2\pi}L^4\sum_{j=0}^{\infty}\frac{(-1)^{j+1}c^{2j+3}}{(2j+5)!(2j+3)}\left(\frac{-3(\frac{1}{2}-\varepsilon)^2+(2j+5)(\frac{1}{2}-\varepsilon)^3}{3}\right.$$
$$\left.-\frac{2j+5}{j+3}(\tfrac{1}{2}-\varepsilon)^{2j+6}+(\tfrac{1}{2}-\varepsilon)^{2j+7}+(\tfrac{1}{2}-\varepsilon)^2(\tfrac{1}{2}+\varepsilon)^{2j+5}\right)+O_\varepsilon(TL^3).$$

By the mean value theorem

$$\left|(\tfrac{1}{2}\pm\varepsilon)^k-(\tfrac{1}{2})^k\right|\leqslant\varepsilon k(\tfrac{1}{2}+\varepsilon)^{k-1}<\varepsilon k$$

for $k\geqslant 0$ and $0<\varepsilon<\frac{1}{2}$, so if we replace the quantities $(\frac{1}{2}\pm\varepsilon)$ above by $\frac{1}{2}$, we introduce an error term of size $\ll\varepsilon TL^4$. Thus, we have

$$\mathcal{M}_2(c,T)=\frac{3}{\pi^2}\frac{T}{2\pi}L^4\sum_{j=0}^{\infty}\frac{(-1)^jc^{2j+3}}{(2j+5)!(2j+3)}\left(\frac{1-2j}{6}+\frac{j+2}{2^{2j+4}(j+3)}\right)+O(\varepsilon TL^4).$$

On the other hand, by (5.5) with $\eta=\frac{1}{2}-\varepsilon$ we have

$$\mathcal{M}_1(T)=\frac{1}{\pi^2}TL^4\frac{5}{32}+O(\varepsilon TL^4).$$

Now as we have seen in section 1, any value of c for which

$$\mathcal{M}_2(c,2T)-\mathcal{M}_2(c,T)<\mathcal{M}_1(2T)-\mathcal{M}_1(T)$$

serves as a lower bound for $\pi\lambda$. Thus we wish to find as large a value of c as is practical for which

$$\frac{\mathcal{M}_2(c,2T)-\mathcal{M}_2(c,T)}{\mathcal{M}_1(2T)-\mathcal{M}_1(T)}=\frac{24}{5\pi}\sum_{j=0}^{\infty}\frac{(-1)^jc^{2j+3}}{(2j+5)!(2j+3)}\left(\frac{1-2j}{3}+\frac{j+2}{2^{2j+3}(j+3)}\right)+O(\varepsilon)$$

is less than 1. By a machine calculation we can show that the main term here is less than 1 if $c=2{\cdot}68\pi$. Since ε may be taken arbitrarily small, it follows that $\lambda>2{\cdot}68$.

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