

THE FOURTH MOMENT OF DERIVATIVES OF THE RIEMANN ZETA-FUNCTION

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1. Introduction

IN THIS paper we prove an asymptotic formula for

$$I(P_1, P_2, P_3, P_4; T) := \int_1^T A_1 A_2 \left(\frac{1}{2} + it\right) A_3 A_4 \left(\frac{1}{2} - it\right) dt \quad (1)$$

where

$$A_i(s) = P_i \left(-\frac{1}{L} \frac{d}{ds} \right) \zeta(s) \quad (1 \leq i \leq 4) \quad (2)$$

with polynomials P_i and where we use

$$L = \log \frac{T}{2\pi}.$$

What we prove is

THEOREM. *We have*

$$I(P_1, P_2, P_3, P_4; T) \sim c(P_1, P_2, P_3, P_4) \frac{TL^4}{\pi^2/6}$$

where $c = c(P_1, P_2, P_3, P_4)$ is equal to

$$\int_{\mathcal{R}} (P_1(\alpha + \beta) P_2(\gamma + \delta) P_3(\alpha + \gamma) P_4(\beta + \delta) + P_1(1 - \alpha - \beta) P_2(1 - \gamma - \delta) P_3(1 - \alpha - \gamma) P_4(1 - \beta - \delta)) d\alpha d\beta d\gamma d\delta \quad (3)$$

with $\mathcal{R} = \{(\alpha, \beta, \gamma, \delta) \in [0, 1]^4 \mid \alpha + \beta + \gamma + \delta \leq 1\}$.

Of course, Ingham [3] proved that

$$\int_1^T |\zeta(\frac{1}{2} + it)|^4 dt \sim \frac{TL^4}{2\pi^2}. \quad (4)$$

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To this we add

COROLLARY. *We have*

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^2 |\zeta'(\tfrac{1}{2} + it)|^2 dt \sim \frac{2}{15\pi^2} TL^6, \quad (5)$$

$$\int_1^T |\zeta'(\tfrac{1}{2} + it)|^4 dt \sim \frac{61}{1680\pi^2} TL^8, \quad (6)$$

and if we let

$$C_{k,m} = \lim_{T \rightarrow \infty} T^{-1} L^{-k^2 - 2km} \int_1^T |\zeta^{(m)}(\tfrac{1}{2} + it)|^{2k} dt \quad (7)$$

for $k = 1, 2$, then

$$\frac{\pi^2}{6} C_{2,m} \sim \frac{1}{16m^4} \quad (8)$$

as $m \rightarrow \infty$.

These results should be compared with the second moment case which follows easily from Ingham's work [3]. With the above notation,

$$\int_1^T A_1(\tfrac{1}{2} + it) A_2(\tfrac{1}{2} - it) dt \sim c(P_1, P_2) TL \quad (9)$$

with

$$c(P_1, P_2) = \int_0^1 P_1(\theta) P_2(\theta) d\theta. \quad (10)$$

In particular, with $P_1(\theta) = P_2(\theta) = (-1)^m \theta^m$ we see that

$$c_{1,m} = \frac{1}{2m+1}. \quad (11)$$

In an effort to find an asymptotic formula for a first moment of a Dirichlet series (no such non-trivial formula is known) we considered

$$I = \int_1^T |\zeta(\tfrac{1}{2} + it) \zeta'(\tfrac{1}{2} + it)| dt \quad (12)$$

and we can show that

$$0.533 \dots = \left(\frac{1}{4} + \left(\frac{e^2 - 5}{4\pi} \right)^2 \right)^{\frac{1}{2}} \leq \frac{I}{TL^2} \leq 1/\sqrt{3} = 0.577 \dots \quad (13)$$

as $T \rightarrow \infty$. Here we can obtain estimates for

$$J = \int_1^T |\zeta(\tfrac{1}{2} + it)^3 \zeta'(\tfrac{1}{2} + it)| dt. \quad (14)$$

For an upper bound we have

$$J \leq \left(\int_1^T |\zeta(\tfrac{1}{2} + it)|^4 dt \right)^{\frac{1}{2}} \left(\int_1^T |\zeta \zeta'(\tfrac{1}{2} + it)|^2 dt \right)^{\frac{1}{2}} \sim \frac{TL^5}{\sqrt{(15)\pi^2}} \quad (15)$$

by (4) and (5). On the other hand, since $\sqrt{1+x} \geq 1 + \frac{1}{2}x - \frac{1}{8}x^3$ for all $x \geq 0$, we have

$$\begin{aligned} J &= \int_1^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \operatorname{Re} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right| \\ &\quad \times \left(1 + \left(\operatorname{Im} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) / \operatorname{Re} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right)^2 \right)^{\frac{1}{2}} dt \\ &\geq J_1 + \frac{1}{2}J_2 - \frac{1}{8}J_3 \end{aligned} \quad (16)$$

where, because of

$$\operatorname{Re} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) = -\frac{1}{2} \log \frac{t}{2\pi} + O\left(\frac{1}{|t|}\right) \quad (17)$$

for $|t| \gg 1$, we have

$$J_1 = \int_1^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \operatorname{Re} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right| dt \sim \frac{TL^5}{4\pi^2}, \quad (18)$$

$$\begin{aligned} J_2 &= \int_1^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \operatorname{Im} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right|^2 \left| \operatorname{Re} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right|^{-1} dt \\ &= \int_1^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right|^2 \left| \operatorname{Re} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right|^{-1} dt \\ &\quad - \int_1^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \operatorname{Re} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right| dt \\ &\sim \frac{4TL^5}{15\pi^2} - \frac{TL^5}{4\pi^2} = \frac{TL^5}{60\pi^2}, \end{aligned} \quad (19)$$

and

$$\begin{aligned}
 J_3 &= \int_1^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \operatorname{Im} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right|^4 \left| \operatorname{Re} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right|^{-3} dt \\
 &= \int_1^T |\zeta|^4 \left(\left| \frac{\zeta'}{\zeta} \right|^4 - \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right|^4 - 2 \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right|^2 \left| \operatorname{Im} \frac{\zeta'}{\zeta} \right|^2 \right) \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right|^{-3} dt \\
 &= \int_1^T |\zeta|^4 \left(\left| \frac{\zeta'}{\zeta} \right|^4 + \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right|^4 - 2 \left| \frac{\zeta'}{\zeta} \right|^2 \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right|^2 \right) \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right|^{-3} dt \\
 &= \int_1^T \left(|\zeta'|^4 \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right|^{-3} - 2 |\zeta|^2 |\zeta'|^2 \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right|^{-1} + |\zeta|^4 \left| \operatorname{Re} \frac{\zeta'}{\zeta} \right| \right) dt \\
 &\sim \left(\frac{61}{210} - \frac{8}{15} + \frac{1}{4} \right) \frac{TL^5}{\pi^2} = \frac{TL^5}{140\pi^2}.
 \end{aligned} \tag{20}$$

Thus we find that

$$\frac{173}{672} \ll \frac{\pi^2 J}{TL^5} \ll \frac{1}{\sqrt{15}}. \tag{21}$$

Note that $173/672 = 0.25744047\dots$ and $15^{-\frac{1}{2}} = 0.25819889\dots$ so that the upper and lower constants differ by less than 0.0008.

As a further application we mention the following results involving

$$Z(t) = \chi(1-s)^{\frac{1}{2}} \zeta(s) \tag{22}$$

for $s = \frac{1}{2} + it$. As is well known Z is a real function of t , $|Z(t)| = |\zeta(\frac{1}{2} + it)|$, and Z changes sign when t is the ordinate of a zero of $\zeta(s)$ of odd multiplicity. It is not hard to show that

$$\frac{Z'}{Z}(t) = \operatorname{Im} \frac{\zeta'}{\zeta}(\tfrac{1}{2} - it). \tag{23}$$

Now from (5), (6), and (17) it is easy to deduce that

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \operatorname{Im} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right|^2 dt = \int_1^T |Z(t)Z'(t)|^2 dt \sim \frac{TL^6}{120\pi^2}, \tag{24}$$

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \operatorname{Im} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right|^4 dt = \int_1^T |Z'(t)|^4 dt \sim \frac{TL^8}{1120\pi^2}, \tag{25}$$

and

$$\frac{TL^5\sqrt{21}}{180\pi^2} \ll \int_1^T |(\tfrac{1}{2} + it)|^4 \left| \operatorname{Im} \frac{\zeta'}{\zeta}(\tfrac{1}{2} + it) \right| dt = \int_1^T |Z(t)^3 Z'(t)| dt \ll \frac{TL^5}{4\sqrt{15}\pi^2}. \quad (26)$$

In (26) the upper bound is a consequence of the Cauchy–Schwarz inequality, (4), and (24), while the lower bound follows from (24), (25), and Hölder’s inequality in the form

$$\int |fg| \geq \frac{\left(\int |fg|\right)^2}{\left(\int |f^4 g|\right)^{\frac{1}{2}}} \quad (27)$$

with $g = Z^4$ and $f = Z'/Z$. We note that for any $\varepsilon > 0$

$$\begin{aligned} \int_1^T |Z(t)^3 Z'(t)| dt &= \frac{1}{2} \sum_{0 < \gamma < T} |Z(t_\gamma)|^4 + O_\varepsilon(T^\varepsilon) \\ &= \frac{1}{2} \sum_{0 < \gamma < T} |\zeta(\tfrac{1}{2} + it_\gamma)|^4 + O_\varepsilon(T^\varepsilon) \end{aligned} \quad (28)$$

where under the assumption of the Riemann Hypothesis t_γ is the unique number between adjacent zeros γ and γ^+ of $Z(t)$ at which $Z'(t_\gamma) = 0$. Equivalently, the t_γ are the numbers at which $|\zeta(\frac{1}{2} + it)|$ has local maxima. Thus, from (26) and (28) we deduce that, on RH ,

$$\frac{TL^5\sqrt{21}}{90\pi^2} \ll \sum_{0 < \gamma < T} |\zeta(\tfrac{1}{2} + it_\gamma)|^4 \ll \frac{TL^5}{2\sqrt{15}\pi^2}. \quad (29)$$

Thus we have determined the correct order of magnitude of the fourth moment of the zeta function at its relative extrema on the critical line. For comparison, we note the corresponding second moment of Conrey–Ghosh [2], on RH ,

$$\sum_{0 < \gamma < \zeta T} |\zeta(\tfrac{1}{2} + it_\gamma)|^2 \sim \frac{e^2 - 5}{4\pi} TL^2. \quad (30)$$

Finally we mention that for the proof of our theorem we use Motohashi’s method [4] to derive approximate functional equations for the A_i and then we use Ingham’s method [3] for deriving (4) to complete the proof.

2. Lemmas

We begin by proving an approximate functional equation for $A_t(s)$. Actually, we change notation slightly and let $A_t(s) = P_t\left(-\frac{1}{l} \frac{d}{ds}\right) \zeta(s)$ where $l = \log \frac{t}{2\pi}$. This makes no difference in the asymptotic formulae as is easily seen by an integration by parts.

Recall that the approximate functional equation for the Riemann zeta function asserts that (see Titchmarsh [5])

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \chi(s) \sum_{n \leq y} n^{s-1} + O(x^{-\sigma} + |t|^{\frac{1}{2}-\sigma} y^{\sigma-1}) \tag{31}$$

uniformly for $x, y \gg 1$, $xy = t/2\pi$, and $0 < \sigma < 1$. We may use Cauchy's theorem to find that

$$\begin{aligned} \zeta^{(m)}(\tfrac{1}{2} + it) &= \sum_{n \leq x} \frac{(-1)^m (\log n)^m}{n^{\frac{1}{2} + it}} + \left(\frac{d}{ds}\right)^m \left(\chi(s) \sum_{n \leq y} n^{s-1}\right) \Big|_{s=\frac{1}{2}+it} \\ &\quad + O((x^{-\frac{1}{2}} + y^{-\frac{1}{2}}) \log^m |t|) \end{aligned} \tag{32}$$

for $|t| \gg 1$. Now

$$\begin{aligned} \frac{\chi'}{\chi}(s) &= -\log \frac{s}{2\pi i} + O\left(\frac{1}{|s|}\right) \\ &= -\log \frac{t}{2\pi} + O\left(\frac{1}{|t|}\right) \end{aligned} \tag{33}$$

for $|t| \gg 1$, $|\sigma| \ll 1$. Therefore,

$$\chi^{(r)}(s) = \chi(s) \left((-l)^r + O\left(\frac{l^{r-1}}{|t|}\right) \right) \quad (|\sigma| \ll 1) \tag{34}$$

for $r = 1, 2, \dots$ where we use the notation

$$l = l(t) = \log \frac{t}{2\pi}. \tag{35}$$

Thus, since $\chi(\frac{1}{2} + it) \ll 1$, it follows that

$$\begin{aligned} \frac{(-1)^m \zeta^{(m)}(\frac{1}{2} + it)}{l^m} &= \sum_{n \leq x} \left(\frac{\log n}{l}\right)^m n^{-\frac{1}{2}-it} \\ &\quad + \chi(\tfrac{1}{2} + it) \sum_{n \leq y} \frac{\left(1 - \frac{\log n}{l}\right)^m}{n^{\frac{1}{2}-it}} + O(x^{-\frac{1}{2}} + y^{-\frac{1}{2}}). \end{aligned} \tag{36}$$

This leads to

LEMMA 1. *Let P be a polynomial and $l = \log \frac{t}{2\pi}$, $|t| \gg 1$. Then for*

$s = \frac{1}{2} + it$ we have

$$P\left(-\frac{1}{l} \frac{d}{ds}\right) \zeta(s) = \sum_{n \leq x} P\left(\frac{\log n}{l}\right) n^{-s} + \chi(s) \sum_{n \leq y} P\left(1 - \frac{\log n}{l}\right) n^{s-1} + E(x) \quad (37)$$

where $xy = t/2\pi$, $x \gg 1$, $y \gg 1$ and $E(x) \ll x^{-\frac{1}{2}} + y^{-\frac{1}{2}}$.

In much the same way we can derive a "functional equation" for $A_i(s)$. Thus, $\zeta(s) = \chi(s)\zeta(1-s)$ implies that for $s = \frac{1}{2} + it$,

$$\begin{aligned} (-1)^m \zeta^{(m)}(s) &= \sum_{r=0}^m \binom{m}{r} \chi^{(r)}(s) \left(\frac{d}{ds}\right)^{m-r} \zeta(1-s) (-1)^m \\ &= \chi(s) \sum_{r=0}^m \binom{m}{r} \left((-l)^r + O\left(\frac{l^{r-1}}{|t|}\right)\right) \left(\frac{d}{ds}\right)^{m-r} \zeta(1-s) (-1)^m \\ &= \chi(s) \left(l - \frac{d}{ds}\right)^m \zeta(1-s) + O\left(\frac{l^m}{|t|^{\frac{1}{2}}}\right) \end{aligned} \quad (38)$$

using the bound $\zeta(\frac{1}{2} + it) \ll t^{\frac{1}{2}}$, say. Hence we have

LEMMA 2. For $s = \frac{1}{2} + it$, $|t| \gg 1$ we have

$$P\left(-\frac{1}{l} \frac{d}{ds}\right) \zeta(s) = \chi(s) P\left(1 - \frac{1}{l} \frac{d}{ds}\right) \zeta(1-s) + O(|t|^{-\frac{1}{2}}). \quad (39)$$

Next we apply Motohashi's method [4] to derive an approximate functional equation for $A_1(\frac{1}{2} + it)$ $A_2(\frac{1}{2} + it)$. We let

$$a_i(n) = a_i(n, t) = P_i\left(\frac{\log n}{l}\right), \quad b_i(n, t) = P_i\left(1 - \frac{\log n}{l}\right), \quad (40)$$

$$A_i(s) = P_i\left(-\frac{1}{l} \frac{d}{ds}\right) \zeta(s), \quad B_i(s) = P_i\left(1 + \frac{1}{l} \frac{d}{ds}\right) \zeta(s) \quad (41)$$

and we let $U = t/2\pi$, $u = U^{\frac{1}{2}}$. Then by Lemma 1,

$$\begin{aligned} \sum_{m \leq u} \frac{a_1(m) a_2(n)}{m^s n^s} &= \sum_{m \leq u} \frac{a_1(m)}{m^s} \sum_{n \leq U/m} \frac{a_2(n)}{n^s} \\ &\quad + \sum_{n \leq u} \frac{a_2(n)}{n^s} \sum_{m \leq U/n} \frac{a_1(m)}{m^s} - \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_1(m) a_2(n)}{m^s n^s} \\ &= \sum_{m \leq u} \frac{a_1(m)}{m^s} \left(A_2(s) - \chi(s) \sum_{n \leq m} \frac{b_2(n)}{n^{1-s}} + E(m) \right) \\ &\quad + \sum_{n \leq u} \frac{a_2(n)}{n^s} \left(A_1(s) - \chi(s) \sum_{m \leq n} \frac{b_1(m)}{m^{1-s}} + E(n) \right) \\ &\quad - \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_1(m) a_2(n)}{m^s n^s}. \end{aligned} \quad (42)$$

Similarly, by Lemmas 1 and 2,

$$\begin{aligned}
 & \chi(s)^2 \sum_{mn \leq U} \frac{b_1(m)}{m^{1-s}} \frac{b_2(n)}{n^{1-s}} \\
 &= \chi(s)^2 \sum_{m \leq u} \frac{b_1(m)}{m^{1-s}} \left(B_2(1-s) - \chi(1-s) \sum_{n \leq m} \frac{a_2(n)}{n^s} + E(m) \right) \\
 &+ \chi(s)^2 \sum_{n \leq u} \frac{b_2(n)}{n^{1-s}} \left(B_1(1-s) - \chi(1-s) \sum_{m \leq n} \frac{a_1(m)}{m^s} + E(n) \right) \\
 &- \chi(s)^2 \sum_{\substack{m \leq u \\ n \leq u}} \frac{b_1(m)}{m^{1-s}} \frac{b_2(n)}{n^{1-s}}. \\
 &= \chi(s) \sum_{m \leq u} \frac{b_1(m)}{m^{1-s}} \left(A_2(s) + O(t^{-\frac{1}{2}}) - \sum_{n \leq m} \frac{a_2(n)}{n^s} + \chi(s)E(m) \right) \\
 &+ \chi(s) \sum_{n \leq u} \frac{b_2(n)}{n^{1-s}} \left(A_1(s) + O(t^{-\frac{1}{2}}) - \sum_{m \leq n} \frac{a_1(m)}{m^s} + \chi(s)E(n) \right) \\
 &- \chi(s)^2 \sum_{\substack{m \leq u \\ n \leq u}} \frac{b_1(m)}{m^{1-s}} \frac{b_2(n)}{n^{1-s}}. \tag{43}
 \end{aligned}$$

Since $a_i(m), b_i(m) \ll 1$ and $\chi(s) \ll 1$ the contributions of the O -terms and the E -terms in (42) and (43) are

$$\ll \sum_{m \leq u} m^{-\frac{1}{2}}(t^{-\frac{1}{2}} + m^{-\frac{1}{2}}) \ll \log t. \tag{44}$$

Adding together corresponding sides of equations (42) and (43) we obtain for the right hand side the quantity

$$\begin{aligned}
 & (A_1(s) + O(t^{-\frac{1}{2}}))A_2(s) + A_1(s)(\chi(s)B_2(1-s) + O(t^{-\frac{1}{2}})) \\
 &- \chi(s) \left(\sum_{\substack{m \leq u \\ n \leq u}} \frac{b_1(m)}{m^{1-s}} \frac{a_2(n)}{n^s} - \sum_{n \leq u} \frac{b_1(n)a_2(n)}{n} \right) \\
 &+ \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_1(m)}{m^s} \frac{b_2(n)}{n^{1-s}} - \sum_{m \leq u} \frac{a_1(m)b_2(m)}{m} \\
 &- \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_1(m)a_2(n)}{m^s n^s} - \chi(s)^2 \sum_{\substack{m \leq u \\ n \leq u}} \frac{b_1(m)b_2(n)}{m^{1-s}n^{1-s}} \\
 &= A_1(s)A_2(s) + O(\log t). \tag{45}
 \end{aligned}$$

Thus, we have proven

LEMMA 3. *With the above notation we have*

$$A_1 A_2(s) = \sum_{n \leq t/2\pi} \frac{a_1 * a_2(n, t)}{n^s} + \chi(s)^2 \sum_{n \leq t/2\pi} \frac{b_1 * b_2(n, t)}{n^{1-s}} + O(\log t) \quad (46)$$

for $s = \frac{1}{2} + it$ where $*$ denotes Dirichlet convolution.

Next we state a form of the mean value theorem for Dirichlet polynomials which allows for “polynomials” whose length varies with t . This is

LEMMA 4. *Suppose that $f_n(t)$ satisfies*

$$|f_n(t)| < \delta d(n)$$

and

$$\text{var}_{[T_1, T_2]} f_n(t) < \delta d(n)$$

and similarly for $g_n(t)$, $1 \leq n \leq N$. Then

$$\int_{T_1}^{T_2} \sum_{m=1}^N \frac{f_m(t)}{m^{\frac{1}{2}+it}} \sum_{n=1}^N \frac{\overline{g_n(t)}}{n^{\frac{1}{2}-it}} dt = \sum_{n=1}^N \frac{1}{n} \int_{T_1}^{T_2} f_n(t) \overline{g_n(t)} dt + O(\delta^2 N \log^3 N). \quad (47)$$

The proof is exactly as that of Lemma 5 of Conrey [1] except that we use

$$\sum_{\substack{m, n \leq N \\ m \neq n}} \frac{d(m) d(n)}{m^{\frac{1}{2}} n^{\frac{1}{2}} |\log m/n|} \ll N \log^3 N \quad (48)$$

which is Lemma B3 of Ingham [3].

Next we have a lemma about estimating a sum.

LEMMA 5. *Let $a_i(n) = a_i(n, T) = P_i\left(\frac{\log n}{L}\right)$ for $1 \leq i \leq 4$ where the P_i are polynomials and $L = \log \frac{T}{2\pi}$. Then*

$$S := \sum_{N \leq T/2\pi} \frac{a_1 * a_2(N) a_3 * a_4(N)}{N} \sim \frac{6L^4}{\pi^2} \int_{\mathfrak{A}} P_1(\alpha + \beta) P_2(\gamma + \delta) P_3(\alpha + \gamma) P_4(\beta + \delta) d\alpha d\beta d\gamma d\delta \quad (49)$$

where $\mathfrak{A} = \{(\alpha, \beta, \gamma, \delta) \in [0, 1]^4: \alpha + \beta + \gamma + \delta \leq 1\}$.

Proof. The idea is to approximate $a_1 * a_2(N)a_3 * a_4(N) =: f(N)$ by

$$\sum_{\alpha\beta\gamma\delta=N} a_1(\alpha\beta)a_2(\gamma\delta)a_3(\alpha\gamma)a_4(\beta\delta) =: g(N). \quad (50)$$

In fact, if N is squarefree then it can be verified easily that $f(N) = g(N)$ (for any arithmetical functions a_i). It is an easy matter to relate the mean $\sum_{N \leq T/2\pi} g(N)/N$ to the integral on the right of (49). Thus, since the proportion of squarefree numbers is $6/\pi^2$, the statement of the lemma is not too surprising.

We let

$$\alpha_i(n) = n^{-\varepsilon_i} \quad (51)$$

for $1 \leq i \leq 4$ where ε_i is a complex number with $|\varepsilon_i| \leq \frac{1}{4}L$. Then

$$a_i(n) = P_i \left(-\frac{1}{L} \frac{d}{d\varepsilon_i} \right) \alpha_i(n) \Big|_{\varepsilon_i=0}.$$

Thus we initially consider

$$S_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = S_1 := \sum_{N \leq T/2\pi} \frac{\alpha_1 * \alpha_2(N)\alpha_3 * \alpha_4(N)}{N}. \quad (52)$$

Clearly α_i is a multiplicative function whence $\alpha_1 * \alpha_2(n)\alpha_3 * \alpha_4(n) =: f_1(N)$ is multiplicative. Let

$$g_1(N) := \sum_{\alpha\beta\gamma\delta=N} \alpha_1(\alpha\beta)\alpha_2(\gamma\delta)\alpha_3(\alpha\gamma)\alpha_4(\beta\delta). \quad (53)$$

Then g_1 is multiplicative, since if $(M, N) = 1$, then $\alpha\beta\gamma\delta = MN$ implies that $\alpha = \alpha'\alpha''$ where $\alpha' | M$, and $\alpha'' | N$ and similarly for the other variables and we see easily that $g_1(MN) = g_1(M)g_1(N)$. Moreover,

$$\begin{aligned} f_1(p) &= g_1(p) = \alpha_1(p)\alpha_3(p) + \alpha_1(p)\alpha_4(p) \\ &\quad + \alpha_2(p)\alpha_3(p) + \alpha_2(p)\alpha_4(p) \\ &= p^{-\varepsilon_1-\varepsilon_3} + p^{-\varepsilon_1-\varepsilon_4} + p^{-\varepsilon_2-\varepsilon_3} + p^{-\varepsilon_2-\varepsilon_4}. \end{aligned} \quad (54)$$

Now we calculate the generating function of $f_1(n)$. We have

$$\begin{aligned} F_1(s) &:= \sum_{n=1}^{\infty} f_1(n)/n^s \quad (\sigma > 1 + L^{-1}) \\ &= \prod_p \sum_{k=0}^{\infty} \frac{f_1(p^k)}{p^{ks}} \\ &= \prod_p \left(\sum_{k=0}^{\infty} \frac{g_1(p^k)}{p^{ks}} \right) \left(1 + \sum_{k=2}^{\infty} \frac{h_1(p^k)}{p^{ks}} \right) = G_1(s)H_1(s) \end{aligned} \quad (55)$$

say, where the product for $H_1(s)$ is absolutely convergent for $\sigma > \frac{1}{2} + L^{-1}$. Of course, F_1 , G_1 , and H_1 depend on the ε_i and we note that when $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$ we have $F_1(s)$ and $G_1(s)$ are equal to

$$\sum_{N=1}^{\infty} \frac{d(N)^2}{N^s} = \frac{\zeta(s)^4}{\zeta(2s)} \quad \text{and} \quad \sum_{N=1}^{\infty} \frac{d_4(N)}{N^s} = \zeta(s)^4,$$

respectively. In general,

$$G_1(s) = \zeta(s + \varepsilon_1 + \varepsilon_3)\zeta(s + \varepsilon_1 + \varepsilon_4)\zeta(s + \varepsilon_2 + \varepsilon_3)\zeta(s + \varepsilon_2 + \varepsilon_4). \quad (56)$$

Now by Perron's formula,

$$S_1 = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} G_1(s+1)H_1(s+1)U^s \frac{ds}{s} + O_\varepsilon(T^{\varepsilon-1}) \quad (57)$$

for any $\varepsilon > 0$ and where $c = 2/L$ and $U = T/2\pi$. In $\sigma > -\frac{1}{2} + L^{-1}$ the integrand has poles at $0, -\varepsilon_1 - \varepsilon_3, -\varepsilon_1 - \varepsilon_4, -\varepsilon_2 - \varepsilon_3, -\varepsilon_2 - \varepsilon_4$ and nowhere else. In particular, all poles are in $|s| \leq (2L)^{-1}$. If we move the path of integration to $\sigma = -\frac{1}{4}$, then we easily see that

$$S_1 = \frac{1}{2\pi i} \int_{|s|=L^{-1}} G_1(s+1)H_1(s+1) \frac{U^s}{s} ds + O(T^{-\frac{1}{4}}). \quad (58)$$

Now for $|s| \leq L^{-1}$ we have

$$H_1(s+1) = H_1(1) + O(L^{-1}) = \frac{6}{\pi^2} + O(L^{-1}) \quad (59)$$

uniformly. Also, by (56),

$$G(s+1) \ll L^4 \quad (60)$$

for $|s| = L^{-1}$ and trivially

$$\frac{U^s}{s} \ll L \quad (61)$$

for $|s| = L^{-1}$. Thus $H_1(s+1)$ may be replaced by $H_1(1)$ in (58) with the introduction of an error term which is $\ll L^3$. Thus, we easily conclude from another application of Perron's formula that

$$S_1 = \frac{6}{\pi^2} \sum_{N \leq U} \frac{g_1(N)}{N} + O(L^3) \quad (62)$$

uniformly in ε_j with $|\varepsilon_j| \leq (4L)^{-1}$. Now for each N , $g_1(N)$ is analytic in all

of the ε_j . Also

$$S = P_1\left(-\frac{1}{L} \frac{d}{d\varepsilon_1}\right) P_2\left(-\frac{1}{L} \frac{d}{d\varepsilon_2}\right) P_3\left(-\frac{1}{L} \frac{d}{d\varepsilon_3}\right) \\ \times P_4\left(-\frac{1}{L} \frac{d}{d\varepsilon_4}\right) S_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \Big|_{\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=0} \quad (63)$$

so that by Cauchy's integral formula for the n th derivative (used on the contours $|\varepsilon_j| = (2^j L)^{-1}$, for instance) we obtain

$$S = \frac{6}{\pi^2} \sum_{N \leq U} \frac{g(N)}{N} + O(L^3). \quad (64)$$

The sum over N here is

$$= \sum_{\alpha\beta\gamma\delta \leq U} \frac{a_1(\alpha\beta)a_2(\gamma\delta)a_3(\alpha\gamma)a_4(\beta\delta)}{\alpha\beta\gamma\delta} \quad (65)$$

which is

$$\sim \int_{1 \leq \alpha\beta\gamma\delta \leq U} P_1\left(\frac{\log \alpha\beta}{L}\right) P_2\left(\frac{\log \gamma\delta}{L}\right) P_3\left(\frac{\log \alpha\gamma}{L}\right) P_4\left(\frac{\log \beta\delta}{L}\right) \frac{d\alpha}{\alpha} \frac{d\beta}{\beta} \frac{d\gamma}{\gamma} \frac{d\delta}{\delta} \quad (66)$$

and after a change of variables, this gives (49).

3. Proof of theorem

By Lemma 3 we have

$$A_1 A_2 \left(\frac{1}{2} + it\right) A_3 A_4 \left(\frac{1}{2} - it\right) = \sum_{j=1}^9 S_j, \quad (67)$$

say, where, since $\chi(s)\chi(1-s) = 1$, we have

$$S_1 = \sum_{m, n \leq t/2\pi} \frac{a_1 * a_2(m) a_3 * a_4(n)}{m^{\frac{1}{2}} n^{\frac{1}{2}}} \left(\frac{n}{m}\right)^{it}, \\ S_2 = \sum_{m, n \leq t/2\pi} \frac{b_1 * b_2(m) b_3 * b_4(n)}{m^{\frac{1}{2}} n^{\frac{1}{2}}} \left(\frac{m}{n}\right)^{it}, \\ S_3 = \chi\left(\frac{1}{2} - it\right)^2 \sum_{m, n \leq t/2\pi} \frac{a_1 * a_2(m) b_3 * b_4(n)}{m^{\frac{1}{2}} n^{\frac{1}{2}}} (mn)^{-it}, \\ S_4 = \chi\left(\frac{1}{2} - it\right)^2 \sum_{m, n \leq t/2\pi} \frac{a_3 * a_4(m) b_3 * b_4(n)}{m^{\frac{1}{2}} n^{\frac{1}{2}}} (mn)^{it}, \\ S_5 = O\left(\left|\sum_{n \leq t/2\pi} \frac{a_1 * a_2(n)}{n^{\frac{1}{2}-it}}\right| \log t\right),$$

$$S_6 = O\left(\left|\sum_{n \leq t/2\pi} \frac{b_1 * b_2(n)}{n^{\frac{1}{2} + it}}\right| \log t\right),$$

$$S_7 = O\left(\left|\sum_{n \leq t/2\pi} \frac{a_3 * a_4(n)}{n^{\frac{1}{2} - it}}\right| \log t\right),$$

$$S_8 = O\left(\left|\sum_{n \leq t/2\pi} \frac{b_3 * b_4(n)}{n^{\frac{1}{2} + it}}\right| \log t\right),$$

and

$$S_9 = O(\log^2 t).$$

Recall that $a_i(n) = a_i(n, t) = P_i\left(\frac{\log n}{l}\right)$ and $b_i(n) = P_i\left(1 - \frac{\log n}{l}\right)$ where $l = \log \frac{t}{2\pi}$, so that the a_i and b_i depend on n and t . We let

$$I_j = \int_1^T S_j dt. \tag{68}$$

To estimate I_1 and I_2 we use Lemma 4 with $N = [T/2\pi]$. For I_1 we let $f_n(t) = a_1 * a_2(n)\chi_{[0, t/2\pi]}(n)$, where $\chi_{[a, b]}$ is the characteristic function of the interval $[a, b]$, and we let $g_n(t) = a_3 * a_4(n)\chi_{[0, t/2\pi]}(n)$. Since $a_i(n) \ll 1$, it follows that

$$f_n(t), g_n(t) \ll d(n). \tag{69}$$

Also, letting $\chi(n)$ denote $\chi_{[0, t/2\pi]}(n)$, we have

$$\begin{aligned} \text{var } f_n(n) |_{T_1}^{T_2} &= \text{var} \sum_{d|n} a_1(d)a_2(n/d)\chi(n) |_{T_1}^{T_2} \\ &\leq \sum_{d|n} (\text{var } a_1(d) |_{T_1}^{T_2} |a_2(n/d)| |\chi(n)| + |a_1(d)| \text{var } a_2(n/d) |_{T_1}^{T_2} \chi(n) \\ &\quad + |a_1(d)| |a_2(n/d)| \text{var } \chi(n) |_{T_1}^{T_2}) \\ &\ll d(n) \end{aligned} \tag{70}$$

since, for example,

$$\begin{aligned} \chi(n) \text{var } a_1(n) |_{T_1}^{T_2} &= \int_{T_1}^{T_2} \left| \frac{d}{dt} a_1(n) \right| \chi(n) dt \\ &\ll \int_{T_1}^{T_2} \left| P_1' \left(\frac{\log n}{\log \frac{t}{2\pi}}, \frac{\log n}{\log^2 \frac{t}{2\pi}} \right) \right| \frac{\chi(n)}{t} dt \\ &\ll 1. \end{aligned} \tag{71}$$

Thus, Lemma 4 yields

$$I_1 = \sum_{n \leq t/2\pi} \frac{1}{n} \int_1^T a_1 * a_2(n) a_3 * a_4(n) \chi(n) dt + O(TL^3). \quad (72)$$

Note that for a fixed polynomial Q ,

$$\int_1^T Q\left(\frac{\log n}{\log t/2\pi}\right) \chi(n) dt = \int_{2\pi n}^T Q\left(\frac{\log n}{\log t/2\pi}\right) dt \quad (73)$$

and by integration by parts the latter integral is

$$\begin{aligned} &= tQ\left(\frac{\log n}{\log t/2\pi}\right) \Big|_{2\pi n}^T + \int_{2\pi n}^T Q'\left(\frac{\log n}{\log t/2\pi}\right) \frac{\log n}{\log^2 t/2\pi} dt \\ &= TQ\left(\frac{\log n}{L}\right) + O(n) + O(T/L). \end{aligned} \quad (74)$$

Thus,

$$\begin{aligned} I_1 &= T \sum_{n \leq T/2\pi} \frac{a_1 * a_2(n, T) a_3 * a_4(n, T)}{n} \\ &\quad + O\left(\sum_{n \leq T} d(n)^2 + TL^{-1} \sum_{n \leq T} d(n)^2 n^{-1}\right) \\ &= T \sum_{n \leq T/2\pi} \frac{a_1 * a_2(n, T) a_3 * a_4(n, T)}{n} + O(TL^3). \end{aligned} \quad (75)$$

Similarly,

$$I_2 = T \sum_{n \leq T/2\pi} \frac{b_1 * b_2(n, T) b_3 * b_4(n, T)}{n} + O(TL^3). \quad (76)$$

Moreover,

$$\begin{aligned} I_5 &\ll \left(\int_1^T \log^2 t dt\right)^{\frac{1}{2}} \left(\int_1^T \left|\sum_{n \leq t/2\pi} \frac{a_1 * a_2(n)}{n^{\frac{1}{2}+it}}\right|^2 dt\right)^{\frac{1}{2}} \\ &\ll (TL^2)^{\frac{1}{2}} (TL^4)^{\frac{1}{2}} \ll TL^3 \end{aligned} \quad (77)$$

and similarly for I_6 , I_7 , and I_8 . Trivially $I_9 \ll TL^2$.

Finally, to estimate I_3 and I_4 , we note that

$$\chi\left(\frac{1}{2} - it\right)^2 = \left(\frac{t}{2\pi e}\right)^{2it} (1 + O(1/|t|)) \quad (|t| \gg 1). \quad (78)$$

Thus,

$$I_3 = \int_1^T \left(\frac{t}{2\pi e} \right)^{2u} \left(1 + O\left(\frac{1}{t} \right) \right) \sum_{m, n \leq t/2\pi} \frac{a_1 * a_2(m) b_3 * b_4(n)}{m^{\frac{1}{2}+u} n^{\frac{1}{2}}} (mn)^{-it} dt. \quad (79)$$

The $O(1/t)$ -term can be taken care of by the Cauchy–Schwarz inequality, the above estimates, and integration by parts. Its contribution to I_3 is

$$\begin{aligned} &\ll \left(\int_1^T \left| \sum_{m < t/2\pi} \frac{a_1 * a_2(m)}{m^{\frac{1}{2}+u}} \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_1^T \left| \sum_{n < t/2\pi} \frac{b_3 * b_4(n)}{n^{\frac{1}{2}+u}} \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\ll L^5. \end{aligned} \quad (80)$$

Thus,

$$I_3 = \sum_{m, n \leq T/2\pi} m^{-\frac{1}{2}} n^{-\frac{1}{2}} \int_{T'}^T a_1 * a_2(m) b_3 * b_4(n) \left(\frac{t}{e\xi} \right)^{2it} dt + O(L^5) \quad (81)$$

where $T' = 2\pi \max\{m, n\}$ and $\xi = 2\pi\sqrt{mn}$. By integration by parts of Lemma B4 of Ingham, for $T \geq T_0 \geq \xi > 2\pi$,

$$\int_{T_0}^T \left(\frac{t}{e\xi} \right)^{2it} (\log t/2\pi)^{-k} dt \ll \min \left(\sqrt{T}, \frac{1}{\log T_0/\xi} \right) (\log T_0)^{-k}. \quad (82)$$

Now $\xi \leq T'$ and $\xi < T'$ unless $m = n$. Also, $\log T'/\xi = |\frac{1}{2} \log(m/n)|$, $a_1 * a_2(m, T_0) \ll d(m)$, and $b_3 * b_4(n, T_0) \ll d(n)$. Thus by (48),

$$I_3 \ll \sum_{m, n \leq T/2\pi} \frac{d(m)d(n)}{m^{\frac{1}{2}} n^{\frac{1}{2}}} |\log m/n|^{-1} \ll TL^3.$$

A similar result holds for I_4 so that we have shown

$$\begin{aligned} \int_1^T A_1 A_2(\tfrac{1}{2} + it) A_3 A_4(\tfrac{1}{4} - it) dt &= T \sum_{n \leq T/2\pi} \frac{a_1 * a_2(n, T) a_3 * a_4(n, T)}{n} \\ &\quad + T \sum_{n \leq T/2\pi} \frac{b_1 * b_2(n, T) b_3 * b_4(n, T)}{n} \\ &\quad + O(TL^3). \end{aligned} \quad (83)$$

Now the theorem is a consequence of (83) and Lemma 5.

To prove (8) we must show that

$$\begin{aligned} &\int_{\mathfrak{O}} (\alpha + \beta)^m (\gamma + \delta)^m (\alpha + \gamma)^m (\beta + \delta)^m \\ &\quad + (1 - \alpha - \beta)^m (1 - \gamma - \delta)^m (1 - \alpha - \gamma)^m (1 - \beta - \delta)^m d\alpha d\beta d\gamma d\delta \\ &\sim \frac{1}{16m^4} \end{aligned} \quad (84)$$

as $m \rightarrow \infty$, where \mathcal{R} is as in the theorem. Since $\alpha + \beta + \gamma + \delta \leq 1$, the maximum in \mathcal{R} of $(\alpha + \beta)(\gamma + \delta)(\alpha + \gamma)(\beta + \delta)$ is $\frac{1}{16}$. Thus, the first term in the integral is $\ll 16^{-m}$. For the second term of the integral, note that if $x > 5(\log m)/m$, then

$$(1-x)^m \leq \left(1 - \frac{5 \log m}{m}\right)^m \ll m^{-5}.$$

Thus the integral in (84) is

$$\begin{aligned} &= \int_{[0, 5 \log m/m]^4} [(1-\alpha-\beta)(1-\gamma-\delta)(1-\alpha-\gamma) \\ &\quad \times (1-\beta-\delta)]^m d\alpha d\beta d\gamma d\delta + O(m^{-5}) \\ &= m^{-4} \int_{[0, 5 \log m/m]^4} \left[\left(1 - \frac{\alpha+\beta}{m}\right) \left(1 - \frac{\gamma+\delta}{m}\right) \right. \\ &\quad \left. \times \left(1 - \frac{\alpha+\gamma}{m}\right) \left(1 - \frac{\beta+\delta}{m}\right) \right]^m d\alpha d\beta d\gamma d\delta + O(m^{-5}) \\ &\sim m^{-4} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-2(\alpha+\beta+\gamma+\delta)} d\alpha d\beta d\gamma d\delta = \frac{1}{16m^4}, \end{aligned} \quad (85)$$

as claimed.

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