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ON THE LOCATION OF THE ZEROS OF THE DERIVATIVE OF A POLYNOMIAL

J. BRIAN CONREY AND LEE A. RUBEL

ABSTRACT. We show that nonreal zeros of a real polynomial $P(z)$ sometimes induce nonreal zeros of $P'(z)$.

1. Introduction. We prove two results in the geometry of the zeros of polynomials [2]. In particular, we study how the distribution of the zeros of a real polynomial may force (or prevent) the existence of a nonreal zero of its derivative. Our investigation was originally motivated by certain considerations about the zeros of Riemann's xi-function (and of its derivative) which arise in analytic number theory (see [1]). We note that our method applies also to real entire functions of order $\leq 1$.

2. Notation. For simplicity we will work with polynomials though the proofs are valid in a more general situation. We assume henceforth that $P$ is a polynomial with real coefficients. Generic real zeros of $P$ will be denoted $r$ and complex conjugate pairs are $w$ and $\overline{w}$ where $w = u + iv$. (To indicate specific zeros we use subscripts.) We associate to $w$ the Jensen disc

$$J_w = \{ z : |z - u| \leq v \}.$$

A well-known theorem of Jensen [2] asserts that if $P'(z) = 0$ then either $z$ is real or $z \in J_w$ for some $w$. This follows immediately from the fact that if $f(z) = (z - w)(z - \overline{w}) = (z - u)^2 + v^2$, then

$$\text{Im} \frac{f'}{f}(z) = -2y \frac{|z - u|^2 - v^2}{|(z - u)^2 + v^2|^2}$$

(so that $\text{Im} P'(z)/P(z) \neq 0$ if $z$ is nonreal and outside every Jensen disc).

We also write $z = x + iy$.

3. Statement of results. Our first theorem asserts that if a real polynomial $P$ has real zeros "near" to the projection of a complex zero $w_0$ of $P$ onto the real axis, then $P'$ has a nonreal zero, provided that $w_0$ is, in a certain sense, isolated. By "near" we mean within a distance of the order of magnitude of $|\text{Im} w_0|$.

THEOREM 1. Let $n \geq 1$ be fixed and let

$$c_n = \begin{cases} 0.5 & \text{if } n = 1, \\ \frac{1}{2}(\sqrt{9n^2 - 8n} - n)^{1/2} & \text{if } n > 1. \end{cases}$$

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Suppose that \( P(w_0) = P(r_1) = \cdots = P(r_n) = 0 \) and that \( |u_0 - r_i| < c_n |v_0| \) for \( 1 \leq i \leq n \). Suppose further that \( J_w \cap J_{w_0} = \emptyset \) where \( w \) ranges over the nonreal zeros of \( P \) with \( w \neq w_0, \bar{w}_0 \). Then \( P' \) has a nonreal zero inside \( J_{w_0} \).

We can obtain a more precise result if the zeros of \( P \) are symmetric about the \( y \)-axis.

**Theorem 2.** Suppose that \( P \) is an even function, \( P(0) \neq 0 \), and \( P(iv_0) = 0 \) where \( v_0 > 0 \). Let \( r_0 \) be the smallest real zero of \( P \) such that \( r_0 \geq v_0 \). Suppose that \( J_w \cap [-r_0, r_0] = \emptyset \) where \( w \) ranges over the nonreal zeros of \( P \) with \( w \neq \pm iv_0 \).

Define
\[
K = \sum_r \frac{1}{r^2} + \sum_w \frac{u^2 - v^2}{(u^2 + v^2)^2}.
\]

Then \( P' \) has a nonreal zero inside \( J_{iv_0} \) if and only if \( K > 0 \).

It follows from this theorem that \( c_n \) of Theorem 1 is asymptotically best possible.

For if the zeros of \( P \) consist of \( n \) (where \( n \) is even) real zeros all at \( \pm r_0 \) and a pair of complex zeros at \( \pm iv_0 \) then Theorem 2 asserts that \( P' \) has only real zeros if
\[
K = \frac{n}{r_0^2} - \frac{2}{v_0^2} < 0,
\]
that is, if \( r_0 > \sqrt{n/2v_0} \); Theorem 1 asserts that \( P' \) has a nonreal zero if \( r_0 < c_nv_0 \).

Since \( c_n \sim \sqrt{n/2} \) as \( n \to \infty \), Theorem 1 is nearly sharp.

4. **Proofs.** We now prove Theorem 1. We may assume without loss of generality that \( u_0 = 0 \) so that \( w_0 = iv_0, v_0 > 0 \). Then
\[
P(z) = (z^2 + v_0^2)(z - r_1) \cdots (z - r_n)Q(z)
\]
where \( Q \) is a polynomial with real coefficients. Consider the contour
\[
C_{\epsilon} = \{z: |z - i\epsilon| = v_0 \text{ and } y > \epsilon\} \cup \{z = x + i\epsilon: -v_0 \leq x \leq v_0\} = C_U + C_{\epsilon}^L
\]
where \( \epsilon \) is a small positive number. We will show that if \( \epsilon \) is small enough then \( \text{Im } P'(z)/P(z) < 0 \) on \( C_{\epsilon} \). Then the change in the argument of \( P'(z)/P(z) \) as \( z \) varies around \( C_{\epsilon} \) is 0. Hence by the argument principle the number of zeros of \( P' \) inside \( C_{\epsilon} \) is equal to the number of zeros of \( P \) inside \( C_{\epsilon} \), namely one (\( P(iv_0) = 0 \)), which shows the result.

By an easy calculation, \( \text{Im } P'(z)/P(z) = -yF(z) + \text{Im } Q'(z)/Q(z) \) where
\[
F(z) = 2 \frac{|z|^2 - v_0^2}{|z^2 + v_0^2|^2} + \sum_{i=1}^{n} \frac{1}{|z - r_i|^2}.
\]
By the hypothesis about the Jensen discs of \( P \), \( \text{Im } Q'(z)/Q(z) < 0 \) for \( z \) on \( C_{\epsilon} \), if \( \epsilon \) is small enough. Also, \( F(z) > 0 \) for \( z \) on \( C_{\epsilon}^U \) for any \( \epsilon > 0 \). Finally we will show that \( F(z) > 0 \) for \( z \) on \( C_{\epsilon}^L \) which implies that \( F(z) > 0 \) for \( z \) on \( C_{\epsilon}^L \), if \( \epsilon > 0 \) is sufficiently small. Hence we must show that
\[
F(z) = F(x, r_1, r_2, \ldots, r_n) = 2 \frac{x^2 - v_0^2}{(x^2 + v_0^2)^2} + \sum_{i=1}^{n} \frac{1}{(x - r_i)^2} > 0
\]
for $|x| \leq v_0$ and $|r_i| \leq c_nv_0$. We consider the minimum $M$ of $F$ on this set. By symmetry, a minimum value of $F$ will occur in $x \geq 0$. Since $\partial F/\partial r_i$ is never 0, the minimum for $F$ occurs when each $r_i = \pm c_nv_0$. Therefore,

$$M = \min_{0 \leq x \leq v_0} \frac{2x^2 - v_0^2}{(x^2 + v_0^2)^2} + \frac{n}{(x + c_nv_0)^2}.$$ 

Let $f(x) = 2(x^2 - v_0^2)(x + c_nv_0)^2 + n(x^2 + v_0^2)^2$. Then to prove $M > 0$ it is sufficient to show that $f(x) > 0$ for $0 \leq x \leq v_0$ or, equivalently, that

$$g(x) = 2(x^2 - 1)(x + c_n)^2 + n(x^2 + 1)^2 > 0$$

for $0 \leq x \leq 1$. For $n = 1$ and $c_1 = 0.5$ this is easily shown by Sturm's Theorem (see [3]). For $n > 1$ we observe that

$$g(x) = (n + 2)x^4 + 4cnx^3 + 2(n - 1 + c_n^2)x^2 - 4cnx + (n - 2c_n^2)$$

$$\geq 2(n - 1 + c_n^2)x^2 - 4cnx + (n - 2c_n^2)$$

for $x \geq 0$ if $c_n \geq 0$. The last expression is positive for all $x$ if

$$0 > 16c_n^2 - 8(n - 1 + c_n^2)(n - 2c_n^2) = 8(2c_n^4 + nc_n^2 - n(n - 1))$$

which holds if $c_n^2 \leq (-n + \sqrt{n^2 + 8n(n - 1)})/4$.

To prove Theorem 2 we introduce an auxiliary polynomial

$$P_t(z) = (z^2 + t^2v_0^2) \prod_{r}(z - r) \prod_{u \neq \pm iv_0 \& u \geq v_0} ((z - u)^2 + v^2)$$

so that $P_t$ and $P$ have the same zeros except that $P$ is zero at $\pm iv_0$ while $P_t$ is zero at $\pm itv_0$. Let $C_\epsilon$ be the circle of radius $r_0 + \epsilon$ with center at the origin. Let $\epsilon > 0$ be so small that $[-r_0 - \epsilon, r_0 + \epsilon] \cap J_w = \emptyset$ for all $w \neq \pm iv_0$. Then (as in Jensen's Theorem) $\text{Im}P_t'(z)/P_t(z)$ is zero only at the two real points of $C_\epsilon$, for $0 \leq t \leq 1$. Therefore the change in $\text{arg}P_t'(z)/P_t(z)$ on $C_\epsilon$ is 0 or $\pm 2\pi$ so that, by the argument principle, the number of zeros of $P_t'(z)$ inside $C_\epsilon$ differs from the number of zeros of $P_t(z)$ inside $C_\epsilon$ by at most one, for $0 \leq t \leq 1$. By Rolle's Theorem, $P_t'(x)$ has an odd number of real zeros strictly between consecutive real zeros of $P_t(z)$. To help envision the proof, imagine that we are watching a movie of the zeros of $P_t'$ as $t$ moves from 0 to 1. When $t = 0$, $P_t(z) = P_0(z)$ has only real zeros inside or on $C_0$ (the circle of radius $r_0$ with center the origin). Moreover, by Jensen's Theorem, $P_0'(z)$ has only real zeros inside $C_0$, and by the above remarks $P_0'(z)$ has precisely one real zero (counting multiplicities) between consecutive real zeros of $P_0(z)$, for $-r_0 \leq x \leq r_0$. For if $P_0'(z)$ had three (or more) zeros between consecutive real zeros $r$ and $r'$ of $P_0(z)$ then (since $P_0'$ is an odd function) $P_0'$ would have three or more zeros between $-r'$ and $-r$ and the total number of zeros of $P_0'$ in $[-r_0, r_0]$ would exceed that of $P_0$ by at least three, in contradiction to what we deduced from the argument principle. Thus, when the movie starts the zeros of $P_t'$ inside $C_0$ are real and simple (except if $P_0$ has a zero of multiplicity $m \geq 3$ at some $r \neq 0$ which causes a zero of $P_0'$ of multiplicity $m - 1$ at $r$) and interlace the zeros of $P_0$ (all of which are real). Also $P_0$ has a double zero at the origin.

Now the zeros of $P_t'$ are continuous functions of $t$. Therefore no new nonreal zeros of $P_t'$ can enter $C_0$ as $t$ varies from 0 to 1, for nonreal zeros of $P_t'$ must remain in the various Jensen discs, none of which intersects $C_0$ (except, of course, $J_{\pm iv_0}$).
Also, a real zero of $P'_t$ cannot coalesce with a (previously distinct) real zero of $P$. For if $r \neq 0$ is a zero of $P$ of multiplicity $m > 0$ then $r$ is a zero of $P_t$ of multiplicity $m$ and therefore a zero of $P'_t$ of multiplicity (precisely) $m - 1$. Thus, as the movie progresses ($0 \leq t \leq 1$) no new zeros of $P'_t$ enter or leave $C_0$ and the real zeros of $P'_t$ which are between the consecutive zeros $r$, $r'$ of $P$ with $r < r' < 0$ (or $r > r' > 0$) must remain real and between $r$ and $r'$.

Let $0 < r_1 \leq r_0$ with $P(r_1) = 0$ and $P(x) \neq 0$ for $-r_1 < x < r_1$. The zeros of $P'_t$ in $(-r_1, r_1)$ will behave differently from those described above. When $t = 0$, there are three zeros of $P'_t$ in $(-r_1, r_1)$: one in $(-r_1, 0)$, one in $(0, r_1)$ and one at the origin. The one at the origin remains there for all $t$. The other two will vary with $t$: call them $S^+_t$, with $S^+_0 > 0$. It is clear from what has been so far described that the only way for $P'(z)$ to have nonreal zeros inside $C_0$ is if $S^+_t$ and $S^-_t$ coalesce at the origin and become (and remain) purely imaginary conjugates. We will now show that if $K < 0$ then $S^+_t$ are real while if $K > 0$ then $S^-_t$ are purely imaginary nonreals.

If $S^+_t$ are nonreal then for some $t_0$, $0 < t_0 < 1$, $P'_t(z)$ has a triple zero at the origin. Hence if $P''_t(0) \neq 0$ for $0 < t < 1$ then $S^+_t$ are real. But an easy calculation gives

$$P'_t(z) = P'_t(z) \cdot \frac{P'_t(z)}{P_t(z)} + P_t(z) \frac{d}{dz} \left( \frac{P'_t(z)}{P_t(z)} \right)$$

so that $P''_t(0) = -P_t(0)K(t)$ (since $P'_t(0) = 0$) where

$$K(t) = \frac{2}{t^2v_0^2} + \sum_r \frac{1}{r^2} + \sum_{u \neq \pm iv_0} \frac{u^2 - v^2}{(u^2 + v^2)^2}.$$ 

Since $P_t(0) \neq 0$ it follows that $K(t) \neq 0$ for $0 < t < 1$ implies that $S^+_t$ are real. Clearly $K$ is an increasing function of $t$ which is negative for all sufficiently small $t > 0$. Therefore, if $K = K(1) < 0$ then $K(t) < 0$ for all $t$, $0 < t < 1$. Hence $S^+_t$ are real in the event that $K < 0$, as was asserted in the theorem.

To prove the other assertion we consider

$$P_\infty(z) = \prod_r (z - r) \prod_{u \neq \pm iv_0} ((z - u)^2 + v^2),$$

so that $P_\infty(z) = P(z)/(z^2 + v^2)$. With the same argument we used above we conclude that within $C_\varepsilon$ the zeros of $P'_\infty$ are all real and interlaced with the zeros of $P_\infty$. In particular, if $-r_1 < x < r_1$ then $P'_\infty(x) = 0$ only when $x = 0$. Now if $t$ is sufficiently large (we no longer require $t \leq 1$), then $P'_t$ and $P'_\infty$ will have the same number of zeros in $-r_1 < x < r_1$. This is because

$$\left| \frac{P'_\infty(x)}{P_\infty(x)} - \frac{P'_t(x)}{P_t(x)} \right| = \left| \frac{2z}{z^2 + t^2v_0^2} \right|$$

so that $P'_t(z)/P_t(z)$ is bounded away from 0 when $P'_\infty(x)/P_\infty(x)$ is (for large $t$) while in a small neighborhood of the origin (which is the only zero of $P'_\infty(x)/P_\infty(x)$ in $-r_1 < x < r_1$) $P'_t(x)/P_t$ and $P'_\infty(x)/P_\infty$ have the same number of zeros by Rouché's Theorem. Therefore, for all sufficiently large $t$, the zeros $S^\pm_t$ are not real. Nonreal zeros of $P'_t$ which are not purely imaginary occur in sets of four, $(\pm w, \pm \bar{w})$, since $P'_t$ is an odd, real function. Therefore, as soon as $t$ is large enough so that $S^\pm_t$ are no longer real they must be purely imaginary and the transition occurs at some $t_0$. 


with $-P_{i0}(0)K(t_0) = P_{i0}''(0) = 0$. But if $K(1) = K > 0$ then $K(t) > 0$ for all $t \geq 1$, since $K(t)$ is an increasing function. Therefore, if $K(1) > 0$ then there is a $t_0 < 1$ such that $K(t_0) = 0$ and $S_{i0}^t$ are nonreal for all $t > t_0$. In particular $S_{i1}^t$ are nonreal and are inside $J_{iv_0}$, by Jensen's Theorem. This proves Theorem 2.

5. **An open question.** We wonder to what extent the hypothesis in Theorem 1, $J_w \cap J_{w0} = \emptyset$, is needed. For the application we have in mind we would like to be able to remove this hypothesis in the case that all the zeros of $P(z)$ are in a strip $-c_0 < \text{Im}\{z\} < c_0$

and $|v_0| > \frac{1}{4}c_0$. We pose the question whether or not a similar conclusion to Theorem 1 can be drawn in this case.

**References**


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