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Simple zeros of the Ramanujan τ-Dirichlet series

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§ 1. Introduction

In this paper we consider the zeros of the Dirichlet series

$$L(s)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s} \quad (\sigma=\operatorname{Re} s>13/2) \tag{1}$$

formed with Ramanujan's tau-function which may be defined by

$$\sum_{n=1}^{\infty}\tau(n)\,z^n=z\prod_{n=1}^{\infty}(1-z^n)^{24} \quad (|z|<1). \tag{2}$$

It is known that $L(s)$ has all of its non-real zeros in the strip $11/2<\sigma<13/2$ and it is conjectured that all these zeros are on $\sigma=6$ and are simple. It has been shown (by Hafner [4]) that a positive proportion of the zeros of $L(s)$ are of odd multiplicity and are on $\sigma=6$, but it is unknown whether any of the zeros are simple. Some other Dirichlet series are known to have infinitely many simple zeros. Heath-Brown [6] and Selberg independently observed that the work [7] of Levinson in 1974 implies that a positive proportion of the zeros of the Riemann zeta function are simple. Of course, the same result works for a Dirichlet L-function. More recently, the authors and S. Gonek have shown in [1] that the Dedekind zeta function of a quadratic extension of the rationals has $\gg T^{6/11}$ simple zeros in the region $0<t<T$. The methods of these papers, however, will probably not give any results for $L(s)$. The Levinson method requires a mean value theorem for the square of the modulus of $\zeta(s)$ multiplied by a Dirichlet polynomial; the "length" of this polynomial is critical and it seems unlikely that such a theorem with a long enough polynomial could be proven, at this time, in the case of L. As far as the method of [1] is concerned, we relied on a factorization of the Dedekind zeta-function as a product of the

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Riemann zeta function and a Dirichlet L-function to achieve the result; no such factorization is available here.

Thus, a new method is required to prove

Theorem. *There exist arbitrarily large T such that for any $\varepsilon>0$, $L(s)$ has $\gg_\varepsilon T^{1/6-\varepsilon}$ simple zeros in the region $0<t<T$.*

Our proof also yields the result that if the Lindelöf hypothesis is true for $L(s)$ (i.e. that $L(6+it)\ll_\varepsilon |t|^\varepsilon$ for any $\varepsilon>0$ and $|t|>1$) then there are arbitrarily large T such that for any $\varepsilon>0$, $L(s)$ has $\gg_\varepsilon T^{\frac{1}{2}-\varepsilon}$ simple zeros in $0<t<T$. The savings of 1/3 in the exponent is due to the improvement that the Lindelöf hypothesis would give over Good's result [3] that

$$L(6+it)\ll_\varepsilon |t|^{1/3+\varepsilon} \tag{3}$$

for any $\varepsilon>0$ and $t\gg 1$. We remark that another conditional approach is Montgomery's pair correlation method which implies that a positive proportion of zeros of the Riemann zeta function are simple, assuming the Riemann Hypothesis. However, this method gives no result in the case we are considering.

Finally, we remark that our method here will probably work for a much more general class of Dirichlet series which have Euler products and functional equations with two gamma factors. In fact, for most of the paper we work in a more general setting. The main reason that we state the theorem only for the Ramanujan tau-Dirichlet series $L(s)$ is that our proof hinges on the fact that $L(s)$ does have at least one non-trivial simple zero, a fact that we have to verify in a somewhat ad hoc manner; in particular, at this point we make use of a result of Spira [9] that the function $(2\pi)^{-s}\Gamma(s)L(s)$, which is real for $\sigma=6$, has a sign change on the line $\sigma=6$ for t somewhere between 0 and 10.

§ 2. Notation and well-known results

We let

$$L(s)=\sum_{n=1}^{\infty} a_n n^{-s} \quad \left(\sigma>\frac{k+1}{2}\right) \tag{4}$$

where, for most of the paper we assume only that $k\geqq 1$; later we specialize to the case $k=12$ and $a_n=\tau(n)$. The properties of a_n and L that we assume are given below; these are all known to hold in the case $a_n=\tau(n)$ (see [5] for most of these).

Firstly, L satisfies the functional equation

$$H(s)L(s)=H(k-s)L(k-s) \tag{5}$$

where throughout this paper we use the notation

$$H(s)=(2\pi)^{-s}\Gamma(s). \tag{6}$$

In its asymmetrical form the functional equation is

$$L(s)=X(s)\,L(k-s) \tag{7}$$

with $X(s)=H(k-s)/H(s)$, of course. Also, the coefficients a_n are multiplicative so that $L(s)$ has an Euler product expansion

$$\begin{aligned} L(s)&=\prod_p (1-a_p p^{-s}+p^{k-1-2s})^{-1} \\ &=\prod_p (1-\alpha_p p^{-s})^{-1}(1-\bar{\alpha}_p p^{-s})^{-1} \end{aligned} \tag{8}$$

where by the work of Deligne [2],

$$|\alpha_p|=p^{(k-1)/2}. \tag{9}$$

The logarithmic derivative is given by

$$\frac{L'}{L}(s)=-\sum_{n=1}^{\infty}\frac{\lambda_n}{n^s} \tag{10}$$

where

$$\lambda_n=\begin{cases}(\alpha_p^m+\bar{\alpha}_p^m)\log p & \text{if } n=p^m \\ 0 & \text{otherwise}\end{cases} \tag{11}$$

with p denoting a prime number. Finally, there is an infinite product expansion for the entire function $\xi(s)=H(s)\,L(s)$ given by

$$\xi(s)=e^{A+Bs}\prod_\rho (1-s/\rho)\,e^{s/\rho} \tag{12}$$

where $\rho=\beta+i\gamma$ runs through the non-trivial zeros of $L(s)$, all of which are in the critical strip $\frac{k-1}{2}<\sigma<\frac{k+1}{2}$.

§ 3. Sketch of proof

The basic idea is as follows. We consider a sum over zeros

$$\sum_\rho L'(\rho)(2\pi)^{-\rho}\,\Gamma(\rho)\,e^{i(\pi/2-\delta)\rho}$$

with the idea of showing that it is large as $\delta\to 0^+$ so that $L'(\rho)$ is non-zero for many ρ. This sum can be expressed via Cauchy's theorem by a pair of integrals

$$\frac{1}{2\pi i}\int \frac{L'}{L}(s)\,L'(s)(2\pi)^{-s}\,\Gamma(s)\,e^{i(\pi/2-\delta)s}\,ds,$$

one on the line $\sigma=\frac{k+1}{2}+\varepsilon$, the other (in the opposite direction) on the line $\sigma=\frac{k-1}{2}-\varepsilon$, where $\varepsilon>0$. This first integral gives a coefficient sum

$$\sum_n f(n)\exp(2\pi i n e^{i\delta})$$

where

$$\sum_n \frac{f(n)}{n^s}=\frac{L'}{L}(s)\,L'(s)\quad\left(\sigma>\frac{k+1}{2}\right).$$

The second integral is similar in view of the functional equation satisfied by $L'(s)$. (The reason we consider our original sum with the oscillating factor $\Gamma(s)$ is to make the second integral tractable.) Now (see Lemma 1) using the periodicity of the exponential function, we find that our coefficient sum is (essentially)

$$\sum_n f(n)\exp(-2\pi n\delta)=\frac{1}{2\pi i}\int \frac{L'}{L}(s)\,L'(s)(2\pi)^{-s}\Gamma(s)\,\delta^{-s}\,ds$$

where the integral is over the line $\sigma=\frac{k+1}{2}+\varepsilon$. The latter integral is related via Cauchy's theorem to another sum over zeros:

$$\sum_\rho L'(\rho)(2\pi)^{-\rho}\Gamma(\rho)\,\delta^{-\rho}.$$

If any of the $L'(\rho)$ are non-zero, then this sum should be large as $\delta\to 0^+$. In fact, it should be about as large as $\delta^{-\beta}$ where β is the largest abscissa of a simple zero. This assertion can be proven using Landau's theorem (see Lemma 7), so that our original sum should be large.

The difficulties with the proof outlined above are mainly technical, arising in part from the consideration of the Mellin transforms of the gamma factors that occur when the functional equation for $L(s)$ is differentiated to obtain a functional equation for $L'(s)$ (see Lemmas 2, 3, 4). However, a genuine difficulty arises in that we get an undesired cancellation from the two integrals which give our original sum. To circumvent this problem, we introduce a factor 2^ρ into our original sum. This leads to a factor like $e^{\pi i n}e^{-2\pi n\delta}=(-1)^n e^{-2\pi n\delta}$ in one coefficient sum and a factor like $e^{4\pi i n}e^{-8\pi n\delta}=e^{-8\pi n\delta}$ in the other. The factor $(-1)^n$ is essentially harmless as there are no primitive characters modulo 2. In fact, $\sum_n f(n)(-1)^n n^{-s}$ has essentially the same poles as $\frac{L'}{L}(s)\,L'(s)$ (see Lemma 5). We no longer have the cancellation and the proof goes through. The essential formula is contained in Lemma 6.

We remark that we could have introduced a factor q^ρ, with q an integer greater than 2, into the sum. This leads one to the assertion, "if any twist $L(s,\chi)=\sum_n a(n)\chi(n)n^{-s}$ of $L(s)$ by a Dirichlet character χ has a simple zero in the critical strip, then $L(s)$ has infinitely many simple zeros in the critical strip." We are currently investigating this situation in the more general setting of an arbitrary Dedekind zeta-function.

§ 4. Lemmas

Lemma 1. *Suppose that*

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

where the sum is absolutely convergent for $\sigma>\sigma_0>0$. *Let* $x>0$ *and* $0<\delta<\pi/4$. *If* $c>\sigma_0$, *then*

$$\frac{1}{2\pi i}\int_{(c)} F(s)H(s)x^{-s}e^{i(\pi/2-2\delta)s}\,ds$$

$$=\frac{1}{2\pi i}\int_{(c)} F_x(s)H(s)x^{-s}(2e^{i\delta}\sin\delta)^{-s}\,ds$$

where $H(s)=(2\pi)^{-s}\Gamma(s)$, $e(z)=e^{2\pi iz}$,

$$F_x(s)=\sum_{n=1}^{\infty}\frac{f(n)e(nx)}{n^s},\qquad(\sigma>\sigma_0)$$

and (c) *denotes the straight line path from* $c-i\infty$ *to* $c+i\infty$.

Proof. The proof depends on the fact that

$$\frac{1}{2\pi i}\int_{(c)}\Gamma(s)z^{-s}\,ds=e^{-z}\qquad(c>0,\ |\arg z|<\pi/2)\tag{13}$$

and on the fact that the exponential function is a homomorphism from $\mathbb{C}$ under addition to $\mathbb{C}^x$ under multiplication. The integral on the left in the statement of the lemma is

$$=\sum_{n=1}^{\infty}f(n)\frac{1}{2\pi i}\int_{(c)}\Gamma(s)(-2\pi inxe^{2i\delta})^{-s}\,ds$$

$$=\sum_{n=1}^{\infty}f(n)e^{2\pi inxe^{2i\delta}}=\sum_{n=1}^{\infty}f(n)e(nx)e^{2\pi nx(e^{2i\delta}-1)i}$$

$$=\sum_{n=1}^{\infty}f(n)e(nx)\exp(-2\pi nxe^{i\delta}2\sin\delta)$$

$$=\sum_{n=1}^{\infty}f(n)e(nx)\frac{1}{2\pi i}\int_{(c)}\Gamma(s)(2\pi nxe^{i\delta}2\sin\delta)^{-s}\,ds$$

which is equal to the integral on the right-hand side.

Lemma 2. *Suppose that* $|\arg z| < \pi/2$ *and* $0 < c < k$. *Then*

$$\frac{1}{2\pi i} \int_{(c)} \Gamma'(s) z^{-s} ds = e^{-z} \log z$$

and

$$\frac{1}{2\pi i} \int_{(c)} \Gamma(s) \frac{\Gamma'}{\Gamma}(k-s) z^{-s} ds = e^{-z} \left(C_k - \int_0^1 \frac{e^{tz}-1}{t} (1-t)^{k-1} dt \right)$$

where $C_k = \frac{\Gamma'}{\Gamma}(k)$.

Proof. The first integral is easily evaluated using (13) and integration by parts.

To evaluate the second integral we move the path of integration to the path $(-N-\frac{1}{2})$ where N is a positive integer and we use Cauchy's theorem. It follows from standard estimates that the integral on $(-N-\frac{1}{2})$ approaches 0 as $N \to \infty$. Since $\Gamma(s)$ has a pole at $s = -n$, $n = 0, 1, 2, \ldots$, with residue $(-1)^n/n!$ we find that the integral is

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!} \frac{\Gamma'}{\Gamma}(k+n).$$

In view of $s\Gamma(s) = \Gamma(s+1)$, it follows that

$$\frac{\Gamma'}{\Gamma}(k+n) = \frac{\Gamma'}{\Gamma}(k) + \sum_{l=0}^{n-1} \frac{1}{l+k}$$
$$= C_k + \int_0^1 t^{k-1} \frac{(1-t^n)}{(1-t)} dt.$$

Thus, our second integral is

$$= C_k e^{-z} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!} \int_0^1 t^{k-1} \frac{(1-t^n)}{(1-t)} dt$$
$$= C_k e^{-z} + \int_0^1 \frac{t^{k-1}}{1-t} (e^{-z} - e^{-zt}) dt$$

and the required result follows upon replacing t by $1-t$ in the integral and factoring out e^{-z}.

Lemma 3. *Suppose that* $v \geqq \frac{1}{2}$, $k \geqq 1$ *and* $0 < \delta < \pi/8$. *Define*

$$w(v, \delta) = \frac{1}{2\pi i} \int_{(c)} \frac{X'}{X}(s) H(s) v^{-s} e^{i(\pi/2 - 2\delta)s} ds \qquad (0 < c < k).$$

Then

$$w(v,\delta) \ll e^{-x} \log\frac{v}{\delta} + \min\{1, x^{-k}\}, \qquad (x = 2\pi v \sin 2\delta)$$

and

$$w(v+\tfrac{1}{2},\delta) + w(v,\delta) \ll \left(\frac{1}{v}+\delta\right)\left(e^{-x}\log\frac{v}{\delta} + \min\{1, x^{-k+1}\}\right).$$

Proof. Since $X(s) = (2\pi)^{s-k}\Gamma(k-s)/((2\pi)^{-s}\Gamma(s))$ it follows that

$$\frac{X'}{X}(s) = -\frac{\Gamma'}{\Gamma}(k-s) - \frac{\Gamma'}{\Gamma}(s) + 2\log 2\pi$$

so that

$$w(v,\delta) = \frac{1}{2\pi i}\int_{(c)} \left(2\log 2\pi\,\Gamma(s) - \Gamma'(s) - \frac{\Gamma'}{\Gamma}(k-s)\,\Gamma(s)\right) z^{-s}\,ds$$

where

$$z = z(v,\delta) = -2\pi i v e^{2i\delta} = 2\pi v \sin 2\delta - 2\pi i v \cos 2\delta = x - iy,$$

say. Then by Lemma 2,

$$w(v,\delta) = F(z)$$

where

$$F(z) = e^{-z}\left(2\log 2\pi - C_k - \log z + \int_0^1 \frac{e^{tz}-1}{t}(1-t)^{k-1}\,dt\right) = e^{-z}G(z),$$

say. Since $z = x - iy$ with $0 < x \leqq y$, $y \geqq 1$ and $\frac{y}{x} \ll 1/\delta$,

$$\int_0^1 \frac{e^{tz}-1}{t}(1-t)^{k-1}\,dt \ll \int_0^{1/|z|} |z|\,dt + \int_{1/|z|}^{1/x} \frac{1}{t}\,dt + \int_{1/x}^{1/2} x e^{tx}\,dt$$
$$+ \int_{1/2}^{1} e^{tx}(1-t)^{k-1}\,dt.$$

Temporarily denoting the last integral by I, the left side is

$$\ll 1 + \log\frac{|z|}{x} + e^{x/2} + I.$$

Note that if $0 \leqq a \leqq b \leqq 1$, then for $x \geqq 1$ and fixed $k>0$,

$$\int_a^b e^{tx}(1-t)^{k-1}\,dt \leqq \int_0^1 e^{tx}(1-t)^{k-1}\,dt = e^x \int_0^1 e^{-xt} t^{k-1}\,dt$$

$$\leqq e^x \int_0^\infty e^{-xt} t^{k-1}\,dt = e^x x^{-k}\Gamma(k) \ll e^x x^{-k}$$

uniformly in x. If $0 \leqq x \leqq 1$, then we use the trivial bound

$$\int_a^b e^{tx}(1-t)^{k-1}\,dt \leqq e^x k^{-1} \ll e^x.$$

Thus, $I \ll e^x \min\{1, x^{-k}\}$. Thus $G(z) \ll \log\frac{v}{\delta} + e^x \min\{1, x^{-k}\}$. Since $|e^{-z}| = e^{-x}$, the first estimate in the statement of the lemma follows.

Next,

$$w(v+\tfrac{1}{2}, \delta) + w(v, \delta) = F(z - \pi i e^{2i\delta}) + F(z).$$

Let $\eta = -\pi i e^{2i\delta}$. Then, writing $F(z) = e^{-z} G(z)$, we have

$$F(z+\eta) + F(z) = e^{-(z+\eta)}(G(z+\eta) - G(z)) + G(z)(e^{-z} + e^{-(z+\eta)}).$$

Now

$$e^{-(z+\eta)} + e^{-z} \ll e^{-x}|e^{-\eta} + 1| = e^{-x}|1 - e^{\pi i(e^{2i\delta}-1)}| \ll \delta e^{-x}.$$

Also,

$$G(z+\eta) - G(z) = -\log\left(1 + \frac{\eta}{z}\right) + \int_0^1 e^{tz}\frac{(e^{t\eta}-1)}{t}(1-t)^{k-1}\,dt.$$

An integration by parts shows that the integral here is

$$= -\frac{\eta}{z} - \frac{1}{z}\int_0^1 e^{tz}(1-t)^{k-2}\left[\frac{(1-k)(e^{t\eta}-1)}{t} + \frac{(1-t)(t\eta e^{\eta t} - e^{\eta t} + 1)}{t^2}\right]dt$$

$$\ll \frac{1}{|z|} + \frac{1}{|z|}\int_0^1 e^{tx}(1-t)^{k-2}\,dt$$

$$\ll \frac{1}{|z|} e^x \min\{1, x^{1-k}\}$$

provided that $k>1$. (In the case $k=1$, the estimate is easier to obtain.) Thus

$$F(z+\eta) + F(z) \ll \left(\frac{1}{v} + \delta\right)\left(e^{-x}\log\frac{v}{\delta} + \min\left\{1, \frac{1}{x^{k-1}}\right\}\right)$$

and the second estimate of the lemma follows.

Lemma 4. *If* $\frac{k+1}{2}<c<\frac{k+2}{2}$ *and* $0<\delta<\pi/8$, *then*

$$E_N(\delta)=\frac{1}{2\pi i}\int_{(c)}\frac{X'}{X}(s)\,H(s)\,L'(s)\,N^{-s}e^{i(\pi/2-2\delta)s}\,ds\ll\delta^{-k/2+1/6}\log^4\frac{1}{\delta}$$

for $N=\frac{1}{2}$, *or* 2.

Proof. We expand $L'(s)$ into a Dirichlet series and integrate term-by-term. Using the notation of Lemma 3,

$$E_N(\delta)=-\sum_{n=1}^{\infty}a_n\log n\,w(nN,\delta).$$

We will estimate this by partial summation. Let

$$S_n=\sum_{m=1}^{n}a_m\log m$$

and

$$S_n^-=\sum_{m=1}^{n}(-1)^m a_m\log m.$$

Then it easily follows from a theorem in Perelli [8] and partial summation that

$$S_n,\,S_n^-\ll n^{k/2-1/6}\log^2 n$$

for $n\geqq 2$. Let us consider the case $N=\frac{1}{2}$ first. We have

$$E_{1/2}(\delta)=-\sum_{n=1}^{\infty}S_n^-\left[(-1)^n w\left(\frac{n+1}{2},\delta\right)-(-1)^{n-1}w\left(\frac{n}{2},\delta\right)\right]$$

since by Lemma 3 the series is absolutely convergent and since $S_n^-\,w\left(\frac{n+1}{2},\delta\right)\to 0$ as $n\to\infty$. Thus by Lemma 3,

$$\begin{aligned}E_{1/2}(\delta)&\ll\sum_{n=1}^{\infty}n^{k/2-1/6}\log^2 n\left|w\left(\frac{n+1}{2},\delta\right)+w\left(\frac{n}{2},\delta\right)\right|\\&\ll\sum_{n=1}^{\infty}n^{k/2-1/6}\log^2 n\left(\frac{1}{n}+\delta\right)\left(e^{-n\pi\sin\delta}\log\frac{n}{\delta}+\min\{1,(n\delta)^{-k+1}\}\right)\\&\ll\sum_{n<1/\delta}n^{-1}\,\delta^{-k/2+1/6}\log^3\frac{1}{\delta}+\delta\sum_{n\geqq 1/\delta}n^{k/2-1/6}e^{-n\delta}\log^3\frac{n}{\delta}\\&\quad+\delta^{2-k}\sum_{n\geqq 1/\delta}n^{-k/2+5/6}\log^3 n.\end{aligned}$$

These sums can be estimated easily and give the result of the lemma. In the case that $N=2$ we have

$$E_2(\delta)=\sum_{n=1}^{\infty} S_n(w(2n+2,\delta)-w(2n,\delta)).$$

We observe that by Lemma 3,

$$\begin{aligned} w(2n+2,\delta)-w(2n,\delta) &= (w(2n+2,\delta)+w(2n+3/2,\delta))-(w(2n+3/2,\delta) \\ &\quad +w(2n+1,\delta))+(w(2n+1,\delta)+w(2n,\delta)) \\ &\ll \left(\frac{1}{n}+\delta\right)\left(e^{-4\pi n\sin\delta}\log\frac{n}{\delta}+\min\{1,(n\delta)^{-k+1}\}\right) \end{aligned}$$

and proceed as above to deduce the lemma.

Lemma 5. *Suppose that*

$$\frac{L'}{L}(s)\,L'(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\qquad\left(\sigma>\frac{k+1}{2}\right). \tag{14}$$

Then

$$\sum_{n=1}^{\infty}\frac{(-1)^n f(n)}{n^s}=[1-2\alpha(s)]\,\frac{L'}{L}(s)\,L'(s)-4\alpha'(s)\,L'(s)-2\,\frac{\alpha'}{\alpha}(s)\,\alpha'(s)\,L(s)$$

where

$$\alpha(s)=(1-a_2 2^{-s}+2^{k-1-2s})=(1-\alpha_2 2^{-s})(1-\bar{\alpha}_2 2^{-s}). \tag{15}$$

Proof. First of all,

$$L'(s)=-\sum_{n=1}^{\infty}\frac{a_n\log n}{n^s},$$

and

$$\frac{L'}{L}(s)=-\sum_{n=1}^{\infty}\frac{\lambda_n}{n^s}\qquad\left(\sigma>\frac{k+1}{2}\right)$$

where

$$\lambda_n=\begin{cases}(\alpha_p^m+\bar{\alpha}_p^m)\log p & \text{if } n=p^m\\ 0 & \text{otherwise.}\end{cases}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n f(n)}{n^s} &= \sum_{m,n}\frac{(-1)^{mn}a_m\log m\,\lambda_n}{m^s n^s}\\ &= \sum_{k=1}^{\infty}\frac{\lambda_{2^k}}{2^{ks}}\sum_m\frac{a_m\log m}{m^s}+\sum_{n\,\mathrm{odd}}\frac{\lambda_n}{n^s}\sum_m\frac{(-1)^m a_m\log m}{m^s}.\end{aligned}$$

Since a_m is multiplicative,

$$\sum_m \frac{(-1)^m a_m \log m}{m^s} = \frac{-d}{ds}\sum_m \frac{(-1)^m a_m}{m^s} = \frac{-d}{ds}\left(L(s) - 2\sum_{m\,\mathrm{odd}} \frac{a_m}{m^s}\right)$$

$$= \frac{-d}{ds}\left[L(s) - 2L(s)(1 - a_2 2^{-s} + 2^{k-1-2s})\right]$$

$$= \frac{-d}{ds}\left[L(s)(1-2\alpha(s))\right].$$

Also,

$$\sum_{n\,\mathrm{odd}} \frac{\lambda_n}{n^s} = \sum_n \frac{\lambda_n}{n^s} - \sum_{k=1}^{\infty} \frac{\lambda_{2^k}}{2^{ks}} = -\frac{L'}{L}(s) - \frac{\alpha'}{\alpha}(s).$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^n f(n)}{n^s} = -\frac{\alpha'}{\alpha}(s)\,L'(s) + \left(\frac{L'}{L}(s) + \frac{\alpha'}{\alpha}(s)\right)\frac{d}{ds}\left[L(s)(1-2\alpha(s))\right]$$

$$= (1-2\alpha(s))\frac{L'}{L}(s)\,L'(s) - 4\alpha'(s)\,L'(s) - 2\frac{\alpha'}{\alpha}(s)\,\alpha'(s)\,L(s).$$

Lemma 6. *Suppose that $0 < \delta < \pi/8$ and that ρ denotes a complex (non-trivial) zero of $L(s)$. Then*

$$\sum_\rho (2\pi)^{-\rho}\Gamma(\rho)\,L'(\rho)\sinh\left(\left(\rho - \frac{k}{2}\right)(\log 2 + i(\pi/2 - 2\delta))\right)$$

$$= \sum_{n=1}^{\infty} f(n)\left[(-1)^n K^{-1}\exp(-2\pi n \sin\delta\, e^{i\delta}) - K\exp(-8\pi n \sin\delta\, e^{-i\delta})\right]$$

$$+ O(\delta^{-k/2+1/6}\log^4 1/\delta) \tag{16}$$

where $K = 2^{k/2} e^{i(\pi/2-2\delta)k/2}$ and $f(n)$ is as in Lemma 5.

Proof. We consider

$$I(\delta) = \frac{1}{2\pi i}\int_{(c)} \frac{L'}{L}(k-s)\frac{L'}{L}(s)\,H(s)\,L(s)\sinh\left[(s-k/2)(\log 2 + i(\pi/2 - 2\delta))\right]ds$$

where $c > \frac{k+1}{2}$. The only poles of the integrand in $\sigma > 0$ are at zeros ρ of $L(s)$. We move the path of integration to the path $(k-c)$ and use Cauchy's residue theorem. By the functional equation, the integrand is an odd function of $s - k/2$. Thus

$$2I(\delta) = -\sum_\rho H(\rho)\,L'(\rho)\sinh\left[(\rho - k/2)(\log 2 + i(\pi/2 - 2\delta))\right].$$

Next, we note that

$$2\sinh[(s-k/2)(\log 2+i(\pi/2-2\delta))]=K^{-1}2^s e^{i(\pi/2-2\delta)s}-K2^{-s}e^{-i(\pi/2-2\delta)s}$$

where $K=2^{k/2}e^{i(\pi/2-2\delta)k/2}$ and that

$$\frac{L'}{L}(k-s)=\frac{X'}{X}(s)-\frac{L'}{L}(s).$$

By Lemma 4,

$$2\cdot\frac{1}{2\pi i}\int_{(c)}\frac{X'}{X}(s)H(s)L'(s)\sinh[(s-k/2)(\log 2+i(\pi/2-2\delta))]\,ds$$
$$=K^{-1}E_{1/2}(\delta)-\overline{KE_2(\delta)}\ll\delta^{-k/2+1/6}\log^4 1/\delta.$$

Thus,

$$2I(\delta)=-\frac{1}{2\pi i}\int_{(c)}\frac{L'}{L}(s)H(s)L'(s)(K^{-1}2^s e^{i(\pi/2-2\delta)s}-K2^{-s}e^{-i(\pi/2-2\delta)s})\,ds$$
$$+O(\delta^{-k/2+1/6}\log^4 1/\delta).$$

By Lemma 1,

$$2I(\delta)=-\frac{1}{2\pi i}\int_{(c)}H(s)(2\sin\delta)^{-s}$$
$$\cdot\left[\sum_{n=1}^{\infty}\frac{(-1)^n f(n)}{n^s}2^sK^{-1}e^{-is\delta}-K\sum_{n=1}^{\infty}\frac{f(n)}{n^s}2^{-s}e^{is\delta}\right]ds$$
$$+O(\delta^{-k/2+1/6}\log^4 1/\delta)$$

where $f(n)$ is defined in (14).

The lemma now follows upon integrating term-by-term and using (13).

Lemma 7. *Suppose that* $k=12$ *and* $a_n=\tau(n)$ *and* $\rho=\beta+i\gamma$ *is a nontrivial simple zero of* $L(s)$. *Then there exist arbitrarily large values of* x *such that for any* $\varepsilon>0$

$$G(x)=\sum_{n=1}^{\infty}f(n)[(-1)^n K_1^{-1}e^{-2\pi n/x}-K_1 e^{-8\pi n/x}]\gg_\varepsilon x^{\beta-\varepsilon};$$

here $f(n)$ *is as in Lemmas* 5 *and* 6 *and* $K_1=(2i)^{k/2}$.

Proof. The proof depends on Landau's well-known theorem on the existence of a pole on the real axis at the abscissa of convergence of a Dirichlet series (or integral) with positive coefficients. The only place we use $a_n=\tau(n)$ is in the assertion that $L(s)$ does not vanish on the real axis to the right of $\sigma=\frac{k-1}{2}$.

Suppose, then, that $|G(x)| \leqq x^b$ for all $x \geqq x_0$ where $(k-1)/2 < b < \beta$, and consider

$$g(s) = \int_1^\infty \frac{x^b - G(x)}{x^{s+1}}\, dx.$$

Then, by Landau's theorem, g has a pole on the real line at the abscissa of convergence. We will show that $g(s)$ has a pole at ρ and has no pole on the interval (b, ∞) of the real line; this will be a contradiction since it will imply that ρ is in the half plane where the integral defining g converges.

We see that

$$g(s) = \frac{1}{s-b} - \int_0^\infty \frac{G(x)}{x^{s+1}}\, dx + g_1(s)$$

where

$$g_1(s) = \int_0^1 \frac{G(x)}{x^{s+1}}\, dx = \int_1^\infty G(1/x)\, x^{s-1}\, dx$$

is entire since $G(1/x) \ll e^{-x}$ as $x \to \infty$. Now it is easy to see that

$$\int_0^\infty \frac{G(x)}{x^{s+1}}\, dx = \sum_{n=1}^\infty \frac{f(n)}{n^s} [(-1)^n K_1^{-1} (2\pi)^{-s} - K_1 (8\pi)^{-s}]\, \Gamma(s).$$

Then, by (14) and (15) of Lemma 5, we find that

$$g(s) = \frac{1}{s-b} + g_1(s) - (2\pi)^{-s} \Gamma(s) \Bigg[\frac{L'}{L}(s)\, L'(s) \{(1 - 2\alpha(s))\, K_1^{-1} - 4^{-s} K_1\}$$
$$- K_1^{-1} \left\{ 4\alpha'(s)\, L'(s) + 2\, \frac{\alpha'}{\alpha}(s)\, \alpha'(s)\, L(s) \right\} \Bigg].$$

Now, by (15), $\alpha(s)$ is entire and by (9) the only zeros of $\alpha(s)$ are on $\sigma = (k-1)/2$. Thus, $\frac{\alpha'}{\alpha}(s)\, L(s)$ is regular at $s = \rho$. Moreover, if $k/2$ is even, then

$$(1 - 2\alpha(s))\, K_1^{-1} - 4^{-s} K_1 = -\frac{4^{-s}}{K_1} [2^{2s} - 2(\alpha_2 + \bar{\alpha}_2)\, 2^s + 2^{k+1}].$$

But for any real number r, with $r^2 \leqq 2^{k+3}$,

$$2^{2s} + r\, 2^s + 2^{k+1}$$

can vanish only when $\operatorname{Re} s = \frac{k+1}{2}$. This condition is satisfied with $r = -2(\alpha_2 + \bar{\alpha}_2)$ since $|\alpha_2|^2 = 2^{k-1}$. Since L has no zeros on $\sigma = \frac{k+1}{2}$ it follows that g has a pole at $s = \rho$. Moreover, g is regular on the subset of the real line (b, ∞)

since L does not have a real zero σ with $\sigma > \frac{k-1}{2}$. This last assertion in the case $k=12$, $a_n = \tau(n)$ follows from the formula (see Hardy [5], Section 10.9)

$$\Gamma(s)\,L(s) = \int_0^\infty y^{s-1} h(e^{-y})\,dy$$

where

$$h(x) = x \prod_{n=1}^\infty (1-x^n)^{24};$$

the integrand is non-negative for real s. This concludes the proof of the lemma.

Lemma 8. *If $k=12$ and $a_n = \tau(n)$ then $L(s)$ has at least one non-trivial simple zero.*

Proof. We make use of the infinite product (12) for $\xi(s)$. It follows from (12) that

$$\frac{\xi'}{\xi}(s) = -\frac{\xi'}{\xi}(k-s) = B + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = -B - \sum_\rho \left(\frac{1}{k-s-\rho} + \frac{1}{\rho}\right). \tag{17}$$

In particular, since $k-\rho$ runs through all the complex zeros of L when ρ does, we obtain

$$2 \sum_{\gamma>0} \frac{\beta}{\beta^2+\gamma^2} = -B = \frac{\xi'}{\xi}(k) = \frac{\Gamma'}{\Gamma}(k) + \frac{L'}{L}(k) - \log 2\pi. \tag{18}$$

If we take the Mth derivative of (17) above and set $s=0$ we obtain (using the definition of ξ and the factorization of $1/\Gamma(z)$ as an entire function)

$$-\sum_\rho \frac{1}{\rho^{M+1}} = \sum_{n=0}^\infty (n+k)^{-M-1} + \frac{1}{M!} \sum_n \frac{\lambda_n \log^M n}{n^k} \qquad (M \geqq 1) \tag{19}$$

where λ_n is as in the proof of Lemma 5. Now we have a proposition for estimating the latter sum.

Proposition. *Suppose that $N \geqq \exp(2m/(k+1))$. Then*

$$\left| \sum_{n=N+1}^\infty \frac{\lambda_n \log^{m-1} n}{n^k} \right| \leqq 2N^{-\left(\frac{k-1}{2}\right)} \sum_{j=0}^m \frac{m!}{j!} \left(\frac{2}{k-1}\right)^{m+1-j} \log^j N.$$

Proof. The sum is bounded in absolute value by

$$2 \sum_{N+1}^\infty \frac{\log^m n}{n^{(k+1)/2}} \leqq 2 \int_N^\infty u^{-(k+1)/2} \log^m u\,du;$$

the latter inequality follows since the integrand is decreasing for $u \geqq \exp(2m/(k+1))$. The proposition now follows upon integrating by parts m times.

Now we specialize to the case $k=12$ and $a_n=\tau(n)$. Spira [9] has shown the existence of a zero $\rho_0=6+i\gamma_0$ of $L(s)$ with $9<\gamma_0<10$, such that ρ_0 has odd multiplicity. Let m_0 denote the multiplicity of ρ_0. Now let us take $M=2$ in (19). For the sum over ρ we group together ρ and $\bar{\rho}$. Then the left side of (19) is

$$=2\sum_{\gamma>0}\frac{\beta(3\gamma^2-\beta^2)}{(\beta^2+\gamma^2)^3}>\frac{12m_0(3\gamma_0^2-36)}{(36+\gamma_0^2)^3}+2\sum_{0<\gamma<4}\frac{\beta(3\gamma^2-\beta^2)}{(\beta^2+\gamma^2)^3}$$

since if $\gamma\geqq 4$, then the summand is positive. Since $\gamma_0<10$ this is

$$>0.001259m_0+2\sum_{0<\gamma<4}\frac{\beta(3\gamma^2-\beta^2)}{(\beta^2+\gamma^2)^3}.$$

On the other hand,

$$\sum_{n=0}^{\infty}(n+12)^{-3}=\zeta(3)-\sum_{n=1}^{11}n^{-3}<0.003774$$

and by the proposition with $m=3, N=4$ and the fact that $\lambda_2=-24\log 2$, $\lambda_3=252\log 3$, and $\lambda_4=-3520\log 2$ we see that the second sum on the right of (19) is negative. Thus,

$$0.001259m_0+2\sum_{0<\gamma<4}\frac{\beta(3\gamma^2-\beta^2)}{(\beta^2+\gamma^2)^2}<0.003774.$$

We conclude that if L has no zeros with $0<\gamma<4$ then $m_0=1$, i.e. $6+i\gamma_0$ is simple, since m_0 is odd. To finish our proof we turn to equation (18) with $k=12$. Then any zero $\rho=\beta+i\gamma$ satisfies $11/2<\beta<13/2$. Suppose that there is a zero $\rho_1=\beta_1+i\gamma_1$ of $L(s)$ with $0\leqq\gamma_1<4$ and let m_1 denote its multiplicity. Then the left side of (18) is

$$>\frac{2m_1\beta_1}{\beta_1^2+\gamma_1^2}+\frac{12m_0}{36+\gamma_0^2}>\frac{13m_1}{(13/2)^2+4^2}+\frac{12m_0}{136}$$
$$>0.22m_1+0.08m_0$$

since $x/(x^2+y^2)$ is a decreasing function of x for $x>y$. In view of $\frac{\Gamma'}{\Gamma}(s)=\frac{1}{s-1}+\frac{\Gamma'}{\Gamma}(s-1)$ and $\frac{\Gamma'}{\Gamma}(1)=-C_0=-0.577\ldots$ the right side of (18) is

$$=-\log 2\pi+\sum_{n=1}^{11}\frac{1}{n}-C_0+\frac{L'}{L}(12)$$

$$<0.61+\frac{L'}{L}(12).$$

By the proposition with $N=2$, $m=1$ and since $\tau(2)=-24$, we have $\frac{L'}{L}(12)<0.02$.

Thus $0.22m_1+0.08m_0<0.63$. Since m_0 and m_1 are positive integers and m_0 is odd it follows that either $m_0=1$ or $m_1=1$ and so $L(s)$ does have a simple zero. This completes the proof of the lemma.

§ 5. Proof of the theorem

Let

$$\theta=\sup\{\beta:\rho=\beta+i\gamma \text{ is a simple zero of } L(s)\}. \tag{20}$$

Let $G_1(\delta)$ denote the sum on the right side of equation (16) in Lemma 6. Then it is not hard to show that as $\delta\to 0^+$

$$G_1(\delta)-G(1/\delta)\ll_\varepsilon \delta^{-\frac{(k-1)}{2}-\varepsilon} \tag{21}$$

for any $\varepsilon>0$, where G is defined in Lemma 7. In fact, this follows easily from the estimates

$$|K-K_1|\ll\delta; \tag{22}$$

$$|e^{-A\sin\delta e^{i\delta}}-e^{-A\delta}|=e^{-A\delta}|e^{-A(\sin\delta e^{i\delta}-\delta)}-1|\ll e^{-A\delta}A\delta^2 \tag{23}$$

for $A\delta^2\ll 1$; and, for any $\varepsilon>0$,

$$\sum_{n\leqq x}|f(n)|\ll_\varepsilon x^{\frac{k+1}{2}+\varepsilon}; \tag{24}$$

equation (24) is an easy consequence of (9), (11), and (14). Thus, by Lemma 7 and (21), there are arbitrarily large values of $T=1/\delta$ such that for any $\varepsilon>0$

$$G_1(1/T)\gg_\varepsilon T^{\theta-\varepsilon}. \tag{25}$$

We now consider the sum on the left side of equation (16) of Lemma 6 with $\delta=1/T$. From Good's work [3] and from standard estimates for the Γ function we have for any $\varepsilon>0$

$$\Gamma(s)L'(s)\ll_\varepsilon e^{-\frac{\pi}{2}|t|}|t|^{\frac{\sigma}{3}+\frac{k}{3}-\frac{1}{6}+\varepsilon} \tag{26}$$

for $\frac{k}{2}\leqq\sigma\leqq\frac{k+1}{2}$ and $|t|\gg 1$. Also,

$$\sinh\left[\left(s-\frac{k}{2}\right)(\log 2+i(\pi/2-2\delta))\right]\ll e^{\left(\frac{\pi}{2}-2\delta\right)|t|} \tag{27}$$

for $|t|\gg 1, |\sigma|\ll 1$. Then it follows easily from (26), (27), and the fact that the number of zeros of $L(s)$ in $0<t<T$ is $\ll T\log T$ that the sum on the left of (16) is

$$\ll_\varepsilon \sum_{\substack{|\gamma|<T^{1+\varepsilon}\\ \rho=\beta+i\gamma \text{ simple}}} T^{\frac{\theta}{3}+\frac{k}{3}-\frac{1}{6}+\varepsilon} \tag{28}$$

for any $\varepsilon > 0$. Since the zeros of L are symmetric about $\sigma = k/2$ it must be the case that $\theta \geqq k/2$. Then, the Theorem follows from (25) and (28).

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References

1. Conrey, J.B., Ghosh, A., Gonek, S.M.: Simple zeros of the zeta function of a quadratic number field, I. Invent. Math. **86**, 563–576 (1986)
2. Deligne, P.: La conjecture de Weil, I. Publ. Math. Inst. Hautes Etud. Sci. **43**, 273–307 (1974)
3. Good, A.: The square mean of Dirichlet series associated with cusp forms. Mathematika **29**, 278–295 (1982)
4. Hafner, J.: Zeros on the critical line for Dirichlet series attached to certain cusp forms. Math. Ann. **264**, 21–37 (1983)
5. Hardy, G.H.: Ramanujan, Third Edition. New York: Chelsea 1978
6. Heath-Brown, D.R.: Simple zeros of the Riemann zeta-function on the critical line. Bull. London Math. Soc. **11**, 17–18 (1979)
7. Levinson, N.: More than one-third of zeros of Riemann's zeta-function are on $\sigma = 1/2$. Adv. Math. **13**, 383–436 (1974)
8. Perelli, A.: On some exponential sums connected with Ramanujan's τ-function. Mathematika **31**, 150–158 (1984)
9. Spira, R.S.: Calculation of the Ramanujan τ-Dirichlet series. Math. Comput. **27**, 379–385 (1973)

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