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More than two fifths of the zeros of the Riemann zeta function are on the critical line

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1. Introduction

In this paper we show that at least 2/5 of the zeros of the Riemann zeta-function are simple and on the critical line. Our method is a refinement of the method Levinson [11] used when he showed that at least 1/3 of the zeros are on the critical line (and are simple, as observed by Heath-Brown [10] and, independently, by Selberg). The main new element here is the use of a mollifier of length $y=T^{\theta}$ with $\theta=4/7-\varepsilon$ whereas in Levinson's theorem the mollifier has $\theta=1/2-\varepsilon$. The work [6] of Deshouillers and Iwaniec on averages of Kloosterman sums is what allows us to use a longer mollifier. In fact, in their paper [7], they obtain an upper bound of essentially the right magnitude for an integral of the modulus squared of the zeta function multiplied by a mollifier of length $T^{4/7}$.

In order to obtain our result, we need asymptotic formulas; obtaining these involves technical but familiar details. In fact, this paper is essentially a synthesis of three papers: Balasubramanian, Conrey, and Heath-Brown [2], Conrey [3], and Deshouillers and Iwaniec [7].

The first paper has the analytic machinery which reduces the integral in question to a main term involving a sum of coefficients of the mollifier and an error term involving sums of incomplete Kloosterman sums; the second paper has the arithmetic machinery for giving an asymptotic formula for the main term; the third paper has the key lemma for bounding the error term.

We mention also that in [3], it is shown that at least 0.365 of the zeros are on the critical line. This paper uses a mollifier with $\theta=1/2-\varepsilon$ as in Levinson, but the coefficients are more elaborate. Levinson uses

$$b(n)=\mu(n)\left(\frac{\log y/n}{\log y}\right)$$

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whereas in [3] and here we will use

$$b(n, P) = \mu(n)\, P\left(\frac{\log y/n}{\log y}\right) \tag{1}$$

where P is a polynomial with $P(0)=0$, $P(1)=1$ which can be chosen optimally by the calculus of variations at the end of the argument. Also in [3], we start from a somewhat more general situation than in Levinson [11]. Levinson's first step is to observe that if the proportion of zeros of

$$\zeta(s) + a(s)\, \zeta'(s)$$

to the right of the critical line is $\leqq p$, then the proportion of zeros of $\zeta(s)$ on the critical line is $\geqq 1-2p$. Here $a(s)$ is a simple function which is essentially $\left(\log \frac{s}{2\pi}\right)^{-1}$. In [3] we make the same observation about a more general combination

$$\sum a_n(s)\, \zeta^{(n)}(s)$$

with simple functions a_n which can be chosen with a certain amount of freedom. Here we use this more general treatment, but we approach it in an easier way as outlined in [5].

2. Background material and statement of theorem

We recall some basic information about the Riemann zeta-function $\zeta(s)$, where $s=\sigma+it$ (see Titchmarsh [15]). It is defined for $\sigma>1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and has a meromorphic continuation to the whole plane with its only pole, a simple pole at $s=1$ with residue 1. It satisfies the functional equation

$$\xi(s) = \xi(1-s) \tag{2}$$

where the entire function $\xi(s)$ is defined by

$$\xi(s) = H(s)\, \zeta(s) \tag{3}$$

with

$$H(s) = (1/2)\, s(s-1)\, \pi^{-s/2}\, \Gamma(s/2). \tag{4}$$

In asymmetrical form, the functional equation is

(5) $$\zeta(s)=\chi(s)\,\zeta(1-s)$$

where, because of familiar properties of the Γ-function,

(6) $$\chi(s)^{-1}=\chi(1-s)=2(2\pi)^{-s}\,\Gamma(s)\cos\frac{\pi s}{2}.$$

Because of the Euler-product representation (for $\sigma>1$),

(7) $$\zeta(s)=\prod_p(1-p^{-s})^{-1},$$

$\zeta(s)$ has no zeros in $\sigma>1$. By (5) and (6) it is seen that $\zeta(s)$ has simple zeros at

$$s=-2,\ -4,\ -6,\dots$$

and nowhere else in $\sigma<0$. Hadamard and de la Vallée-Poussin showed, independently in 1885, that $\zeta(s)$ has no zeros on $\sigma=1$; hence all the non-real zeros of $\zeta(s)$ are in the critical strip $0<\sigma<1$. The zeros of $\zeta(s)$ in the critical strip are denoted by $\varrho=\beta+i\gamma$. Von Mangoldt proved that

(8) $$\begin{aligned} N(T) &= \#\{\varrho: 0<\gamma<T\} \\ &= \frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T). \end{aligned}$$

If we let

(9) $$N_0(T)=\#\{\varrho: 0<\gamma<T \quad \text{and} \quad \beta=1/2\},$$

then Riemann [13] conjectured that $N_0(T)=N(T)$ for all T; i.e., that all the zeros are on $\sigma=1/2$. Hardy was the first to show that $N_0(T)$ goes to infinity with T; later he and Littlewood showed [9] that $N_0(T)\gg T$. Selberg [14] was the first to prove that $N_0(T)\gg T\log T$ i.e., that

(10) $$\kappa=\liminf_{T\to\infty}\frac{N_0(T)}{N(T)}>0.$$

We call κ the proportion of zeros on the critical line. Let now

(11) $$N_0^*(T)=\#\{\varrho: 0<\gamma<T,\ \beta=1/2,\ \zeta'(\varrho)\neq 0\}$$

be the number of simple zeros on the critical line up to a height T, and let

(12) $$\kappa^*=\liminf_{T\to\infty}\frac{N_0^*(T)}{N(T)}.$$

Then the work of Levinson [11] implies that

$$\kappa^* \geqq 0.3474. \tag{13}$$

In Conrey [3] it was shown that

$$\kappa \geqq 0.3658 \tag{14}$$

and in [4] that

$$\kappa^* \geqq 0.3485. \tag{15}$$

Anderson showed in [1] that

$$\kappa^* \geqq 0.3532. \tag{16}$$

Here we prove

Theorem 1. *With the above notation,* $\kappa \geqq 0.4077$ *and* $\kappa^* \geqq 0.401$. *In particular, at least 2/5 of the zeros of* $\zeta(s)$ *are simple and on the critical line.*

3. Beginning of the proof

It follows from (2) that $\xi^{(n)}(s)$ is real for $s=1/2+it$ when n is even and is purely imaginary when n is odd. Let g_n, $n \geqq 0$, be complex numbers with g_n real if n is odd and g_n purely imaginary if n is even. Let $g \neq 0$ be real. Let T be a large parameter (we will be working with the strip $0<t\leqq T$) and let

$$L = \log T. \tag{17}$$

Now define

$$\eta(s) = g\,\xi(s) + \sum_{n=0}^{N} g_n\, \xi^{(n)}(s)\, L^{-n} \tag{18}$$

for some fixed N. Then, for $s=1/2+it$,

$$g\,\xi(s) = \operatorname{Re} \eta(s) \tag{19}$$

so that $\xi(s)=0$ on $\sigma=1/2$ if and only if $\operatorname{Re}\eta(s)=0$. The idea, then, is to show that $\operatorname{Re}\eta(1/2+it)$ vanishes at least

$$(\kappa + o(1))\, \frac{T}{2\pi} \log T \tag{20}$$

times for t in $[0, T]$. We do this by showing that the change in argument of $\eta(1/2+it)$ as t varies from 0 to T is at least

$$(\kappa + o(1)) \frac{T}{2} \log T \tag{21}$$

since for every change of π in the argument of some function $f(z)$ it must be the case that $\mathrm{Re}\, f(z)$ has at least one zero.

To estimate the change in argument of $\eta(s)$ on the 1/2-line, we let $\eta(s) = H(s)\, V_1(S)$ where H is defined in (4) and

$$V_1(s) = g\zeta(s) + \sum_{n=0}^{N} \frac{g_n}{L^n} \sum_{k=0}^{n} \binom{n}{k} \frac{H^{(n-k)}(s)}{H(s)} \zeta^{(k)}(s). \tag{22}$$

By Lemma 1 of [3], for $|t| \geqq 2$,

$$\frac{H^{(m)}}{H}(s) = (1/2 \log s/(2\pi))^m \,(1 + O(1/|t|)) \tag{23}$$

so that

$$\begin{aligned} V_1(s) &= g\zeta(s) + \sum_{n=0}^{N} g_n \left(\frac{\log s/(2\pi)}{2L} + \frac{1}{L}\frac{d}{ds} \right)^n \zeta(s)\,(1 + O(1/|t|)) \\ &= \left(Q_1 \left(\frac{\log s/(2\pi)}{2L} + \frac{1}{L}\frac{d}{ds} \right) \zeta(s) \right) (1 + O(1/|t|)) \end{aligned} \tag{24}$$

where

$$Q_1(x) = g + \sum_{n=0}^{N} g_n x^n. \tag{25}$$

A useful approximation to $V_1(s)$ in $0 < t < T$, $|\sigma| \ll 1$, is given by

$$V(s) = Q\left(-\frac{1}{L}\frac{d}{ds} \right) \zeta(s) \tag{26}$$

where

$$Q(x) = Q_1(1/2 - x). \tag{27}$$

Note that for $\sigma > 1$,

$$V(s) = \sum_{n=1}^{\infty} \frac{Q(\log n/L)}{n^s}. \tag{28}$$

Also, the condition that g_{2n} is imaginary and g_{2n+1} is real is equivalent to $\mathrm{Re}\, Q_1(ix) = g$ for real x or $\mathrm{Re}\, Q(1/2 + ix) = g$ for real x. Equivalently,

$$Q(z) + \bar{Q}(1 - z) \equiv g. \tag{29}$$

By Lemma 1 of [3],

$$\arg H(1/2+it)=\frac{t}{2}\log\frac{|t|}{2\pi e}+O(1) \tag{30}$$

so that

$$\Delta \arg \eta(1/2+it)|_2^T=\frac{T}{2}\log T+\Delta \arg V_1(1/2+it)|_2^T+O(T). \tag{31}$$

Now if $Q(0)=1$ (i.e. $Q_1(1/2)=1$), then it is not hard to show (see Conrey [3], Section 4) using the argument principle and standard estimates that

$$\Delta \arg V_1(1/2+it)|_2^T=-2\pi N_{V_1}^*(T)+O(T) \tag{32}$$

where $N_{V_1}^*(T)$ is the number of zeros of $V_1(s)$ with $\sigma\geqq 1/2$ and $0<t<T$ counted with multiplicity, except that zeros on $\sigma=1/2$ only count with weight 1/2 (times their multiplicity). We need an upper bound for $N_{V_1}^*(T)$.

We introduce the mollifier

$$B(s,P)=\sum_{n\leqq y}\frac{b(n,P)n^{\sigma_0-1/2}}{n^s} \tag{33}$$

where $\sigma_0=1/2-R/L$ with $R>0$ a free parameter; also

$$y=T^\theta \tag{34}$$

(and eventually $\theta=4/7-\varepsilon$) and $b(n,P)$ is defined in (1). (Recall that P denotes a polynomial with $P(0)=0$ and $P(1)=1$ so that $b(1,P)=1$ and $b(n,P)\to 0$ as $n\to y$.)

Then

$$N_{V_1}^*(T)\leqq N_{V_1B}^*(T) \tag{35}$$

where $N_{V_1B}^*(T)$ is the number of zeros of V_1B in $0<t\leqq T$, $\sigma\geqq\frac{1}{2}$ counted with multiplicity, except that zeros on the 1/2-line have only one-half their usual weight.

Now, as above

$$\sigma_0=1/2-R/L \tag{36}$$

where R is a positive real number, $R\ll 1$. Then by Littlewood's lemma and the arithmetic-mean, geometric-mean inequality exactly as in Levinson [11] or Conrey [3],

$$2\pi N_{V_1B}(T)\leqq\frac{TL}{2R}\log\left(\frac{1}{T}\int_1^T|V_1B(\sigma_0+it)|^2\,dt\right)+O(T) \tag{37}$$

where $N_{V_1B}(T)$ is the number of zeros of $BV_1(s)$ in $0<t\leqq T$, $\sigma\geqq 1/2$ counted with multiplicity (and no convention about zeros on the 1/2-line). This leads to

$$\kappa\geqq 1-\frac{1}{R}\log\left(\frac{1}{T}\int_1^T|V_1B(\sigma_0+it)|^2\,dt\right)+o(1). \tag{38}$$

We will prove an asymptotic formula for the integral here. In fact, such a formula will follow by an integration by parts from an asymptotic formula for

$$\int_1^T |VB(\sigma_0+it)|^2\,dt.$$

Thus,

$$\kappa \geqq 1 - \frac{1}{R}\log\left(\frac{1}{T}\int_1^T |BV(\sigma_0+it)|^2\,dt\right) + o(1) \tag{39}$$

where $R>0$, $\sigma_0 = 1/2 - R/L$, $V(s) = Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s)$ where Q is a polynomial satisfying $Q(0)=1$ and $Q(z)+\bar{Q}(1-z)\equiv g$ for some real number g; and $B(s)=\sum_{n\leqq y} b(n,P)n^{-s+\sigma_0-1/2}$ with $b(n,P)=\mu(n)\,P\left(\frac{\log y/n}{\log y}\right)$ where P is a polynomial with $P(0)=0$ and $P(1)=1$.

Regarding simple zeros, we note that our argument thus far has shown that

$$\begin{aligned}\frac{1}{\pi}\Delta \arg\eta(1/2+it)|_2^T &= \frac{TL}{2\pi} - 2N^*_{V_1}(T) + O(T)\\ &\geqq \frac{TL}{2\pi} - 2N^*_{V_1B}(T) + O(T)\\ &\geqq \frac{TL}{2\pi} - 2N_{V_1B}(T) + N_{0,V_1}(T) + O(T)\end{aligned} \tag{40}$$

where $N_{0,V_1}(T)$ is the number of zeros of $V_1(s)$ (or equivalently of $\eta(s)$) on the half-line. Thus, we actually have

$$\begin{aligned}\frac{1}{\pi}\Delta \arg\eta(1/2+it)|_2^T - N_{0,\eta}(T) &\geqq \frac{TL}{2\pi} - 2N_{V_1B}(T) + O(T)\\ &\gtrsim \frac{TL}{2\pi}\left(1 - \frac{1}{R}\log\frac{1}{T}\int_1^T |B\,V(\sigma_0+it)|^2\,dt\right).\end{aligned} \tag{41}$$

Now the left hand side here is a lower bound for the number of t in $(2, T)$ for which $\operatorname{Re}\eta(1/2+it)=0$ but $\eta(1/2+it)\neq 0$. In the event that

$$\eta(s) = g\,\xi(s) + g_0\,\xi(s) + g_1\,\xi'(s)L^{-1} \tag{42}$$

we have $\operatorname{Re}\eta(1/2+it) = g\,\xi(1/2+it)$ so that $\operatorname{Re}\eta(1/2+it)=0$ but $\eta(1/2+it)\neq 0$ implies that $\xi(1/2+it)=0$ but $\xi'(1/2+\mathrm{it})\neq 0$, i.e., that $1/2+it$ is a simple zero of ξ.

Hence we have

$$\kappa^* \geqq 1 - \frac{1}{R} \log \left(\frac{1}{T} \int_2^T |VB(\sigma_0 + it)^2 \, dt \right) + o(1) \tag{43}$$

subject to all the conditions on R, V, B, and Q mentioned above and with the additional condition that Q be a polynomial of degree 1.

4. The mean value theorem

Apart from some numerical calculations, Theorem 1 will be a consequence of the following mean value theorem.

Theorem 2. *Let $B(s, P)$ be as in* (1), (33), *and* (34). *Suppose that* $R \ll 1$ *and* $\sigma_0 = 1/2 - R/L$. *Let* $V(s) = Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s)$ *for some polynomial Q. If $\theta < 4/7$, then*

$$\int_2^T |VB(\sigma_0 + it)|^2 \, dt \sim c(P, Q, R)\, T$$

as $T \to \infty$ where

$$c(P, Q, P) = |P(1)\, Q(0)|^2 + \frac{1}{\theta} \int_0^1 \int_0^1 \left| \frac{d}{du} \left(e^{R(y+\theta u)} Q(y + \theta u)\, P(x+u) \right) \Big|_{u=0} \right|^2 dx\, dy$$

$$= |P(1)\, Q(0)|^2 + \frac{1}{\theta} \int_0^1 \int_0^1 e^{2Ry} |Q(y)\, P'(x) + \theta Q'(y)\, P(x) + \theta R Q(y)\, P(x)|^2 \, dx\, dy.$$

To deduce Theorem 1 from Theorem 2, we make our choices for P, Q, and R. For the moment, let

$$w(y) = e^{Ry} Q(y). \tag{44}$$

Then

$$c(P, Q, R) = |w(0)|^2 + \frac{1}{\theta} \int_0^1 \int_0^1 \left| \frac{d}{du} \left(w(y + \theta u)\, P(x+u) \right) \Big|_{u=0} \right|^2 dx\, dy \tag{45}$$

$$= |w(0)|^2 + \frac{1}{\theta} \int_0^1 \int_0^1 |w(y)\, P'(x) + \theta w'(y)\, P(x)|^2 \, dx\, dy.$$

The double integral here is

$$I(P)=A\int_0^1 |P'(x)|^2\,dx+2\,\mathrm{Re}\,(B\int_0^1 P'(x)\,\bar{P}(x)\,dx)+C\int_0^1 |P(x)|^2\,dx$$

where

$$(46)\qquad A=\int_0^1 |w(y)|^2\,dy,\quad B=\theta\int_0^1 w(y)\,\bar{w}'(y)\,dy,\quad C=\theta^2\int_0^1 |w'(y)|^2\,dy.$$

By the Euler-Lagrange equations, $I(P)$ will be minimized by a function P satisfying

$$(47)\qquad AP''-(B-\bar{B})P'-CP=0,\quad P(0)=0,\quad P(1)=1.$$

By an integration by parts,

$$\int_0^1 |P'(x)|^2\,dx=\bar{P}P'(x)|_0^1-\int_0^1 \bar{P}(x)\,P''(x)\,dx.$$

We use (47) here to substitute for P''; then by (46) and an easy calculation,

$$I(P)=AP'(1)+\bar{B}\,2\,\mathrm{Re}\int_0^1 P(x)\,\bar{P}'(x)\,dx.$$

But

$$\int_0^1 P(x)\,\bar{P}'(x)\,dx+\int_0^1 \bar{P}(x)\,P'(x)=P(x)\,\bar{P}(x)|_0^1=1$$

whence

$$I(P)=AP'(1)+\bar{B}.$$

The solution of (47) is easily seen to be

$$P(x)=\frac{e^{rx}-e^{sx}}{e^r-e^s}$$

where r and s are roots of $Az^2-(B-\bar{B})z-C=0$. (Although P is not a polynomial it can be uniformly approximated by polynomials of the right sort.) Solving for r and s we may write

$$P(x)=\frac{e^{i\beta x}}{e^{i\beta}}\,\frac{e^{\alpha x}-e^{-\alpha x}}{e^{\alpha}-e^{-\alpha}}$$

where

$$(48)\qquad \alpha=\frac{((B-\bar{B})^2+4AC)^{1/2}}{2A},\quad i\beta=\frac{B-\bar{B}}{2A}.$$

Then

$$P'(1)=\alpha\,\frac{e^{\alpha}+e^{-\alpha}}{e^{\alpha}-e^{-\alpha}}+i\beta$$

from which we see from (48) that

$$I(P) = AP'(1) + \bar{B} = A\alpha \coth\alpha + \frac{B+\bar{B}}{2}$$

$$= A\alpha \coth\alpha + \frac{\theta}{2}(|w(1)|^2 - |w(0)|^2)$$

since

$$2\,\mathrm{Re}\,B = \theta\left(\int_0^1 w\bar{w}' + \int_0^1 \bar{w}w'\right) = \theta|w|^2\Big|_0^1.$$

By (45) we now have

$$c = c(Q, R) = 1/2(|w(0)|^2 + |w(1)|^2) + \frac{A\alpha\coth\alpha}{\theta}. \tag{49}$$

We illustrate the computation in the case of simple zeros. We are forced here to take

$$Q(z) = 1 + \lambda z$$

for some real number λ. Then

$$A = \int_0^1 |w(y)|^2\,dy = \int_0^1 e^{2Ry}(1+\lambda y)^2\,dy = \lambda^2 I_2 + 2\lambda I_1 + I_0$$

where

$$I_n = \int_0^1 e^{2Ry} y^n\,dy.$$

Also, $B = 0$,

$$C = \theta^2 \int_0^1 e^{2Ry}(R(1+\lambda y)+\lambda)^2\,dy = \theta^2((R+\lambda)^2 I_0 + 2R\lambda(R+\lambda)I_1 + R^2\lambda^2 I_2)$$

and

$$\alpha = (C/A)^{1/2}.$$

Then with $\theta = 4/7$, $R = 1.2$, $\lambda = -1.02$ we get (using $I_0 = (e^{2R}-1)/(2R)$, $I_1 = (e^{2R} - I_0)/(2R)$, and $I_2 = (e^{2R} - 2I_1)/(2R)$), $I_0 = 4.17\ldots$, $I_1 = 2.85\ldots$, $I_2 = 2.21\ldots$, $A = 0.66\ldots$, $B = 0$, $C = 0.71\ldots$, $\alpha = 1.04\ldots$, $\coth\alpha = 1.28\ldots$,

$$c = (1/2)\,(1 + e^{2R}(1+\lambda)^2) + \frac{A}{\theta}\alpha\coth\alpha = 2.05\ldots,$$

and $1 - (\log c)/R = 0.4013\ldots$. The 0.4077 result arises from choosing $Q(y)$ as the $\varphi(y)$ in Conrey [3], Section 7, with $m = 0$.

5. The proposition

In this section, we deduce our mean value theorem, Theorem 2, from the following

Proposition. *Let $a, b \in \mathbb{C}$ with $a, b \ll 1$, and put $\alpha = a/L$, $\beta = b/L$ where $L = \log T$. Let $s_0 = 1/2 + iw$ with $T \leqq w \leqq 2T$. Suppose that $\delta > 0$, $\Delta = T^{1-\delta}$ and that $y = T^{\theta}$ with $0 < \theta < 4/7$. Let*

$$g(a, b, w, P_1, P_2) = \frac{1}{i\Delta\pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta(s+\alpha)\,\zeta(1-s+\beta)\,\mathcal{B}(s, P_1)\,\mathcal{B}(1-s, P_2)\,ds$$

where (c) denotes the straight line path from $c - i\infty$ to $c + i\infty$ and where

$$\mathcal{B}(s, P_i) = \sum_{n \leqq y} \frac{b(n, P_i)}{n^s} = B(s + \sigma_0 - 1/2, P_i)$$

with $P_i(0) = 0$ for $i = 1, 2$. Then

$$g(a, b, w, P_1, P_2) = \frac{1}{\theta} \int_0^1 e^{-(b+a)y}\,dy\,\frac{\partial}{\partial u}\frac{\partial}{\partial v}\,e^{-a\theta u - b\theta v} \int_0^1 P_1(x+u)\,P_2(x+v)\,dx\Big|_{u=v=0} + P_1(1)\,P_2(1) + o_\delta(1)$$

uniformly in a, b, and w.

Proof of Theorem 2. To prove Theorem 2 it suffices to show, in the notation of Theorem 2, that

$$\frac{1}{\Delta\pi^{1/2}} \int_{-\infty}^{\infty} e^{-(t-w)^2\Delta^{-2}} |VB(\sigma_0 + it)|^2\,dt = c(P, Q, R) + o_\delta(1) \tag{50}$$

uniformly for $T \leqq w \leqq 2T$, with $\Delta = T^{1-\delta}$. For then Theorem 2 follows exactly as in Section 3 of Balasubramanian, Conrey, and Heath-Brown [2]. To prove (50) we write the left side as a complex integral

$$\frac{1}{i\Delta\pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}}\,VB(s + \sigma_0 - 1/2)\,\overline{VB}(1 - s + \sigma_0 - 1/2)\,ds$$

where $s_0 = 1/2 + iw$; by (26) this is

$$= Q\left(\frac{-d}{da}\right)\bar{Q}\left(\frac{-d}{db}\right)\left(\frac{1}{i\Delta\pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta(s+\alpha)\,\zeta(1-s+\beta)\,\mathcal{B}(s, P)\,\mathcal{B}(1-s, \bar{P})\,ds\Big|_{a=b=-R}\right)$$

where $\alpha = a/L$ and $\beta = b/L$. In the notation of the proposition, this is

$$= Q\left(\frac{-d}{da}\right) \bar{Q}\left(\frac{-d}{db}\right) g(a, b, w, P, \bar{P})|_{a=b=-R}.$$

Thus, by the proposition, the left side of (50) is

$$= Q\left(\frac{-d}{da}\right) \bar{Q}\left(\frac{-d}{db}\right)$$

$$\left(|P(1)|^2 + \frac{1}{\theta} \frac{\partial}{\partial u} \frac{\partial}{\partial v} \int_0^1 e^{-a(y+\theta u) - b(y+\theta v)} dy \int_0^1 P(x+u) \bar{P}(x+v) dx|_{u=v=0} + o_\delta(1)\right)|_{a=b=-R}.$$

Clearly, g is analytic in the complex variables a and b if $a, b \ll 1$. Thus, we may use Cauchy's integral formula and the fact that

$$Q\left(\frac{-d}{da}\right) e^{-ay}|_{a=-R} = Q(y) e^{Ry} \tag{51}$$

to conclude that the left side of (50) is

$$= |Q(0) P(1)|^2$$

$$+ \frac{1}{\theta} \frac{\partial}{\partial u} \frac{\partial}{\partial v} \int_0^1 \int_0^1 P(x+u) \bar{P}(x+v) e^{2Ry} Q(y+\theta u) \bar{Q}(y+\theta v) dx dy|_{u=v=0} + o_\delta(1).$$

Equation (50) easily follows from this.

6. Initial Lemmas

In this section we prove that the proposition is a consequence of the following two lemmas.

Lemma 1. *Suppose that* $y = T^\theta$, $0 < \theta < 1$, P_1 *and* P_2 *are polynomials with* $P_1(0) = P_2(0) = 0$,

$$b(n, P) = \mu(n) P\left(\frac{\log y/n}{\log y}\right),$$

$L = \log T$, $\alpha = a/L$, $\beta = b/L$ *with* $a, b \ll 1$ *and*

$$\Sigma(\alpha, \beta, P_1, P_2) = \sum_{h,k \leqq y} \frac{b(h, P_1)\, b(k, P_2)}{h^{1+\alpha} k^{1+\beta}} (h, k)^{1+\alpha+\beta}.$$

Then, as $T \to \infty$

$$\Sigma(\alpha, \beta, P_1, P_2) \sim \frac{1}{\theta L} \frac{\partial}{\partial u} \frac{\partial}{\partial v} e^{a\theta u + b\theta v} \int_0^1 P_1(x+u) \cdot P_2(x+v)\, dx|_{u=v=0}$$

$$= \frac{1}{\theta L} \int_0^1 (P_1'(x) + a\theta P_1(x))\,(P_2'(x) + b\theta P_2(x))\, dx.$$

Proof. We give a sketch, as this sort of mean is worked out in Conrey [3], Section 6, using Lemmas 10 and 11 of that paper. We write

$$(h, k)^{1+\alpha+\beta} = \sum_{\substack{d|h \\ d|k}} \sum_{e|d} \mu(e) \left(\frac{d}{e}\right)^{1+\alpha+\beta}$$

$$= \sum_{\substack{d|h \\ d|k}} d^{1+\alpha+\beta} F(d, 1+\alpha+\beta)$$

where

$$F(d, s) = \prod_{p|d} (1 - p^{-s}).$$

Next we change the order of summation, so that the sum over d is on the outside and on the inside we have a product of a sum over $h' = h/d$ and a sum over $k' = k/d$. The sums over h' and k' are evaluated using Lemma 10 and the result is evaluated using Lemma 11. After some simplification, we have our result. (More details may be found in Section 6 of Conrey [3].)

Lemma 2. *Let g be as in the proposition and let Σ be as in Lemma* 1. *Assume the hypotheses of the proposition. Then*

$$g(a, b, w, P_1, P_2) = \frac{\Sigma(b, a, P_1, P_2) - e^{-a-b}\Sigma(-a, -b, P_1, P_2)}{\alpha+\beta} + o_\delta(1)$$

uniformly in a, b, and w.

Proof of the proposition. Let

$$\sigma(a, b, P_1, P_2) = \frac{1}{\theta} \int_0^1 (P_1'(x) + a\theta P_1(x))\,(P_2'(x) + b\theta P_2(x))\, dx$$

so that by Lemma 1,

$$\Sigma(a, b, P_1, P_2) \sim \frac{1}{L} \sigma(a, b, P_1, P_2). \tag{52}$$

Then

$$\sigma(a, b, P_1, P_2) - \sigma(-b, -a, P_1, P_2) = (a+b) \int_0^1 (P_1'(x)\, P_2(x) + P_1(x)\, P_2'(x))\, dx \tag{53}$$

$$= (a+b)\, P_1(x)\, P_2(x)|_0^1 = (a+b)\, P_1(1)\, P_2(1)$$

for any a and b. Thus, by Lemma 2 and (52) and (53),

$$g \sim (a+b)^{-1}\,[\sigma(-a,\,-b)+(a+b)\,P_1(1)\,P_2(1)-e^{-a-b}\sigma(-a,\,-b)]$$

$$=\frac{1-e^{-a-b}}{a+b}\,\sigma(-a,\,-b,\,P_1,\,P_2)+P_1(1)\,P_2(1)$$

$$=\int_0^1 e^{-(a+b)y}\,dy\,\frac{1}{\theta}\,\frac{\partial}{\partial u}\,\frac{\partial}{\partial v}\,e^{-a\theta u-b\theta v}\int_0^1 P_1(x+u)\,P_2(x+v)\,dx\big|_{u=v=0}+P_1(1)\,P_2(1)$$

as stated in the proposition.

7. The main term

In this section, we produce the main term of g in Lemma 2 after some preparatory lemmas.

Lemma 3. *Suppose that* $1<c<2$ *and as usual,*

$$\chi(1-s)=2(2\pi)^{-s}\,\Gamma(s)\cos\frac{\pi s}{2}\,.$$

Let

$$J(y,\,s_0,\,\delta,\,\Delta)=\frac{1}{i\Delta\pi^{1/2}}\int_{(c)} e^{(s-s_0)^2\Delta^{-2}}\chi(1-s+\beta)\,y^{-s}\,ds.$$

Then

$$J=y^{-\beta}\int_0^\infty v^{s_0-\beta}\exp\left(-\frac{\Delta^2\log^2 v}{4}\right)(e(-yv)+e(yv))\,\frac{dv}{v}$$

for any $y\neq 0$, $\Delta>0$, s_0 *and* β *with* $\operatorname{Re}\beta<c$.

Proof. By a change of variables,

$$J=\frac{y^{-\beta}}{i\Delta\pi^{1/2}}\int_{(c-\operatorname{Re}\beta)} e^{(s+\beta-s_0)^2\Delta^{-2}}\chi(1-s)\,y^{-s}\,ds.$$

The lemma now follows from Lemma 2 of Balasubramanian, Conrey, and Heath-Brown [2].

Lemma 4. *Let*

$$D(s,\,\alpha,\,\beta,\,H/K)=\sum_{m,n} m^{-s-\alpha}n^{-s-\beta}e(mnH/K)$$

where H, K *are integers* ($K\geqq 1$) *with* $(H,\,K)=1$ *and* $\alpha,\,\beta,\,s\in\mathbb{C}$. *Then*

$$D(s, \alpha, \beta, H/K) - K^{1-\alpha-\beta-2s}\zeta(s+\alpha)\,\zeta(s+\beta)$$

is an entire function of s. Also, D satisfies the functional equation

$$D(s, \alpha, \beta, H/K) = -2\left(\frac{K}{2\pi}\right)^{1-2s-\alpha-\beta} \Gamma(1-s-\alpha)\,\Gamma(1-s-\beta)$$

$$\left[\cos\frac{\pi}{2}(2s+\alpha+\beta)\,D\left(1-s, -\alpha, -\beta, \frac{-\bar{H}}{K}\right) - \cos\frac{\pi}{2}(\alpha-\beta)\,D\left(1-s, -\alpha, -\beta, \frac{\bar{H}}{K}\right)\right].$$

Moreover, if $\alpha, \beta \ll (\log K)^{-1}$, *then* $D(0, \alpha, \beta, H/K) \ll_\varepsilon K^{1+\varepsilon}$ *for any* $\varepsilon > 0$.

All of these assertions are easily proven using the techniques of Estermann's original paper [8]. Basically, one uses the fact that

$$D(s, \alpha, \beta, H/K) = \sum_{a=1}^{K} \zeta(s+\alpha, a, K)\,\zeta(s+\beta, aH/K)$$

where

$$\zeta(s, a, K) = \sum_{n \equiv a \bmod K} n^{-s}$$

and

$$\zeta(s, a/K) = \sum_{n=1}^{\infty} e(an/K)\,n^{-s}.$$

These functions satisfy the functional equations

$$\zeta(s, a, K) = G(s)\,K^{-s}\left[e^{\pi i s/2}\zeta(1-s, a/K) - e^{-\pi i s/2}\zeta(1-s, -a/K)\right]$$

and

$$\zeta(s, a/K) = G(s)\,K^{1-s}\left[e^{\pi i s/2}\zeta(1-s, -a, K) - e^{-\pi i s/2}\zeta(1-s, a, K)\right]$$

where

$$G(s) = -i(2\pi)^{s-1}\,\Gamma(1-s).$$

The details are in Estermann's paper [8].

Lemma 5. *Let H and K be relatively prime integers with* $K > 0$. *Suppose that* $\alpha, \beta, x \in \mathbb{C}$ *with* $\operatorname{Im} x > 0$ *and let*

$$S(x, \alpha, \beta, H/K) = \sum_{m,n} m^{-\alpha} n^{-\beta} e(mnH/K)\,e(mnx).$$

If $c > 1$, *then*

$$S(x, \alpha, \beta, H/K)$$
$$=\zeta(1-\alpha+\beta)K^{-1+\alpha-\beta}z^{-1+\alpha}\Gamma(1-\alpha)+\zeta(1-\beta+\alpha)K^{-1+\beta-\alpha}z^{-1+\beta}\Gamma(1-\beta)$$
$$+D(0, \alpha, \beta, H/K)+\frac{1}{\pi i}\int_{(c)} z^{s-1}\Gamma(1-s)\,\Gamma(s-\alpha)\,\Gamma(s-\beta)\,(K/2\pi)^{2s-1-\alpha-\beta}$$
$$\times[\cos\pi/2(2s+\alpha+\beta)\,D(s, -\alpha, -\beta, -\bar{H}/K)$$
$$+\cos\pi/2(\alpha-\beta)\,D(s, -\alpha, -\beta, \bar{H}/K)]\,ds.$$

Proof. By Mellin's formula,

$$S=\sum_{m,n} m^{-\alpha}n^{-\beta}e(mnH/K)\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\,(-2\pi imnx)^{-s}\,ds$$

where we take $c>1$. Thus,

$$S=\frac{1}{2\pi i}\int_{(c)} D(s, \alpha, \beta, H/K)\,\Gamma(s)z^{-s}\,ds$$

where $z=-2\pi ix$. We move the path of integration to $(1-c)$ and then make the change of variable $s \longrightarrow 1-s$. Thus, by Cauchy's theorem and Lemma 4,

$$S=\zeta(1-\alpha+\beta)K^{-1+\alpha-\beta}z^{-1+\alpha}\Gamma(1-\alpha)$$
$$+\zeta(1-\beta+\alpha)K^{-1+\beta-\alpha}z^{-1+\beta}\Gamma(1-\beta)$$
$$+D(0, \alpha, \beta, H/K)+\frac{1}{\pi i}\int_{(c)} z^{s-1}\Gamma(1-s)\,\Gamma(s-\alpha)\,\Gamma(s-\beta)\,(K/2\pi)^{2s-1-\alpha-\beta}$$
$$\times[\cos\pi/2\,(2s+\alpha+\beta)\,D(s, -\alpha, -\beta, -\bar{H}/K)$$
$$+\cos\pi/2(\alpha-\beta)\,D(s, -\alpha, -\beta, \bar{H}/K)]\,ds$$

which completes the proof of the lemma.

Lemma 6. *Let w be real with $T \leqq w \leqq 2T$ and let $s_1=1/2+\beta+iw$ where $\beta=b/L$ with $b\in\mathbb{C}$, $b\ll 1$. Let $\delta>0$, $\pi/2>\lambda>0$, $\Delta=T^{1-\delta}$, and $\alpha=a/L$ with $a\in\mathbb{C}$, $a\ll 1$. Define*

$$r(s_1, a)=\int_{L_\lambda} v^{s_1}\exp(-\Delta^2(\log^2 v)/4)\,(v-1)^{-1+\alpha}\,dv/v,$$

where L_λ is the half-line $L_\lambda=\{re^{i\lambda}: r\geqq 0\}$. Then

$$r(s_1, a)=-\pi ie^{-a}+o_\delta(1)$$

as $T\longrightarrow\infty$, uniformly in a and s_1.

Proof. We change the path of integration to the positive real axis except for a small semicircular indentation into the upper half-plane centered at $v=1$. Now let $v=e^x$. Writing $w=\omega T$, $\alpha=a/L$, $\beta=b/L$, and $\Delta=T^{1-\delta}$, we have

$$r(s_1, a) = \int_{\mathscr{C}} e^{bx/L} e^{i\omega T x} \exp\left(-\frac{T^{2-2\delta}x^2}{4}\right)(e^x - 1)^{a/L} \frac{dx}{2\sinh x/2}$$

where $\mathscr{C}$ is the path consisting of the entire real axis from $-\infty$ to ∞ apart from a small semicircular indentation into the upper half plane centered at $x=0$. Now let

$$R(s_1, a) = \int_{\mathscr{C}} e^{bx/L} e^{i\omega T x} \exp\left(-\frac{T^{2-2\delta}x^2}{4}\right) x^{a/L}\, dx/x,$$

and consider $r(s_1, a) - R(s_1, a)$. Since

$$q(a, x, T) = \frac{(e^x-1)^{a/L}}{2\sinh x/2} - \frac{x^{a/L}}{x} \ll |x|^{a/L}$$

as $|x| \to 0$, it follows that

$$R(s_1, a) - r(s_1, a) = \lim_{\eta \to 0^+} \int_{-\infty}^{-\eta} + \int_{-\eta}^{\infty} e^{bx/L} e^{i\omega T x} \exp\left(-\frac{T^{2-2\delta}x^2}{4}\right) q(a, x, T)\, dx.$$

The convergence is uniform in T, ω, b, and a for $T \geqq 1$, and $a, b, \omega \ll 1$. Taking the limit as $T \to \infty$, we get 0 for the right side whence

$$r(s_1, a) = R(s_1, a) + o_\delta(1)$$

uniformly in ω, b, and a. In the integral defining R let $y = Tx$. Then

$$R(s_1, a) = e^{-a} \int_{\mathscr{C}} e^{by/TL} e^{i\omega y} \exp\left(-\frac{T^{2-2\delta}x^2}{4}\right) y^{a/L}\, dy/y.$$

Again, for fixed $\delta > 0$ the convergence is uniform in a, b, and $\omega \ll 1$. Letting $T \to \infty$ we have, by the residue theorem,

$$\lim_{T\to\infty} R(s_1, a) = e^{-a} \int_{\mathscr{C}} e^{i\omega y}\, dy/y = e^{-a} \int_{\mathscr{C}} e^{iy}\, dy/y = -\pi i e^{-a}.$$

Thus, $R(s_1, a) = -\pi i e^{-a} + o_\delta(1)$ whence the lemma follows.

Now we begin the proof of Lemma 2. First of all we move the path of integration in the definition of g to (c) where $c = 1 + \eta$ with $\eta > 0$ small and fixed. Since $\alpha, \beta \ll 1/L$ it will be the case that $|\alpha|, |\beta| < \eta$ if T is sufficiently large. Thus, in moving the path of integration we cross a pole at $s = 1 - \alpha$. The contribution from the residue is negligible since for $s \ll 1$,

$$\exp((s - s_0)^2 \Delta^{-2}) \ll \exp(T^{-2\delta}) \ll T^{-20} \tag{54}$$

because of the definition of Δ and s_0. We use the functional equation (5) on $\zeta(1 - s + \beta)$; then we interchange summation and integration and have

$$g=\sum_{h,k\leqq y}\frac{b(h,P_1)\,b(k,P_2)}{k}\sum_{m,n}m^{-\alpha}n^{\beta}J(mnh/k,s_0,\beta,\Delta)+o_\delta(1)$$

$$=\sum_{h,k\leqq y}\frac{b(h,P_1)\,b(k,P_2)}{k^{1-\beta}h^{\beta}}\sum_{m,n}m^{-\alpha-\beta}\int_0^\infty v^{s_0-\beta}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)$$

$$\times\left(e\left(\frac{mnhv}{k}\right)+e\left(-\frac{mnhv}{k}\right)\right)\frac{dv}{v}+o_\delta(1).$$

We express the integral as a sum of two integrals and use Cauchy's theorem to move one path to L_λ and the other to $L_{-\lambda}$ where $\lambda>0$ is small and L_λ is the half-line $\{re^{i\lambda}: r\geqq 0\}$. We interchange summation over m and n with the integration and have

$$g=\sum_{h,k\leqq y}\frac{b(h,P_1)\,b(k,P_2)}{k^{1-\beta}h^{\beta}}(I_1+I_2)+o_\delta(1) \tag{55}$$

where, in the notation of Lemma 5,

$$I_1=\int_{L_\lambda}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)S(h/k(v-1),\alpha+\beta,0,h/k)\frac{dv}{v} \tag{56}$$

and

$$I_2=\int_{L_{-\lambda}}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)S(-h/k(v-1),\alpha+\beta,0,h/k)\frac{dv}{v}. \tag{57}$$

Then, by Lemma 5, with $H=h/(h,k)$ and $K=k/(h,k)$,

$$I_1=M_1+R_1+E_1 \tag{58}$$

where

$$M_1=\int_{L_\lambda}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right) \tag{59}$$

$$\times\left[\zeta(1-\alpha-\beta)\,\Gamma(1-\alpha-\beta)\,K^{-1+\alpha+\beta}\left(-2\pi i\frac{h}{k}(v-1)\right)^{-1+\alpha+\beta}\right.$$

$$\left.+\zeta(1-\alpha-\beta)\,K^{-1-\alpha-\beta}(-2\pi ih/k(v-1))^{-1}\right]\frac{dv}{v},$$

$$R_1=D(0,\alpha+\beta,H/K)\int_{L_\lambda}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)\frac{dv}{v}, \tag{60}$$

and

$$E_1=\int_{L_\lambda}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)F_1(v)\frac{dv}{v} \tag{61}$$

with

$$(62)\qquad F_1(v)=\frac{1}{\pi i}\int_{(c)}(-2\pi ih/k(v-1))^{s-1}\,\Gamma(1-s)\,\Gamma(s-\alpha-\beta)\,\Gamma(s)\left(\frac{K}{2\pi}\right)^{2s-1-\alpha-\beta}$$
$$\times[\cos(\pi/2(2s+\alpha+\beta))\,D(s,\,-\alpha-\beta,0,\,-\bar{H}/K)$$
$$+\cos(\pi/2(\alpha+\beta))\,D(s,\,-\alpha-\beta,0,\,\bar{H}/K)]\,ds.$$

There are similar expressions for $I_2=M_2+R_2+E_2$.

Now in the notation of Lemma 6,

$$(63)\qquad M_1=\zeta(1-\alpha-\beta)\,\Gamma(1-\alpha-\beta)\,(-2\pi i)^{-1+\alpha+\beta}H^{-1+\alpha+\beta}r(s_1,\alpha+\beta)$$
$$+\zeta(1+\alpha+\beta)\,(-2\pi i)^{-1}H^{-1}K^{-\alpha-\beta}r(s_1,1).$$

Now $\zeta(s)\sim 1/(s-1)$ for s near 1 and $\Gamma(1)=1$. Thus, by Lemma 6,

$$(64)\qquad \sum_{h,k\leq y}\frac{b(h,P_1)\,b(k,P_2)}{k^{1-\beta}h^{\beta}}M_1$$
$$=\frac{-1}{2(\alpha+\beta)}(e^{-a-b}\Sigma(-a,\,-b,P_1,P_2)-\Sigma(b,a,P_1,P_2))\,(1+o(1)).$$

We get exactly the same expression for the sum of the M_2. Thus, from the terms with M_1 and M_2 we get the main term of g in Lemma 2.

8. The error terms

In this section we complete the proof of Lemma 2 which completes the proofs of the theorems. This section is where the important work of Deshouiller and Iwaniec on averages of Kloosterman sums enters.

It remains to bound

$$(65)\qquad \sum_{h,k\leq y}\frac{b(h,P_1)\,b(k,P_2)}{h^{\beta}k^{1-\beta}}(R_i+E_i)$$

for $i=1$ and 2. As the situation is identical for $i=2$ we deal with $i=1$ only. As far as R_i is concerned, since, by Lemma 4 of [2]

$$(66)\qquad \int_{L_\lambda}v^{s_1}\exp\left(-\Delta^2\frac{\log^2 v}{4}\right)\frac{dv}{v}=\frac{2\pi^{1/2}}{\Delta}\exp\left(\frac{s_1^2}{\Delta^2}\right)\ll\exp(T^{-2\delta})$$

it follows from Lemma 4 that

$$R_1\ll T^{-20}$$

whence the contribution to (65) from R_1 is $\ll T^{-10}$.

Now by (61) and (62), the part of (65) which involves E_i may be written as a sum of two terms, one of which is

$$(67)\qquad Z = \int_{L_\lambda} \int_{(c)} G(\alpha+\beta, v, s_1, \Delta, s)\, \mathcal{M}(\alpha, \beta, s)\, ds\, dv$$

where

$$(68)\qquad G(\alpha, v, s_1, \Delta, s) = -i v^{s_1} \exp\left(-\Delta^2 \frac{\log^2 v}{4}\right) \Gamma(s)\, \Gamma(1-s)\, \Gamma(s-\alpha)\, (2\pi)^{\alpha-s}$$
$$\times \cos(\pi/2(2s+\alpha))\, e^{-\pi i s/2} (v-1)^{s-1} v^{-1}$$

and

$$(69)\qquad \mathcal{M}(\alpha, \beta, s) = \sum_{m,n} m^{\alpha+\beta-s} n^{-s} \sum_{h,k \leqq y} \frac{b(h, P_1)\, b(k, P_2)}{h^{1-s+\beta} k^{1-s+\alpha}} (h, k)^{1-2s+\alpha+\beta}\, e\left(\frac{mn\bar{H}}{K}\right)$$

where $H = h/(h, k)$ and $K = k/(h, k)$. The other term is slightly less complicated and may be treated the same way as this one will be. Replacing mn by n and arranging the sums over h and k according to the g.c.d. of h and k we get

$$(70)\qquad \mathcal{M}(\alpha, \beta, s) = \sum_{g \leqq y} 1/g \sum_{N, U, V} \mathcal{M}(N, U, V, \alpha, \beta, g, s)$$

where

$$(71)\qquad \mathcal{M}(N, U, V, \alpha, \beta, g, s) = \sum_{n \sim N} \frac{\delta(n)}{n^s} \sum_{u \sim U} \sum_{\substack{v \sim V \\ (u,v)=1}} \frac{b(ug, P_1)\, b(vg, P_2)}{u^{1-s+\beta} v^{1-s+\alpha}}\, e\left(\frac{n\bar{u}}{v}\right)$$

with

$$(72)\qquad \delta(n) = \sum_{d \mid n} d^{\alpha+\beta}$$

and where the notation $x \sim X$ means $X < x \leqq 2X$, and the sums on U and V have $\ll \log y$ terms with $U, V \ll y/g$ and the sum on N is for $N = 2^J$, $J = 0, 1, 2, \ldots$. Now Z is a sum of terms of the shape

$$(73)\qquad Z(N, U, V) = \int_{L_\lambda} \int_{(c)} G(\alpha+\beta, v, s_1, \Delta, s)\, \mathcal{M}(N, U, V, \alpha, \beta, g, s)\, ds\, dv.$$

If $UV \geqq TN$, then we move the s path of integration to $s = \eta + it$; otherwise we leave it at $s = c + it = 1 + \eta + it$. (Recall that $\eta > 0$ is fixed, to be chosen at the end of the proof in terms of ε.) In moving the path of integration, we cross a pole at $s = 1$ with residue

$$(74)\qquad \Gamma(1-\alpha)\, (2\pi)^{\alpha-1} \cos(\pi/2(2+\alpha))\, \mathcal{M}(N, U, V, \alpha, \beta, g, 1)$$
$$\int_{L_\lambda} v^{s_1} \exp\left(-\Delta^2 \frac{\log^2 v}{4}\right) \frac{dv}{v} \ll_\delta T^{-10}$$

by (66). To complete the proof we require two lemmas.

Lemma 7. *Let G be as in* (68) *with the usual conventions about s_1, Δ, and α. Suppose that $c=\eta$ or $c=1+\eta$ where $\eta>0$ is a small fixed number. Let $\lambda=1/T$. Then*

$$\int_{L_\lambda}\int_{(c)}(1+|s|)\,|G(\alpha,v,s_1,\Delta,s)\,ds\,dv|\ll_{\varepsilon,\eta}\Delta^{-c-5/2}T^{5/2+\eta+\varepsilon}$$

for any $\varepsilon>0$, uniformly in α and s_1.

The proof is exactly the same as that of Lemma 5 of [2].

Lemma 8. *Let $\mathcal{M}(N,U,V,\alpha,\beta,g,s)$ be as in* (71). *Suppose that $y\leqq T^{8/13}$; $1\leqq U$, $V\leqq y$; $\eta>0$ and $s=c+it$ with $c=\eta$ if $UV\geqq TN$ and $c=1+\eta$ if $UV<TN$. Then*

$$\mathcal{M}(N,U,V,\alpha,\beta,g,s)\ll_{\varepsilon,\eta}(1+|s|)\,(TN)^{\varepsilon}y^{2\eta}T^{c}N^{-\eta}(T^{-1/2}y^{7/8}+T^{-1}y^{7/4})$$

uniformly for $a,b\ll 1$, all t, and all $g\ll y/V$.

Before giving the proof of Lemma 8 we complete the proof of Lemma 2. We have by Lemmas 7 and 8

$$Z=\sum_{g\leqq y}\frac{1}{g}\sum_{N,U,V}Z(N,U,V) \tag{75}$$
$$\ll_{\varepsilon,\eta}\Delta^{-7/2}T^{5/2+2\varepsilon}y^{2\eta+\varepsilon}(T^{1/2}y^{7/8}+y^{7/4})\sum_{N,U,V}N^{\varepsilon-\eta}+\sum_{\substack{N,U,V\\ NT\leqq UV}}T^{-1}$$
$$\ll_{\varepsilon}\Delta^{-7/2}T^{5/2+2\varepsilon}y^{3\varepsilon}(T^{1/2}y^{7/8}+y^{7/4})$$

on taking $\eta=\varepsilon/2$. Since $\theta<7/4$, this is $o_\varepsilon(1)$ as $T\to\infty$ if δ and ε are sufficiently small.

Proof of Lemma 8. Initially we use the fact that the variable g can be separated from u and from v since, for example,

$$P\left(\frac{\log y/(ug)}{\log y}\right)$$

is a sum of $\ll 1$ terms of the shape a constant times

$$\left(1-\frac{\log u+\log g}{\log y}\right)^{k}$$

which is itself a sum of $\ll 1$ terms of the shape

$$\left(\frac{\log u}{\log y}\right)^{k_1}\left(\frac{\log g}{\log y}\right)^{k_2}.$$

Also, $\mu(ug)=\mu(u)\,\mu(g)$ if $(u,g)=1$ and $\mu(ug)=0$ if $(u,g)>1$. Thus, $\mathscr{M}(N, U, V, \alpha, \beta, g, s)$ is a sum of $\ll 1$ terms that are themselves

$$\ll N^{-c}(UV)^{1-c}\,|S|$$

where

$$S=\sum_{n\sim N} r(n) \sum_{\substack{u\sim U\\(u,vg)=1}} \sum_{v\sim V} \mu(u)\, r^*(u)\, r(v)\, e\left(\frac{n\bar{u}}{v}\right). \tag{76}$$

Here the functions r may be different at each occurrence; but they all may be described as follows: $r(\ )$ depends on its argument as well as $g, s, \alpha, \beta, N, U$, and V and $r(n)\ll_\varepsilon n^\varepsilon$ for any $\varepsilon>0$ uniformly in $g, s, \alpha, \beta, N, U$, and V. In addition, r^* is an r function which is smooth in its dependency on u, satisfying

$$\frac{d}{du} r^*(u) \ll (1+|s|)\, u^{-1} r(u) \tag{77}$$

for some $r(u)$, and having the property of separability, i.e.,

$$r^*(ab)=r^*(a)\, r^*(b) \tag{78}$$

where the r^*'s here are not necessarily the same at each occurrence.

We now use Vaughan's identity to get a new expression for $\mu(n)$; equating coefficients on both sides of the identity $1/\zeta = 1/\zeta(1-\zeta M)^2 + 2M - \zeta M^2$ where

$$M=M(s)=\sum_{n\leqq W} \mu(n) n^{-s}, \quad W=U^{1/4} \tag{79}$$

we find that

$$\mu(u)=c_1(u)+c_2(u)+c_3(u) \tag{80}$$

where

$$c_1(u)=\sum_{\substack{\alpha\beta\gamma=u\\ \alpha\geqq W,\, \beta\geqq W}} \mu(\gamma)\, c_4(\alpha)\, c_4(\beta) \tag{81}$$

with

$$c_4(\alpha)=-\sum_{\substack{d_1 d_2=\alpha\\ d_1\leqq W}} \mu(d_1); \tag{82}$$

$$c_2(u)=\begin{cases} 2\mu(u) & \text{if } u\leqq W,\\ 0 & \text{if } u>W; \end{cases} \tag{83}$$

$$c_3(u)=-\sum_{\substack{\alpha\beta\gamma=u\\ \alpha\leqq W\\ \beta\leqq W}} \mu(\alpha)\,\mu(\beta). \tag{84}$$

This leads to $S = S_1 + S_2 + S_3$ where

(85) $$S_i = \sum_{n \sim N} r(n) \sum_{\substack{u \sim U \\ (u, vg) = 1}} \sum_{v \sim V} c_i(u)\, r^*(u)\, r(v)\, e\left(\frac{n\bar{u}}{v}\right)$$

for $i = 1$, 2, and 3. We treat each of these sums in a slightly different way. We note also that it suffices to show that for any $\varepsilon > 0$,

(86) $$S_i \ll_\varepsilon (yN)^\varepsilon \max(TN, UV)\, y^{7/8}\, T^{-1/2};$$

the lemma then follows with a different value of ε.

We start with S_2 which is trivially estimated by

(87) $$\begin{aligned} S_2 &\ll_\varepsilon (yN)^\varepsilon NWV \ll_\varepsilon (yN)^\varepsilon (TN)\, T^{-1} U^{1/4} V \\ &\ll_\varepsilon (yN)^\varepsilon (TN)\, T^{-1} y^{5/4}. \end{aligned}$$

Thus S_2 satisfies (86) since $y \leqq T^{4/3}$. Next we consider S_1. Grouping together γ and the larger of α and β in (81) into a variable b and calling the other variable a, we see that S_1 can be split into $\ll_\varepsilon y^\varepsilon$ sums of the shape

(88) $$S_1' = \sum_{n \sim N} r(n) \sum_{\substack{a \sim A \\ b \sim B \\ (ab, v) = 1}} \sum_{v \sim V} r(a)\, r(b)\, r(v)\, e\left(\frac{n\bar{a}\bar{b}}{v}\right)$$

where $U \ll AB \ll U$ and $W \leqq A \leqq B$. Now we have the following lemma, which is a case of Lemma 1 of Deshouillers and Iwaniec [7].

Lemma 9. *Suppose that $|c(a, n)| \leqq 1$ and $U \ll AB \ll U$. Then for any $\varepsilon > 0$*

$$\sum_{\substack{v \sim V \\ (b, v) = 1}} \sum_{b \sim B} \left| \sum_{n \sim N} \sum_{\substack{a \sim A \\ (a, v) = 1}} c(a, n)\, e\left(\frac{n\bar{a}\bar{b}}{v}\right) \right|$$

$$\ll_\varepsilon (NUV)^{1/2+\varepsilon} \{(UVA^{-1})^{1/2} + (A+N)^{1/4} [UVA^{-1}(N+A)(V+A^2) + NU^2]^{1/4}\}.$$

Using the fact that $x^a + y^a \ll (x+y)^a \ll x^a + y^a$ for $a \geqq 0$ and $x, y \geqq 1$ it is not hard to see that the right hand side of the relation in the proposition is

(89) $$\ll_\varepsilon (Ny)^\varepsilon \Big(\sum_{(a, n, u, v) \in E} A^a N^n U^u V^v \Big)^{1/4}$$

for any $\varepsilon > 0$ where

(90) $$\begin{aligned} E = \{&(-2, 2, 4, 4), (-1, 4, 3, 4), (1, 4, 3, 3), (1, 2, 3, 4), \\ &(3, 2, 3, 3), (0, 4, 4, 2), (1, 3, 4, 2)\}. \end{aligned}$$

Clearly, the fact that our coefficients $r(n)$ satisfy

$$r(n) \ll_\varepsilon n^\varepsilon$$

for any $\varepsilon > 0$ does not affect the use of the bound (89) for S_2'. We now show how to bound $A^a N^n U^u V^v$ for $(a, n, u, v) \in E$, and $U^{1/4} \ll A \ll U^{1/2}$.

We have two cases to deal with: $a \geqq 0$ and $a < 0$. If $a \geqq 0$ then

$$(91) \qquad \begin{aligned} A^a N^n U^u V^v &\ll (TN)^n (UV)^{4-n} T^{-n} U^{a/2+n+u-4} V^{n+v-4} \\ &\ll (\max\{TN, UV\})^4 T^{-n} y^{2n+u+v-8+a/2} \end{aligned}$$

since for all $(a, n, u, v) \in E$, $n+u \geqq 4$ and $n+v \geqq 4$ so that we may use $U, V \ll y$. Now we have to show that

$$(92) \qquad T^{-n} y^{2n+u+v-8+a/2} \ll T^{-2} y^{7/2}$$

for all $(a, n, u, v) \in E$ with $a \geqq 0$. The terms we get for the left side are

$$T^{-4} y^{13/2}, \quad T^{-2} y^{7/2}, \quad T^{-2} y^{7/2}, \quad T^{-4} y^6, \quad \text{and} \quad T^{-3} y^{9/2}.$$

Clearly (92) holds since $y \ll T^{2/3}$. For $a < 0$ we use

$$(93) \qquad \begin{aligned} A^a N^n U^u V^v &\ll (TN)^n (UV)^{4-n} T^{-n} U^{a/4+n+u-4} V^{n+v-4} \\ &\ll (\max\{TN, UV\})^4 T^{-n} y^{2n+u+v-8+a/4} \end{aligned}$$

since for all $(a, n, u, v) \in E$ with $a < 0$ we have $a/4 + n + u - 4 \geqq 0$. Now we have to show that

$$(94) \qquad T^{-n} y^{2n+u+v-8+a/4} \ll T^{-2} y^{7/2}$$

for both $(a, n, u, v) \in E$ with $a < 0$. The terms we get for the left side are

$$(95) \qquad T^{-2} y^{7/2} \quad \text{and} \quad T^{-4} y^{27/4}$$

and so (94) holds since $y \ll T^{8/13}$. It follows now that

$$(96) \qquad S_1' \ll_\varepsilon \max(TN, UV)\, (yN)^\varepsilon T^{-1/2} y^{7/8}.$$

Finally, we consider S_3. Grouping together α and β into a variable a and replacing γ by b we see that S_3 can be split into $\ll_\varepsilon y^\varepsilon$ sums of the shape

$$(97) \qquad S_3' = \sum_{v \sim V} r(v) \sum_{\substack{b \sim B \\ (b, vg) = 1}} r^*(b) \sum_{n \sim N} r(n) \sum_{\substack{a \sim A \\ (a, v) = 1}} r(a)\, e\left(\frac{n\bar{a}\bar{b}}{v}\right)$$

where $U \ll AB \ll U$ and $A \ll W^2 = U^{1/2}$. Now if $A \gg U^{1/4}$, then the treatment is exactly as with S_2 above using Lemma 9. If $A \ll U^{1/4}$, then we sum over b first using Weil's bound for the Kloosterman sum. Thus, Weil's bound implies that

$$(98) \qquad \sum_{\substack{B \leq b < B+x \\ (b, vg)=1}} e\left(\frac{l\bar{b}}{v}\right) \ll_\varepsilon v^{1/2} (vg)^\varepsilon (l, v) (1 + Bv^{-1})$$

so that by a summation by parts (using the bound in (77) for $\frac{d}{dx} r^*(x)$) we get

$$(99) \qquad \begin{aligned} S_3' &\ll_\varepsilon (1+|s|)(yN)^\varepsilon AV^{1/2}(1+BV^{-1}) \sum_{n \sim N} \sum_{v \sim V} (n, v) \\ &\ll_\varepsilon (1+|s|)(yN)^\varepsilon ANV^{1/2}(V+B) \\ &\ll_\varepsilon (1+|s|)(yN)^\varepsilon (ANV^{3/2} + UNV^{1/2}). \end{aligned}$$

If $A \ll U^{1/4}$, then

$$\begin{aligned} ANV^{3/2} + UNV^{1/2} &\ll Ny^{7/4} \\ &\ll \max\{TN, UV\} T^{-1} y^{7/4}. \end{aligned}$$

Thus, in any event

$$S_3 \ll_\varepsilon (1+|s|) \max\{TN, UV\} (yN)^\varepsilon (T^{-1} y^{7/4} + T^{-1/2} y^{7/8}).$$

This completes the proof of the lemma and the theorems.

Note added in proof. We can improve Theorem 1 slightly. With $R = 1.28$ and

$$\begin{aligned} Q(x) = 0.492 + 0.602\,(1-2x) - 0.08\,(1-2x)^3 \\ - 0.06\,(1-2x)^5 + 0.046\,(1-2x)^7 \end{aligned}$$

we have $\kappa \geqq 0.4088$.

References

[1] *R. J. Anderson*, Simple zeros of the Riemann zeta-function, J. Number Theory **17** (1983), 176—182.
[2] *R. Balasubramanian, J. B. Conrey* and *D. R. Heath-Brown*, Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial, J. reine angew. Math. **357** (1985), 161—181.
[3] *J. B. Conrey*, Zeros of derivatives of Riemann's xi-function on the critical line, J. Number Theory **16** (1983), 49—74.
[4] *J. B. Conrey*, Zeros of derivatives of Riemann's xi-function on the critical line, II, J. Number Theory **17** (1983), 71—75.
[5] *J. B. Conrey*, On the distribution of the zeros of the Riemann zeta-function, Topics in Analytic Number Theory, Austin 1985.
[6] *J.-M. Deshouillers* and *H. Iwaniec*, Kloosterman sums and Fourier coefficients of cusp forms, Invent. Math. **70** (1982), 219—288.

[7] *J.-M. Deshouillers* and *H. Iwaniec*, Power mean values of the Riemann zeta function II, Acta Arith. **48** (1984), 305—312.
[8] *T. Estermann*, On the representation of a number as the sum of two products, P.L.M.S. (2) **31** (1930), 123—133.
[9] *G. H. Hardy* and *J. E. Littlewood*, The zeros of Riemann's zeta-function on the critical line, M.Z. **10** (1921), 283—317.
[10] *D. R. Heath-Brown*, Simple zeros of the Riemann zeta-function on the critical line, B.L.M.S. **11** (1979), 17—18.
[11] *N. Levinson*, More than one third of the zeros of Riemann's zeta-function are on $\sigma = 1/2$, Adv. Math. **13** (1974), 383—436.
[12] *N. Levinson*, Deduction of semi-optional mollifier for obtaining lower bounds for $N_0(T)$ for Riemann's zeta function, Proc. Nat. Acad. Sci. USA **72** (1975), 294—297.
[13] *B. Riemann*, Über die Anzahl der Primzahlen unter eine gegebener Grösse, Monatsber. Akad. Berlin (1859), 671—680.
[14] *A. Selberg*, On the zeros of Riemann's zeta-function, Skr. Norske Vid. Akad. Oslo **10** (1942), 1—59.
[15] *E. C. Titchmarsh*, The Theory of the Riemann Zeta-Function, Second Ed., Oxford 1986.

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