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An Extension of Hecke's Converse Theorem

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1 Introduction and statement of results

By a "converse theorem," we mean a uniqueness and existence statement about a class of Dirichlet series. A typical converse theorem asserts that the only Dirichlet series with a given list of properties are among those which have already been discovered. The first converse theorem, proven by Hamburger [H] in 1922, states that the Riemann ζ-function is characterized by its functional equation.

Hecke showed that the L-functions associated with holomorphic modular forms of even integral weight for the full modular group satisfy certain functional equations, and conversely, the only Dirichlet series satisfying these functional equations are L-functions associated with modular forms. This is the source of the term "converse theorem." We give some notation and then describe Hecke's result.

Throughout the paper, we let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

We assume that $F(s)$ converges in some right half-plane and continues to an entire function such that $\Gamma(s)F(s)$ is entire and bounded in vertical strips. This condition is denoted EBV. We say that F satisfies a functional equation of degree 2, level N, and weight k, if

$$\begin{aligned}\Phi(s) &= \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)F(s) \\ &= \pm(-1)^{k/2}\Phi(k-s).\end{aligned}$$

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If g is a function on the complex upper half-plane $\mathcal{H} = \{z \in \mathbb{C} : y > 0\}$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{R})^+$, then we define the function $g|_k\gamma$ by

$$\left(g|_k\gamma\right)(z) = (\det\gamma)^{k/2}(cz+d)^{-k} g\left(\frac{az+b}{cz+d}\right).$$

The Hecke congruence group of level N is defined by

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N|c\right\}.$$

A function $g : \mathcal{H} \to \mathbb{C}$ is called a cusp form of weight k for $\Gamma_0(N)$ if $g|_k\gamma = g$ for all $\gamma \in \Gamma_0(N)$, and g vanishes at all cusps of $\Gamma_0(N)$. The space of cusp forms of weight k is denoted $S_k(\Gamma_0(N))$.

Associated to the Dirichlet series $F(s)$ is a function on $\mathcal{H}$,

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz).$$

Hecke's Converse Theorem relates properties of $F(s)$ to properties of $f(z)$.

Hecke's Converse Theorem. Suppose $N = 1, 2, 3$, or 4. If $F(s)$ is EBV and satisfies a functional equation of degree 2, level N, and weight k, then $f|_k\gamma = f$ for all $\gamma \in \Gamma_0(N)$. □

In fact, it holds that $f \in S_k(\Gamma_0(N))$, but for now we are only concerned with the transformation properties of $f(z)$.

Proof. By the Mellin inversion formula, if $y > 0$,

$$f(iy) = \frac{1}{2\pi i}\int_{(c)} N^{-s/2}\Phi(s)y^{-s}\, ds$$
$$= \frac{1}{2\pi i}\int_{(c)} N^{(s-k)/2}\Phi(k-s)y^{s-k}\, ds,$$

for $c > 0$. The functional equation $\Phi(s) = \pm(-1)^{k/2}\Phi(k-s)$ gives the transformation rule $f(iy) = \pm N^{k/2}(iNy)^{-k} f(-1/iNy)$. Since f is holomorphic, this holds for y with positive real part. In other words, $f|_k H_N = \pm f$, where

$$H_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}.$$

Since $f(z) = f(z+1)$, f is invariant under

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \quad \text{and} \quad H_N \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} H_N^{-1} = \begin{pmatrix} 1 & \\ N & 1 \end{pmatrix}.$$

If $N = 1, 2, 3$, or 4, those two matrices generate $\Gamma_0(N)$, proving the theorem. ■

If $N \geq 5$, then the above argument fails, and in fact the space of functions satisfying the given conditions is infinite-dimensional. In order to get the desired conclusion that f is invariant under $\Gamma_0(N)$, we must put further restrictions on F. Weil [W] conceived of the important idea of requiring that the twists of F by Dirichlet characters satisfy an appropriate functional equation. Later versions by Razar [Raz] and Li [Li] reduced the number of twists to a finite number depending on N. All subsequent converse theorems for higher-rank groups [JP-SS], [P-S] were built on the idea of requiring a functional equation for F and also for various twists of F.

In this paper, we have partial success at replacing the assumption on twists of F by the assumption of F having an Euler product of the appropriate form. We say that F has an Euler product of degree 2, level N, and weight k, if

$$F(s) = \prod_{p \text{ prime}} F_p(s),$$

where

$$F_p(s) = (1 - a_p p^{-s} + p^{k-1-2s})^{-1} \qquad \text{if } p \nmid N,$$

$$F_q(s) = (1 - q^{k/2-1-s})^{-1} \qquad \text{if } q \| N,$$

$$F_q(s) = 1 \qquad \text{if } q^2 | N.$$

The following theorem is our result.

Theorem 1. Let $5 \leq N \leq 12$, or $14 \leq N \leq 17$, or $N = 23$, and suppose $F(s)$ is EBV and has both a functional equation and an Euler product of degree 2, level N, and weight k. Then $f|_k\gamma = f$ for all $\gamma \in \Gamma_0(N)$. □

The proof only makes use of the Euler product at a finite number of places, depending on N.

Corollary 1. Under the conditions of Theorem 1, $f \in S_k(\Gamma_0(N))$. □

The deduction that f vanishes at the cusps of $\Gamma_0(N)$, that is, $f \in S_k(\Gamma_0(N))$, is described in Section 5.

This work is motivated by the Selberg class of Dirichlet series [Se]. One is led to believe that admission to this class is reserved for very special L-functions, and probably all are associated with automorphic forms. For instance, Conrey and Ghosh [CG] show that the only elements of the Selberg class of degree 1 are the Riemann ζ-function and the Dirichlet L-functions associated with primitive Dirichlet characters. One would like to show that the only elements of the Selberg class of degree 2 are the L-functions associated to cusp forms, both holomorphic and nonholomorphic, which are eigenvalues of the Hecke operators T_p for $p \nmid N$ and of the Atkin-Lehner operators U_q for $q \mid N$. This would require using the Euler product condition of the Selberg axioms in place of the twists required by Weil's theorem. Our result makes use of the Euler product, but requires it to be of a special form.

Our results are stated in the case where $F(s)$ "looks like" the Dirichlet series associated to a holomorphic cusp form. All of the methods work equally well in the case that $F(s)$ looks like the Dirichlet series associated to a GL_2 Maass form. The only extra step is verifying that the conclusion of Lemma 5 holds when $f(z)$ is an eigenfunction of the hyperbolic Laplacian. A proof of this is given by Böckle [Bo].

The paper is organized as follows. In Section 2 we provide more background information and then derive results based on the shape of the local factor of $F(s)$ at $p = 2$. In Section 3 we present some additional general methods. In Section 4 we present ad-hoc methods for which we have not found an appropriate generalization. In Section 5 we use the shape of the Euler product of $F(s)$ to show that $f(z)$ vanishes at the cusps of $\Gamma_0(N)$.

2 The local factor at p = 2

Recall that for p prime, the Hecke operator T_p is defined by

$$T_p = \begin{pmatrix} p & \\ & 1 \end{pmatrix} + \sum_{a=0}^{p-1} \begin{pmatrix} 1 & a \\ & p \end{pmatrix},$$

and for q prime the Atkin-Lehner operator U_q is defined by

$$U_q = \sum_{a=0}^{q-1} \begin{pmatrix} q & a \\ & q \end{pmatrix}.$$

We also put

$$H_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \qquad W_N = \begin{pmatrix} 1 & \\ N & 1 \end{pmatrix}.$$

If a cusp form of weight k on $\Gamma_0(N)$ is an eigenfunction of H_N, then its associated L-function will have a functional equation of degree 2, level N, and weight k. The reverse implication is also true, as we saw in the proof of Hecke's theorem. If the cusp form is an eigenfunction of each T_p for $p \nmid N$ and each U_q for $q|N$, then the L-function will have an Euler product of degree 2, level N, and weight k. It is easy to see that the reverse holds also.

Lemma 1. If $F(s)$ has a functional equation of degree 2, level N, and weight k, then $f|_k P = f$ and $f|_k H_N = \pm f$, and so $f|_k W_N = f$. If $F(s)$ has an Euler product of degree 2, level N, and weight k, then

$$f|_k T_p = a_p f \qquad \text{if } p \nmid N,$$

$$f|_k U_q = f|_k \begin{pmatrix} q & \\ & 1 \end{pmatrix} \qquad \text{if } q \| N,$$

$$f|_k U_q = 0 \qquad \text{if } q^2 | N. \qquad \square$$

All of the information about the Dirichlet series $F(s)$ has been translated to equivalent information about the function $f(z)$. We will use this to deduce that $f(z)$ is invariant under $\Gamma_0(N)$.

It is convenient to introduce

$$\Omega_f = \{\omega \in \mathbb{C}[GL_2(\mathbb{R})^+] \ : \ f|_k \omega = 0\}.$$

Note that Ω_f is a right ideal in the group ring $\mathbb{C}[GL_2(\mathbb{R})^+]$. The goal of showing that $f(z)$ is invariant under $\Gamma_0(N)$ can be rewritten as showing $\gamma \equiv 1 \bmod \Omega_f$ for all $\gamma \in \Gamma_0(N)$, or equivalently, for a set of γ which generate $\Gamma_0(N)$. From now on, all congruences are assumed to be mod Ω_f.

For each value of N mentioned in Theorem 1, we will exhibit a set of matrices which generate $\Gamma_0(N)$. These were found by using the generators of $\Gamma(1)$ and the coset representatives of $\Gamma_0(N)$ in $\Gamma(1)$ to find a (large) generating set which was then reduced down to a manageable size. Generators for $\Gamma_0(N)$ are also given by Chuman [Ch]; see also the preprint by Ingle, Moore, and Wichert [IMW].

The next three lemmas demonstrate how to use the shape of the Euler product at the prime $p = 2$ to produce additional matrices for which $f(z)$ is invariant. The lemmas naturally correspond to the three cases $2 \nmid N$, $2 \| N$, and $4 | N$.

A useful calculation that will be used repeatedly is

$$H_N \begin{pmatrix} a & b \\ cN & d \end{pmatrix} H_N \equiv \begin{pmatrix} d & -c \\ -bN & a \end{pmatrix}.$$

Let

$$M_2 = \begin{pmatrix} 2 & 1 \\ N & (N+1)/2 \end{pmatrix}.$$

Lemma 2. If $P \equiv 1$, $H_N \equiv \pm 1$, and $T_2 \equiv a_2$ for some $a_2 \in \mathbb{C}$, then $M_2 \equiv 1$. $\square$

Proof. We are given

$$\begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix} \equiv a_2.$$

Left-multiplying and right-multiplying by H_N gives

$$\begin{pmatrix} 1 & \\ & 2 \end{pmatrix} + \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 2 & \\ N & 1 \end{pmatrix} \equiv a_2 .$$

This new congruence is valid because $H_N \equiv \pm 1$, and Ω_f is a right ideal. Subtract the two congruences to get

$$\begin{pmatrix} 2 & \\ N & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix} .$$

Right-multiply by $\left(\begin{smallmatrix} 2 & 1 \\ & 1 \end{smallmatrix}\right)$ and use $P \equiv 1$ to get $M_2 \equiv 1$. ■

Proof of Theorem 1 for $N = 5$, 7, and 9. For those values of N, the group $\Gamma_0(N)$ has generators

$$\Gamma_0(N) = \langle P,\ W_N,\ M_2 \rangle .$$

By Lemmas 1 and 2, $f(z)$ is invariant under each of those matrices. ■

Lemma 3. If $H_N \equiv \pm 1$ and $U_2 \equiv \left(\begin{smallmatrix} 2 & \\ & 1 \end{smallmatrix}\right)$, then

$$\begin{pmatrix} -2 & 1 \\ N & -(N+2)/2 \end{pmatrix} \equiv -1 .$$ □

Proof. We are given

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} \equiv \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} .$$

Multiply on the right by H_{2N} and use the relation

$$\begin{aligned} H_{2N} &= H_N \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} \\ &\equiv \pm \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} , \end{aligned}$$

to get

$$\pm \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} H_{2N} \equiv \pm \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} .$$

Combine the two congruences to get

$$\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} H_{2N} \equiv \mp \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix}.$$

Right-multiply by $\left(\begin{smallmatrix} 2 & 1 \\ & 2 \end{smallmatrix}\right)^{-1}$ and left-multiply by H_N to get the stated relation. ■

Proof of Theorem 1 for $N = 6$ and 10. Let A denote the matrix in Lemma 3, so $A \equiv -1$. We have the following lists of generators:

$$\Gamma_0(6) = \left\langle P,\ W_6,\ A^{-1}W_6A \right\rangle,$$

$$\Gamma_0(10) = \left\langle P,\ W_{10},\ (W_{10}A)^2,\ H_{10}(W_{10}A)^2H_{10},\ A^{-1}W_{10}^{-1}AP^{-1} \right\rangle.$$

By Lemmas 1 and 3, $f(z)$ is invariant under those matrices. ■

Lemma 4. If $U_2 \equiv 0$, then

$$\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} \equiv -1. \qquad \square$$

Proof. Trivial. ■

Proof of Theorem 1 for $N = 8$, 12, and 16. Let B denote the matrix in Lemma 4, so that $B \equiv -1$. We have the following lists of generators:

$$\Gamma_0(8) = \left\langle P,\ W_8,\ B^{-1}W_8B \right\rangle,$$

$$\Gamma_0(12) = \left\langle P,\ W_{12},\ BW_{12}^{-1}B,\ H_{12}B^{-1}W_{12}B^{-1}H_{12},\ BH_{12}BW_{12}^{-1}BH_{12}B \right\rangle,$$

$$\Gamma_0(16) = \left\langle P,\ W_{16},\ BW_{16}^{-1}B,\ (BH_{16})^4,\ (B^{-1}H_{16})^4 \right\rangle.$$

By Lemmas 1 and 4, $f(z)$ is invariant under each of those matrices. ■

We have seen that for each N, the local factor of $F(s)$ at $p = 2$ can be used to deduce invariance properties of $f(z)$ which are not obtainable from Hecke's method. For certain small N, this is sufficient to deduce the invariance of $f(z)$ under all of $\Gamma_0(N)$. In the next two sections we make use of the local factors at other primes to deduce further invariance properties of $f(z)$.

3 General methods

We begin with a generalization of Lemma 2. Let

$$R_n = \sum_{1 \le a \le n}{}' \begin{pmatrix} n & a \\ 0 & n \end{pmatrix},$$

where $\sum'$ means that the sum is over $(a,n)=1$. Note that

$$T_p\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = R_p + \begin{pmatrix} p^2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

Theorem 2. If $P \equiv 1$, and for each $p \mid n$ we have $T_p \equiv \alpha$ for some $\alpha \in \mathbb{C}$, then $H_N R_n H_{n^2N} \equiv R_n$. □

Proof. First consider the case where $n = p^\lambda$. Using the fundamental identity $T_{p^\lambda} T_p \equiv T_{p^{\lambda+1}} + p^{k-1} T_{p^{\lambda-1}}$, it is not difficult to prove by induction that

$$\begin{aligned} T_{p^\lambda} &= \sum_{j=0}^{\lambda} \sum_{b=0}^{p^j-1} \begin{pmatrix} p^{\lambda-j} & b \\ 0 & p^j \end{pmatrix} \\ &\equiv \alpha_\lambda, \end{aligned}$$

for some $\alpha_\lambda \in \mathbb{C}$. Multiplying $T_{p^{\lambda-1}} \equiv \alpha_{\lambda-1}$ by $\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$ gives

$$\sum_{j=0}^{\lambda-1} \sum_{b=0}^{p^j-1} \begin{pmatrix} p^{\lambda-j} & b \\ 0 & p^j \end{pmatrix} \equiv \alpha_{\lambda-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix}.$$

Subtracting this relation from $T_{p^\lambda} \equiv \alpha_\lambda$, we obtain

$$T_{p^\lambda} - T_{p^{\lambda-1}} \begin{pmatrix} p & \\ & 1 \end{pmatrix} = \sum_{b=0}^{p^{\lambda-1}} \begin{pmatrix} 1 & b \\ & p^\lambda \end{pmatrix} \equiv \alpha_\lambda - \alpha_{\lambda-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix}.$$

Multiplying by $\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$ gives

$$\sum_{b=0}^{p^\lambda-1} \begin{pmatrix} p^\lambda & b \\ & p^\lambda \end{pmatrix} \equiv \alpha_\lambda \begin{pmatrix} p^\lambda & \\ & 1 \end{pmatrix} - \alpha_{\lambda-1} \begin{pmatrix} p^{\lambda+1} & \\ & 1 \end{pmatrix}.$$

Now,

$$R_{p^\lambda} = \sum_{b=0}^{p^\lambda-1} \begin{pmatrix} p^\lambda & b \\ & p^\lambda \end{pmatrix} - \sum_{b=0}^{p^{\lambda-1}-1} \begin{pmatrix} p^{\lambda-1} & b \\ & p^{\lambda-1} \end{pmatrix}.$$

Therefore, by use of the previous relation twice,

$$R_{p^\lambda} \equiv \alpha_\lambda \begin{pmatrix} p^\lambda & \\ & 1 \end{pmatrix} - \alpha_{\lambda-1} \begin{pmatrix} p^{\lambda+1} & \\ & 1 \end{pmatrix} - \alpha_{\lambda-1} \begin{pmatrix} p^{\lambda-1} & \\ & 1 \end{pmatrix} + \alpha_{\lambda-2} \begin{pmatrix} p^\lambda & \\ & 1 \end{pmatrix}.$$

Now,

$$H_N \begin{pmatrix} x & \\ & y \end{pmatrix} H_{Np^{2\lambda}} \equiv \begin{pmatrix} p^{2\lambda} y & \\ & x \end{pmatrix}.$$

From this, it is easy to see that the right side of the above is invariant under left-multiplication by H_N and right-multiplication by $H_{Np^{2\lambda}}$. This concludes the proof in the case that n is a prime power.

To handle the general case, just note that if $(m_1, m_2) = 1$, then

$$R_{m_1 m_2} \equiv R_{m_1} R_{m_2},$$

and

$$\begin{pmatrix} m_1 & \\ & 1 \end{pmatrix} R_{m_2} \equiv R_{m_2} \begin{pmatrix} m_1 & \\ & 1 \end{pmatrix}.$$

■

Now we put Theorem 2 into a more usable form. Let

$$\beta(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

Corollary 2. If $P \equiv 1$, $H_N \equiv \pm 1$, and for each $p \mid n$, $T_p \equiv \alpha$ for some $\alpha \in \mathbb{C}$, then

$$\sum_{b(m)}{}' \left(1 - \begin{pmatrix} m & -b \\ -Nc & n \end{pmatrix} \right) \beta\left(\frac{b}{m}\right) \equiv 0,$$

where the sum is over a set of reduced residues b modulo m, and where c and n are integers depending on m, b, and N such that $mn - bcN = 1$. □

Proof. Let

$$\gamma(b, c) = \begin{pmatrix} m & -b \\ -Nc & n \end{pmatrix} \in \Gamma_0(N).$$

It is an easy calculation to check that

$$\begin{aligned} \beta\left(\frac{c}{m}\right) H_{Nm^2} &= H_N \gamma(b, c) \beta\left(\frac{b}{m}\right) \begin{pmatrix} N & \\ & N \end{pmatrix} \\ &\equiv \pm\gamma(b, c)\beta\left(\frac{b}{m}\right). \end{aligned}$$

Now,

$$R_m \equiv \sum_{c(m)}{}' \beta\left(\frac{c}{m}\right).$$

Therefore, by Theorem 2,

$$\sum_{c(m)}{}' \beta\left(\frac{c}{m}\right) \equiv \pm \sum_{c(m)}{}' \beta\left(\frac{c}{m}\right) H_{Nm^2}$$
$$\equiv \sum_{c(m)}{}' \gamma(c,b)\beta\left(\frac{c}{m}\right).$$

This relation implies Corollary 2. ■

If $m = 2$, then Corollary 2 is equivalent to Lemma 2.

The reduction from Corollary 2 to invariance properties of $f(z)$ uses ideas from the proof of Weil's converse theorem as described in Ogg's book [Ogg]. We quote Proposition 3 from that book for convenience.

Lemma 5. Suppose f is holomorphic in $\mathcal{H}$ and $\varepsilon \in GL_2(\mathbb{R})^+$ is elliptic. If $f|_k\varepsilon = f$, then either ε has finite order, or f is constant. □

The typical way we apply Lemma 5 is to use Corollary 2 to first prove

$$1 - \gamma \equiv (1-\gamma)\varepsilon$$

for some $\gamma \in SL_2(\mathbb{Z})$ and some elliptic $\varepsilon \in GL_2(\mathbb{R})^+$ which is not of finite order. We then conclude $\gamma \equiv 1$.

We illustrate the method in the case $N = 11$. By Corollary 2, we have

$$\left(1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}\right)\beta(1/3) + \left(1 - \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\right)\beta(-1/3) \equiv 0$$

and

$$\left(1 - \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix}\right)\beta(1/4) + \left(1 - \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}\right)\beta(-1/4) \equiv 0.$$

Therefore,

$$1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \equiv -\left(1 - \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\right)\beta\left(-\frac{2}{3}\right)$$
$$= \left(1 - \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix}\right)\begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\beta\left(-\frac{2}{3}\right)$$
$$\equiv -\left(1 - \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}\right)\beta\left(-\frac{2}{4}\right)\begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\beta\left(-\frac{2}{3}\right)$$
$$= \left(1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}\right)\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}\beta\left(-\frac{2}{4}\right)\begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\beta\left(-\frac{2}{3}\right).$$

However,

$$\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \beta\left(-\frac{2}{4}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) = \begin{pmatrix} 1 & -2/3 \\ 11/2 & -8/3 \end{pmatrix}$$

is elliptic but not of finite order. So,

$$\begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \equiv 1.$$

We will use the above calculation to prove Theorem 1 for $N = 11$, and then we will describe a generalization of the method.

Let

$$M_{m,b} = M_{m,b}(N) = \begin{pmatrix} m & b \\ cN & d \end{pmatrix},$$

where $0 < 2|c| < |m|$. Also put $M_m = M_{m,1}$. This is ambiguous only if $m = \pm 2$, in which case we use our previous definition of M_2, and put $M_{-2} = W_N^{-1} M_2 P^{-1}$.

Proof of Theorem 1 for $N = 11$. We have generators

$$\Gamma_0(11) = \langle P,\ M_2,\ M_3 \rangle\,.$$

By Lemmas 1 and 2, and Corollary 2 and the above calculation, $f(z)$ is invariant under the given matrices. ■

We summarize the above argument as follows. First of all, for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

let

$$\gamma' = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Clearly, γ and γ' have the same determinant and belong to the same group $\Gamma_0(N)$.

Now let m and n be two of the integers 3, 4, and 6. (These are the integers with $\varphi(m) = \varphi(n) = 2$.) Let

$$\gamma = \begin{pmatrix} m & 1 \\ N & n \end{pmatrix} \in \Gamma_0(N).$$

Thus, N will be one of 8, 11, 15, 17, 23, or 35. Corollary 2 implies

$$(1-\gamma)\beta(-1/m) + (1-\gamma')\beta(1/m) \equiv 0,$$

and also

$$(1-\gamma'^{-1})\beta(-1/n) + (1-\gamma^{-1})\beta(1/n) \equiv 0.$$

Therefore,

$$\begin{aligned} 1-\gamma &\equiv -(1-\gamma')\beta(2/m) \\ &= (1-\gamma'^{-1})\gamma'\beta(2/m) \\ &\equiv -(1-\gamma^{-1})\beta(2/n)\gamma'\beta(2/m) \\ &= (1-\gamma)\gamma^{-1}\beta(2/n)\gamma'\beta(2/m). \end{aligned}$$

But

$$\gamma^{-1}\beta(2/n)\gamma'\beta(2/m) = \begin{pmatrix} 1 & \frac{2}{m} \\ -\frac{2N}{n} & \frac{4}{mn} - 3 \end{pmatrix}$$

is elliptic of infinite order. Hence,

$$\gamma \equiv 1,$$

and similarly for γ'.

What we have shown is the following.

Corollary 3. Under the conditions of Corollary 2, if $\varphi(m) = \varphi((N+1)/m) = 2$, then $M_m \equiv 1$ and $M_{-m} \equiv 1$. □

Proof of Theorem 1 for $N = 17$. We have generators,

$$\Gamma_0(17) = \langle P,\ W_{17},\ M_2,\ M_3,\ M_6 \rangle\,.$$

By Lemmas 1 and 2, and Corollary 3, $f(z)$ is invariant under the above matrices. ■

4 Ad-hoc methods

The previous two sections gave general methods for finding matrices $\gamma \in \Gamma_0(N)$ such that $f|_k\gamma = f$. This was sufficient to generate $\Gamma_0(N)$ for a few N. In this section we use ad-hoc methods to deduce the invariance of $f(z)$ for various other matrices. It may be that discovering a general scheme behind these seemingly ad-hoc methods could lead to a proof of the general case.

The main weakness in our method of using Corollary 2 is its intractability when $\varphi(m) > 2$. The next two proofs exhibit a "bootstrap" feature, where expressions with $\varphi(m) > 2$ are first reduced down to simpler expressions, and then these simpler expressions are used in a manner similar to the proof of Corollary 3.

Proof of Theorem 1 for $N = 14$. We have generators,

$$\Gamma_0(14) = \langle P,\ W_{14},\ M_3,\ M_{-3},\ M_{13,6} \rangle .$$

By Lemma 1, f is invariant under P and W_{14}. Writing the conclusion of Lemma 3 as $A \equiv -1$, we have that f is invariant under

$$A^{-1} W_{14}^{-1} A P = M_{13,6}.$$

It remains to show invariance under the two other generators.

Corollary 2, with $m = 3$, gives

$$\left(1 - \begin{pmatrix} 3 & -1 \\ -14 & 5 \end{pmatrix}\right) \beta\left(\frac{1}{3}\right) + \left(1 - \begin{pmatrix} 3 & 1 \\ 14 & 5 \end{pmatrix}\right) \beta\left(-\frac{1}{3}\right) \equiv 0,$$

which we write as

$$(1 - \gamma_1)\, \beta\left(\frac{1}{3}\right) + (1 - \gamma_2)\, \beta\left(-\frac{1}{3}\right) \equiv 0,$$

temporarily putting $\gamma_1 = M_{-3}$ and $\gamma_2 = M_3$. Corollary 2, with $m = 5$, gives

$$\left(1 - \begin{pmatrix} 5 & -2 \\ 28 & -11 \end{pmatrix}\right) \beta\left(\frac{2}{5}\right) + \left(1 - \begin{pmatrix} 5 & -1 \\ -14 & 3 \end{pmatrix}\right) \beta\left(\frac{1}{5}\right)$$
$$+ \left(1 - \begin{pmatrix} 5 & 1 \\ 14 & 3 \end{pmatrix}\right) \beta\left(-\frac{1}{5}\right) + \left(1 - \begin{pmatrix} 5 & 2 \\ -28 & -11 \end{pmatrix}\right) \beta\left(-\frac{2}{5}\right) \equiv 0.$$

We can reduce the $m = 5$ expression by noting the following:

$$(AP)^2 = \begin{pmatrix} 5 & 2 \\ -28 & -11 \end{pmatrix}$$
$$\equiv 1$$

and

$$(W_{14}A)^2 = \begin{pmatrix} 5 & -2 \\ 28 & -11 \end{pmatrix}$$
$$\equiv 1.$$

So the $m = 5$ expression implies

$$(1 - \gamma_2^{-1})\, \beta\left(\frac{1}{5}\right) + (1 - \gamma_1^{-1})\, \beta\left(-\frac{1}{5}\right) \equiv 0.$$

Now proceed exactly as in the proof of Corollary 3:

$$\begin{aligned}
1-\gamma_1 &\equiv -\left(1-\gamma_2\right)\,\beta\left(-\frac{2}{3}\right)\\
&= \left(1-\gamma_2^{-1}\right)\,\gamma_2\,\beta\left(-\frac{2}{3}\right)\\
&\equiv -\left(1-\gamma_1^{-1}\right)\,\beta\left(-\frac{2}{5}\right)\,\gamma_2\,\beta\left(-\frac{2}{3}\right)\\
&= \left(1-\gamma_1\right)\,\gamma_1^{-1}\,\beta\left(-\frac{2}{5}\right)\,\gamma_2\,\beta\left(-\frac{2}{3}\right)\\
&= \left(1-\gamma_1\right)\begin{pmatrix} 1 & -2/3 \\ 28/5 & -41/15 \end{pmatrix}.
\end{aligned}$$

That last matrix is elliptic of infinite order, so $\gamma_1 \equiv 1$. Therefore, $\gamma_2 \equiv 1$. This finishes the proof of Theorem 1 for $N = 14$. ■

Note that for the above proof we used the local factors of $F(s)$ at $p = 2$, 3, and 5, while in all other cases we only used the factors of $F(s)$ at $p = 2$ and 3.

Proof of Theorem 1 for $N = 15$. We have generators,

$$\Gamma_0(15) = \langle P,\ W_{15},\ M_2,\ M_4,\ M_{11,4}\rangle\,.$$

From Lemmas 1 and 2, and Corollary 3, we have that $f(z)$ is invariant under the first four generators. It remains to prove invariance under $M_{11,4}$.

Corollary 2, with $m = 8$, gives

$$\begin{aligned}
&\left(1-\begin{pmatrix} 8 & -3 \\ -45 & 17 \end{pmatrix}\right)\beta\left(\frac{3}{8}\right)+\left(1-\begin{pmatrix} 8 & -1 \\ -15 & 2 \end{pmatrix}\right)\beta\left(\frac{1}{8}\right)\\
&+\left(1-\begin{pmatrix} 8 & 1 \\ 15 & 2 \end{pmatrix}\right)\beta\left(-\frac{1}{8}\right)+\left(1-\begin{pmatrix} 8 & 3 \\ 45 & 17 \end{pmatrix}\right)\beta\left(-\frac{3}{8}\right)\equiv 0.
\end{aligned}$$

We can reduce this expression by noting that

$$\begin{aligned}
M_2^{-1} &= \begin{pmatrix} 8 & -1 \\ -15 & 2 \end{pmatrix}\\
&\equiv 1
\end{aligned}$$

and

$$PM_2^{-1}W_{15} = \begin{pmatrix} 8 & 1 \\ 15 & 2 \end{pmatrix}$$
$$\equiv 1.$$

And we also have

$$PM_2^{-1}\begin{pmatrix} 8 & 3 \\ 45 & 17 \end{pmatrix} = M_{11,4}$$

and

$$M_2^{-1}W_{15}\begin{pmatrix} 8 & -3 \\ -45 & 17 \end{pmatrix} = M_{11,4}^{-1}.$$

So the $m = 8$ relation gives

$$\begin{aligned} 1 - M_{11,4} &\equiv -(1 - M_{11,4}^{-1})\beta\left(\frac{3}{4}\right) \\ &= (1 - M_{11,4})M_{11,4}^{-1}\,\beta\left(\frac{3}{4}\right) \\ &= (1 - M_{11,4})\begin{pmatrix} 11 & 17/4 \\ -30 & -23/2 \end{pmatrix}. \end{aligned}$$

The last matrix is elliptic of infinite order, so $M_{11,4} \equiv 1$. This finishes the proof of Theorem 1 for $N = 15$. ■

Proof of Theorem 1 for $N = 23$. We have generators,

$$\Gamma_0(23) = \langle P,\ W_{23},\ M_2,\ M_3,\ M_4,\ M_6\rangle\,.$$

By Lemmas 1 and 2, and Corollary 3, we obtain the invariance of $f(z)$ under each of the above matrices except M_3.

By Corollary 2, with $m = 3$, we get

$$\left(1 - \begin{pmatrix} 3 & 1 \\ 23 & 8 \end{pmatrix}\right)\beta\left(\frac{-1}{3}\right) + \left(1 - \begin{pmatrix} 3 & -1 \\ 23 & -8 \end{pmatrix}\right)\beta\left(\frac{1}{3}\right) \equiv 0,$$

which we can rewrite as

$$(1 - M_3)\beta\left(\frac{-1}{3}\right) + (1 - M_{-3})\beta\left(\frac{1}{3}\right) \equiv 0.$$

Let

$$\gamma = \begin{pmatrix} 10 & 3 \\ 23 & 7 \end{pmatrix}.$$

One can check that $W_{23}M_{-2}\gamma = M_3$ and $M_{-6}M_6M_{-3}\gamma = I$. In particular, $\gamma \equiv M_3$ and $M_{-3}\gamma \equiv 1$. Consequently,

$$1 - M_3 \equiv -(1 - M_{-3})\gamma.$$

Combining the two expressions gives

$$1 - M_3 \equiv (1 - M_3)\beta(-2/3)\gamma.$$

But

$$\beta(-2/3)\gamma = \begin{pmatrix} -16/3 & -5/3 \\ 23 & 7 \end{pmatrix}$$

is elliptic of infinite order. So, $M_3 \equiv 1$ as desired. This finishes the proof of Theorem 1 for $N = 23$. ■

This finishes the proof of Theorem 1. It remains to show that $f(z)$ is actually a cusp form.

5 Vanishing at the cusps

We have assumed that $F(s)$ converges in some right half-plane. It follows from this, and from the invariance of $f(z)$ under $\Gamma_0(N)$, that $f(z)$ is holomorphic at the cusps of $\Gamma_0(N)$. If one assumes further that $F(s)$ converges for $\sigma > k - \delta$, for some $\delta > 0$, then it follows that $f(z)$ vanishes at the cusps of $\Gamma_0(N)$. See Proposition 1 and the Lemma on page V-14 of Ogg's book [Ogg]. Note that the analog of the Ramanujan conjecture gives bounds on the a_n which are much stronger than is required for this method. We will show that no such assumption is needed. The vanishing of $f(z)$ at the cusps of $\Gamma_0(N)$ follows from the previously assumed functional equation and Euler product of $F(s)$.

The following, in combination with Lemma 1, gives Corollary 1 as a consequence of Theorem 1.

Theorem 3. Let $N = 2^k N'$, with $0 \le k \le 3$ and N' odd and squarefree, and put $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$. If $f|_k\gamma = f$ for all $\gamma \in \Gamma_0(N)$, and $f|_k H_N = \pm f$, and

$$f|_k U_q = f|_k \begin{pmatrix} q & \\ & 1 \end{pmatrix} \qquad \text{if } q \| N,$$

$$f|_k U_q = 0 \qquad \text{if } q^2|N,$$

then $f(z)$ vanishes at the cusps of $\Gamma_0(N)$. □

The values of N in Theorem 3 are exactly those for which $\{1/r \ : \ r|N\}$ is a set of cusps for $\Gamma_0(N)$. Note that in this case the cusp $1/r$ has width N/r.

The following lemma follows from the Chinese remainder theorem. It is implicit in the classification of the cusps of $\Gamma_0(N)$.

Lemma 6. If $r|N$ and $(ar, N/r) = 1$, then there exists $\gamma \in \Gamma_0(N)$ such that

$$\gamma \begin{pmatrix} 1 & \\ ar & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ -r & d \end{pmatrix}$$

for some $b, d \in \mathbb{Z}$. □

Proof of Theorem 3. Suppose $r|N$. We show that $f(z)$ vanishes at the cusp $1/r$.

Put $N = rQ$. By taking linear combinations of U_q for $q|Q$, we can obtain, for some $\beta_q \in \mathbb{Z}$,

$$\sum_{1\le a\le Q}{}' \begin{pmatrix} 1 & a/Q \\ & 1 \end{pmatrix} \equiv \sum_{q|Q} \beta_q \begin{pmatrix} q & \\ & 1 \end{pmatrix}.$$

Left-multiply and right-multiply by H_N to get

$$\sum_{1\le a\le Q}{}' \begin{pmatrix} 1 & \\ ar & 1 \end{pmatrix} \equiv \sum_{q|Q} \beta_q \begin{pmatrix} 1 & \\ & q \end{pmatrix}.$$

So by Lemma 6 we have

$$\sum_{1\le a\le Q}{}' \begin{pmatrix} 1 & b_a \\ -r & d_a \end{pmatrix} \equiv \sum_{q|Q} \beta_q \begin{pmatrix} 1 & \\ & q \end{pmatrix},$$

which can be rewritten as

$$\begin{pmatrix} 1 & \\ -r & 1 \end{pmatrix} \sum_{1\le a\le Q}{}' \begin{pmatrix} 1 & b_a \\ & 1 \end{pmatrix} \equiv \sum_{q|Q} \beta_q \begin{pmatrix} 1 & \\ & q \end{pmatrix}.$$

By definition, that congruence says

$$\left(f\Big|_k \begin{pmatrix} 1 & \\ -r & 1 \end{pmatrix} \sum_{1\le a\le Q}{}' \begin{pmatrix} 1 & b_a \\ & 1 \end{pmatrix} \right)(z) = \left(f\Big|_k \sum_{q|Q} \beta_q \begin{pmatrix} 1 & \\ & q \end{pmatrix} \right)(z)$$
$$= \sum_{n=1}^{\infty} c_n e(nz/Q),$$

for some $c_n \in \mathbb{C}$. Now, if α_0 is the constant term in the expansion of $f|_k\left(\begin{smallmatrix} 1 & \\ -r & 1\end{smallmatrix}\right)$, then the constant term on the left side of the above expression is $\varphi(Q)\alpha_0$. The expression on the right side above has no constant term, so $\alpha_0 = 0$. In other words, $f(z)$ vanishes at the cusp $1/r$. This proves Theorem 3. ■

The only values of N in Theorem 1 which are not covered by Theorem 3 are $N = 9$ and 16. Using the equality

$$H_{M^2}\begin{pmatrix} 1 & \\ M & 1 \end{pmatrix} = \begin{pmatrix} -M & -1 \\ M^2 & \end{pmatrix}$$
$$\equiv \begin{pmatrix} 1 & \\ -M & 1 \end{pmatrix}\begin{pmatrix} 1 & 1/M \\ & 1 \end{pmatrix},$$

it is possible to modify the above proof to give the conclusion of Theorem 3 for $N = 9$ and 16 in the particular case $f|_k H_N = f$, that is, when $F(s)$ has sign $+1$ in its functional equation. If $F(s)$ has sign -1 in its functional equation, then the above method fails and we must make an assumption on the growth of the coefficients of $F(s)$ to obtain Corollary 1. This extra assumption cannot be eliminated, as demonstrated by the following example. The Dirichlet series $L(s, \chi_3)L(s-1, \chi_3)$, where χ_3 is the Dirichlet character mod 3, is entire and has a functional equation and Euler product of degree 2, weight 2, and level 9. Its functional equation has sign -1, and it is associated to an Eisenstein series of weight 2 on $\Gamma_0(9)$. Similar examples exist for the other values of N not covered by Theorem 3.

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