A NOTE ON THE FOURTH POWER MOMENT OF THE RIEMANN ZETA-FUNCTION

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Dedicated to Professor Heini Halberstam

Heath-Brown has evaluated the 4th power moment of the Riemann zeta-function on the critical line via

$$\int_0^T |\zeta(1/2+it)|^4 \, dt = TP_4\left(\log \frac{T}{2\pi}\right) + O(T^{7/8+\epsilon})$$

where P_4 is a 4th degree polynomial. Ingham had previously discovered the leading coefficient of P_4; Heath-Brown gave an explicit formula for the second coefficient of P_4 and expressed the other coefficients as infinite sums of infinite integrals. In this note we find an explicit expression for P_4 in terms of special values of $\zeta(s)$. In particular, we show that the coefficients of P_4 are in the field

$$Q(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \zeta(2), \zeta'(2), \zeta''(2))$$

where the γ's are the coefficients in the Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \dots$$

of $\zeta(s)$ at $s=1$.

Theorem. *With P_4 defined implicitly above, we have*

$$P_4(x) = g_1(x) + g_0(x)$$

where

$$g_1(x) = 2 \operatorname*{Res}_{s=0} \frac{x^s \zeta(s+1)^4}{s(s+1)\zeta(2s+2)}$$

and

$$g_0(x) = \left(\frac{d}{ds}\right)^2 \frac{(xe^{2\gamma_0})^s(\frac{1}{2}\zeta(s+1)^2 - \frac{1}{s}\zeta(2s+1) - \zeta(2s+2))}{(s+1)\zeta(s+2)}\Bigg|_{s=0}.$$

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Proof. We refer to Heath-Brown's paper. There he shows that P_4 is naturally expressed as a sum of g_1+g_0 where g_1 arises from the diagonal terms and is exactly as above. The other term, g_0, arises from the off diagonal terms and it is the explicit calculation of g_0 that is the purpose of this paper. Following Heath-Brown [H-B] (see pages 404, 406, and 407), we write

$$\sum_{n\le x} d(n)d(n+r) = m(x,r)+E(x,r)$$

where $m(x,r)/x$ is a polynomial in $\log x$ of degree 2 in x with coefficients depending on r. In fact,

$$m(x,r)=\operatorname*{Res}_{s=1} D(s,r)\frac{x^s}{s}$$

with

$$D(s,r)=\sum_{n=1}^\infty \frac{d(n)d(n+r)}{n^s}.$$

Then, as Heath-Brown shows,

$$g_0\left(\frac{T}{2\pi}\right)=\frac{2}{T}\sum_{r=1}^\infty\frac{1}{r}\int_0^{T/(2\pi)} m'(x,r)\sin(Tr/x)\ dx.$$

Lemma. *With the above notation,*

$$m(x,r)=x\sum_{s=1}^\infty\frac{\mu(s)}{s^2}\sum_{d|r}\frac{1}{d}\left(\left(\log\frac{x}{d^2s^2}+2\gamma_0-1\right)^2+1\right)$$

and

$$m'(x,r)=\sum_{s=1}^\infty\frac{\mu(s)}{s^2}\sum_{d|r}\frac{1}{d}\left(\log\frac{x}{d^2s^2}+2\gamma_0\right)^2$$

Proof. According to Heath-Brown, $m(x,r)$ may be calculated by considering the main terms in

$$2\sum_{q\le x^{1/2}}R(x,q,r)-\sum_{q\le x^{1/2}}R(qx^{1/2},q,r)$$

where

$$R(x,q,r)=\frac{x}{q^2}\sum_{d|(q,r)}\sum_{\delta|q/d} d\delta\mu(q/d\delta)\left(\log\frac{x\delta^2}{q^2}+2\gamma_0-1\right).$$

Writing $q=sd\delta$ we have

$$\sum_{q\le x^{1/2}}R(x,q,r)=x\sum_{d|r}\frac{1}{d}\sum_{s\le x^{1/2}/d}\frac{\mu(s)}{s^2}\left(\log\frac{x}{d^2s^2}+2\gamma_0-1\right)\sum_{\delta\le x^{1/2}/sd}\frac{1}{\delta}.$$

Using

$$\sum_{\delta\le y}\frac{1}{\delta}=\log y+\gamma_0+O(1/y)$$

(and we note

$$\sum_{\delta \le y} \frac{\log \delta}{\delta} = \frac{1}{2} \log^2 y - \gamma_1 + O(\log y/y),$$

for future reference) we find that the above sum is

$$= \frac{x}{2} \sum_{d|r} \frac{1}{d} \sum_{s \le x^{1/2}/d} \frac{\mu(s)}{s^2} \left(\log \frac{x}{d^2 s^2} + 2\gamma_0 - 1\right) \left(\log \frac{x}{d^2 s^2} + 2\gamma_0\right) + O\left(x^{1/2} \sum_{d|r} \sum_{s \le x^{1/2}} \frac{1}{s}\right).$$

Extending the first sum over s to ∞, we find that the above is

$$= \frac{x}{2} \sum_{s=1}^{\infty} \frac{\mu(s)}{s^2} \sum_{d|r} \frac{1}{d} \left(\log \frac{x}{d^2 s^2} + 2\gamma_0 - 1\right) \left(\log \frac{x}{d^2 s^2} + 2\gamma_0\right) + O(x^{1/2} \log x d(r)).$$

Similarly, using

$$\sum_{\delta \le y} 1 = y + O(1)$$

and

$$\sum_{\delta \le y} \log \delta = y \log y - y + O(\log y)$$

we find that the main part of

$$\sum_{q \le x^{1/2}} R(x^{1/2} q, q, r)$$

is given by

$$x \sum_{s=1}^{\infty} \frac{\mu(s)}{s^2} \sum_{d|r} \frac{1}{d} \left(\log \frac{x}{d^2 s^2} + 2\gamma_0 - 2\right).$$

Putting these two calculations together, we obtain the formula for $m(x, r)$. Differentiating with respect to x gives the rest of the lemma.

Now we compute

$$g_0 = g_0(T/(2\pi)) = \frac{2}{T} \sum_{r=1}^{\infty} \frac{1}{r} \int_0^{T/(2\pi)} m'(x, r) \sin(Tr/x)\, dx.$$

To begin with, we change variables, letting $y = Tr/x$. Then, substituting the expression for $m'(x, r)$ from the lemma gives

$$g_0 = 2 \sum_{r=1}^{\infty} \int_{2\pi r}^{\infty} \sum_{s=1}^{\infty} \frac{\mu(s)}{s^2} \sum_{d|r} \frac{1}{d} \left(\log \frac{Tr}{s^2 d^2 y} + 2\gamma_0\right)^2 \frac{\sin y}{y^2}\, dy.$$

We bring the sum over s to the front, and interchange the summation over r and the integration, and replace d by n and write $r = mn$. Then we have

$$g_0 = 2\sum_{s=1}^{\infty} \frac{\mu(s)}{s^2} \int_{2\pi}^{\infty} \sum_{mn \le y/2\pi} \frac{1}{n} \left(\log \frac{Tm}{s^2 ny} + 2\gamma_0\right)^2 \frac{\sin y}{y^2}\, dy.$$

Next, we interchange the sum over m and the integration and find that

$$g_0 = 2\sum_{s=1}^{\infty} \frac{\mu(s)}{s^2} \sum_{m=1}^{\infty} \int_{2\pi m}^{\infty} \sum_{n \le y/2\pi m} \frac{1}{n} \left(\log \frac{Tm}{s^2 ny} + 2\gamma_0\right)^2 \frac{\sin y}{y^2}\, dy.$$

A change of variables in the integration, $u = y/2\pi m$, gives

$$g_0 = 2\sum_{s=1}^{\infty} \frac{\mu(s)}{s^2} \sum_{m=1}^{\infty} \int_{1}^{\infty} \sum_{n \le u} \frac{1}{n} \left(\log \frac{T/2\pi}{s^2 un} + 2\gamma_0\right)^2 \frac{\sin 2\pi um}{2\pi um}\, \frac{du}{u}.$$

Now,

$$-\sum_{m=1}^{\infty} \frac{\sin 2\pi um}{\pi m} = u - [u] - 1/2 = ((u)).$$

Thus, bringing the sum over m to the inside, we obtain

$$g_0 = -\sum_{s=1}^{\infty} \frac{\mu(s)}{s^2} \int_{1}^{\infty} \sum_{n \le u} \frac{1}{n} \left(\log \frac{T/2\pi}{s^2 un} + 2\gamma_0\right)^2 ((u))\, \frac{du}{u^2}.$$

Interchanging the sum over n and the integration, and changing s into m leads to

$$g_0 = -\sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} \sum_{n=1}^{\infty} \int_{n}^{\infty} \frac{1}{n} \left(\log \frac{T/2\pi}{m^2 un} + 2\gamma_0\right)^2 ((u))\, \frac{du}{u^2}.$$

Now we express the logs via differentiation:

$$g_0 = -\left(\frac{d}{ds}\right)^2 \left(\sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2+s}} \left(\frac{Te^{2\gamma_0}}{2\pi}\right)^s \sum_{n=1}^{\infty} \frac{1}{n^{1+s}} \int_{n}^{\infty} ((u))\, \frac{du}{u^{2+s}}\right|_{s=0}.$$

It is well known that

$$-\int_{n}^{\infty} \frac{((u))}{u^{s+2}}\, du = \frac{1}{s+1} \left(\zeta(s+1) - \frac{1}{sn^s} - \sum_{m=1}^{n} \frac{1}{m^{s+1}} - \frac{1}{2n^{s+1}}\right).$$

Thus,

$$g_0 = \lim_{N\to\infty} -\left(\frac{d}{ds}\right)^2 \left(\left(\frac{Te^{2\gamma_0}}{2\pi}\right)^s \frac{1}{(s+1)\zeta(s+2)} \right.$$
$$\left. \sum_{n=1}^{N} \frac{1}{n^{1+s}} \left(\zeta(s+1) - \frac{1}{sn^s} - \sum_{m=1}^{n} \frac{1}{m^{s+1}} - \frac{1}{2n^{s+1}}\right)\right|_{s=0}.$$

Now

$$\sum_{n=1}^{N} \frac{1}{n^{s+1}} \sum_{m=1}^{n} \frac{1}{m^{s+1}} = \sum_{m=1}^{N} \frac{1}{m^{s+1}} \sum_{n=m}^{N} \frac{1}{n^{s+1}}$$

so that

$$2\sum_{n=1}^{N} \frac{1}{n^{s+1}} \sum_{m=1}^{n} \frac{1}{m^{s+1}} = \left(\sum_{n=1}^{N} \frac{1}{n^{s+1}}\right)^2 + \sum_{n=1}^{N} \frac{1}{n^{2s+2}}.$$

Thus, the sum over n in the above expression is

$$= \zeta(s+1) \sum_{n=1}^{n} \frac{1}{n^{s+1}} - \frac{1}{s} \sum_{n=1}^{N} \frac{1}{n^{2s+1}} - \frac{1}{2}\left(\sum_{n=1}^{N} \frac{1}{n^{s+1}}\right)^2 - \sum_{n=1}^{N} \frac{1}{n^{2s+2}}.$$

If we assume that $s > 0$, we can let $N \to \infty$ here, getting

$$\frac{1}{2}\zeta(s+1)^2 - \frac{1}{s}\zeta(2s+1) - \zeta(2s+2).$$

Inserting this into our prior expresion for g_0, we obtain the Theorem.

References

[H-B] D.R. Heath-Brown, *The fourth power moment of the Riemann zeta function*, J. London Math. Soc (3) **38** (1979), 385-422.

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