SIMPLE ZEROS OF THE RIEMANN ZETA-FUNCTION

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1. *Introduction and notation*

Let $\zeta(s)$ denote the Riemann zeta-function and $\rho = \beta + i\gamma$ a zero of $\zeta(s)$. In the process of evaluating $N(T)$, the number of zeros of ζ with $0 < \gamma \leqslant T$, each zero is counted with multiplicity, denoted by $m(\rho)$, to give the von Mangoldt formulae

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

$$S(T) = \frac{1}{\pi} \arg \zeta(\tfrac{1}{2} + iT) \ll \log T.$$

It is not known if $m(\rho) = 1$ for all ρ but it is so conjectured. Indeed, all the zeros located so far, more than 1.5×10^9 of them, are simple. Support for this conjecture is further strengthened by the fact that Montgomery's pair correlation conjecture (see [**12**]) implies that almost all the zeros are simple.

It is known that at least two-fifths of the zeros are simple. (See Conrey [**3**, **4**] for a discussion of results in this direction). Assuming the Riemann Hypothesis (RH), Montgomery [**12**] has shown that at least two-thirds of the zeros are simple, and later, with Taylor [**13**], the constant two-thirds was improved to

$$\frac{3}{2} - \frac{\sqrt{2}}{2} \cot \frac{\sqrt{2}}{2} = 0.6725\ldots .$$

Cheer and Goldston [**1**] have recently improved upon this result slightly by showing that RH implies that at least 0.672753 of the zeros are simple.

A problem related to the simplicity of the zeros of $\zeta(s)$ is the enumeration of distinct zeros (that is, to count each zero precisely once without regard to its multiplicity).

Let us define

$$N_d(T) = |\{\rho = \beta + i\gamma\colon\ 0 < \gamma \leqslant T,\ \zeta(\rho) = 0\}|,$$

$$N_s(T) = |\{\rho = \beta + i\gamma\colon\ 0 < \gamma \leqslant T,\ \zeta(\rho) = 0,\ m(\rho) = 1\}|.$$

We are interested in obtaining bounds for the constants C_d and C_s defined by

$$C_d = \liminf_{T \to \infty} N_d(T)/N(T),$$

$$C_s = \liminf_{T \to \infty} N_s(T)/N(T).$$

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Of course, it is clear that $C_d = 1$ if and only if $C_s = 1$. To get a bound for C_s, Montgomery proceeded by observing that

$$N_s(T) \geqslant \sum_{\gamma \leqslant T} (2 - m(\gamma)) = 2N(T) - \sum_{\gamma \leqslant T} m(\gamma)$$

(where all sums over zeros are repeated according to multiplicity and we use $m(\gamma)$ in place of $m(\rho)$ on assuming RH), and he showed in [**12**] that

$$\sum_{\gamma \leqslant T} m(\gamma) \leqslant (C_0 + o(1))N(T) \quad (C_0 = \tfrac{4}{3},\ T \to \infty).$$

One gets a bound for C_d by observing that

$$2N_s(T) \leqslant \sum_{\gamma \leqslant T} \frac{(m(\gamma) - 2)(m(\gamma) - 3)}{m(\gamma)} = \sum_{\gamma \leqslant T} m(\gamma) - 5N(T) + 6N_d(T),$$

so that

$$C_d \geqslant \tfrac{1}{6}(5 + 2C_s - C_0). \tag{1.1}$$

Due to the relationship between C_0 and C_s in Montgomery's method, we have

$$C_d \geqslant \tfrac{1}{6}(9 - 3C_0) \geqslant \tfrac{5}{6} \quad \text{if } C_0 = \tfrac{4}{3}. \tag{1.2}$$

(The constants are improved very slightly by using the better value for C_0 in [**13**] or in [**1**].) If one could devise a method for evaluating C_s independently of C_0, then (1.1) would be essentially better than (1.2).

In this paper we develop a new method to obtain lower bounds for C_s and C_d. Our method requires the assumption of the Riemann Hypothesis as well as an assumption of an upper bound for averages of sixth moments of Dirichlet L-functions $L(s, \chi)$. This sixth moment hypothesis is implied by the Generalized Lindelöf Hypothesis, which is the assumption that for any $\varepsilon > 0$,

$$L(s, \chi) \ll_\varepsilon (q(1 + |t|))^\varepsilon$$

for $\sigma \geqslant \frac{1}{2}$, where χ is a character modulo q. The Generalized Lindelöf Hypothesis is weaker than the Generalized Riemann Hypothesis which conjectures that $L(s, \chi) \neq 0$ if $\sigma > \frac{1}{2}$. For convenience, we state our main result as follows.

THEOREM 1. *Assuming the Riemann Hypothesis (RH) and the Generalized Lindelöf Hypothesis (GLH), we have*

$$C_s \geqslant \tfrac{19}{27}, \quad C_d \geqslant \tfrac{5}{6} + \tfrac{1}{81}.$$

REMARK. In an earlier version of this paper, we had omitted the assumption of GLH. We thank the referee for pointing out this mistake. Our paper [**5**], which refers to the methods here, requires this additional assumption of GLH as well. Also, we would like to thank especially a second referee who gave a considerable simplification for the evaluation of our main terms in § 8. We have included this simplification here.

1.1. *Notation*

We shall use $*$ to denote Dirichlet convolution of arithmetic functions. Thus if α and β are arithmetic functions, then

$$(\alpha * \beta)(n) = \sum_{d|n} \alpha(d)\beta\left(\frac{n}{d}\right).$$

Further, we define operators T_s and L^k by

$$(T_s\alpha)(n) = n^s\alpha(n),$$
$$(L^k\beta)(n) = (\log n)^k\beta(n).$$

Note that

$$T_s(\alpha * \beta) = (T_s\alpha) * (T_s\beta)$$

and

$$L(\alpha * \beta) = (L\alpha) * \beta + \alpha * (L\beta).$$

Also, observe that if k is squarefree and α is multiplicative (that is, $\alpha(mn) = \alpha(m)\alpha(n)$ wherever $(m,n) = 1$), then

$$\alpha(\beta * \gamma)(k) = (\alpha\beta * \alpha\gamma)(k).$$

We use τ_r to denote the r-fold divisor function (often denoted $d_r(\ )$). Often this has an unspecified but bounded positive integer subscript r, not necessarily the same at each occurrence. We make frequent use of the inequalities $\tau_r(n)\tau_s(n) \leqslant \tau_{rs}(n)$ and $\tau_r(mn) \leqslant \tau_r(m)\tau_r(n)$ for positive integers r, s, m and n. We use I for the arithmetic function which is the identity for Dirichlet convolution, that is, $I(1) = 1$ and $I(n) = 0$ for $n > 1$. Also, we use 1 for the arithmetic function for which $1(n) = 1$ for all n.

Throughout $\mathscr{L}$ shall denote $\log(T/2\pi)$, and $e(x)$ denotes $e^{2\pi i x}$.

2. *Sketch of the proof for C_s*

We note that ρ is a simple zero of $\zeta(s)$ if and only if $\zeta'(\rho) \neq 0$. By Cauchy's inequality, it follows that

$$\left|\sum_{\gamma \leqslant T} B\zeta'(\rho)\right|^2 \leqslant N_s(T) \sum_{\gamma \leqslant T} |B\zeta'(\rho)|^2, \tag{2.1}$$

where $B(s)$ is any regular function. We shall take

$$B(s) = \sum_{k \leqslant y} b(k)k^{-s}, \tag{2.2}$$

where

$$y = T^\theta, \tag{2.3}$$

and

$$b(k) = \mu(k) P\left(\frac{\log y/k}{\log y}\right), \tag{2.4}$$

where $P(x)$ is a polynomial with real coefficients which satisfies

$$P(0) = 0, \quad P(1) = 1.$$

Then, if $\theta < \frac{1}{2}$, we shall prove the following theorem.

THEOREM 2. *Assuming GLH, we have*

$$S_1 := \sum_{\gamma \leqslant T} B\zeta'(\rho) \sim \left(\tfrac{1}{2} + \theta \int_0^1 P(x)\, dx\right) \frac{T\mathscr{L}^2}{2\pi} \tag{2.5}$$

and

$$S_2 := \sum_{\gamma \leqslant T} B\zeta'(\rho) B\zeta'(1-\rho) \sim \lambda \frac{T\mathscr{L}^3}{2\pi}, \tag{2.6}$$

where

$$\lambda = \tfrac{1}{3} + \left(\theta \int_0^1 P(x)\, dx\right)^2 + \theta \int_0^1 P(x)\, dx + \frac{1}{12\theta} \int_0^1 P'(x)^2\, dx. \tag{2.7}$$

Assuming the Riemann Hypothesis, we have $S_2 = \sum_{\gamma \leqslant T} |B\zeta'(\rho)|^2$. (This is the only place we need RH.) Our estimate for C_s is then obtained by choosing the polynomial $P(x)$ optimally. By the calculus of variations, the optimal choice is

$$P(x) = -\theta x^2 + (1+\theta)x.$$

Then

$$\int_0^1 P(x)\, dx = \tfrac{1}{2} + \tfrac{1}{6}\theta \quad \text{and} \quad \int_0^1 P'(x)^2\, dx = \tfrac{1}{3}\theta^2 + 1,$$

so that with $\theta = \frac{1}{2} - \varepsilon$ $(\varepsilon \to 0^+)$ we obtain

$$S_1 \sim \tfrac{19}{24} N(T)\mathscr{L} \quad \text{and} \quad S_2 \sim \tfrac{57}{64} N(T)\mathscr{L}^2.$$

Theorem 1 then follows.

To obtain the formulae (2.5) and (2.6), we use Cauchy's residue theorem to write

$$S_1 = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{\zeta'}{\zeta}(s) \zeta'(s) B(s)\, ds \tag{2.8}$$

and

$$S_2 = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{\zeta'}{\zeta}(s) \zeta'(s) \zeta'(1-s) B(s) B(1-s)\, ds, \tag{2.9}$$

where $\mathscr{C}$ is the positively oriented rectangle with vertices at $1-c+i$, $c+i$, $c+iT$, and $1-c+iT$,

$$c = 1 + \mathscr{L}^{-1}, \tag{2.10}$$

and T is chosen so that the distance from T to the nearest γ is $\gg \mathscr{L}^{-1}$. (It is easy to see that this assumption on T involves no loss of generality.)

Remark. There is a fair amount of work involved in evaluating S_1 and S_2, even if one considers only main terms without regard to error terms. Therefore, let us motivate this work by a suggestion as to what we might expect, by considering averages over all t instead of just ordinates of zeros of $\zeta(s)$. In this case we would start with the function $G(s)B(s)$ instead of $\zeta'(s)B(s)$, where (see (3.2) for the definition of $\chi(s)$)

$$G(s) = \zeta'(s) - \frac{\chi'}{\chi}(s)\zeta(s)$$

and for P we take $P(x) = x$. Note that G and ζ' agree at $s = \rho$. Then it is trivial to see that

$$\int_1^T GB(\tfrac{1}{2} + it)\, dt \sim T\mathscr{L}$$

by writing the integral as a complex integral and moving the line of integration to $\sigma = c > 1$. Also, we know that

$$\int_1^T |GB(\tfrac{1}{2} + it)|^2\, dt \sim \tfrac{4}{3} T\mathscr{L}^2$$

since Levinson evaluates this integral in his work [**10**]. (He considers the integral more generally on the a-line where $\frac{1}{2} - a \ll (\log T)^{-1}$.) Thus, consideration of integral means leads one to suspect that our method might lead to a proof that three quarters of the zeros are simple. However, one can see from the formulae (2.5), (2.6), and (2.7) that the average at zeros is slightly different from the average over all t.

3. *Preliminary transformations*

The ensuing analysis is much simpler if, instead of (2.8) and (2.9), we use slightly different expressions.

The functional equation for $\zeta(s)$ is given by

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{3.1}$$

where

$$\chi(1-s) = \chi(s)^{-1} = 2(2\pi)^{-s}\Gamma(s)\cos\tfrac{1}{2}\pi s. \tag{3.2}$$

We differentiate to obtain

$$\zeta'(s) = -\chi(s)\left(\zeta'(1-s) - \frac{\chi'}{\chi}(s)\zeta(1-s)\right). \tag{3.3}$$

We first consider S_1. It follows from (2.5) and (3.3) that

$$S_1 = -\sum_{\gamma \leqslant T} \chi(\rho)\zeta'(1-\rho)B(\rho)$$

$$= \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\zeta'}{\zeta}(1-s)\chi(s)\zeta'(1-s)B(s)\, ds. \tag{3.4}$$

Let $s = \sigma + it$ be on $\mathscr{C}$. We shall use the following well-known estimates:

$$\chi(s) \ll |t|^{1/2-\sigma}, \tag{3.5}$$

$$B(s) \ll y^{1-\sigma}\mathscr{L}, \tag{3.6}$$

$$\zeta'(1-s) \ll |t|^{\sigma/2}\mathscr{L}^2, \tag{3.7}$$

and

$$\frac{\zeta'}{\zeta}(1-s) \ll \mathscr{L}^2 \tag{3.8}$$

for s on $\mathscr{C}$; the last estimate holds if the distance from t to the nearest γ is $\gg \mathscr{L}^{-1}$ which we have assumed. Hence, the contribution to S_1 from the integral in (3.4) taken along the horizontal sides of $\mathscr{C}$ is

$$\ll T^{c/2}y^c\mathscr{L}^5 \ll T^{1/2}y\mathscr{L}^5. \tag{3.9}$$

To estimate the contribution to S_1 from the right-hand side of the contour $\mathscr{C}$, we use (3.3) to replace $\chi(s)\zeta'(1-s)$ in (3.4). Also, by (3.1) and (3.2),

$$\frac{\zeta'}{\zeta}(s) + \frac{\zeta'}{\zeta}(1-s) = \frac{\chi'}{\chi}(s) = -\log\frac{|t|}{2\pi} + O\left(\frac{1}{1+|t|}\right). \tag{3.10}$$

Thus, the integral along the right-hand side of $\mathscr{C}$ is equal to

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT}\left(\frac{\chi'}{\chi}(s)^2\zeta(s) - 2\frac{\chi'}{\chi}(s)\zeta'(s) + \frac{\zeta'}{\zeta}(s)\zeta'(s)\right)B(s)\,ds$$
$$= \frac{T}{2\pi}\mathscr{L}^2 + O(T\mathscr{L}), \tag{3.11}$$

since only the term $n = 1$ from the Dirichlet series for $\zeta(s)B(s)$ contributes anything to the main term.

To evaluate the contribution to S_1 from the left-hand side of the contour $\mathscr{C}$, we make the change of variable $s \to 1-s$ in (3.4). Then, the integral along the left-hand side is equal to $-\bar{I}_1$, where

$$I_1 = \frac{1}{2\pi i}\int_{c+i}^{c+iT}\chi(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)B(1-s)\,ds. \tag{3.12}$$

So far, we have shown by (3.4), (3.9), (3.11), and (3.12) that

$$S_1 = \frac{T}{2\pi}\mathscr{L}^2 - \bar{I}_1 + O(T\mathscr{L}) + O(T^{1/2}y\mathscr{L}^5). \tag{3.13}$$

We now consider S_2. By (2.6), (3.3), and Cauchy's theorem,

$$S_2 = -\sum_{\gamma\leqslant T}\chi(1-\rho)\zeta'(\rho)^2B(1-\rho)B(\rho)$$
$$= -\frac{1}{2\pi i}\int_{\mathscr{C}}\chi(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)^2B(s)B(1-s)\,ds. \tag{3.14}$$

By the estimates (3.5)–(3.8), the contribution to S_2 of the integral over the horizontal sides of the contour is

$$\ll T^{1/2}y\mathscr{L}^8. \tag{3.15}$$

The integral along the left-hand side of $\mathscr{C}$ is, by (3.3) and (3.10), equal to

$$\frac{1}{2\pi i}\int_{1-c+i}^{1-c+iT}\left(\frac{\chi'}{\chi}(s)-\frac{\zeta'}{\zeta}(1-s)\right)\chi(s)\left(\frac{\chi'}{\chi}(s)\zeta(1-s)-\zeta'(1-s)\right)^2 B(s)B(1-s)\,ds,$$

which, by the change of variable $s\to 1-s$ and another use of (3.3), is

$$-\frac{1}{2\pi i}\int_{c-i}^{c-iT}\chi(1-s)$$
$$\times\left(\frac{\chi'}{\chi}(s)^3\zeta(s)^2-3\frac{\chi'}{\chi}(s)\zeta'(1-s)\chi(s)-\frac{\zeta'}{\zeta}(s)\zeta'(s)^2\right)B(s)B(1-s)\,ds. \tag{3.16}$$

We multiply this out and rewrite it as a sum of three integrals. For the first two of these we move the line of integration to the line $\sigma=\frac{1}{2}$ with the same error term as in (3.15). Thus the above is equal to

$$\mathscr{M}_1+\mathscr{M}_2-\bar{I}_2+O(T^{1/2}y\mathscr{L}^8), \tag{3.17}$$

where

$$\mathscr{M}_1=\frac{1}{2\pi}\int_1^T\frac{\chi'}{\chi}(\tfrac{1}{2}-it)^3|B\zeta(\tfrac{1}{2}+it)|^2\,dt, \tag{3.18}$$

$$\mathscr{M}_2=-\frac{3}{2\pi}\int_1^T\frac{\chi'}{\chi}(\tfrac{1}{2}-it)|B\zeta'(\tfrac{1}{2}+it)|^2\,dt, \tag{3.19}$$

and

$$I_2=\frac{1}{2\pi i}\int_{c+i}^{c+iT}\chi(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)^2B(s)B(1-s)\,ds. \tag{3.20}$$

The integral in (3.14) taken along the right-hand side of $\mathscr{C}$ is equal to $-I_2$. Thus, by (3.15) and (3.17), we see that

$$S_2=\mathscr{M}_1+\mathscr{M}_2-2\Re I_2+O(T^{1/2}y\mathscr{L}^8). \tag{3.21}$$

The integrals in $\mathscr{M}_1$ and $\mathscr{M}_2$ have been evaluated (implicitly) by Conrey in [**2**]. Let

$$F(s)=\sum_{n=0}^{N}(-1)^n a_n\zeta^{(n)}(s)\mathscr{L}^{-n}=\sum_{m=1}^{\infty}Q\left(\frac{\log m}{\mathscr{L}}\right)m^{-s}, \tag{3.22}$$

where

$$Q(x)=\sum_{n=0}^{N}a_nx^n$$

is a polynomial with real coefficients. Let

$$\mathscr{M}=\int_T^{T+U}|BF(\tfrac{1}{2}+it)|^2\,dt, \tag{3.23}$$

where

$$U=T\mathscr{L}^{-10}$$

and $B(s)$ is as in (2.2). Then

$$\mathcal{M} = U\bigg(Q(0)^2 + \theta \int_0^1 \int_0^1 \bigg(P(u)Q'(v) + \frac{1}{\theta} P'(u)Q(v)\bigg)^2 du\, dv\bigg) + O(U\mathcal{L}^{-1}(\log \mathcal{L})^5), \tag{3.24}$$

provided $\theta < \frac{1}{2}$. Moreover, for $T \leqslant t \leqslant T + U$, we have

$$\frac{\chi'}{\chi}(\tfrac{1}{2} - it) = -\mathcal{L} + O(\mathcal{L}^{-10}). \tag{3.25}$$

Thus, we can replace χ'/χ by $-\mathcal{L}$ and sum integrals of length U to obtain estimates for $\mathcal{M}_1$ and $\mathcal{M}_2$. For $\mathcal{M}_1$ we take $Q_1(x) = 1$, and for $\mathcal{M}_2$ we take $Q_2(x) = -x$. We then obtain

$$\mathcal{M}_1 = \frac{-T\mathcal{L}^3}{2\pi}\bigg(1 + \frac{1}{\theta}\int_0^1 P'(u)^2\, du\bigg) + O(T\mathcal{L}^2(\log \mathcal{L})^5) \tag{3.26}$$

and

$$\mathcal{M}_2 = \frac{3T\mathcal{L}^3}{2\pi}\bigg(\frac{1}{2} + \theta \int_0^1 P(u)^2\, du + \frac{1}{3\theta}\int_0^1 P'(u)^2\, du\bigg) + O(T\mathcal{L}^2(\log \mathcal{L})^5). \tag{3.27}$$

We have shown that (3.26) and (3.27) are valid for $\theta < \frac{1}{2}$ as $T \to \infty$. It is worth remarking that (3.26) and (3.27) are valid for $\theta < \frac{4}{7}$, by the work in [**4**].

It remains to estimate I_1 and I_2 in (3.12) and (3.20). Indeed, this forms the main area of difficulty and requires a bit of preparation, in the form of lemmas, which we state in the next section.

4. *Preliminary lemmas*

LEMMA 1. *Let $r > 0$. Then, for any $c_0 > 0$,*

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s) r^{-s}\, ds = \begin{cases} e^{-2\pi i r} + E(r,c) r^{-c} & \text{if } r \leqslant T/2\pi, \\ E(r,c) r^{-c} & \text{otherwise}, \end{cases} \tag{4.1}$$

uniformly for $c_0 \leqslant c \leqslant 2$, where

$$E(r,c) \ll T^{c-1/2} + \frac{T^{c+1/2}}{|T - 2\pi r| + T^{1/2}}. \tag{4.2}$$

This is provided implicitly by Gonek in [**8**, Lemma 2].

LEMMA 2. *Suppose that $A(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ for $\sigma > 1$, where*

$$a(n) \ll \tau_{k_1}(n)(\log n)^{l_1}$$

for some non-negative integers k_1 and l_1. Let $B(s) = \sum_{n \leqslant y} b(n) n^{-s}$, where

$$b(n) \ll \tau_{k_2}(n)(\log n)^{l_2}$$

for non-negative integers k_2 and l_2 and where

$$T^{\epsilon} \ll y \ll T$$

for some $\epsilon > 0$. *Also, let*

$$I = \frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s)B(1-s)A(s)\,ds,$$

where $c = 1 + 1/\log T$. *Then we have*

$$I = \sum_{n\leqslant y} \frac{b(n)}{n} \sum_{m\leqslant nT/2\pi} a(m)e(-m/n) + O(T^{1/2}y(\log T)^A)$$

for some A.

REMARK. An admissible value for A is $k_1 + k_2 + l_1 + l_2$.

Proof. By Lemma 1 it suffices to show that

$$\sum_{n\leqslant y} |b(n)| \sum_m \frac{|a(m)|}{m^c} (T^{1/2} + f_{m,n}(T)) \ll yT^{1/2}(\log T)^A,$$

where

$$f_{m,n}(T) = \frac{T^{3/2}}{|T - 2\pi m/n| + T^{1/2}}.$$

Now by a theorem of Shiu [**15**], for any $\epsilon > 0$,

$$\sum_{X-Y<n\leqslant X} \tau_k(n)(\log n)^l \ll_\epsilon Y(\log X)^{k+l-1} \tag{4.3}$$

provided that $Y \gg X^\epsilon$. It follows easily that

$$\sum_{n\leqslant y} |b(n)| \sum_m \frac{|a(m)|}{m^c} \ll y(\log T)^A$$

with $A = k_1 + k_2 + l_1 + l_2 - 1$. Thus, the term with the $T^{1/2}$ is acceptable. Next, we break up the sum involving $f_{m,n}$ into three parts. The terms with $|T - 2\pi m/n| > \frac{1}{2}T$ have $f_{m,n}(T) \ll T^{1/2}$, so this case is like the one we have just discussed. Now consider the terms for which

$$T^{1/2} \leqslant |T - 2\pi m/n| \leqslant \tfrac{1}{2}T.$$

Assume first that

$$T^{1/2} \leqslant 2\pi m/n - T \leqslant \tfrac{1}{2}T,$$

the other inequality leading to a similar argument. We further refine the sum into $\ll \log T$ sums of the shape

$$T + P < 2\pi m/n < T + 2P$$

where $T^{1/2} \ll P \ll T$. For m and n satisfying this condition, $f_{m,n}(T) \ll P^{-1}$. Also, $m \approx nT$ and m ranges over an interval of length $\approx nP$ so that the contribution from one of these sums is

$$\ll T^{3/2} \sum_{n\leqslant y} |b(n)| \sum_m \frac{|a(m)|}{nT} P^{-1},$$

which, by Shiu's theorem, is

$$\ll yT^{1/2}(\log T)^A$$

for some A. Summing this over the $\ll \log T$ values of P contributes an additional factor of $\log T$.

Finally, we consider the contribution of the terms for which

$$|T - 2\pi m/n| \leqslant T^{1/2}.$$

For such values of m and n we have $f_{m,n}(T) \ll T^{-1/2}$. For a given n the admissible values of m are $\asymp nT$ in size and range over an interval of length $\asymp nT^{1/2}$. Thus, Shiu's theorem is applicable and we find that the total contribution from these terms is

$$\ll T^{3/2} \sum_{n \leqslant y} |b(n)| \sum_m \frac{|a(m)|}{nT} T^{-1/2} \ll yT^{1/2}(\log T)^A$$

for some A. This concludes the proof.

LEMMA 3. *Suppose that*

$$A_j(s) = \sum_{m=1}^{\infty} a_j(m) m^{-s}$$

and that

$$A(s) = \sum_{m=1}^{\infty} a(m) m^{-s} = \prod_{j=1}^{J} A_j(s),$$

for some natural number $J \geqslant 1$. Then for any integer $d > 0$ and any completely multiplicative function f, we have

$$\sum_{m=1}^{\infty} f(m) a(md) m^{-s} = \sum_{d = d_1 \ldots d_J} \prod_{j=1}^{J} \sum_{(m, d_1 \ldots d_{j-1})=1}^{\infty} f(m) a_j(m d_j) m^{-s}.$$

Proof. The assertion for $J > 2$ follows from $J - 1$ applications of the case $J = 2$, which we now prove. This case follows immediately on observing that for any two arithmetical functions α and β,

$$(\alpha * \beta)(md) = \sum_{\substack{g \mid m \\ (h, m/g)=1}} \sum_{h \mid d} \alpha(gh) \beta\left(\frac{md}{gh}\right);$$

for the right-hand side is equal to

$$\sum_{l \mid md} \alpha(l) \beta\left(\frac{md}{l}\right) \sum_{\substack{gh=l \\ (h, m/g)=1}} 1 = \sum_{l \mid md} \alpha(l) \beta\left(\frac{md}{l}\right) \sum_{\substack{g \mid l \\ (l,m)=g}} 1,$$

which is the same as the left-hand side.

LEMMA 4 (Perron's formula). *Suppose that $A(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ for $\sigma > 1$,* *where*

$$|a(n)| \leqslant K\tau_k(n)(\log n)^l.$$

Then for any $\varepsilon > 0$ *we have*

$$R := \sum_{n \leqslant x} a(n) - \frac{1}{2\pi i}\int_{c-iU}^{c+iU} A(s)\,\frac{x^s}{s}\,ds \ll_\varepsilon Kx^\varepsilon(1 + xU^{-1}),$$

where $c = 1 + (\log x)^{-1}$ *and the implied constant is independent of* K*. If* $U \ll x^{1-\varepsilon}$ *for some* $\varepsilon > 0$*, then*

$$R \ll_\varepsilon KxU^{-1}(\log x)^{k+l}.$$

Proof. By the lemma of § 17 of Davenport [**6**],

$$R \ll \sum_{\substack{n=1\\ n\neq x}}^{\infty} \left(\frac{x}{n}\right)^c |a(n)| \min(1, U^{-1}|\log x/n|^{-1}) + |a(x)|,$$

the last term occurring only if x is an integer. The lemma follows easily from this, (4.3), and the fact that $a(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$.

LEMMA 5. *Suppose that* y, $d \ll T$ *with* $\log y \gg \log T$*. Let*

$$\Sigma(d,M) := \sum_{k \leqslant y/d} \frac{b(kd)}{k}(\log k)^M.$$

Then for any $\varepsilon > 0$*,*

$$\Sigma(d,M) = \begin{cases} (-1)^M \dfrac{\mu(d)d}{\phi(d)} P^{(1-M)}\left(\dfrac{\log y/d}{\log y}\right)(\log y)^{M-1} + O_\varepsilon(E_M) & \text{if } 0 \leqslant M \leqslant 1,\\ O_\varepsilon(E_M) & \text{if } M \geqslant 2,\end{cases}$$

where

$$E_M = F_1(d)\mathscr{L}^{M-2+\varepsilon}(1 + \mathscr{L}(d/y)^b)$$

with $b = (C\log\mathscr{L})^{-1}$ *for some constant* $C > 0$ *and*

$$F_1(d) = \prod_{p\mid d}(1 + p^{-3/4}).$$

In all cases,

$$\Sigma(d,M) \ll_\varepsilon \mathscr{L}^{M-1+\varepsilon}F_1(d).$$

This follows from Lemma 10 of Conrey [**2**].

LEMMA 6. *If* Q *is a polynomial, then*

$$\sum_{d \leqslant y} \frac{\mu^2(d)}{\phi(d)} Q\left(\frac{\log y/d}{\log y}\right) = \left(\int_0^1 Q(u)\,du\right)\log y + O(1).$$

This is well-known in the case $Q(x) \equiv 1$ and the general case may be deduced from this particular one.

LEMMA 7. *Suppose that Q is a squarefree positive integer. Then for bounded $r \geqslant 0$ and fixed σ, with $0 < \sigma < 1$, we have*

$$\sum_{d \mid Q} \tau_r(d) d^{-\sigma} \ll \exp(C(\log Q)^{1-\sigma}(\log\log Q)^{-1})$$

and

$$\sum_{d \mid Q} \tau_r(d) d^{-1} \ll (\log\log Q)^C$$

for some positive constant C which is independent of Q.

Proof. Since the summand is multiplicative, we have

$$\log \sum_{d \mid Q} \tau_r(d) d^{-\sigma} = \sum_{p \mid Q} \log(1 + \tau_r(p) p^{-\sigma}) \ll \sum_{p \mid Q} p^{-\sigma}.$$

If the primes $p \mid Q$ are denoted $p_1, \dots, p_n$ and the first n primes are $q_1, \dots, q_n$, then $q_n \leqslant 2 \log Q$ if Q is large enough. Moreover, $p_i^{-\sigma} \leqslant q_i^{-\sigma}$ so that, by the Prime Number Theorem, the above sum is

$$\ll \sum_{q \leqslant 2 \log Q} q^{-\sigma} \ll \int_{2^-}^{2 \log Q} u^{-\sigma} \, d\pi(u)$$

$$\ll \begin{cases} (\log Q)^{1-\sigma}(\log\log Q)^{-1} & \text{if } 0 < \sigma < 1, \\ \log\log\log Q & \text{if } \sigma = 1. \end{cases}$$

The result now follows.

5. *Estimation of I_1 and I_2*

By (3.12) and (3.20), we have

$$I_\nu = \frac{1}{2\pi i} \int_{c+i}^{c+iT} \chi(1-s) A_\nu(s) B(1-s) \, ds, \tag{5.1}$$

where $\nu = 1$ and 2, with

$$A_1(s) = \frac{\zeta'}{\zeta}(s) \zeta'(s) = \sum_1^\infty a_1(m) m^{-s} \tag{5.2}$$

and

$$A_2(s) = \frac{\zeta'}{\zeta}(s) \zeta'(s)^2 B(s) = \sum_1^\infty a_2(m) m^{-s}. \tag{5.3}$$

Thus, by Lemma 2 with $k = 4$ and $l = 3$, we get $I_\nu = \mathscr{M}_\nu + E_\nu$ where

$$\mathscr{M}_\nu = \sum_{k \leqslant y} \sum_{m \leqslant kT/2\pi} \frac{a_\nu(m) b(k)}{k} e\Big(-\frac{m}{k}\Big) \tag{5.4}$$

and

$$E_\nu \ll T^{1/2} y \mathscr{L}^7. \tag{5.5}$$

To evaluate $\mathscr{M}_\nu$ we apply Perron's formula to the sum over m and the main term arises from the residue of the pole at $s = 1$ of the generating function. If we assume the Generalized Riemann Hypothesis, then the error terms are easily handled. For our less conditional treatment, however, we must use Perron's formula at a later stage after some preliminary rearrangements of the sum.

We first express the additive character $e(\ )$ in terms of multiplicative characters. Thus, if $m/k = M/K$ where $(M, K) = 1$, then

$$e\left(-\frac{m}{k}\right) = e\left(-\frac{M}{K}\right) = \frac{1}{\phi(K)} \sum_{\chi \bmod K} \tau(\bar{\chi})\chi(-M) \tag{5.6}$$

where, as usual for a character χ mod K,

$$\tau(\chi) = \sum_{a=1}^{K} \chi(a)e(a/K). \tag{5.7}$$

Note, for use in § 8, that $\tau(\chi_0) = \mu(K)$, where χ_0 is the principal character mod K, so that the contribution of the principal character to this expression for $e(-m/k)$ is just $\mu(K)/\phi(K)$. Now we wish to reduce the sum to a sum over primitive characters. If ψ mod q induces χ mod K, then (see [**6**, p. 67])

$$\tau(\chi) = \mu\left(\frac{K}{q}\right)\psi\left(\frac{K}{q}\right)\tau(\psi). \tag{5.8}$$

Thus, by (5.7),

$$e\left(-\frac{m}{k}\right) = \frac{1}{\phi(K)} \sum_{q\,|\,K} \sum_{\psi}{}^{*} \mu\left(\frac{K}{q}\right)\bar{\psi}\left(\frac{K}{q}\right)\tau(\bar{\psi})\psi(-M) \tag{5.9}$$

since $(M, K) = 1$ implies that $(M, q) = 1$, whence $\chi(-M) = \psi(-M)$; here and elsewhere the * indicates that the sum is over all primitive characters mod q. (The character mod 1 which induces all other principal characters will be included as a primitive character.) Finally, we wish to eliminate the dependence in our formula on $g = \gcd(m, k)$. By the Möbius inversion formula, for any f,

$$f(M, K) = f\left(\frac{m}{g}, \frac{k}{g}\right) = \sum_{d\,|\,g} \sum_{e\,|\,d} \mu\left(\frac{d}{e}\right) f\left(\frac{m}{e}, \frac{k}{e}\right). \tag{5.10}$$

The condition $d\,|\,g$ is equivalent to $d\,|\,m$, $d\,|\,k$. Thus,

$$\begin{aligned}
e\left(-\frac{m}{k}\right) &= \sum_{\substack{d\,|\,m\\ d\,|\,k}} \sum_{e\,|\,d} \frac{\mu(d/e)}{\phi(k/e)} \sum_{q\,|\,k/e} \sum_{\psi}{}^{*} \mu\left(\frac{k}{eq}\right)\bar{\psi}\left(\frac{k}{eq}\right)\tau(\bar{\psi})\psi\left(-\frac{m}{e}\right) \\
&= \sum_{q\,|\,k} \sum_{\psi}{}^{*} \tau(\bar{\psi}) \sum_{\substack{d\,|\,m\\ d\,|\,k}} \sum_{\substack{e\,|\,d\\ e\,|\,k/q}} \frac{\mu(d/e)}{\phi(k/e)} \bar{\psi}\left(\frac{-k}{eq}\right)\psi\left(\frac{m}{e}\right)\mu\left(\frac{k}{eq}\right) \\
&= \sum_{q\,|\,k} \sum_{\psi}{}^{*} \tau(\bar{\psi}) \sum_{\substack{d\,|\,m\\ d\,|\,k}} \psi\left(\frac{m}{d}\right)\delta(q, k, d, \psi),
\end{aligned} \tag{5.11}$$

where for a character ψ mod q we define

$$\delta(q,k,d,\psi) = \sum_{\substack{e\,|\,d \\ e\,|\,k/q}} \frac{\mu(d/e)}{\phi(k/e)} \bar{\psi}\left(\frac{-k}{eq}\right)\psi\left(\frac{d}{e}\right)\mu\left(\frac{k}{eq}\right). \tag{5.12}$$

Note that if k is squarefree, then

$$|\delta(q,k,d,\psi)| \leqslant \sum_{\substack{e\,|\,d \\ e\,|\,k/q}} \frac{\phi(e)}{\phi(k)} = \frac{(d,k/q)}{\phi(k)}. \tag{5.13}$$

Now by (5.4) and (5.11), if we replace k by kq and m by md and rearrange, then

$$\begin{aligned}\mathscr{M}_\nu &= \sum_{q\leqslant y} \sideset{}{^*}\sum_{\psi} \tau(\bar{\psi}) \sum_{k\leqslant y/q} \frac{b(kq)}{kq} \sum_{d\,|\,kq} \delta(q,kq,d,\psi) \sum_{m\leqslant kqT/2\pi d} a_\nu(md)\psi(m) \\ &= \sum_{k\leqslant y} \frac{1}{k} \mathscr{N}_\nu\left(\frac{y}{k}, \frac{kT}{2\pi}, k\right),\end{aligned} \tag{5.14}$$

where

$$\mathscr{N}_\nu(\xi,z,k) = \sum_{q\leqslant \xi} \frac{b(kq)}{q} \sideset{}{^*}\sum_{\psi} \tau(\bar{\psi}) \sum_{d\,|\,kq} \delta(q,kq,d,\psi) \sum_{m\leqslant qz/d} a_\nu(md)\psi(m). \tag{5.15}$$

To estimate this we distinguish the cases $q=1$, $1<q\leqslant\eta=\mathscr{L}^A$ for some $A>0$, and $\eta<q\leqslant\xi\leqslant y$. The main term arises from $q=1$; the case $1<q\leqslant\eta$ is treated using Siegel's Theorem, and the case $\eta<q\leqslant\xi$ depends on an identity for A_ν and the large sieve. This is analogous to the proofs of Bombieri's Theorem given by Gallagher [**7**] and Vaughan [**17**].

In § 6 we consider small values of $q>1$; in § 7 we consider larger values of q, and in § 8 we evaluate the main terms.

6. *Small values of q*

By (5.2) and (5.3),

$$a_\nu(md) \ll \tau_4(m)\tau_4(d)(1+(\log m)^3) \quad (\nu = 1,2). \tag{6.1}$$

Therefore, by Perron's formula (Lemma 4) with $U = \exp((\log w)^{1/2})$, $T \ll w \ll T^2$, and $d \ll T$ we have

$$\sum_{m\leqslant w} a_\nu(md)\psi(m) = \frac{1}{2\pi i}\int_{\theta-iU}^{\theta+iU} A_\nu(s,\psi,d) w^s \frac{ds}{s} + O\left(\tau_4(d)\frac{w}{U}\mathscr{L}^C\right), \tag{6.2}$$

where $\theta = 1+(\log w)^{-1}$, $C>0$ is some constant, and

$$A_\nu(s,\psi,d) = \sum_{m=1}^{\infty} \frac{a_\nu(md)\psi(m)}{m^s}, \quad \sigma > 1. \tag{6.3}$$

We apply Lemma 3 to show that $A_\nu(s,\psi,d)$ has an analytic continuation to the left of $\sigma = 1$. We have by (5.2),

$$A_1(s,\psi,d) = \sum_{d_1d_2=d}\left(\sum_{m=1}^{\infty}\frac{\Lambda(md_1)\psi(m)}{m^s}\right)\left(\sum_{\substack{m=1\\(m,d_1)=1}}^{\infty}\frac{\psi(m)\log md_2}{m^s}\right)$$

$$= \sum_{d_1d_2=d}\left(I(d_1)\frac{L'}{L}(s,\psi) - \frac{\Lambda(d_1)}{1-\psi(P)P^{-s}}\right)d_2^s\frac{d}{ds}\left(\frac{L(s,\psi)\Phi(s,\psi,d_1)}{d_2^s}\right), \tag{6.4}$$

where

$$I(d) = \begin{cases} 1 & \text{if } d = 1,\\ 0 & \text{if } d \neq 1,\end{cases}$$

$$\Phi(s,\psi,d) = \prod_{p\mid d}(1-\psi(p)p^{-s}) = \sum_{n\mid d}\frac{\mu(n)\psi(n)}{n^s},$$

and, if $\Lambda(d_1) \neq 0$, then $P = P(d_1)$ is such that $\Lambda(d_1) = \log P$. Similarly, by (5.3),

$$\begin{aligned} A_2(s,\psi,d) = \sum_{d_1d_2d_3d_4=d}&\left(I(d_1)\frac{L'}{L}(s,\psi) - \frac{\Lambda(d_1)}{1-\psi(P)P^{-s}}\right)\\ &\times d_2^s d_3^s\frac{d}{ds}\left(\frac{L(s,\psi)\Phi(s,\psi,d_1)}{d_2^s}\right)\\ &\times\frac{d}{ds}\left(\frac{L(s,\psi)\Phi(s,\psi,d_1d_2)}{d_3^s}\right)B(s,\psi,d_1d_2d_3,d_4),\end{aligned} \tag{6.5}$$

where

$$B(s,\psi,d,e) = \sum_{\substack{(m,d)=1\\ m\leqslant y/e}}\frac{b(me)\psi(m)}{m^s}. \tag{6.6}$$

Thus we see that $A_\nu(s,\psi,d)$ has a meromorphic continuation to the whole plane. If ψ is the principal character mod 1, then $A_\nu(s,\psi,d)$ has a pole at $s=1$. For any other character, $A_\nu(s,\psi,d)$ has at most one (simple) pole in the region

$$\sigma \geqslant \sigma_o(t) := 1 - \frac{c}{\log q(1+|t|)},$$

where c is an absolute constant. By Siegel's theorem, this pole, if it exists, is at a real number β which satisfies

$$1-\beta \gg_\varepsilon q^{-\varepsilon} \tag{6.7}$$

for any $\varepsilon > 0$.

We will move the path of integration in (6.2) to the segment $\sigma = \sigma_o(U)$, $|t| \leqslant U$ and estimate the integral on the new path. In doing so, we cross the pole at $s=1$ if $q=1$ and the pole at $s=\beta$, if it exists. Define

$$R_\nu(w,d) = \operatorname*{Res}_{s=1} A_\nu(s,1,d)\frac{w^s}{s}. \tag{6.8}$$

Let Γ be the path consisting of three line segments $\{\sigma - iU\colon\ \theta \geqslant \sigma \geqslant \sigma_o\}$, $\{\sigma_o + it\colon\ -U \leqslant t \leqslant U\}$, and $\{\sigma + iU\colon\ \sigma_o \leqslant \sigma \leqslant \theta\}$. Then by (6.2), (6.8), and Cauchy's theorem,

$$\sum_{m \leqslant w} a_\nu(md)\psi(m) - I(q)R_\nu(w,d)$$

$$\ll \int_\Gamma \left| A_\nu(s,\psi,d)\,\frac{w^s}{s}\,ds \right| + \left| \operatorname*{Res}_{s=\beta} A_\nu(s,\psi,d)\,\frac{w^s}{s} \right| + \tau_4(d)\,\frac{w}{U}\,\mathscr{L}^C. \tag{6.9}$$

By (6.4), (6.5), (6.6), and standard estimates,

$$A_\nu(s,\psi,d) \ll \tau_4(d)\mathscr{L}^C y^{1-\sigma_o}$$

for $\sigma \geqslant \sigma_o(U)$, $|t| \leqslant U$, $|s-1| \gg \mathscr{L}^{-1}$, and $|s-\beta| \gg \mathscr{L}^{-1}$, where $C>0$ is an absolute constant. Moreover, by the definitions of U, w, and σ_o,

$$w^{\sigma_o(U)} \ll_A w\exp(-C(A)\mathscr{L}^{1/2})$$

for $q \leqslant \eta = \mathscr{L}^A$, where $C(A)$ denotes a positive function of A not necessarily the same at each occurrence. With the choice

$$\varepsilon = 1/2A,$$

it follows from (6.7) that

$$w^\beta \ll_A w\exp(-C(A)\mathscr{L}^{1/2})$$

for $q \leqslant \eta$. Hence, by (6.9),

$$\sum_{m \leqslant w} a_\nu(md)\psi(m) - I(q)R_\nu(w,d) \ll_A \tau_4(d) w \exp(-C(A)\mathscr{L}^{1/2}) \tag{6.10}$$

uniformly for $T \ll w \ll T^2$ and $q \leqslant \eta = \mathscr{L}^A$, where $A>0$ is any fixed constant. We use this in (5.15). Now

$$|\tau(\psi)| = q^{1/2} \tag{6.11}$$

and by (5.13), if kq is squarefree, then

$$|\delta(q,kq,d,\psi)| \leqslant \frac{(d,k)}{\phi(k)\phi(q)}. \tag{6.12}$$

Further, for kQ squarefree, $Q \ll T$, $k \leqslant y$, and $\sigma = \frac{1}{2}$ or $\sigma = 1$, we have

$$\frac{1}{\phi(k)} \sum_{d\,|\,kQ} \tau_r(d) d^{-\sigma}(d,k) = \frac{1}{\phi(k)} \sum_{d\,|\,k} \tau_r(d) d^{1-\sigma} \sum_{e\,|\,Q} \tau_r(e) e^{-\sigma}$$

$$\ll \begin{cases} \tau_r(k)k^{-1}\mathscr{L} & \text{if } \sigma = 1, \\ \tau_r(k)k^{-1/2}\exp(C\mathscr{L}^{1/2}) & \text{if } \sigma = \frac{1}{2}, \end{cases} \tag{6.13}$$

by Lemma 10 and since $k \ll \phi(k)\log\log k$. Here r stands for a non-negative integer, which is bounded above and which is not necessarily the same at each occurrence. Thus, by (5.15) and (6.10)–(6.13), for $w = qz/d$, $z \gg T$, and any

$A > 0$, we find that

$$N_\nu(\eta, z, k) - b(k) \sum_{d \mid k} \delta(1, k, d, 1) R_\nu\Big(\frac{z}{d}, d\Big) \ll_A z \frac{\tau_r(k)}{k} \mathscr{L}^C \sum_{q \leqslant \eta} q^{1/2} \exp(-C(A)\mathscr{L}^{1/2})$$

$$\ll_A \tau_r(k) \frac{z}{k} \exp(-C(A)\mathscr{L}^{1/2}). \tag{6.14}$$

We now consider the larger values of q.

7. *Large sieve estimates*

For larger q we use Perron's formula (Lemma 4) with $U = T^{20}$, $T \ll w \ll T^2$, and $\theta = 1 + (\log w)^{-1}$ to obtain

$$\sum_{m \leqslant w} a_\nu(md)\psi(m) = \frac{1}{2\pi i} \int_{\theta - iU}^{\theta + iU} A_\nu(s, \psi, d) w^s \frac{ds}{s} + O_\varepsilon(w^\varepsilon). \tag{7.1}$$

Then by (5.15), (6.11), and (6.12),

$$N_\nu(\xi, z, k) - N_\nu(\eta, z, k) \ll_\varepsilon \max_{Q \leqslant \xi} \sum_{d \mid kQ} \frac{\mu^2(kQ)(d, k)}{\phi(k)} \int_\eta^\xi x^{-1}\, dS(x) + T^\varepsilon \xi^{1/2}, \tag{7.2}$$

where

$$S(x) = \sum_{q \leqslant x} \frac{q^{1/2}}{\phi(q)} \sum_\psi{}^{*} \left| \int_{\theta - iU}^{\theta + iU} A_\nu(s, \psi, d) \Big(\frac{qz}{d}\Big)^s \frac{ds}{s} \right| \tag{7.3}$$

for $z \ll Ty$. We use an identity for A_ν which corresponds to Vaughan's identity. We have

$$A_\nu = H_\nu + I_\nu \quad (\nu = 1, 2), \tag{7.4}$$

where

$$H_\nu = H_\nu(s, \psi, d) = (A_\nu - F_\nu)(1 - LG) \tag{7.5}$$

and

$$I_\nu = I_\nu(s, \psi, d) = F_\nu - F_\nu LG + A_\nu LG; \tag{7.6}$$

here, F_ν is a partial sum of A_ν,

$$F_\nu = F_\nu(s, \psi, d) = \sum_{m \leqslant u} \frac{a_\nu(md)\psi(m)}{m^s}, \tag{7.7}$$

$L = L(s, \psi)$, and G is a partial sum of $L(s, \psi)^{-1}$,

$$G = G(s, \psi) = \sum_{m \leqslant v} \frac{\mu(m)\psi(m)}{m^s}. \tag{7.8}$$

For the parameters u and v we will take

$$u = x^2, \quad v = T^{1/2}. \tag{7.9}$$

Now we have

$$\int_{\theta - iU}^{\theta + iU} A_\nu(s, \psi, d) \frac{w^s}{s}\, ds = \int_{\theta - iU}^{\theta + iU} (H_\nu + I_\nu)(s, \psi, d) \frac{w^s}{s}\, ds. \tag{7.10}$$

By (6.4) and (6.5), $L(s,\psi)A_\nu(s,\psi,d)$ is regular for $\sigma>0$; hence the same is true of I_ν. Therefore, by Cauchy's theorem, if $q>1$, then

$$\int_{\theta-iU}^{\theta+iU} I_\nu(s,\psi,d)\,\frac{w^s}{s}\,ds = \int_\Gamma I_\nu(s,\psi,d)\,\frac{w^s}{s}\,ds, \tag{7.11}$$

where Γ is the path consisting of three line segments $\{\sigma - iU\colon\ \frac12 \leqslant \sigma \leqslant \theta\}$, $\{\frac12 + it\colon\ -U\leqslant t\leqslant U\}$, and $\{\sigma + iU\colon\ \frac12\leqslant\sigma\leqslant\theta\}$. We can easily bound the integral along the horizontal segments of Γ. By (6.4) and (6.5),

$$\begin{aligned} A_\nu(s,\psi,d)L(s,\psi) \ll \sum_{d_1d_2d_3d_4=d} &(1+|L(s,\psi)|+|L'(s,\psi)|)^3 \\ &\times|\mathscr{B}(s,\psi)|\prod_{p\mid d}(1+p^{-\sigma})^2\log^3 d, \end{aligned} \tag{7.12}$$

where

$$\mathscr{B}(s,\psi)=\sum_{m\leqslant y}\frac{\beta(m)\psi(m)}{m^s} \tag{7.13}$$

for some coefficients β which depend on d_1, d_2, d_3, and d_4, but satisfy

$$\beta(n)\ll 1 \tag{7.14}$$

uniformly in n and the d_i. Thus, for some fixed r,

$$A_\nu L \ll \tau_r(d)\mathscr{L}^3(1+|L|+|L'|)^3|\mathscr{B}| \tag{7.15}$$

for $d\ll T$ and $\sigma\geqslant\frac12$. If $1<q\ll T$ and $\sigma\geqslant\frac12$, then, it is well-known that

$$L(s,\psi)\ll(q|s|)^{1/4},\quad L'(s,\psi)\ll(q|s|)^{1/4}\mathscr{L} \tag{7.16}$$

and it is trivial that

$$F_\nu(s,\psi,d)\ll_\varepsilon y^{1+\varepsilon},\quad G(s,\psi),\ \mathscr{B}(s,\psi)\ll T^{1/4}. \tag{7.17}$$

Since $U=T^{20}$, it follows from these estimates and (7.11) that

$$\int_{\theta-iU}^{\theta+iU} I_\nu(s,\psi,d)\,\frac{w^s}{s}\,ds = \int_{1/2-iU}^{1/2+iU} I_\nu(s,\psi,d)\,\frac{w^s}{s}\,ds + O(T^{-1}). \tag{7.18}$$

Now by (5.15), (6.12), (6.14), (7.2), (7.3), (7.4), (7.10), and (7.18),

$$\begin{aligned} N_\nu(\xi,z,k) &- b(k)\sum_{d\mid k}\delta(1,k,d,1)R_\nu\Big(\frac{z}{d},d\Big) \\ &\ll_{A,\varepsilon}\tau_r(k)\,\frac{z}{k}\exp(-C(A)\mathscr{L}^{1/2})+\xi^{1/2}T^\varepsilon \\ &\quad + z\max_{Q\leqslant\xi}\sum_{d\mid kQ}\frac{g\mu^2(kQ)}{\phi(k)d}\int_\eta^\xi x^{-1}d\mathscr{H}_\nu(x) \\ &\quad + z^{1/2}\max_{Q\leqslant\xi}\sum_{d\mid kQ}\frac{g\mu^2(kQ)}{\phi(k)d^{1/2}}\int_\eta^\xi x^{-1}\,d\mathscr{I}_\nu(x), \end{aligned} \tag{7.19}$$

where $g=(d,k)$,

$$\mathscr{H}_\nu(x)=\sum_{q\leqslant x}\frac{q^{3/2}}{\phi(q)}\sum_\psi{}^*\int_{-U}^{U}|H_\nu(\theta+it)|\frac{dt}{\theta+|t|},\tag{7.20}$$

and

$$\mathscr{I}_\nu(x)=\sum_{q\leqslant x}\frac{q}{\phi(q)}\sum_\psi{}^*\int_{-U}^{U}|I_\nu(\tfrac12+it)|\frac{dt}{\frac12+|t|}.\tag{7.21}$$

To estimate $\mathscr{H}_\nu$ and $\mathscr{I}_\nu$ we can use Hölder's inequality and the large sieve inequality in the form

$$\sum_{q\leqslant x}\frac{q}{\phi(q)}\sum_\psi{}^*\int_{-U}^{U}\left|\sum_n a_n\psi(n)n^{-it}\right|^2\frac{dt}{a+|t|}$$
$$\ll\sum_n(n+x^2\log U)|a_n|^2;\tag{7.22}$$

this holds uniformly for $a\geqslant\frac12$ (see Vaughan [**17**]).

By (6.1), (6.3), (7.7), and (7.22),

$$\sum_{q\leqslant x}\frac{q}{\phi(q)}\sum_\psi{}^*\int_{-U}^{U}|A_\nu(\theta+it)-F_\nu(\theta+it)|^2\frac{dt}{\theta+|t|}$$
$$\ll\sum_{m\geqslant u}(m+x^2\mathscr{L})|a_\nu(md)|^2m^{-2\theta}$$
$$\ll(1+x^2u^{-1})\mathscr{L}^C\tau_4(d)^2\tag{7.23}$$

for some C. Similarly, by (7.8),

$$\sum_{q\leqslant x}\frac{q}{\phi(q)}\sum_\psi{}^*\int_{-U}^{U}|1-LG(\theta+it,\psi)|^2\frac{dt}{\theta+|t|}$$
$$\ll\sum_{m>1}(m+x^2\mathscr{L})\left|\sum_{\substack{e\mid m\\ e\leqslant v}}\mu(e)\psi(e)\psi\left(\frac{m}{e}\right)\right|^2m^{-2\theta}$$
$$\ll(1+x^2v^{-1})\mathscr{L}^C\tag{7.24}$$

for some $C>0$, since the sum over e is 0 if $m\leqslant v$ and is $\ll\tau(m)$ in any case. Hence, by Cauchy's inequality, (7.5), (7.9), (7.20), (7.23) and (7.24),

$$\mathscr{H}_\nu(x)\ll\tau_r(d)\mathscr{L}^C(1+x^2v^{-1})^{1/2}x^{1/2}$$
$$\ll\tau_r(d)x^{1/2}(1+xT^{-1/4})\mathscr{L}^C\tag{7.25}$$

for some $C>0$.

Now for an arbitrary $f=f(s,\psi)$ let

$$\mathscr{A}(f)=\sum_{q\leqslant x}\frac{q}{\phi(q)}\sum_\psi{}^*\int_{-U}^{U}|f(\tfrac12+it,\psi)|\frac{dt}{\frac12+|t|},\tag{7.26}$$

so that $\mathscr{I}_\nu(x) = \mathscr{A}(I_\nu)$. Then by (7.6), (7.15), and Hölder's inequality, we have

$$\mathscr{I}_\nu(x) \ll A(F_\nu^2)^{1/2}\mathscr{A}(1)^{1/2} + \mathscr{A}(F_\nu^2)^{1/2}\mathscr{A}(L^4)^{1/4}\mathscr{A}(G^4)^{1/4}$$
$$+ \tau_4(d)\mathscr{L}^5\mathscr{A}(1+|L|^6+|L'|^6)^{1/2}\mathscr{A}(\mathscr{B}^4)^{1/4}\mathscr{A}(G^4)^{1/4}. \qquad (7.27)$$

By (7.7), (7.8), (7.13), (7.22), and (6.1),

$$A(F_\nu^2) \ll (x^2+u)\mathscr{L}^C\tau_4^2(d),$$
$$A(G^4) \ll (x^2+v^2)\mathscr{L}^C,$$

and

$$A(\mathscr{B}^4) \ll (x^2+y^2)\mathscr{L}^C,$$

for some constant $C > 0$. Also, $\mathscr{A}(1) \ll x^2\mathscr{L}$, and by the method of Montgomery [**11**] (see also Ramachandra [**14**]),

$$\mathscr{A}(|L|^k),\ \mathscr{A}(|L'|^k) \ll_\varepsilon x^2T^\varepsilon \quad (k = 2 \text{ or } 4). \qquad (7.28)$$

We also need (7.28) with $k = 6$ which is our sixth moment assumption referred to in the introduction. This follows from GLH and this is the only place we use GLH.

Thus, by (7.9), (7.27), and the above, we find that

$$\mathscr{I}_\nu(x) \ll_\varepsilon \tau_4(d)T^\varepsilon(x^{1/2}(x^2+u)^{1/2}(x^2+v^2)^{1/4} + x(x^2+y^2)^{1/4}(x^2+v^2)^{1/4})$$
$$\ll_\varepsilon \tau_4(d)T^\varepsilon(x^2 + x^{3/2}T^{1/4} + x^{3/2}y^{1/2} + xT^{1/4}y^{1/2}). \qquad (7.29)$$

Next, by (7.25),

$$\int_\eta^\xi x^{-1}\,d\mathscr{H}_\nu(x) \ll (\eta^{-1/2} + \xi^{1/2}T^{-1/4})\tau_r(d)\mathscr{L}^C. \qquad (7.30)$$

By (7.29),

$$\int_\eta^\xi x^{-1}\,d\mathscr{I}_\nu(x) \ll_\varepsilon (\xi + \xi^{1/2}T^{1/4} + \xi^{1/2}y^{1/2} + T^{1/4}y^{1/2})\tau_4(d)T^\varepsilon. \qquad (7.31)$$

Hence, by (7.19), (7.30), and (7.31),

$$\mathscr{N}_\nu(\xi,z,k) - b(k)\sum_{d\mid k}\delta(1,k,d,1)R_\nu\Big(\frac{z}{d},d\Big)$$
$$\ll_{A,\varepsilon} \tau_r(k)\Big(\frac{z}{k}\Big)\exp(-C(A)L^{1/2}) + \tau_r(k)\Big(\frac{z}{k}\Big)(\eta^{-1/2}+\xi^{1/2}T^{-1/4})\mathscr{L}^C$$
$$+ \tau_r(k)\Big(\frac{z}{k}\Big)^{1/2}(\xi + \xi^{1/2}T^{1/4} + \xi^{1/2}y^{1/2} + T^{1/4}y^{1/2})T^\varepsilon.$$

We use this estimate in (5.14); after substituting $\xi = y/k$, $z = Tk/(2\pi)$ and summing over k, we have

$$\mathscr{M}_\nu - R_\nu \ll_{A,\varepsilon} T\exp(-C(A)\mathscr{L}^{1/2}) + T\eta^{-1/2}\mathscr{L}^C + (T^{3/4}y^{1/2} + T^{1/2}y)T^\varepsilon,$$

for some fixed $C > 0$ and $\eta = \mathscr{L}^A$ for $A > 0$ to be chosen. We take $A = 2(A'+C)$ and have for any $A' > 0$,

$$\mathscr{M}_\nu = R_\nu + O_{A',\varepsilon}(T\mathscr{L}^{-A'} + (T^{3/4}y^{1/2} + T^{1/2}y)T^\varepsilon).$$

This gives $\mathscr{M}_\nu = R_\nu + o(T)$ provided $y \ll T^\theta$ with $\theta < \frac{1}{2}$.

The rest of the paper is devoted to estimating the main term.

8. *The main terms*

In the sequel all equalities should be taken as asymptotic equalities. We also assume throughout that $y = T^{\theta}$ with $\theta < \frac{1}{2}$. Since the contributions from $q > 1$ have been shown to be negligible, only the principal character term, $q = 1$, contributes. This essentially means that for the purposes of obtaining main terms one may replace $e(-m/k)$ by $\mu(K)/\phi(K)$ (compare (5.6), (5.7), and the remark below (5.7)). Hence we may write the expressions for $\mathcal{M}_{\nu}$ more transparently as

$$\mathcal{M}_{\nu} = \sum_{k \leqslant y} \sum_{m \leqslant kT/2\pi} \frac{a_{\nu}(m)b(k)}{k} \frac{\mu(K)}{\phi(K)}.$$

We now describe a simple way of obtaining the main term for $\mathcal{M}_1$. We may easily recast our expression for $\mathcal{M}_1$ as

$$\mathcal{M}_1 = \sum_{l \leqslant y} \sum_{k \leqslant y/l} \frac{b(kl)}{kl} \frac{\mu(k)}{\phi(k)} \sum_{\substack{m \leqslant kT/2\pi \\ (m,k)=1}} a_1(ml). \tag{8.1}$$

Observe that $a_1(n) = \sum_{d|n} \Lambda(d) \log(n/d) = \log^2 n - \sum_{d|n} \Lambda(d) \log d$. We will replace $\sum_{d|n} \Lambda(d) \log d$ by $\sum_{p|n} \log^2 p$; the overall error in evaluating $\mathcal{M}_1$ caused by this is easily seen to be $O(T \log T)$. Thus,

$$\begin{aligned}
\sum_{\substack{m \leqslant kT/2\pi \\ (m,k)=1}} a_1(ml) &= \sum_{\substack{m \leqslant kT/2\pi \\ (m,k)=1}} \left(\log^2(ml) - \sum_{p|l} \log^2 p - \sum_{\substack{p|m \\ (p,l)=1}} \log^2 p \right) \\
&= \phi(k) \frac{T}{2\pi} \left((\log klT)^2 - \sum_{p|l} \log^2 p \right) \\
&\quad - \sum_{\substack{p \leqslant kT/2\pi \\ (p,kl)=1}} \log^2 p \sum_{\substack{m \leqslant kT/2\pi p \\ (m,k)=1}} 1 \\
&= \phi(k) \frac{T}{2\pi} \left((\log klT)^2 - \sum_{p|l} \log^2 p - \tfrac{1}{2}(\log kT)^2 \right) \\
&= \phi(k) \frac{T}{2\pi} \left(\tfrac{1}{2}(\log kT)^2 + 2 \log l \log kT + \log^2 l - \sum_{p|l} \log^2 p \right).
\end{aligned}$$

Employing this in (8.1) we obtain $\mathcal{M}_1 = M_{11} + M_{12} + M_{13} + M_{14}$, with obvious meaning. Using Lemmas 5 and 6 and integration by parts we see that

$$\begin{aligned}
M_{11} &= \frac{T}{2\pi} \sum_{k \leqslant y} \frac{\mu(k)}{k} \frac{(\log kT)^2}{2} \sum_{l \leqslant y/k} \frac{b(kl)}{l} \\
&= \frac{T}{2\pi} \sum_{k \leqslant y} \frac{\mu^2(k)}{\phi(k)} \frac{(\log kT)^2}{2 \log y} P'\left(\frac{\log(y/k)}{\log y} \right)
\end{aligned}$$

$$= \frac{T}{2\pi}\frac{\log^2 y}{2}\int_0^1 P'(1-\alpha)(1/\theta+\alpha)^2\,d\alpha$$

$$= \frac{T}{2\pi}\frac{\log^2 y}{2}\left(\theta^{-2}+\int_0^1 (2/\theta+2\alpha)P(1-\alpha)\,d\alpha\right).$$

Similarly,

$$M_{12} = \frac{2T}{2\pi}\sum_{k\leqslant y}\frac{\mu(k)}{k}(\log kT)\sum_{l\leqslant y/k}\frac{b(kl)}{l}\log l$$

$$= -\frac{2T}{2\pi}\sum_{k\leqslant y}\frac{\mu^2(k)}{\phi(k)}(\log kT)P\left(\frac{\log(y/k)}{\log y}\right)$$

$$= -\frac{2T}{2\pi}\log^2 y\int_0^1 (1/\theta+\alpha)P(1-\alpha)\,d\alpha.$$

It is trivial from Lemma 5 that $M_{13}=0$. Finally,

$$M_{14} = -\frac{T}{2\pi}\sum_{k\leqslant y}\frac{\mu(k)}{k}\sum_{p\leqslant y/k}\frac{\log^2 p}{p}\sum_{l\leqslant y/kp}\frac{b(klp)}{l}$$

$$= -\frac{T}{2\pi}\sum_{k\leqslant y}\frac{\mu^2(k)}{\phi(k)}\sum_{p\leqslant y/k}\frac{\mu(p)\log^2 p}{(p-1)\log y}P'\left(\frac{\log(y/kp)}{\log y}\right)$$

$$= \frac{T}{2\pi}\log^2 y\int_0^1\int_0^{1-\alpha}\beta P'(1-\alpha-\beta)\,d\beta\,d\alpha$$

$$= \frac{T}{2\pi}\log^2 y\int_0^1\int_0^{1-\alpha}P(1-\alpha-\beta)\,d\beta\,d\alpha$$

$$= \frac{T}{2\pi}\log^2 y\int_0^1 \alpha P(1-\alpha)\,d\alpha.$$

Adding the expressions for M_{11}, M_{12}, M_{13} and M_{14} above, we obtain

$$\mathscr{M}_1 \sim \frac{T}{2\pi}\left(\tfrac{1}{2}\mathscr{L}^2 - \int_0^1 P(x)\,dx\,\mathscr{L}\log y\right),$$

so that by (3.13), and (5.5) we have the required estimate

$$S_1 = \frac{T}{2\pi}\left(\tfrac{1}{2}\mathscr{L}^2+\mathscr{L}\log y\int_0^1 P(x)\,dx\right)+O_\varepsilon(T^{1/2+\varepsilon}y)+o(T\mathscr{L}^2).$$

We now turn to the task of evaluating $\mathscr{M}_2$. As in (8.1) we may recast our expression for $\mathscr{M}_2$ as

$$\mathscr{M}_2 = \sum_{l\leqslant y}\sum_{k\leqslant y/l}\frac{b(kl)}{kl}\frac{\mu(k)}{\phi(k)}\sum_{\substack{m\leqslant kT/2\pi\\(m,k)=1}}a_2(ml)$$

$$= \sum_{l\leqslant y}\sum_{k\leqslant y/l}\frac{b(kl)}{kl}\frac{\mu(k)}{\phi(k)}S(kT/2\pi;\,l,\,k), \tag{8.2}$$

say. Let $\Lambda_j(n)$ be defined by $\sum_n \Lambda_j(n)n^{-s} = (-\zeta'(s)/\zeta(s))^j$ and $g(n)$ by

$$\sum_n g(n)n^{-s} = (-\zeta'(s)/\zeta(s))\zeta'(s)^2 = (-\zeta'(s)/\zeta(s))^3\zeta^2(s).$$

LEMMA A. *Suppose a and b are coprime, squarefree integers and that $0 \leqslant j \leqslant 3$ is an integer. Let $g_j(n;\, a,b)$ be defined by the relation*

$$\sum_{n=1}^{\infty} g_j(n;\, a,b)n^{-s} = \Bigg(\sum_{(n,ab)=1} \Lambda(n)n^{-s}\Bigg)^j\Bigg(\sum_{(m,b)=1} d(ma)m^{-s}\Bigg).$$

Then

$$G_j(x;\, a,b) := \sum_{n\leqslant x} g_j(n;\, a,b) \sim x\,\frac{(\log x)^{j+1}}{(j+1)!}\,\frac{\phi(b)^2}{b^2}\,\delta(a),$$

where $\delta(a) = \Pi_{p\,|\,a}(2-1/p)$. Further,

$$G(x;\, a,b) := \sum_{\substack{n\leqslant x\\(n,b)=1}} g(an) \sim x\,\frac{\phi(b)^2}{b^2}\sum_{j=0}^{3}\binom{3}{j}\beta_j(a)\delta(a)\,\frac{(\log x)^{j+1}}{(j+1)!},$$

where $\beta_j(a) := \sum_{d\,|\,a}\Lambda_{3-j}(d)/\delta(d)$.

Proof. By standard ideas, the main term is the residue at $s = 1$ of

$$\frac{x^s}{s}\Bigg(-\frac{\zeta'}{\zeta}(s) + \sum_{p\,|\,ab}\frac{\log p}{p^s-1}\Bigg)^j\zeta^2(s)\prod_{p\,|\,b}(1-p^{-s})^2\prod_{p\,|\,a}((1-p^{-s})^2(2+3p^{-s}+\ldots)),$$

which is asymptotically equal to the right-hand side of the first statement. The second assertion follows from the first upon noting that

$$\Bigg(\sum_{(n,b)=1}\Lambda(n)n^{-s}\Bigg)^3 = \sum_{j=0}^{3}\binom{3}{j}\Bigg(\sum_{(n,ab)=1}\Lambda(n)n^{-s}\Bigg)^j\Bigg(\sum_{(n,a)>1}\Lambda(n)n^{-s}\Bigg)^{3-j}.$$

From the definition of $a_2(n)$ we easily see that

$$\begin{aligned} S(kT/2\pi;\, l,k) &= -\sum_{\substack{m\leqslant y\\(m,k)=1}} b(m)G(kT(l,m)/(2\pi m);\, l/(l,m),k)\\ &= -\sum_{l_1\,|\,l}\ \sum_{\substack{m\leqslant yl_1/l\\(m,kl)=1}} b(ml/l_1)G(kT/(2\pi m);\, l_1,k). \end{aligned}$$

By Lemma A and Lemma 5, the right-hand side above is

$$\sim -\frac{T\phi^2(k)}{2\pi k}\sum_{j=0}^{3}\binom{3}{j}\sum_{l_1\,|\,l}\delta(l_1)\beta_j(l_1)\sum_{\substack{m\leqslant yl_1/l\\(m,kl)=1}}\frac{b(ml/l_1)}{m}\,\frac{(\log(kT/m))^{j+1}}{(j+1)!}$$

$$\sim -\frac{T\phi^2(k)}{2\pi k}\sum_{j=0}^{3}\binom{3}{j}\frac{1}{(j+1)!}\sum_{l_1\,|\,l}\delta(l_1)\beta_j(l_1)\mu(l/l_1)\,\frac{kl}{\phi(kl)}$$

$$\times\left(\frac{(\log kT)^{j+1}}{\log y}P'\left(\frac{\log(yl_1/l)}{\log y}\right)+(j+1)(\log kT)^jP\left(\frac{\log(yl_1/l)}{\log y}\right)\right)$$

$$\sim -\frac{\phi(k)lT}{2\pi\phi(l)}\sum_{j=0}^{3}\binom{3}{j}\frac{1}{(j+1)!}\sum_{l_1\,|\,l}\delta(l_1)\beta_j(l_1)\mu(l/l_1)(\log kT)^j$$

$$\times\left(\frac{\log kT}{\log y}P'\left(\frac{\log(yl_1/l)}{\log y}\right)+(j+1)P\left(\frac{\log(yl_1/l)}{\log y}\right)\right).$$

Using this in (8.2), making obvious substitutions, and interchanging summations, we see that

$$\mathscr{M}_2\sim -\frac{T}{2\pi}\sum_{j=0}^{3}\binom{3}{j}\frac{1}{(j+1)!}M_2(j), \tag{8.3}$$

where

$$M_2(j):=\sum_{l_1\leqslant y}\beta_j(l_1)\frac{\delta(l_1)}{\phi(l_1)}\sum_{k\leqslant y/l_1}\frac{\mu(k)}{k}(\log kT)^j\sum_{l\leqslant y/kl_1}\frac{\mu(l)}{\phi(l)}b(lkl_1)$$
$$\times\left(\frac{\log kT}{\log y}P'\left(\frac{\log(y/l)}{\log y}\right)+(j+1)P\left(\frac{\log(y/l)}{\log y}\right)\right). \tag{8.4}$$

A straightforward argument shows that the innermost sum over l is equal to

$$\mu(kl_1)\sum_{\substack{l\leqslant y/kl_1\\(l,kl_1)=1}}\frac{\mu^2(l)}{\phi(l)}P\left(\frac{\log y/kll_1}{\log y}\right)\left(\frac{\log kT}{\log y}P'\left(\frac{\log y/l}{\log y}\right)+(j+1)P\left(\frac{\log y/l}{\log y}\right)\right)$$

$$\sim \mu(kl_1)\frac{\phi(kl_1)}{kl_1}\log y\left(\frac{\log kT}{\log y}Q_0\left(\frac{\log y/kl_1}{\log y}\right)+(j+1)Q_1\left(\frac{\log y/kl_1}{\log y}\right)\right),$$

where for $i=0,\ 1,\ Q_i(\alpha):=\int_0^\alpha P(\alpha-\beta)P^{(1-i)}(1-\beta)\,d\beta$. Employing this in (8.4) we obtain

$$M_2(j)\sim \log y\sum_{l_1\leqslant y}\delta(l_1)\beta_j(l_1)\frac{\mu(l_1)}{l_1}\sum_{\substack{k\leqslant y/l_1\\(k,l_1)=1}}\frac{\mu^2(k)}{k^2}\phi(k)(\log kT)^j$$
$$\times\left(\frac{\log kT}{\log y}Q_0\left(\frac{\log y/kl_1}{\log y}\right)+(j+1)Q_1\left(\frac{\log y/kl_1}{\log y}\right)\right). \tag{8.5}$$

A simple calculation reveals that the sum over k above is

$$\sim (\log y)^{j+1}R_j\left(\frac{\log y/l_1}{\log y}\right)\frac{\prod_p((1+(p-1)/p^2)(1-p^{-1}))}{\prod_{p\,|\,l_1}(1+(p-1)/p^2)}$$
$$=(\log y)^{j+1}R_j\left(\frac{\log y/l_1}{\log y}\right)\frac{C}{C(l_1)},$$

where C and $C(l_1)$ have their natural meaning and

$$R_j(\alpha)=\int_0^\alpha(1/\theta+\beta)^j((1/\theta+\beta)Q_0(\alpha-\beta)+(j+1)Q_1(\alpha-\beta))\,d\beta.$$

Substituting this in (8.5) and recalling the definition of $\beta_j(l_1)$, we have

$$M_2(j) \sim C(\log y)^{j+2} \sum_{l_1 \leqslant y} \delta(l_1)\beta_j(l_1) \frac{\mu(l_1)}{l_1 C(l_1)} R_j\left(\frac{\log y/l_1}{\log y}\right)$$

$$= C(\log y)^{j+2} \sum_{d \leqslant y} \frac{\mu(d)\Lambda_{3-j}(d)}{dC(d)} \sum_{\substack{l_1 \leqslant y/d \\ (l_1,d)=1}} \frac{\mu(l_1)\delta(l_1)}{l_1 C(l_1)} R_j\left(\frac{\log(y/l_1 d)}{\log y}\right). \quad (8.6)$$

Mimicking the argument of Lemma 10 of Conrey [**2**] we see that

$$\sum_{\substack{l_1 \leqslant y/d \\ (l_1,d)=1}} \frac{\mu(l_1)\delta(l_1)}{l_1 C(l_1)} R_j\left(\frac{\log(y/l_1 d)}{\log y}\right) \sim C^{-1} \prod_{p \mid d} \frac{p^2+p-1}{p^2-p} \frac{1}{\log^2 y} R_j''\left(\frac{\log(y/d)}{\log y}\right).$$

(It is an easily verified crucial point in this argument that $R_j(0) = R_j'(0) = 0$.) Plugging the above display into the right-hand side of (8.6), we obtain easily

$$M_2(j) \sim (\log y)^j \sum_{d \leqslant y} \frac{\mu(d)\Lambda_{3-j}(d)}{d} R_j''\left(\frac{\log(y/d)}{\log y}\right).$$

Using the above display and the familiar distribution of the $\Lambda_{3-j}(n)$ (and integration by parts) we arrive at

$$\frac{2\pi \mathscr{M}_2}{T(\log y)^3} = -\sum_{j=0}^{3} \binom{3}{j} \frac{M_2(j)}{(j+1)!} \frac{1}{(\log y)^3}$$

$$= \int_0^1 \tfrac{1}{2}\alpha^2 R_0''(1-\alpha)\, d\alpha - \frac{3}{2}\int_0^1 \alpha R_1''(1-\alpha)\, d\alpha$$

$$+ \frac{3}{6}\int_0^1 R_2''(1-\alpha)\, d\alpha - \tfrac{1}{24} R_3''(1)$$

$$= \int_0^1 R_0(\alpha)\, d\alpha - \tfrac{3}{2}R_1(1) + \tfrac{1}{2}R_2'(1) - \tfrac{1}{24}R_3''(1).$$

An easy computation shows that the expression in the right-hand side above multiplied by θ^3 equals

$$-\frac{1}{24\theta}\int_0^1 P'(x)^2\, dx + \frac{1}{12} + \frac{3\theta}{2}\int_0^1 P(x)^2\, dx - \frac{\theta}{2}\int_0^1 P(x)\, dx - \frac{\theta^2}{2}\left(\int_0^1 P(x)\, dx\right)^2.$$

This completes our treatment of $\mathscr{M}_2$.

The estimate (2.7) now follows from (3.21), (3.26), (3.27) and the above, and so proves our theorem.

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