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A Conjecture for the Sixth Power Moment of the Riemann Zeta-function

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In 1918, Hardy and Littlewood [2] proved that

$$\int_1^T |\zeta(1/2+it)|^2\, dt \sim T\log T;$$

and in 1926, Ingham [4] showed that

$$\int_1^T |\zeta(1/2+it)|^4\, dt \sim \frac{1}{2\pi^2} T\log^4 T.$$

In general, it is conjectured that if $k > 0$, then there exists a $c_k > 0$ such that

$$\int_1^T |\zeta(1/2+it)|^{2k}\, dt \sim c_k T\log^{k^2} T.$$

No value has been suggested for c_k if k is different from 0,1, or 2.

In this paper, we present evidence to support the following conjecture.

Conjecture. As $T \to \infty$,

$$\int_1^T |\zeta(1/2+it)|^6\, dt \sim \frac{42}{9!}\prod_p \left\{\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)\right\} T\log^9 T. \qquad \square$$

Sketch of basic argument

We recall the functional equation of the zeta-function:

$$\zeta(s) = \chi(s)\zeta(1-s),$$

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where

$$\chi(1-s)=\chi(s)^{-1}=2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}.$$

We also require an "approximate" functional equation for $\zeta(s)^2$:

$$\zeta(s)^2=D(s)+\chi(s)^2D(1-s),$$

where

$$D(s)=\sum_{n\le\frac{|t|}{2\pi}}\frac{d(n)}{n^s}+E(s)$$

with $d(n)$ the usual divisor function and with $E(s)$ a suitable error term. The estimate

$$E(1/2+it)\ll\log(2+|t|)$$

is known to hold (see [3]). However, for the purposes of this paper, we will not be concerned with $E(s)$.

The beginnings of our argument are as follows.

$$\begin{aligned}\int_0^T|\zeta(1/2+it)|^6\,dt&=\frac{1}{i}\int_{(1/2)}^{(1/2)+iT}\zeta(s)^3\zeta(1-s)^3\,ds\\&=\frac{1}{i}\int_{(1/2)}^{(1/2)+iT}\chi(1-s)\zeta(s)^4\zeta(1-s)^2\,ds\\&=\frac{1}{i}\int_{(1/2)}^{(1/2)+iT}\chi(1-s)\zeta(s)^4\left(D(1-s)+\chi(1-s)^2D(s)\right)\,ds\\&=\frac{1}{i}\int_{(1/2)}^{(1/2)+iT}\chi(1-s)\zeta(s)^4D(1-s)\,ds\\&\quad+\frac{1}{i}\int_{(1/2)}^{(1/2)+iT}\chi(1-s)^3\zeta(s)^4D(s)\,ds\\&=I_1+I_2,\end{aligned}$$

say. Now,

$$\begin{aligned}I_2&=\frac{1}{i}\int_{(1/2)}^{(1/2)+iT}\chi(1-s)^3\chi(s)^4\zeta(1-s)^4D(s)\,ds\\&=\frac{1}{i}\int_{(1/2)}^{(1/2)+iT}\chi(s)\zeta(1-s)^4D(s)\,ds\\&=\frac{1}{i}\int_{(1/2)-iT}^{(1/2)}\chi(1-s)\zeta(s)^4D(1-s)\,ds\\&=\overline{I_1}.\end{aligned}$$

Thus,

$$\int_1^T|\zeta(1/2+it)|^6\,dt=2\Re\frac{1}{i}\int_{(1/2)}^{(1/2)+iT}\chi(1-s)\zeta(s)^4D(1-s)\,ds.$$

Theorem from "Mean-Values III"

We appeal to Theorem 2 of Conrey and Ghosh [1] to evalute this integral. We first set up some notation so that we can state a special case of that theorem. Define $D_N(s, P)$ by

$$D_N(s,P) = \sum_{n \le N} \frac{d(n)}{n^s} P(\log n / \log N),$$

where P is any real polynomial. Let

$$\begin{aligned} K_N(T) &= \int_1^T |\zeta(1/2+it)|^2 \zeta(1/2+it)^2 D_N(1/2-it, P)\ dt \\ &= \frac{1}{i} \int_{(1/2)}^{(1/2)+iT} \chi(1-s)\zeta(s)^4 D_N(1-s, P)\ ds. \end{aligned}$$

Theorem. If $N = T^\theta$ with $0 < \theta < 1/2$, then

$$K_N(T) \sim T(\log N)^9 \frac{a_3}{720\theta^3} \int_0^1 P(\alpha)\alpha^5 {}_2F_1(-2,-3,6,-\alpha\theta)\ d\alpha$$

as $T \to \infty$, where ${}_2F_1$ is the usual hypergeometric function and

$$a_3 = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right\}. \qquad \square$$

The hypergeometric function simplifies to

$$1 - \alpha\theta + (\alpha\theta)^2/7.$$

We are interested in the case where $P = 1$. We find in this case that

$$\begin{aligned} K_N(T) &\sim \frac{a_3}{\theta^3 3!5!} \int_0^1 \alpha^5 \left(1 - \alpha\theta + \frac{(\alpha\theta)^2}{7}\right)\ d\alpha T \log^9 N \\ &= \frac{\theta^6 a_3}{720} \left(\frac{1}{6} - \frac{\theta}{7} + \frac{\theta^2}{56}\right) T \log^9 T. \end{aligned}$$

This formula is valid for $\theta < 1/2$. To apply it to our formula from the last section, we would need it to hold for $\theta = 1$. In our paper [1], we expressed the belief that the theorem from which the above is taken is actually valid for all $\theta \le 1$.

If we assume that we can take $\theta = 1$ in this theorem, then we are immediately led to

$$\int_0^T |\zeta(1/2+it)|^6\ dt \sim 42 \frac{a_3}{9!} T \log^9 T.$$

Sketch of another method

We remark that we can arrive at the same conclusion by another method, which we briefly sketch.

With $s = 1/2 + it$, we have

$$\begin{aligned}\int_0^T |\zeta(1/2+it)|^6 \, dt &= \int_0^T |\zeta(s)|^2 \left|\zeta(s)^2\right|^2 \, dt \\ &= \int_0^T |\zeta(s)|^2 \left|D(s) + \chi(s)^2 D(1-s)\right|^2 \, dt \\ &= 2\int_0^T |\zeta(s)|^2 |D(s)|^2 \, dt + 2\Re \int_0^T |\zeta(s)|^2 \chi(1-s)^2 D(s)^2 \, dt\end{aligned}$$

since $|\chi(1/2 - it)| = 1$.

To evaluate the first integral here, we appeal to a special case of Theorem 1 of Conrey and Ghosh [1].

Theorem. Let

$$J_N(T) = \int_1^T |\zeta(1/2+it)|^2 |D_N(1/2+it, P)|^2 \, dt.$$

If $N = T^\theta$ for some θ with $0 < \theta < 1/2$, and if P is a real polynomial, then

$$J_N(T) \sim T(\log N)^9 \frac{a_3}{24} \int_0^1 \alpha^3 \left(\frac{1}{\theta} h'(\alpha)^2 + 4h(\alpha)h'(\alpha)\right) \, d\alpha$$

as $T \to \infty$, where

$$h(\alpha) = \int_\alpha^1 (\beta - \alpha)^2 P(\beta) \, d\beta. \qquad \square$$

Again, this theorem can be proven for $\theta < 1/2$, and again we expressed the belief in [1] that it actually holds true for $\theta \le 1$. Assuming the formula for $\theta = 1$ leads to

$$2\int_0^T |\zeta(s)|^2 |D(s)|^2 \, dt \sim 28 \frac{a_3}{9!} T \log^9 T \, dt.$$

To handle the second integral, we appeal again to the approximate functional equation for $\zeta(s)^2$. We find that the second term above is

$$\begin{aligned}&= 2\Re \int_0^T \chi(1-s)^3 \zeta(s)^2 D(s)^2 \, ds \\ &= 2\Re \int_0^T \chi(1-s)^3 \left(D(s) + \chi(s)^2 D(1-s)\right) D(s)^2 \, ds \\ &= 2\Re \int_0^T \chi(1-s)^3 D(s)^3 \, ds + 2\Re \int_0^T \chi(1-s) D(1-s) D(s)^2 \, ds.\end{aligned}$$

The first integral here is not expected to contribute to the main term, essentially because

$$\chi(1/2 - it)^3 = \exp\left(3it \log \frac{t}{2\pi e}\right)$$

is "spinning" too fast. To evaluate the second integral, we proceed as in the proof of Theorem 2 in [1]. If the $D(1-s)$ were replaced by $D_N(1-s, 1)$ with $N = T^\theta$ and $\theta < 1/2$, then, in a way similar to the proof of Theorem 2, we could obtain an asymptotic evaluation. Again we assume that this asymptotic evaluation is actually correct for all $\theta \leq 1$. In this way, we obtain

$$2\Re \int_0^T \chi(1-s)D(1-s)D(s)^2 \, ds \sim 14\frac{a_3}{9!} T \log^9 T.$$

This argument again leads to the conjecture of this paper.

Final remarks

Conrey and Gonek, in a work in progress, have arrived at exactly the same conjecture using a third method. Their method is to consider the asymptotic behavior of "long" Dirichlet polynomials, based on techniques developed by Goldston and Gonek. Their method supposes that asymptotic formulae exist for sums

$$\sum_{n \leq x} d_3(n) d_3(n+h),$$

where d_3 can be defined by

$$\zeta(s)^3 = \sum_{n=1}^{\infty} \frac{d_3(n)}{n^s}.$$

Also, the asymptotic formulae for these sums have smooth main terms and error terms which are bounded on average over h by $x^{1/2+\epsilon}$ for h up to $x^{1/2+\epsilon}$.

Finally, we mention that another possibility for testing our conjecture would be through the method recently developed in the thesis of Jose Gaggero Jara (under the direction of S. M. Gonek at the University of Rochester). In that work, Jara develops an asymptotic formula for

$$\int_0^T |\zeta(1/2+it)|^4 \left|\sum_{n=1}^{N} a_n n^{it}\right|^2 \, dt$$

for arbitrary positive coefficients a_n provided that $N = T^\theta$ with $\theta < 4/589$. It is probably the case that the formula should actually hold for all $\theta < 1/2$, and possibly even for all $\theta < 1$. With $\theta = 1/2$ and $a_n = 1$, the result should give one-half of the sixth moment. With $\theta = 1$ and $a_n = 1$, it should give all of the sixth moment.

Generally, by taking an approximate functional equation for $\zeta(1/2+it)$ with "uneven" lengths t^θ and $t^{1-\theta}$, the sum of the results of Jara's theorem with $a_n = 1$ and $N = T^\theta$ and $N = T^{1-\theta}$ (for any $\theta < 1$) should also give the sixth moment.

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