ON THE NON-VANISHING OF $L_f(s)$ AT THE CENTER OF THE CRITICAL STRIP

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Abstract. We determine some conditions which imply the nonvanishing of $L_f(k/2)$ for f a cusp form of weight k for the full modular group. These conditions are verified for weights $k \leq 500$.

1. Introduction and statement of results

Let $\mathcal{S}_k(1)$ be the space of holomorphic cusp forms of weight k for the full modular group. This space has a basis of Hecke eigenforms

$$B = \{f_1, \dots, f_d\},$$

satisfying $T_n f_j = a_j(n) f_j$ for all n, where T_n is the nth Hecke operator. The $f_j(z)$ have Fourier expansions

$$f_j(z) = \sum_{n=1}^{\infty} a_j(n) e(nz)$$

where $e(z) = \exp(2\pi i z)$.

A Dirichlet series is associated to $f_j(z)$ by

$$L_{f_j}(s) = \sum_{n=1}^{\infty} \frac{a_j(n)}{n^s}.$$

In this paper we study the "critical value" $L_{f_j}(k/2)$. Since $L_{f_j}(s)$ has functional equation

$$(2\pi)^{-s}\Gamma(s) L_{f_j}(s) = (-1)^{k/2}(2\pi)^{k-s}\Gamma(k-s) L_{f_j}(k-s),$$

it follows that $L_{f_j}(k/2) = 0$ if $k \equiv 2 \bmod 4$. If $k \equiv 0 \bmod 4$, then $L_{f_j}(k/2) = 0$ implies that $L_{f_j}(s)$ has a zero of order at least 2 at $s = k/2$. In this paper we make progress toward showing that $L_{f_j}(k/2) \neq 0$ for $k \equiv 0 \bmod 4$.

Our motivation for this work came from a recent conjecture of Duke and Imamoglu [DI] concerning Siegel modular forms. They conjecture a lift of a modular form f of weight $2k - n$ to a Siegel modular form of degree n and weight k. The lift is a theta series provided $L_f((2k-n)/2) \neq 0$.

Our main result is

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Theorem 1. *Suppose $k \equiv 0 \bmod 4$. If, for some n, the characteristic polynomial of T_n acting on $\mathcal{S}_k(1)$ is irreducible, then $L_{f_j}(k/2) \neq 0$.*

We have verified the irreducibility of the characteristic polynomial of T_2 acting on $\mathcal{S}_k(1)$ for $k \leq 500$, yielding

Theorem 2. *$L_{f_j}(k/2) \neq 0$ for $k \leq 500$, $k \equiv 0 \bmod 4$.*

Questions concerning the characteristic polynomials of Hecke operators have received considerable attention lately. For example, if N is large then the characteristic polynomial of T_p acting on $S_2(\Gamma_0(N))$ will have lots of linear factors corresponding to elliptic curves, but one can look for irreducible factors of a large degree (see [K]). At the other extreme, one expects the characteristic polynomial of T_p acting on $S_k(\Gamma(1))$ to be irreducible, so Buzzard [B] began a study of its Galois group. We applied the method of Buzzard to show

Theorem 3. *For $k \leq 500$, $k \equiv 0 \bmod 4$, with $d = [k/12]$ prime, the Galois group of the characteristic polynomial of T_2 acting on $\mathcal{S}_k(1)$ is the symmetric group S_d.*

2. Background and Notation

Our methods involve the Hecke operators T_n. These can be defined as linear maps $T_n : \mathcal{S}_k(1) \to \mathcal{S}_k(1)$ given by

$$T_n\left(\sum_{m=1}^{\infty} a(m)e(mz)\right) = \sum_{m=1}^{\infty}\left(\sum_{d|(n,m)} d^{k-1}a\left(\frac{mn}{d^2}\right)\right)e(mz)$$

We make extensive use of the fact that $\mathcal{S}_k(1)$ has a basis of functions with rational integer Fourier coefficients. Set $B_{\mathbb{Q}} = \{F_1, \ldots, F_d\}$ where

$$\begin{aligned} F_j &= \Delta^j E_4^{\frac{k}{4}-3j} \\ &= \sum_{m=j}^{\infty} A_j(m)e(mz), \qquad\qquad A_j(m) \in \mathbb{Z}, \end{aligned}$$

say, where $\Delta \in S_{12}(1)$ is the cusp form of weight 12, and $E_k \in M_k(1)$ is the Eisenstein series of weight k for the full modular group. We denote by $T_{n,k}(x)$ the characteristic polynomial of T_n acting on $\mathcal{S}_k(1)$. The matrix representing T_n in the basis $B_{\mathbb{Q}}$ has integer entries, so $T_{n,k}(x) \in \mathbb{Z}[x]$.

We require some information about the field $\mathbb{K}$ obtained by adjoining to $\mathbb{Q}$ all of the Fourier coefficients $a_j(n)$ of all of the $f_j(z)$. This is the same as the splitting field of all of the $T_{n,k}(x)$.

Lemma 1. *$\mathbb{K} = \mathbb{Q}(\{a_j(n) \ : \ 1 \leq j, n \leq d\})$. In particular, $\mathbb{K}$ is a finite Galois extension of $\mathbb{Q}$.*

Proof. Since $B_{\mathbb{Q}}$ is a basis, we can write

$$f_j = \sum_{m=1}^{d} \alpha_{j,m} F_m \tag{2.1}$$

The $\alpha_{j,m}$ can be found by solving the "upper triangular" system

$$a_j(n) = \sum_{m=1}^{d} \alpha_{j,m} A_m(n), \qquad a \le j \le d.$$

Thus, the $\alpha_{j,m}$ are in the field $\mathbb{Q}(\{a_j(n) \ : \ 1 \le j,\, n \le d\})$. But then by (2.1) all of the Fourier coefficients of the f_j are in that field, so $\mathbb{K}$ is as described.

The Galois group $Gal(\mathbb{K}/\mathbb{Q})$ acts on functions with Fourier coefficients in $\mathbb{K}$ by

$$\sigma\left(\sum_{m=1}^{\infty} a(m)e(mz)\right) = \left(\sum_{m=1}^{\infty} \sigma\left(a(m)\right)e(mz)\right).$$

Lemma 2. *$Gal(\mathbb{K}/\mathbb{Q})$ acts on $B = \{f_1, \dots, f_d\}$. The action is transitive if $T_{n,k}(x)$ is irreducible for some n.*

Proof. $Gal(\mathbb{K}/\mathbb{Q})$ acts on B because the elements of $Gal(\mathbb{K}/\mathbb{Q})$ commutes with each Hecke operator T_n, and the f_j are the simultaneous eigenfunctions of the T_n.

If $T_{n,k}(x)$ is irreducible then $Gal(\mathbb{K}/\mathbb{Q})$ acts transitively on $\{a_j(n) : 1 \le j \le d\}$, which is the set of nth Fourier coefficients of the f_j. This forces the action to be transitive on B.

In the next section we relate the transitive action of $Gal(\mathbb{K}/\mathbb{Q})$ on B to the nonvanishing of $L_{f_j}(k/2)$. In the following section we describe our calculations which show that $T_{2,k}(x)$ is irreducible for small k.

3. Proof of Theorem 1

Let

$$\begin{aligned} f(z) &= \sum_{m=1}^{\infty} a(m)e(mz) \\ &= f_j(z) \end{aligned}$$

for some j. Let

$$\sigma_\ell(n) = \sum_{d|n} d^\ell$$

and set

$$F(s,\ell) = \sum_{n=1}^{\infty} \frac{a(n)\sigma_{\ell-1}(n)}{n^s}.$$

Proposition 1. *$F(k-1, k/2) = 0$ if and only if $L_f(k/2) = 0$.*

Proof. By standard methods,

$$F(s,\ell) = \frac{L_f(s)L_f(s+1-\ell)}{\zeta(2s+2-k-\ell)}.$$

In particular,

$$F(k-1, k/2) = \frac{L_f(k-1)L_f(k/2)}{\zeta(k/2)}.$$

Finally, note that $k-1$ is strictly within the plane of absolute convergence for the Euler product for $L_f(s)$, so $L_f(k-1) \ne 0$.

The next Proposition uses the notation of the Petersson inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}_k(1)$.

Proposition 2. *$\langle E_{k/2}^2, f\rangle = (4\pi)^{1-k}\Gamma(k)F(k-1,k/2)$. In particular, $\langle E_{k/2}^2, f\rangle = 0$ if and only if $L_f(k/2) = 0$.*

Proof. Let

$$I(s) = \int_0^\infty \int_0^1 \overline{f(z)} E_\ell(z) y^{s-1}\ dx\ dy$$

where $E_\ell(z)$ is the holomorphic Eisenstein series of weight ℓ:

$$E_\ell(z) = \sum_{\gamma\in\Gamma/\Gamma_\infty} j(\gamma,z)^\ell = \sum_{m=0}^\infty \sigma_{\ell-1}(m)e(mz)$$

for a suitable definition of $\sigma_{\ell-1}(0)$.

By substituting the Fourier series for $E_\ell(z)$ into the equation for $I(s)$ and integrating term-by-term, we find that

$$I(s) = (4\pi)^{-s}\Gamma(s)F(s,\ell-1).$$

But by the Rankin-Selberg unfolding method, we also have

$$I(s) = \iint_{\mathcal{D}} \overline{f(z)} E_\ell(z) \sum_\gamma \overline{j(\gamma,z)}^k j(\gamma,z)^\ell y(\gamma z)^{s+1} d\mu(z),$$

where $\mathcal{D}$ is a fundamental domain for the full modular group. The sum over γ equals $y(z)^k E_{k-\ell}(s+1-k,z)$, which involves the non-holomorphic Eisenstein series of weight $k-\ell$. In the notation of the Petersson inner product, we have

$$(4\pi)^{-s}\Gamma(s)F(s,\ell-1) = \langle E_\ell(\cdot)E_{k-\ell}(s+1-k,\cdot), f(\cdot)\rangle.$$

If we take $s = k-1$ here, and use the fact that

$$E_{k-\ell}(0,z) = E_{k-\ell}(z),$$

then, with $\ell = k/2$, we obtain

$$(4\pi)^{1-k}\Gamma(k)F(k-1,k/2) = \langle E_{k/2}^2, f\rangle,$$

as claimed.

Proof of Theorem 1. A Hecke eigenbasis for the space of modular forms of weight k is given by

$$\{f_1, f_2, \ldots, f_d, E_k\}$$

where $d = dim(\mathcal{S}_k(1))$. Since $E_{k/2}^2$ is a modular form of weight k, it follows that there is a unique way to write

$$E_{k/2}^2 = E_k + \sum_{j=1}^d c_j f_j \tag{3.1}$$

for some coefficients $c_j \in \mathbb{K}$. In particular,

$$\langle E_{k/2}^2, f\rangle = c_1\langle f, f\rangle,$$

so by Proposition 1, $L_{f_j}(k/2) = 0$ if and only if $c_j = 0$. We will show that $c_j \neq 0$.

Applying $\sigma \in Gal(\mathbb{K}/\mathbb{Q})$ to (3.1) gives

$$E_{k/2}^2 = E_k + \sum_{j=1}^{d} \sigma(c_j) f_{\overline{\sigma}}, \tag{3.2}$$

where $\overline{\sigma}$ is the permutation of $\{1, 2, \ldots, d\}$ induced by the action of σ on $B = \{f_1, \ldots, f_d\}$. Since the expression (3.1) is unique, it follows that $\sigma(c_j) = c_{\overline{\sigma}}$.

Suppose $Gal(\mathbb{K}/\mathbb{Q})$ acts transitively on B, and c_j=0 for some j. Then by the previous observation we find that $c_j = 0$ for all j. This implies $E_{k/2}^2 = E_k$, which is a contradiction if $k > 4$. So by Lemma 2, if $T_{n,k}(x)$ is irreducible for some n, then $c_j \neq 0$. This proves Theorem 1.

4. Calculation of the Hecke operators

In this section we describe some calculations involving the Hecke operators T_n acting on $\mathcal{S}_k(1)$. Our goal are to show that the the characteristic polynomial $T_{n,k}(x)$ of T_n is irreducible and has Galois group S_d, where $d = [k/12] = dim(\mathcal{S}_k(1))$.

Direct calculation of the polynomial $T_{n,k}(x)$ is quite difficult for large k because of the huge size of the coefficients ($T_{n,k}(x)$ is a polynomial of degree d, with roots of size d^6). To verify that $T_{n,k}(x)$ is irreducible we calculate $T_{n,k}(x)$ mod p for small p and check if the reduced polynomial is irreducible mod$p$. Having established that the polynomial is irreducible, we use the following Lemma of Buzzard [B] to show that it has Galois group S_d:

Lemma 3. *Suppose $f(x) \in \mathbb{Z}[x]$ is a monic irreducible polynomial of degree d, with d prime. If there is a prime q such that the* mod q *reduction $\bar{f} \in \mathbb{F}_q[x]$ satisfies $\bar{f} = f_0 f_1 \cdots f_j$ for $f_i \in \mathbb{F}_q[x]$ distinct irreducible polynomials with f_0 having degree 2 and f_i having odd degree for $i \geq 1$, then the Galois group of f is the full symmetric group S_d.*

For fixed k, if one $T_{n,k}(x)$ is irreducible then all of them are, and they all have the same Galois group. So we restrict our attention to $T_{2,k}(x)$. The steps in our calculations are:

Step 1. Generate the basis $B_{\mathbb{Q}} = \{F_1, \ldots, F_d\}$ by taking products of the weight 4 Eisenstein series E_4 and the weight 12 cusp form Δ. We generate and store the first 200 terms in the Fourier expansion.

Step 2. Form the matrix $M_2 = \mathcal{A} - x\mathcal{B}$, where $\mathcal{A} = (a_{j,l})$ with $a_{j,l} = A_j(2l) + 2^{k-1}A_j(l/2)$ and $\mathcal{B} = (A_j(l))$

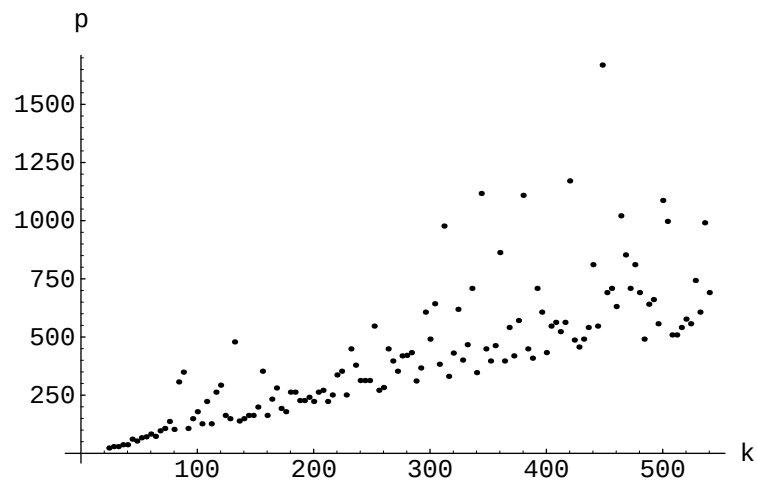
Note that M_2 has integer entries, and the eigenvalues of T_2 are the values of x for which M_2 is singular. In particular, $\det(M_2)$ is the characteristic polynomial of T_2.

Step 3. Reduce M_2 mod p, compute the characteristic polynimial of $\mathcal{A}\mathcal{B}^{-1}$, and factor that polynomial mod$p$.

Step 4. Check if the polynomial from Step 3 is irreducible and if it meets the conditions in Lemma 3.

Steps 3 and 4 are repeated until we find a prime p for which the polynomial is irreducible, and a prime q for which it meets the conditions in Lemma 3. We checked the conditions in Lemma 3 for all k, even though the Lemma only applies when $[k/12]$ is prime.

The following graph shows the smallest prime p for which the polynomial $T_{2,k}(x)$ is irreducible mod$p$.



The program was written in Mathematica on a Sun Sparc ??. The program took 867 CPU minutes for the range $24 \leq k \leq 400$, and an additional 2103 minutes for the range $404 \leq k \leq 500$.

References

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