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Transition Mean Values of Real Characters[1]

J. B. Conrey, D. W. Farmer, and K. Soundararajan

American Institute of Mathematics, Palo Alto, California 94306; and School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

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We evaluate the real character sum $\sum_m \sum_n (m/n)$ where the two sums are of approximately the same length. The answer is surprising. © 2000 Academic Press

1. INTRODUCTION AND STATEMENT OF RESULTS

Let

$$S(X, Y) := \sum_{\substack{m \leqslant X \\ m \text{ odd}}} \sum_{\substack{n \leqslant Y \\ n \text{ odd}}} \left(\frac{m}{n}\right),$$

where (m/n) is the Jacobi symbol. Our goal is to obtain an asymptotic formula for $S(X, Y)$. We will see that this is straightforward except when X and Y are of comparable size.

First we give asymptotic formulas valid for $Y = o(X/\log X)$ or $X = o(Y/\log Y)$. An easy application of the Pólya–Vinogradov inequality shows that

$$\sum_{\substack{m \leqslant X \\ m \text{ odd}}} \left(\frac{m}{n}\right) = \begin{cases} \dfrac{X}{2} \dfrac{\varphi(n)}{n} + O(X^{\varepsilon}) & \text{if} \quad n = \square \\ O(n^{1/2} \log n) & \text{if} \quad n \neq \square \end{cases}$$

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and

$$\sum_{\substack{n \leqslant Y \\ n \text{ odd}}} \left(\frac{m}{n}\right) = \begin{cases} \dfrac{Y}{2} \dfrac{\varphi(m)}{m} + O(Y^{\varepsilon}) & \text{if} \quad m = \square \\ O(m^{1/2} \log m) & \text{if} \quad m \neq \square, \end{cases}$$

where $\square$ represents the square of a rational integer.

It follows that

$$S(X, Y) = \sum_{\substack{n \leqslant Y \\ n = \text{odd } \square}} \left(\frac{X}{2} \frac{\varphi(n)}{n} + O(X^{\varepsilon})\right) + O(Y^{3/2} \log Y)$$

$$= \frac{2}{\pi^2} XY^{1/2} + O(Y^{3/2} \log Y + Y^{1/2} X^{\varepsilon} + X \log Y), \qquad (1.1)$$

and similarly,

$$S(X, Y) = \frac{2}{\pi^2} X^{1/2} Y + O(X^{3/2} \log X + X^{1/2} Y^{\varepsilon} + Y \log X). \qquad (1.2)$$

Equation (1.1) provides an asymptotic formula for $S(X, Y)$ when $Y = o(X/\log X)$, and (1.2) when $X = o(Y/\log Y)$. The range when X and Y are of comparable size marks a transition in the behavior of $S(X, Y)$, and our object here is to understand this transitory phase.

THEOREM 1. *Uniformly for all large X and Y, we have*

$$S(X, Y) = \frac{2}{\pi^2} C\left(\frac{Y}{X}\right) X^{3/2} + O((XY^{7/16} + YX^{7/16}) \log XY),$$

where for $\alpha \geqslant 0$ we define

$$C(\alpha) = \sqrt{\alpha} + \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\alpha} \sqrt{y} \left(1 - \cos\left(\frac{2\pi k^2}{y}\right) + \sin\left(\frac{2\pi k^2}{y}\right)\right) dy.$$

An alternate expression for $C(\alpha)$ is

$$C(\alpha) = \alpha + \alpha^{3/2} \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{1/\alpha} \sqrt{y} \sin\left(\frac{\pi k^2}{2y}\right) dy.$$

To assist the reader in understanding the function $C(\alpha)$, graphs of $C(\alpha)$ and $C'(\alpha)$ are presented in Section 6.

The first expression for $C(\alpha)$ shows, upon integrating by parts, that

$$C(\alpha) = \sqrt{\alpha} + \frac{\pi}{18}\alpha^{3/2} + O(\alpha^{5/2}) \qquad \text{as} \quad \alpha \to 0.$$

Similarly, the second expression for $C(\alpha)$ gives the limiting behavior

$$C(\alpha) = \alpha + O(\alpha^{-1}) \qquad \text{as} \quad \alpha \to \infty.$$

Note that in these limiting cases, the value of $C(\alpha)$ approaches that given by the $n = \square$ terms (as $\alpha \to 0$) and the $m = \square$ terms (as $\alpha \to \infty$). From these limiting behaviors (or (1.1) and (1.2)) we see that $C(Y/X)\, X^{3/2}$ is of size $XY^{1/2} + YX^{1/2}$, so that the error term of Theorem 1 is always smaller than the main term. We shall leave to the reader the problem of showing that our two expressions for $C(\alpha)$ agree: this is an exercise in the Poisson summation formula.

If quadratic reciprocity said $(m/n) = (n/m)$ for all m and n, then we would have a functional equation $C(\alpha) = \alpha^{3/2} C(1/\alpha)$. Plainly this does not hold, although the above expressions do show a relationship between $C(\alpha)$ and $\alpha^{3/2} C(1/\alpha)$. If the sum defining $S(X, Y)$ had been restricted to $m \equiv n \equiv 1 \pmod 4$, then such a functional equation would hold.

Differentiating our second expression for $C(\alpha)$ term by term we obtain

$$(\alpha^{-3/2} C(\alpha))' = -\frac{1}{2}\alpha^{-3/2} - \alpha^{-5/2} f\left(\frac{\alpha}{2}\right),$$

where

$$f(x) = \frac{1}{\pi} \sum_{k \neq 0} \frac{1}{k^2} \sin(\pi k^2 x).$$

The function $f(x)$ is commonly called "Riemann's nondifferentiable function," so named because Weierstrass reported that Riemann suggested it as an example of a continuous function which is not differentiable. A considerable amount of work has been done investigating the differentiability properties of $f(x)$. Hardy [H] showed that it is not differentiable at $x = s$ if s is irrational or if $s = p/q$ with p or q even. Gerver [G] gave a long elementary proof that $f'(p/q) = -1$ if p and q are odd, so Riemann's assertion is not quite correct. For an interesting survey on Riemann's function, see Duistermaat [Du]. In Section 6 we show that determining the differentiability of $f(x)$ at a rational point is a straightforward exercise in the Poisson summation formula; our approach appears to be similar to that of Smith [Sm]. It seem surprising that the asymptotics of a natural object like $S(X, Y)$ should involve non-smooth functions!

In the following section we explain our motivation for studying the sum $S(X, Y)$, and we give a generalization of Theorem 1. In Section 3 we do some preliminary reductions and identify the main terms and error terms in the sum. These are evaluated and estimated in Sections 4 and 5. In Section 6 we present graphs of $C(\alpha)$ and $C'(\alpha)$, and we determine at which rationals $C'(\alpha)$ is differentiable.

2. MOTIVATION: MOLLIFYING $L(\frac{1}{2}, \chi_d)$

The motivation for studying $S(X, Y)$ came from the third author's proof [S] that $L(\frac{1}{2}, \chi_d) \neq 0$ for a positive proportion of fundamental discriminants d, where χ_d is the real primitive character to the modulus d. Jutila [J] showed that there exist positive constants c_1 and c_2 such that

$$\sum_{|d|<X} L(\tfrac{1}{2}, \chi_d) \sim c_1 X \log X$$

and

$$\sum_{|d|<X} |L(\tfrac{1}{2}, \chi_d)|^2 \sim c_2 X \log^3 X,$$

where both sums range over fundamental discriminants. It follows from the above formulas and Cauchy's inequality that the number of $|d| < X$ with $L(\frac{1}{2}, \chi_d) \neq 0$ exceeds $X/\log X$.

The approach used to obtain the nonvanishing of $L(\frac{1}{2}, \chi_d)$ for a positive proportion of d was to consider a "mollified" sum of $L(\frac{1}{2}, \chi_{8d})$. Let

$$M(d) = \sum_{\ell \leqslant M} \lambda(\ell) \sqrt{\ell} \left(\frac{8d}{\ell}\right),$$

where $\lambda(\ell)$ is chosen so that

$$S_1 := \sum_{|8d|<X} L(\tfrac{1}{2}, \chi_{8d})\, M(d) \qquad \text{and} \qquad S_2 := \sum_{|8d|<X} |L(\tfrac{1}{2}, \chi_{8d})\, M(d)|^2$$

are both of size X. By Cauchy's inequality this implies that $L(\frac{1}{2}, \chi_{8d}) \neq 0$ for a positive proportion of d. The optimal choice for $\lambda(\ell)$ is determined in [S]. The answer is complicated, so suffice it to say that $\lambda(\ell)$ is supported on the odd integers, where

$$\lambda(\ell) \text{ is roughly proportional to } \frac{\mu(\ell)}{\ell} \frac{\log^2(M/\ell)}{\log^2 M} \frac{\log(X^{3/2}M^2\ell)}{\log M}.$$

This leads to the result that $L(\frac{1}{2}, \chi_{8d}) \neq 0$ for at least 7/8 of all squarefree integers d.

The most difficult part of the above argument is the evaluation of a certain "off-diagonal" contribution to the main term. This involves finding an asymptotic formula for an expression of the form

$$\Sigma_\ell(X) := \sum_{m,\,n} F(m, n, \ell, X) \left(\frac{m}{\ell n}\right),$$

for some explicit function F. In this paper we suppress the function F, and we find that the resulting sum retains the interesting features of the corresponding sum considered in [S].

Motivated by the sum $\Sigma_\ell(X)$ we also consider the slightly more general sum

$$S_\ell(X, Y) := \sum_{\substack{m \leqslant X \\ m \text{ odd}}} \sum_{\substack{n \leqslant Y \\ n \text{ odd}}} \left(\frac{m}{\ell n}\right).$$

THEOREM 2. *Let ℓ be an odd squarefree integer. There exists a constant $C_\ell(\alpha)$ such that, uniformly for all large X and Y,*

$$S_\ell(X, Y) \sim \frac{1}{\sigma(\ell)} \frac{2}{\pi^2} C_\ell\left(\frac{Y}{X}\right) X^{3/2},$$

where $\sigma(\ell)$ is the divisor sum function. We have $C_\ell(\alpha) = C(\alpha\ell)$, where $C(\alpha)$ is given in Theorem 1.

The proof of Theorem 2 will be omitted because it closely follows the proof of Theorem 1.

3. INITIAL REDUCTIONS

When $Y \leqslant X^{16/17}$ or $X \leqslant Y^{16/17}$, Theorem 1 follows immediately from (1.1) and (1.2) and the limiting behaviors of $C(\alpha)$. We assume below that $Y^{16/17} \leqslant X \leqslant Y^{17/16}$. In place of $S(X, Y)$ it is technically easier to consider the smoothed sum

$$\mathcal{S}(X, Y) := \sum_{m \text{ odd}} \sum_{n \text{ odd}} \left(\frac{m}{n}\right) H\left(\frac{m}{X}\right) \Phi\left(\frac{n}{Y}\right).$$

Here H and Φ are smooth functions supported in $(0, 1)$, satisfying $H(t) = \Phi(t) = 1$ for $t \in (1/U, 1 - 1/U)$, and such that $H^{(j)}(t)$, $\Phi^{(j)}(t) \ll_j U^j$ for all integers $j \geqslant 0$. The parameter U will later be chosen to equal $(XY)^{2/5} (X + Y)^{-3/5}$.

Using the Pólya–Vinogradov inequality in a way similar to the argument described in the Introduction, it is easy to see that

$$|S(X, Y) - \mathcal{S}(X, Y)| \ll \frac{X^{3/2} + Y^{3/2}}{U} \log XY. \tag{3.1}$$

With our choice of U this is seen to be smaller than the error term.

We evaluate $\mathcal{S}(X, Y)$ by applying the Poisson summation formula to the sum over m in $\mathcal{S}(X, Y)$. For all odd integers n and all integers k, we introduce the Gauss-type sums

$$\tau_k(n) := \sum_{a \,(\mathrm{mod}\, n)} \left(\frac{a}{n}\right) e\left(\frac{ak}{n}\right) =: \left(\frac{1+i}{2} + \left(\frac{-1}{n}\right)\frac{1-i}{2}\right) G_k(n),$$

where $e(x) := e^{2\pi i x}$ as usual. We quote Lemma 2.3 of [S] which determines $G_k(n)$.

Lemma 1. *If $(m, n) = 1$ then $G_k(mn) = G_k(m)\, G_k(n)$. Suppose that p^a is the largest power of p dividing k (put $a = \infty$ if $k = 0$). Then for $b \geqslant 1$ we have*

$$G_k(p^b) = \begin{cases} 0 & \text{if } b \leqslant a \text{ is odd} \\ \varphi(p^b) & \text{if } b \leqslant a \text{ is even} \\ -p^a & \text{if } b = a+1 \text{ is even} \\ \left(\frac{k/p^a}{p}\right) p^a \sqrt{p} & \text{if } b = a+1 \text{ is odd} \\ 0 & \text{if } b \geqslant a+2. \end{cases}$$

By Poisson summation we have (see Section 2.4 of [S] for details):

$$\sum_{m \text{ odd}} \left(\frac{m}{n}\right) H\left(\frac{m}{X}\right) = \frac{X}{2n}\left(\frac{2}{n}\right) \sum_{k=-\infty}^{\infty} (-1)^k \tau_k(n)\, \widehat{H}\left(\frac{kX}{2n}\right).$$

Expressing τ_k in terms of G_k, using the relation $G_k(n) = (-1/n)\, G_{-k}(n)$, and recombining the k and $-k$ terms we may rewrite the above as

$$\frac{X}{2n}\left(\frac{2}{n}\right) \sum_{k=-\infty}^{\infty} (-1)^k G_k(n)\, \widetilde{H}\left(\frac{kX}{2n}\right),$$

where

$$\widetilde{H}(\xi) := \frac{1+i}{2}\widehat{H}(\xi) + \frac{1-i}{2}\widehat{H}(-\xi).$$

These manipulations show that

$$\mathscr{S}(X, Y) = \frac{X}{2} \sum_{k=-\infty}^{\infty} \sum_{n \text{ odd}} (-1)^k \left(\frac{2}{n}\right) \frac{G_k(n)}{n} \Phi\left(\frac{n}{Y}\right) \tilde{H}\left(\frac{kX}{2n}\right)$$
$$= M + R, \tag{3.2}$$

where M isolates the terms when $k = 2\Box$, and R includes the remaining terms. That is,

$$M := \frac{X}{2} \sum_{k=0}^{\infty} \sum_{n \text{ odd}} \frac{G_{2k^2}(n)}{n} \left(\frac{2}{n}\right) \Phi\left(\frac{n}{Y}\right) \tilde{H}\left(\frac{k^2 X}{n}\right)$$
$$= \frac{X}{2} \sum_{k=0}^{\infty} M_k,$$

say, and

$$R := \frac{X}{2} \sum_{\substack{k=-\infty \\ k \neq 2\Box}}^{\infty} (-1)^k \sum_{n \text{ odd}} \frac{G_k(n)}{n} \left(\frac{2}{n}\right) \Phi\left(\frac{n}{Y}\right) \tilde{H}\left(\frac{kX}{2n}\right).$$

We will see that M gives the main term and R is an error term.

4. THE REMAINDER TERM R

We will require some simple estimates on $\widehat{H}(t)$ and $\tilde{H}(t)$. These follow by integration by parts and our assumptions on H and Φ. We have

$$|\widehat{H}(t)|, \qquad |\tilde{H}(t)|, \qquad |(\tilde{H}(t))'| \ll_j U^{j-1} |t|^{-j} \tag{4.1}$$

for all integers $j \geqslant 1$, and all real t, and

$$\tilde{H}(\xi) = \frac{1 - \cos(2\pi\xi) + \sin(2\pi\xi)}{2\pi\xi} + O\left(\frac{1}{U}\right). \tag{4.2}$$

We handle the remainder term R using the following Lemma which exhibits cancellation in the sum $G_k(n)(2/n)$ when $2k \neq \Box$.

Lemma 2. *If $k \neq 2\Box$ then*

$$\sum_{\substack{n \leqslant x \\ n \text{ odd}}} \frac{G_k(n)}{\sqrt{n}} \left(\frac{2}{n}\right) \ll |k|^{1/4} \log(2|k|)\, d(k^2)\, x^{1/2},$$

where $d(k^2)$ is the number of divisors of k^2.

Before proving the Lemma we note the bound it gives for R. By partial summation and Lemma 2 we have

$$\sum_{n \text{ odd}} \frac{G_k(n)}{n} \left(\frac{2}{n}\right) \Phi\left(\frac{n}{Y}\right) \tilde{H}\left(\frac{kX}{2n}\right)$$
$$\ll |k|^{1/4} \log(2\,|k|)\, d(k^2) \cdot \int_0^Y \sqrt{t} \left| \left(\frac{1}{\sqrt{t}} \Phi\left(\frac{t}{Y}\right) \tilde{H}\left(\frac{kX}{2t}\right)\right)' \right| dt,$$

and using (4.1) with $j=3$ this is

$$\ll |k|^{1/4} \log(2\,|k|)\, d(k^2) \frac{U^2 Y^2}{k^2 X^2}.$$

Summing over all $k \neq 2\square$ we obtain

$$R \ll \frac{U^2 Y^2}{X}. \tag{4.3}$$

Proof of Lemma 2. We write $n = rs$ where r and s are odd with s coprime to k and r divisible only by primes dividing k. By Lemma 1, $G_k(n) = G_k(r)\, G_k(s) = G_k(r)\, \mu^2(s) \sqrt{s}(k/s)$. Further note that $G_k(r) = 0$ unless $r \mid k^2$, and at any rate $|G_k(r)| \leqslant r$. Thus our desired sum is

$$\ll \sum_{\substack{r \mid k^2 \\ r \leqslant x}} \sqrt{r} \left| \sum_{s \leqslant x/r} \mu^2(s) \left(\frac{2k}{s}\right) \right|.$$

Expressing $\mu^2(s) = \sum_{d^2 \mid s} \mu(d)$, and using the Pólya–Vinogradov inequality (since $(2k/\cdot)$ is a non-principal character with conductor $\leqslant 8\,|k|$) we obtain

$$\sum_{s \leqslant x/r} \mu^2(s) \left(\frac{2k}{s}\right) \ll \sum_{d \leqslant \sqrt{x/r}} \left| \sum_{t \leqslant x/rd^2} \left(\frac{2k}{t}\right) \right|$$
$$\ll \sum_{d \leqslant \sqrt{x/r}} \min\left(\frac{x}{rd^2}, |k|^{1/2} \log(2\,|k|)\right)$$
$$\ll |k|^{1/4} (\log\ |2k|)\, x^{1/2} r^{-1/2}.$$

Using this in our previous display we obtain Lemma 2.

5. THE MAIN TERM M

First consider the case $k=0$. It follows straight from the definition that $G_0(n)=\varphi(n)$ if $n=\square$ and $G_0(n)=0$ otherwise. Thus

$$M_0=\widehat{H}(0)\sum_{n=\text{odd }\square}\frac{\varphi(n)}{n}\,\Phi\left(\frac{n}{Y}\right)=\widehat{H}(0)\,\frac{2\sqrt{Y}}{\pi^2}\int_0^\infty\frac{\Phi(y)}{\sqrt{y}}\,dy+O(\log Y).$$

The second step follows from

$$\sum_{\substack{n\leqslant x\\ n=\text{odd }\square}}\frac{\varphi(n)}{n}=\frac{4}{\pi^2}\sqrt{x}+O(\log x),$$

and then using partial summation.

Now suppose $k\geqslant 1$. Since n is odd, by changing variables in the sum defining G_k we have $G_{2k}(n)(2/n)=G_k(n)$. This gives

$$M_k=\sum_{n\text{ odd}}\frac{G_{k^2}(n)}{n}\,\Phi\left(\frac{n}{Y}\right)\tilde{H}\left(\frac{k^2X}{n}\right).$$

Using Lemma 1, and a straightforward calculation, we obtain

$$\sum_{\substack{n\leqslant x\\ n\text{ odd}}}\frac{G_{k^2}(n)}{\sqrt{n}}=\frac{4}{\pi^2}\,x+O(\log x).$$

Hence, by partial summation,

$$\begin{aligned}M_k&=\frac{4}{\pi^2}\int_0^\infty\Phi\left(\frac{t}{Y}\right)\tilde{H}\left(\frac{k^2X}{t}\right)\frac{dt}{\sqrt{t}}\\&\quad+O\left(\log Y\int_0^Y\left|\left(\Phi\left(\frac{t}{Y}\right)\tilde{H}\left(\frac{k^2X}{t}\right)\frac{1}{\sqrt{t}}\right)'\right|\,dt\right).\end{aligned}$$

Making the change of variable $t=yY$ and using the bounds (4.1) (with $j=1$ or 2), we have

$$M_k=\frac{4}{\pi^2}\sqrt{Y}\int_0^\infty\Phi(y)\,\tilde{H}\left(\frac{k^2X}{yY}\right)\frac{dy}{\sqrt{y}}+O\left(\frac{U\sqrt{Y}}{k^2X}\log Y\right).$$

Combining this with our earlier expression for M_0 we conclude that

$$M=\frac{X\sqrt{Y}}{\pi^2}\int_0^\infty\frac{\Phi(y)}{\sqrt{y}}\sum_{k=-\infty}^{\infty}\tilde{H}\left(\frac{k^2X}{yY}\right)dy+O((X+U\sqrt{Y})\log Y).\quad(5.1)$$

Using (4.1) (with $j=1$) when $|k| \geqslant \sqrt{UY}/\sqrt{X}$, and (4.2) for smaller k we deduce that

$$\sum_{k=-\infty}^{\infty} \tilde{H}\left(\frac{k^2X}{yY}\right) = 1 + \frac{yY}{2\pi X} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^2}$$

$$\times \left(1 - \cos\left(\frac{2\pi k^2 X}{yY}\right) + \sin\left(\frac{2\pi k^2 X}{yY}\right)\right) + O\left(\frac{\sqrt{Y}}{\sqrt{UX}}\right).$$

Using this in (5.1) we obtain

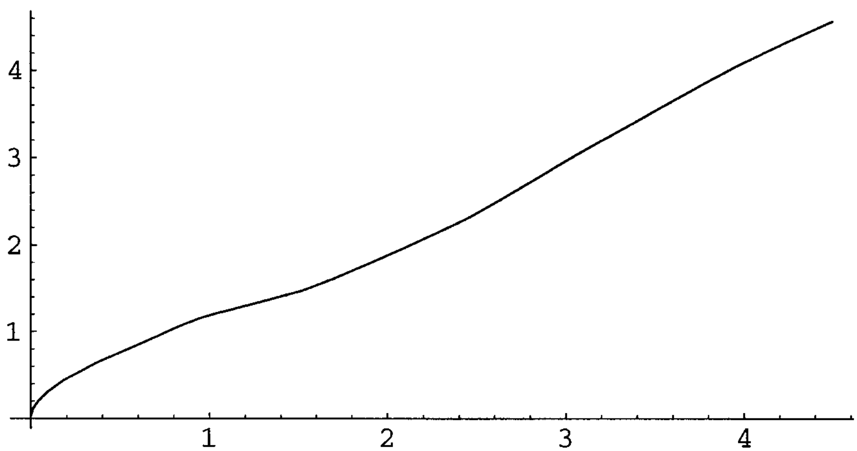
$$M = \frac{2X^{3/2}}{\pi^2} C\left(\frac{Y}{X}\right) + O\left(\frac{X^{3/2}+Y^{3/2}}{\sqrt{U}} + (X+U\sqrt{Y})\log Y\right).$$

We combine this with (3.1), (3.2) and (4.3), and choose $U=(XY)^{2/5}(X+Y)^{-3/5}$ to obtain Theorem 1.

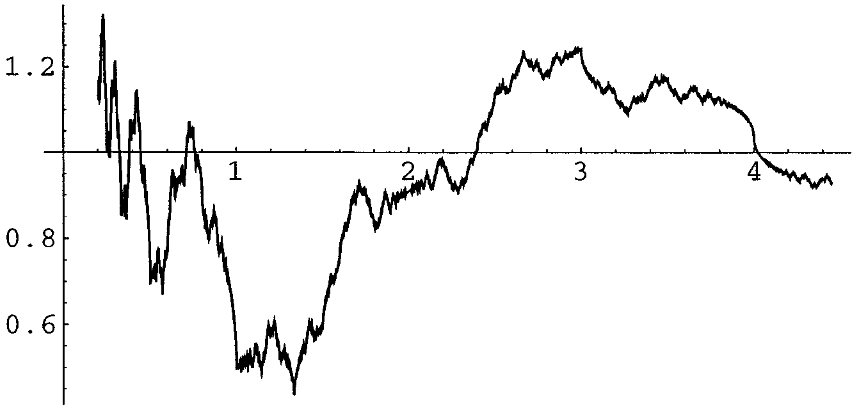
6. SOME GRAPHS OF $C(\alpha)$

In this Section we present graphs of $C(\alpha)$ and $C'(\alpha)$, and we prove that $C'(\alpha)$ is differentiable at $\alpha \in \mathbb{Q}$ if and only if $\alpha = 2p/q$ with p and q both odd.

The following graphs show $C(\alpha)$,



and $C'(\alpha)$:



It certainly appears from the graph that $C'(\alpha)$ is not everywhere differentiable.

PROPOSITION. *$C'(\alpha)$ is differentiable at $\alpha\mathbb{Q}$ if and only if $\alpha = 2p/q$ with p and q both odd.*

The differentiability of $C'(\alpha)$ at $\alpha = 2$ can be seen in the above graph.

Proof. Let

$$f(\alpha) = \frac{1}{\pi} \sum_{k \neq 0} \frac{\sin(\pi k^2 \alpha)}{k^2}.$$

By the second expression for $C(\alpha)$ in Theorem 1 we have

$$(\alpha^{-3/2} C(\alpha))' = -\frac{1}{2} \alpha^{-3/2} - \alpha^{-5/2} f\left(\frac{\alpha}{2}\right), \tag{6.1}$$

the term-by-term differentiation being justified by the uniform absolute convergence of the resulting sum. We will show that $f(\alpha)$ is differentiable at $p/q \in \mathbb{Q}$ if and only if p and q are both odd.

The following Lemma is an exercise in the Poisson summation formula.

LEMMA. *Suppose $(p, q) = 1$. As $\alpha \to 0^{\pm}$ we have*

$$f\left(\frac{p}{q} + \alpha\right) = f(p/q) - \alpha \pm \sqrt{|2\alpha|} \left(\frac{1}{2q} \sum_{v \bmod 2q} \cos \frac{\pi p v^2}{q} \mp \sin \frac{\pi p v^2}{q} \right)$$
$$+ O(\alpha^{3/2}).$$

In particular, $f(\alpha)$ is differentiable at $\alpha = p/q$ if and only if $G(p/q) = 0$, where

$$G(p/q) := \sum_{v \bmod 2q} e^{\pi i p v^2/q}$$

The value of $G(p/q)$ is well known and can be found in Chapter 2 of Davenport [D]. We find that $G(p/q) = 0$ if and only if p and q are both odd. Combining this with formula (6.1) completes the proof of the Proposition. Note that $C'(\alpha)$ is right- or left-differentiable at certain other p/q, such as the odd integers. This can be determined by considering the real and imaginary parts of $G(p/q)$.

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