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ZEROS OF FEKETE POLYNOMIALS

by **B. CONREY, A. GRANVILLE,**
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1. Introduction.

Dirichlet noted that, from the formula

$$\Gamma(s) = n^s \int_0^\infty x^{s-1} e^{-nx} dx = n^s \int_0^1 (-\log t)^{s-1} t^{n-1} dt,$$

we may obtain the identity

$$\begin{aligned} \Gamma(s) L\left(s, \left(\frac{\cdot}{p}\right)\right) &= \Gamma(s) \sum_{n \geq 1} \frac{(n/p)}{n^s} = \int_0^1 (-\log t)^{s-1} \sum_{n \geq 1} \left(\frac{n}{p}\right) t^{n-1} dt \\ (1.1) \qquad \qquad &= \int_0^1 \frac{(-\log t)^{s-1}}{t} \frac{f_p(t)}{1-t^p} dt. \end{aligned}$$

Here $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol and

$$(1.2) \qquad f_p(t) := \sum_{a=0}^{p-1} \left(\frac{a}{p}\right) t^a.$$

Equation (1.1) allowed Dirichlet to define $L(s, \left(\frac{\cdot}{p}\right))$ as a regular function for all complex s . Fekete observed that if $f_p(t)$ has no real zeros t with

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$0 < t < 1$, then $L(s, (\frac{\cdot}{p}))$ has no real zeros $s > 0$; and the $f_p(t)$ are thus now known as *Fekete polynomials*. Indeed, if $L(s, (\frac{\cdot}{p})) = 0$ then by (1.1) and the mean value theorem there is a t in $(0, 1)$ with $\frac{(-\log t)^{s-1} f_p(t)}{1-t^p} = 0$, and so $f_p(t) = 0$ here.

Among small primes p , there are only a few for which the Fekete polynomial $f_p(t)$ has a real zero t in the range $0 < t < 1$. In fact, we may verify computationally that there are just 23 primes up to 1000 for which f_p has a zero in $(0, 1)$. This implies that there are no positive real zeros of $L(s, (\frac{\cdot}{p}))$ for most such primes p , and in particular no *Siegel zeros* (that is, real zeros “especially close to 1”). It is interesting to note that for those primes $p \equiv 3 \pmod{4}$ for which $f_p(t)$ does have a zero in $(0, 1)$, the class number of $Q(\sqrt{-p})$ is surprisingly small (for example $p = 43, 67, 163, \dots$). Unfortunately this trend does not persist: Indeed Baker and Montgomery [1] proved that $f_p(t)$ has a large number of zeros in $(0, 1)$ for almost all primes p (that is, the number of such zeros $\rightarrow \infty$ as $p \rightarrow \infty$, and it seems likely that there are, in fact, $\asymp \log \log p$ such zeros).

In this paper we shall study the complex zeros of $f_p(t)$. Using zero locating software one finds that, for primes p up to 1000, about half of the zeros lie on the unit circle; leading one to expect this to be the general phenomenon. It turns out to be fairly easy to prove that *at least* half of the zeros of $f_p(t)$ are on the unit circle (that is $|t| = 1$): First note that

$$F_p(z) := z^{-p/2} f_p(z) = \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \left(z^{a-p/2} + \left(\frac{-1}{p}\right) z^{p/2-a} \right)$$

by combining the a and $p-a$ terms¹. Taking $z = e^{2i\pi t}$ we have

$$(1.3) \quad F_p(e^{2i\pi t}) = \begin{cases} 2 \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \cos((2a-p)\pi t) & \text{if } p \equiv 1 \pmod{4} \\ 2i \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \sin((2a-p)\pi t) & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Define $H_p(t) = F_p(e^{2i\pi t})$ if $p \equiv 1 \pmod{4}$, and $H_p(t) = -iF_p(e^{2i\pi t})$ if $p \equiv 3 \pmod{4}$. By (1.3) we see that $H_p(t)$ is a periodic, continuous real-valued function when t is real.

Now if $\zeta_p = e^{2i\pi/p}$ then, for all k not divisible by p , $f_p(\zeta_p^k)$ is a *Gauss sum* and has absolute value \sqrt{p} (see Section 2 of [2]); therefore

¹ Here $z = e^{2i\pi t}$ with $0 \leq t < 1$, so that there is no ambiguity in the meaning of $z^{-p/2}$.

$|F_p(\zeta_p^k)| = \sqrt{p}$. Moreover

$$\begin{aligned} F_p(\zeta_p^k) &= (\zeta_p^k)^{-p/2} \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^{ak} = (-1)^k \left(\frac{k}{p}\right) \sum_{a=1}^{p-1} \left(\frac{ak}{p}\right) \zeta_p^{ak} \\ &= (-1)^k \left(\frac{k}{p}\right) F_p(\zeta_p). \end{aligned}$$

Therefore if $(k/p) = ((k+1)/p)$ then $H_p(k/p)$ and $H_p((k+1)/p)$ have different signs. Since $H_p(t)$ is real-valued and continuous, it must have a zero in-between k/p and $(k+1)/p$, by the intermediate value theorem. Thus the number of zeros of $H_p(t)$ in $[0, 1)$ (and so of $F_p(z)$ on the unit circle) is

$$\geq \#\left\{k : 1 \leq k \leq p-2 \text{ and } \left(\frac{k}{p}\right) = \left(\frac{k+1}{p}\right)\right\} = \frac{p-3}{2},$$

as we shall see in Lemma 2.

Other than possible zeros at $z = -1$ and at $z = 1$, this accounts for all the zeros on the unit circle for each prime $p < 500$. So the question is, is this all, for all p ? The answer is “no” and indeed one finds more zeros when $p = 661$. In general one has the following:

THEOREM 1. — *There exists a constant κ_0 , $1 > \kappa_0 > 1/2$ such that*

$$\#\{z : |z| = 1 \text{ and } f_p(z) = 0\} \sim \kappa_0 p \text{ as } p \rightarrow \infty.$$

We determine κ_0 in terms of another constant κ_1 defined as follows:

THEOREM 2. — *Let \mathcal{F}_J be the set of rational functions*

$$g(x) = \frac{1}{x} + \frac{1}{1-x} + \sum_{\substack{|j| < J \\ j \neq 0, -1}} \frac{\delta_j}{x+j}$$

where we allow each δ_j to take value $+1$ or -1 . There exists a constant κ_1 , $1/2 > \kappa_1 > 0$, such that

$$\#\{g \in \mathcal{F}_J : g(x) = 0 \text{ for some } x \in (0, 1)\} \sim \kappa_1 \#\{g \in \mathcal{F}_J\}$$

as $J \rightarrow \infty$.

The constants κ_0 and κ_1 are related as follows:

THEOREM 1 $\frac{1}{2}$. — *In fact $\kappa_0 = 1/2 + \kappa_1$.*

It is still an open question to determine the value of κ_0 . It is known that a “random” trigonometric polynomial of degree p has $p/\sqrt{3}$ zeros in

$[0, 1)$ (see [7]), so one might guess that $\kappa_0 = 1/\sqrt{3} \approx 0.5773\dots$. However this is not the case. We will show

$$0.500813 > \kappa_0 > 0.500668.$$

While it is theoretically easy to find the value of κ_0 , we do not know a good practical way of achieving this.

As well as determining precisely the proportion, κ_0 , of the zeros of $f_p(t)$ which lie on the unit circle, we would also like to understand the distribution of the set of zeros in the complex plane. There are several easy remarks to make: By (1.2) we have

$$t^p f_p(1/t) = \left(\frac{-1}{p}\right) f_p(t)$$

and so the zeros of $f_p(t)$, other than $t = 0$, are symmetric about the unit circle (i.e. they come in pairs other than at $t = 0, \pm 1$). We also note that, for $|t| > 1$,

$$|f_p(t)/t^{p-1}| = \left| \sum_{a=0}^{p-1} \left(\frac{a}{p}\right) \frac{1}{t^{p-1-a}} \right| \geq 1 - \sum_{a=0}^{p-2} \frac{1}{|t|^{p-1-a}} > 1 - \frac{1}{|t| - 1}.$$

However if $|t| \geq 2$ then $1 - 1/(|t| - 1) \geq 0$, and so $f_p(t)$ has no zeros in $|t| \geq 2$. By symmetry it has no zeros in $|t| \leq 1/2$ except 0. Thus

PROPOSITION 1. — *The zeros of $f_p(t)$, other than at 0, 1 and -1 come in pairs $\alpha, 1/\alpha$. Moreover, other than 0, they all lie in the annulus $\{r \in \mathbb{C} : 1/2 < |r| < 2\}$.*

As for the distribution of the arguments of the roots of $f_p(t)$ we can use a beautiful result of Erdős and Turán (Theorem 1 of [3]), which immediately implies that, for any $0 \leq \alpha < \beta < 1$,

$$(1.4) \quad \#\{\tau \in \mathbb{C} : f_p(\tau) = 0, \alpha < \arg(\tau)/2\pi < \beta\} = (\beta - \alpha)p + O(\sqrt{p \log p}).$$

The arguments above, and those used in proving Theorems 1 and 2, focus on determining which arcs $(\zeta_p^K, \zeta_p^{K+1})$ of the unit circle contain a zero of $f_p(t)$. Evidently (1.4) cannot be used so precisely. However we can show that there are zeros of $f_p(t)$ near to such an arc, so long as $f_p(t)$ gets “small” on that arc.

THEOREM 3. — *Suppose that $\epsilon > 0$ is a sufficiently small constant. If p is a sufficiently large prime and K an integer such that there exists a value of t on the unit circle in the arc from ζ_p^K to ζ_p^{K+1} with $|f_p(t)| < \epsilon\sqrt{p}$, then there exists $\tau = r\zeta_p^{K+\theta}$ with $f_p(\tau) = 0$ where $0 < \theta < 1$ and $1 - \epsilon^{1/3}/p < r \leq 1$.*

Remark. — Applying Proposition 1 we also have $f_p((1/r)\zeta_p^{K+\theta}) = 0$.

As we have already discussed, Gauss sums $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^{ak}$ (and many generalizations) have the surprising property that they have absolute value exactly equal to \sqrt{p} . It is, we think, of interest to ask what happens when we replace the primitive p -th root of unity ζ_p^k in the expression for a Gauss sum above, by some primitive $2p$ -th root of unity. These may be written as $\zeta_p^{k+1/2}$ or ζ_{2p}^{2k+1} , or $-\zeta_p^k$; so we must consider the values of $f_p(-\zeta_p^k)$. Do these all take on the same absolute value? The answer we now see is “no”, as we evaluate the distribution of these absolute values:

THEOREM 4. — *For any fixed real number ρ*

$$\#\left\{k : 1 \leq k \leq p \text{ such that } H_p\left(\frac{k + \frac{1}{2}}{p}\right) < \rho\sqrt{p}\right\} \sim c_\rho p$$

as $p \rightarrow \infty$ where

$$c_\rho = \frac{1}{2} + \frac{1}{\pi} \int_{x=0}^{\infty} \sin(\rho\pi x) \prod_{\substack{n \geq 1 \\ n \text{ odd}}} \cos^2\left(\frac{2x}{n}\right) \frac{dx}{x}.$$

Moreover $c_{-\rho}$ and $1 - c_\rho = \exp(-\exp(\pi\rho/2 + O(1)))$ for positive ρ .

After proving this in Section 6, we indicate how our proof may be modified to establish several related results. First, to show that $\max_{|z|=1} |f_p(z)| \gg \sqrt{p} \log \log p$, so re-establishing a result of Montgomery [5]. Second to understand the distribution of the values of the Fekete polynomial at $(p-1)$ -st roots of unity.

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2. First results.

Let χ be any character $(\bmod p)$ and let k be an integer not divisible by p . Note that

$$(2.1) \quad \sum_{a=1}^{p-1} \chi(a) \zeta_p^{ak} = \bar{\chi}(k) \sum_{a=1}^{p-1} \chi(ak) \zeta_p^{ak} = \bar{\chi}(k) \sum_{b=1}^{p-1} \chi(b) \zeta_p^b.$$

In particular we see that $f_p(\zeta_p^k) = \left(\frac{k}{p}\right)f_p(\zeta_p)$, whereas in contrast $f_p(1) = 0$. Recall that for a non-principal character $\chi \pmod{p}$, the Gauss sum $\tau(\chi)$ is $\sum_{a=1}^{p-1} \chi(a)\zeta_p^a$. Thus $f_p(\zeta_p)$ is the Gauss sum $\tau\left(\left(\frac{\cdot}{p}\right)\right)$. It is easy to determine the magnitude of $|f_p(\zeta_p)|$: Note that

$$\begin{aligned} (p-1)f_p(\zeta_p)^2 &= \sum_{k=0}^{p-1} f_p(\zeta_p^k)^2 = \sum_{k=0}^{p-1} \sum_{a,b=0}^{p-1} \left(\frac{ab}{p}\right) \zeta_p^{(a+b)k} \\ &= \sum_{a,b=1}^{p-1} \left(\frac{ab}{p}\right) \sum_{k=0}^{p-1} \zeta_p^{(a+b)k} = p \sum_{\substack{a=1 \\ b=p-a}}^{p-1} \left(\frac{ab}{p}\right) = p \left(\frac{-1}{p}\right) (p-1). \end{aligned}$$

Hence we have $f_p(\zeta_p)^2 = (-1/p)p$, and so $|f_p(\zeta_p)| = \sqrt{p}$. Gauss showed more and determined that

$$f_p(\zeta_p) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \pmod{4}, \\ i\sqrt{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Since $f_p(\zeta_p^k) = (k/p)f_p(\zeta_p)$, for $1 \leq k \leq p-1$, and $f_p(1) = 0$, we get by Lagrangian interpolation

$$f_p(z) = \sum_{k=0}^{p-1} f_p(\zeta_p^k) \prod_{\substack{j=0 \\ j \neq k}}^{p-1} \left(\frac{z - \zeta_p^j}{\zeta_p^k - \zeta_p^j} \right).$$

Note that

$$\prod_{\substack{j=0 \\ j \neq k}}^{p-1} (z - \zeta_p^j) = \frac{z^p - 1}{z - \zeta_p^k},$$

and that

$$\prod_{\substack{j=0 \\ j \neq k}}^{p-1} (\zeta_p^k - \zeta_p^j) = \zeta_p^{k(p-1)} \prod_{j=1}^{p-1} (1 - \zeta_p^j) = p\zeta_p^{-k}.$$

Hence

$$(2.2) \quad \frac{p}{f_p(\zeta_p)} \frac{f_p(z)}{z^p - 1} = \frac{p}{f_p(\zeta_p)} \frac{z^{-p/2} f_p(z)}{z^{p/2} - z^{-p/2}} = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \frac{\zeta_p^k}{z - \zeta_p^k}.$$

If $|z| = 1$ then note that $z^{p/2} - z^{-p/2} \in i\mathbb{R}$, and from (1.3) and $f_p(\zeta_p)^2 = (-1/p)p$ we have $z^{\frac{-p}{2}} f_p(z)/f_p(\zeta_p) \in \mathbb{R}$. Thus the right side of (2.2) $\in i\mathbb{R}$

for all $|z| = 1$. To facilitate studying $f_p(z)$ as z goes around the unit circle from ζ_p^K to ζ_p^{K+1} , we write $z = \zeta_p^{K+x} = \zeta_p^K e^{2i\pi x/p}$ and then let

$$(2.3) \quad \begin{aligned} g_{p,K}(x) &:= i \left(\frac{K}{p} \right) \frac{p}{f_p(\zeta_p)} \frac{f_p(z)}{z^p - 1} \Big|_{z=\zeta_p^{K+x}} \\ &= i \left(\frac{K}{p} \right) \sum_{k=K-(\frac{p-1}{2})}^{K+(\frac{p-1}{2})} \left(\frac{k}{p} \right) \frac{1}{\zeta_p^{K-k+x} - 1}. \end{aligned}$$

Thus $g_{p,K}(x)$ is a real valued function of $x \in [0, 1]$.

PROPOSITION 2. — If $0 \leq K \leq p-1$ is an integer with $\left(\frac{K}{p}\right) = \left(\frac{K+1}{p}\right)$ then $g_{p,K}(x)$ has exactly one zero in $(0, 1)$. Equivalently, $f_p(z)$ has exactly one zero on the arc of the unit circle from ζ_p^K to ζ_p^{K+1} . If $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$ then $g_{p,K}$ has either no zeros, or exactly two zeros in $(0, 1)$. Equivalently, $f_p(z)$ has exactly 0 or 2 zeros on the arc from ζ_p^K to ζ_p^{K+1} .

Remark. — In the above proposition, and henceforth, we count zeros with multiplicity.

Before proving the proposition, we evaluate $\sum_{k=1}^{p-1} \frac{1}{\sin^2(\pi k/p)}$.

LEMMA 1. — For all integers $p \geq 2$,

$$\sum_{k=1}^{p-1} \frac{1}{\sin^2(\frac{\pi k}{p})} = \frac{p^2 - 1}{3}.$$

Proof. — Put $A(z) = \prod_{k=1}^{p-1} (z - \zeta_p^k)$. Logarithmic differentiation shows that

$$\left\{ z \left(\frac{A'(z)}{A(z)} \right)' + \frac{A'(z)}{A(z)} \right\} \Big|_{z=1} = - \sum_{k=1}^{p-1} \frac{\zeta_p^k}{(1 - \zeta_p^k)^2} = \frac{1}{4} \sum_{k=1}^{p-1} \frac{1}{\sin^2\left(\frac{\pi k}{p}\right)}.$$

However, $A(z) = \frac{z^p - 1}{z - 1} = z^{p-1} + z^{p-2} + \dots + 1$ and using this to evaluate the left side above, we get the lemma. \square

Proof of Proposition 2. — Note that with $g = g_{p,K}$, we have $\lim_{x \rightarrow 0+} g(x) = \infty$, and $\lim_{x \rightarrow 1-} g(x) = -\left(\frac{K}{p}\right)\left(\frac{K+1}{p}\right)\infty$. Further observe that

$$\begin{aligned}
 g'(x) &= \frac{2\pi}{p} \left(\frac{K}{p} \right) \sum_{|k-K| < p/2} \left(\frac{k}{p} \right) \frac{\zeta_p^{K-k+x}}{(\zeta_p^{K-k+x} - 1)^2} \\
 &= -\frac{\pi}{2p} \left(\frac{K}{p} \right) \sum_{|k-K| < p/2} \left(\frac{k}{p} \right) \frac{1}{\sin^2(\frac{\pi}{p}(K-k+x))}.
 \end{aligned}$$

If $(\frac{K}{p}) = (\frac{K+1}{p})$ then, by Lemma 1,

$$\begin{aligned}
 (2.4) \quad |g'(x)| &\geq \frac{\pi}{2p} \left(\frac{1}{\sin^2(\frac{\pi}{p}x)} + \frac{1}{\sin^2(\frac{\pi}{p}(1-x))} - \sum_{\substack{j \neq 0,1 \\ |j| < p/2}} \frac{1}{\sin^2(\frac{\pi}{p}(x-j))} \right) \\
 &\geq \frac{\pi}{2p} \left(\frac{2}{\sin^2(\frac{\pi}{2p})} - \frac{p^2-1}{3} \right) > 0,
 \end{aligned}$$

since the sum of the first two terms is minimized when $x = 1/2$. Hence $g'(x) \neq 0$ for all $x \in (0, 1)$, so that g is monotone decreasing in $[0, 1]$ going from ∞ to $-\infty$. Thus g has exactly one zero in this interval.

Moreover

$$g''(x) = \frac{\pi^2}{p^2} \left(\frac{K}{p} \right) \sum_{|k-K| < p/2} \left(\frac{k}{p} \right) \frac{\cos(\frac{\pi}{p}(K-k+x))}{\sin^3(\frac{\pi}{p}(K-k+x))}.$$

Now if $(\frac{K}{p}) = -(\frac{K+1}{p})$ then

$$|g''(x)| \geq \frac{\pi^2}{p^2} \left(\frac{\cos(\frac{\pi}{p}x)}{\sin^3(\frac{\pi}{p}x)} + \frac{\cos(\frac{\pi}{p}(1-x))}{\sin^3(\frac{\pi}{p}(1-x))} - \sum_{\substack{j \neq 0,1 \\ |j| < p/2}} \frac{\cos(\frac{\pi}{p}(j-x))}{|\sin(\frac{\pi}{p}(j-x))|^3} \right).$$

Let μ be the minimum of $\cot(\frac{\pi}{p}t)$ over $t = x, 1-x$. Since $\cot t$ decreases rapidly as t goes from 0 to $\pi/2$ we see that the above is

$$\geq \frac{\pi^2}{p^2} \mu \left(\frac{1}{\sin^2(\frac{\pi}{p}x)} + \frac{1}{\sin^2(\frac{\pi}{p}(1-x))} - \sum_{\substack{j \neq 0,1 \\ |j| < p/2}} \frac{1}{\sin^2(\frac{\pi}{p}(x-j))} \right) > 0,$$

as in (2.4). Thus $g'(x)$ is monotone increasing in $(0, 1)$ going from $-\infty$ to $+\infty$. Thus there is a unique x_0 in $(0, 1)$ with $g'(x_0) = 0$, and the minimum value of $g(x)$ is attained at x_0 . Plainly g has 0 or 2 zeros depending on whether $g(x_0) > 0$, or $g(x_0) \leq 0$. This proves the proposition. \square

From Proposition 2 we know that $f_p(z)$ has at least as many zeros on $|z| = 1$, as there are values $1 \leq K \leq p-1$ with $(\frac{K}{p}) = (\frac{K+1}{p})$. We next determine the number of such values K .

LEMMA 2 (Gauss). — For any non-principal character $\chi \pmod{p}$, we have

$$(2.5) \quad \sum_{b=1}^{p-1} \chi(b) \bar{\chi}(b+k) = \begin{cases} p-1 & \text{if } p \mid k \\ -1 & \text{if } p \nmid k. \end{cases}$$

Hence

$$\# \left\{ b \pmod{p} : \left(\frac{b}{p} \right) = \left(\frac{b+1}{p} \right) \right\} = \frac{p-3}{2},$$

and

$$\# \left\{ b \pmod{p} : \left(\frac{b}{p} \right) = -\left(\frac{b+1}{p} \right) \right\} = \frac{p-1}{2}.$$

Proof. — If $p \mid k$ then the right side of (2.5) is $\sum_{b=1}^{p-1} |\chi(b)|^2 = p-1$. Suppose now that $p \nmid k$, and let $c = (b+k)/b = 1 + k/b$. As b runs over the non-zero residue classes \pmod{p} , note that c runs over all residue classes except the residue class $1 \pmod{p}$. Hence the right side of (2.5) is

$$\sum_{\substack{c \pmod{p} \\ c \not\equiv 1 \pmod{p}}} \bar{\chi}(c) = -1,$$

as desired. \square

If $\left(\frac{K}{p} \right) = -\left(\frac{K+1}{p} \right)$ then we need to determine (in the notation of the proof of Proposition 2) whether $g(x_0) > 0$ or ≤ 0 . This depends heavily on the values of $\left(\frac{k}{p} \right)$ for k neighbouring K . The following lemma shows that these neighbouring values behave like independent random variables.

LEMMA 3 (Weil). — Fix integer J , and then the numbers $\delta_j \in \{-1, 1\}$ for each j with $|j| < J$. We have, uniformly,

$$\# \left\{ x \pmod{p} : \left(\frac{x-j}{p} \right) = \delta_j \text{ for all } |j| < J \right\} = \frac{p}{2^{2J-1}} + O(J\sqrt{p}).$$

Proof. — The above equals

$$\begin{aligned} & \sum_{x=1}^p \frac{1}{2^{2J-1}} \prod_{|j| < J} \left(1 + \delta_j \left(\frac{x-j}{p} \right) \right) + O(J) \\ &= \frac{p}{2^{2J-1}} + O\left(\frac{1}{2^{2J-1}} \sum_{\substack{S \subseteq \{|j| < J\} \\ S \neq \emptyset}} \sum_{x=1}^p \left(\frac{\prod_{j \in S} (x-j)}{p} \right) + J \right). \end{aligned}$$

By Weil's Theorem [8], if $f(x)$ is a squarefree polynomial (mod p) then

$$\left| \sum_{x=1}^p \left(\frac{f(x)}{p} \right) \right| \ll (\text{degree } f) \sqrt{p}.$$

Hence the above is

$$= \frac{p}{2^{2J-1}} + O\left(\frac{\sqrt{p}}{2^{2J-1}} \sum_{m=1}^{2J-1} \binom{2J-1}{m} m + J \right),$$

and the result follows. \square

We conclude this section by determining the order of the zeros of $f_p(z)$ at ± 1 . In fact we shall determine the number of zeros of $f_p(z)$ on the arcs $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$ (which contains -1), and ζ_p^{-1} to ζ_p (which contains 1).

LEMMA 4. — *If $p \equiv 1 \pmod{4}$ then $f_p(z)$ has only a simple zero at $z = -1$, on the arc from $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$, and $f_p(z)$ has only a double zero at $z = 1$, on the arc from ζ_p^{-1} to ζ_p . If $p \equiv 3 \pmod{4}$ then there are no zeros of $f_p(z)$ on the arc from $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$, and $f_p(z)$ has only a simple zero at $z = 1$ on the arc from ζ_p^{-1} to ζ_p .*

Proof. — We make free use of the fact that $(-1/p) = 1$, or -1 depending on whether $p \equiv 1 \pmod{4}$, or $3 \pmod{4}$. Let's begin with the arc from $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$. We take $K = (p-1)/2$ in Proposition 2. Note that $(\frac{K}{p}) = (\frac{K+1}{p})$ if $p \equiv 1 \pmod{4}$, and $(\frac{K}{p}) = -(\frac{K+1}{p})$ if $p \equiv 3 \pmod{4}$. In the first case, Proposition 2 tells us that there's exactly one (simple) zero on this arc. Since

$$f_p(-1) = \sum_{a=1}^{p-1} (-1)^a \left(\frac{a}{p} \right) = \frac{1}{2} \sum_{a=1}^{p-1} (-1)^a \left(\left(\frac{a}{p} \right) - \left(\frac{p-a}{p} \right) \right) = 0$$

for $p \equiv 1 \pmod{4}$, this simple zero is at -1 . Now suppose $p \equiv 3 \pmod{4}$. By Proposition 2, we know that there are 0 or 2 zeros on this arc, depending on whether $\min_x g_{p,K}(x) > 0$ or not. We now show that this minimum is attained at $x = 1/2$, and the minimum value is positive. Putting $j = K - k$ in (2.3) we have

$$\begin{aligned} g_{p,K}(x) &= i \left(\frac{K}{p} \right) \sum_{|j| \leq (p-1)/2} \left(\frac{K-j}{p} \right) \frac{1}{\zeta_p^{j+x} - 1} \\ &= i \left(\frac{K}{p} \right) \sum_{j=0}^{(p-1)/2} \left(\frac{K-j}{p} \right) \left(\frac{1}{\zeta_p^{j+x} - 1} - \frac{1}{\zeta_p^{-j-1+x} - 1} \right), \end{aligned}$$

since $K + j + 1 \equiv -(K - j) \pmod{p}$. Evidently $g_{p,K}(1 - x) = \overline{g_{p,K}(x)}$, so $g_{p,K}(1 - x) = g_{p,K}(x)$ since $g_{p,K}(x)$ is real-valued. However we see that the minimum of $g_{p,K}(x)$ is obtained at a unique point in $(0, 1)$, so that must be at $x = 1/2$. Now

$$f_p(-1) = \sum_{a=1}^{p-1} (-1)^a \left(\frac{a}{p}\right) = \sum_{\substack{a=1 \\ a \text{ even}}}^{p-1} \left(\frac{a}{p}\right) - \sum_{\substack{b=1 \\ b \text{ even}}}^{p-1} \left(\frac{p-b}{p}\right)$$

where $a = p - b$ is odd in the second sum,

$$f_p(-1) = 2 \sum_{d=1}^{(p-1)/2} \left(\frac{2d}{p}\right) = 2 \left(\frac{2}{p}\right) \sum_{d=1}^{(p-1)/2} \left(\frac{d}{p}\right) = 2 \left(2 \left(\frac{2}{p}\right) - 1\right) h(-p),$$

where $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$ (see Section 2 of [2]). By (2.3), and since $f_p(\zeta_p) = i\sqrt{p}$ by Gauss, we have

$$\begin{aligned} g_{p,K}\left(\frac{1}{2}\right) &= -\left(\frac{K}{p}\right) \frac{\sqrt{p}}{2} f_p(-1) = \sqrt{p} \left(-2 \left(\frac{2K}{p}\right) + \left(\frac{K}{p}\right)\right) h(-p) \\ &= \sqrt{p} \left(2 + \left(\frac{K}{p}\right)\right) h(-p) > 0. \end{aligned}$$

This shows that $f_p(z)$ has no zeros on the arc from $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$ when $p \equiv 3 \pmod{4}$.

Now let's consider the arc from ζ_p^{-1} to ζ_p . Take $K = p - 1$, and consider $g_{p,K}(x)$ as defined in (2.3). Usually $g_{p,K}(x)$ would have a discontinuity at 1, but here since $\left(\frac{K+1}{p}\right) = \left(\frac{0}{p}\right) = 0$ we do not have this problem. Thus $g_{p,K}$ is a continuous function on $(0, 2)$, and we may study $f_p(z)$ on the arc from ζ_p^{-1} to ζ_p by studying $g_{p,K}(x)$ on $(0, 2)$. Note that for any p , $f_p(1) = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0$, so that there is at least a simple zero at $z = 1$. Also $f'_p(1) = -i(-1/p)f_p(\zeta_p)g_{p,p-1}(1)$ by (2.3). Since $f_p(z) = (-1/p)z^p f_p(\bar{z})$, we deduce that $g_{p,p-1}(x) = -(-1/p)g_{p,p-1}(2-x)$.

If $p \equiv 1 \pmod{4}$ then $g_{p,p-1}(1) = 0$ and so $f'_p(1) = 0$. Now, as in the proof of (2.4), the first part of the proof of Proposition 2, we have $|g'_{p,K}(x)| > 0$ for all $x \in (0, 2)$. Therefore g has only a simple zero at $x = 1$, and thus f_p has a double zero at 1.

If $p \equiv 3 \pmod{4}$ then, as in the second part of the proof of Proposition 2, $|g'_{p,K}(x)| > 0$ for $x \in (0, 2)$. Thus there is a unique minimum of $g_{p,K}(x)$ on $(0, 2)$, but since $g_{p,p-1}(x) = g_{p,p-1}(2-x)$ this must be attained at $x = 1$. However, by (2.3), and as $f_p(\zeta_p) = i\sqrt{p}$ by Gauss,

$$g_{p,K}(1) = -\frac{f'_p(1)}{\sqrt{p}} = -\frac{1}{\sqrt{p}} \sum_{a=1}^{p-1} a \left(\frac{a}{p}\right) = \sqrt{p} h(-p) > 0,$$

(see [2], Section 2), and so $g_{p,K}(x) > 0$ and thus has no zeros in $(0, 2)$. Therefore f_p has only a simple zero at $z = 1$ on this arc. \square

3. Functions with random coefficients.

If $g \in \mathcal{F}_J$ then, for any $x \in (0, 1)$, we have

$$\begin{aligned}
 \frac{1}{2}g''(x) &= \frac{1}{x^3} + \frac{1}{(1-x)^3} + \sum_{\substack{|j| < J \\ j \neq 0, -1}} \frac{\delta_j}{(x+j)^3} \\
 (3.1) \qquad &\geq \frac{1}{x^3} + \frac{1}{(1-x)^3} - \sum_{\substack{|j| < J \\ j \neq 0, -1}} \frac{1}{(x+j)^3} \\
 &> 2\frac{1}{(1/2)^3} - 2\zeta(3) > 0.
 \end{aligned}$$

Since $\lim_{t \rightarrow 0+} g'(t) = -\infty$ and $\lim_{t \rightarrow 1-} g'(t) = \infty$ we deduce that $g'(x)$ has exactly one zero in $(0, 1)$, call it x_0 . Note that $g(x)$ attains its minimum value at x_0 . If $0 \leq t < 1/\pi$ then

$$-g'(t) \geq \frac{1}{t^2} - 2 \left(\frac{1}{(1/2)^2} + \frac{1}{(3/2)^2} + \frac{1}{(5/2)^2} + \dots \right) = \frac{1}{t^2} - \pi^2 > 0.$$

Similarly if $1 - 1/\pi < t \leq 1$ then $g'(t) > 0$. Thus

$$(3.2) \qquad x_0 \in \left[\frac{1}{\pi}, 1 - \frac{1}{\pi} \right].$$

We now show that few g are small in absolute value, at their minimum x_0 .

PROPOSITION 3. — We have $|g(x_0)| > J^{-1/4}$ for almost all $g \in \mathcal{F}_J$, where $g'(x_0) = 0$, uniformly as $J \rightarrow \infty$.

Proof. — Consider the subset S of \mathcal{F}_J with all the δ_j fixed given values, except when $j \in [I, I + I^{1/2}]$ where $I = J^{1/4}$. Let $f \in S$ with $\delta_j = -1$ for all $j \in [I, I + I^{1/2}]$. Suppose that $f'(x_1) = 0$ and let

$$\gamma = \sum_{\substack{|j| < J \\ j \notin [I, I + I^{1/2}]}} \frac{\delta_j}{x_1 + j}$$

where $\delta_0 = 1$, $\delta_{-1} = -1$. Let g be any element of S with $g'(x_0) = 0$.

By (3.1) note that

$$(3.3) \quad |x_1 - x_0| \ll \left| \int_{x_0}^{x_1} f''(t) dt \right| = |f'(x_0) - f'(x_1)| = |f'(x_0)| \\ = |f'(x_0) - g'(x_0)| \leq 2 \sum_{j \in [I, I+I^{1/2}]} \frac{1}{(x_0 + j)^2} \ll \frac{1}{I}.$$

Hence, keeping in mind $x_0, x_1 \in [1/\pi, 1 - 1/\pi]$,

$$g(x_0) - \gamma = \sum_{j \in [I, I+I^{1/2}]} \frac{\delta_j}{x_0 + j} + O\left(\sum_{\substack{|j| < J \\ j \notin [I, I+I^{1/2}]}} \left| \frac{1}{x_0 + j} - \frac{1}{x_1 + j} \right| \right) \\ = \frac{1}{I} \sum_{j \in [I, I+I^{1/2}]} \delta_j + O\left(\sum_{j \in [I, I+I^{1/2}]} \left| \frac{1}{I} - \frac{1}{x_0 + j} \right| + |x_1 - x_0| \right) \\ = \frac{1}{I} \sum_{j \in [I, I+I^{1/2}]} \delta_j + O\left(\frac{1}{I}\right),$$

since each $|1/I - 1/(x_0 + j)| \ll 1/I^{3/2}$ and there are $I^{1/2}$ such terms. Therefore if $|g(x_0)| \leq 1/I$ then

$$(3.4) \quad \sum_{j \in [I, I+I^{\frac{1}{2}}]} \delta_j = -\gamma I + O(1).$$

Now, the δ_j are independent binomial random variables, so the distribution of their sum tends towards the normal distribution. Therefore the maximum probability for (3.4) to occur happens when $\gamma = 0$; and so (3.4) holds with probability $O(I^{-1/4})$, for any γ , implying Proposition 3. \square

4. Proof of Theorem 2.

Suppose that $g \in \mathcal{F}_J$ and $f \in \mathcal{F}_K$, with $J < K$, such that the δ_j are the same in each for $|j| < J$. Select $x_0, x_1 \in (0, 1)$ so that $g'(x_0) = 0$ and $f'(x_1) = 0$. Now

$$|f(x_1) - f(x_0)| \leq \sum_{|j| < K} \left| \frac{1}{x_1 + j} - \frac{1}{x_0 + j} \right| \ll \sum_{|j| < K} \frac{|x_1 - x_0|}{j^2 + 1} \ll |x_1 - x_0|,$$

since $x_0, x_1 \in [1/\pi, 1 - 1/\pi]$. Arguing exactly as in (3.3), we see that $|x_0 - x_1| \ll 1/J$, and so we have

$$(4.1) \quad |f(x_1) - f(x_0)| \ll \frac{1}{J}.$$

We next consider the mean-square of

$$|f(x_0) - g(x_0)| = \left| \sum_{J \leq |j| < K} \frac{\delta_j}{x_0 + j} \right|.$$

To do so we will need to sum over all $\delta = \{\delta_j\}_{J \leq |j| < K} \in \Delta_{J,K}$, that is the set of all possibilities with each $\delta_j = -1$ or 1 (note that there are 2 possible values for each δ_j so the set $\Delta_{J,K}$ has 2^{2K-2J} elements). With this notation, the mean square is

$$\begin{aligned} & \frac{1}{2^{2K-2J}} \sum_{\delta \in \Delta_{J,K}} \left| \sum_{J \leq |j| < K} \frac{\delta_j}{x_0 + j} \right|^2 \\ &= \sum_{J \leq |j_1|, |j_2| < K} \frac{1}{(x_0 + j_1)(x_0 + j_2)} \frac{1}{2^{2K-2J}} \sum_{\delta \in \Delta_{J,K}} \delta_{j_1} \delta_{j_2} \\ &= \sum_{J \leq |j| < K} \frac{1}{(x_0 + j)^2} \asymp \frac{1}{J}. \end{aligned}$$

Thus if $\psi_J \rightarrow \infty$ as $J \rightarrow \infty$ then

$$(4.2) \quad \left| \sum_{J \leq |j| < K} \frac{\delta_j}{x_0 + j} \right| < \frac{\psi_J}{J^{1/2}},$$

for almost all choices of the δ_j .

Combining (4.1) and (4.2), we see that for almost all choices of δ_j ($J \leq |j| < K$) we have

$$(4.3) \quad |f(x_1) - g(x_0)| \leq |f(x_1) - f(x_0)| + |f(x_0) - g(x_0)| < \frac{2\psi_J}{J^{1/2}}.$$

Taking $\Psi_J = J^{1/4}/2$, and combining this with Proposition 3 we see that for almost all $g \in \mathcal{F}_J$, and almost all extensions f of g to \mathcal{F}_K , $f(x_1)$ has the same sign as $g(x_0)$. Summing up over all $g \in \mathcal{F}_J$ we deduce that $\omega_K = \omega_J + o(1)$, where

$$\omega_J := \frac{\#\{g \in \mathcal{F}_J : g(x) = 0 \text{ for some } x \in (0, 1)\}}{\#\{g \in \mathcal{F}_J\}},$$

and the “ $o(1)$ ” term depends only on J . Therefore $\lim_{J \rightarrow \infty} \omega_J$ exists, and equals κ_1 say.

Strong bounds on κ_1 , which imply those in the statement of Theorem 2, are given in Proposition 6 in Section 8.

Theorem 2 follows.

5. Proofs of Theorems 1 and $1\frac{1}{2}$.

Let $1 \leq K \leq p-1$ be an integer. If $\left(\frac{K}{p}\right) = \left(\frac{K+1}{p}\right)$ then by Proposition 2 there is exactly one zero of $f_p(z)$ on the arc from ζ_p^K to ζ_p^{K+1} ; by Lemma 2 this happens for $\sim p/2$ values of K . Suppose now that $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$ so that $f_p(z)$ has either 0 or 2 zeros on the arc from ζ_p^K to ζ_p^{K+1} depending on whether $\min_{x \in (0,1)} g_{p,K}(x)$ is positive or not. To decide this question we need the following proposition:

PROPOSITION 4. — Suppose $J \leq \sqrt{p}$, and $J \rightarrow \infty$ as $p \rightarrow \infty$. For almost all $1 \leq K \leq p-1$ we have

$$g_{p,K}(x) = \frac{p}{2\pi} \left(\frac{K}{p}\right) \sum_{|j| < J} \left(\frac{K-j}{p}\right) \frac{1}{j+x} + O\left(\frac{p}{J^{1/3}}\right),$$

uniformly for all $x \in (0,1)$.

Proof. — Note that for $J \leq |j| < p/2$,

$$\left| \frac{1}{\zeta_p^{j+x} - 1} - \frac{1}{\zeta_p^j - 1} \right| = \left| \frac{\zeta_p^x - 1}{(\zeta_p^{j+x} - 1)(\zeta_p^j - 1)} \right| \asymp \frac{px}{j(j+x)} \ll \frac{p}{j^2},$$

and, for $|j| < J$,

$$\frac{1}{\zeta_p^{j+x} - 1} = \frac{p}{2i\pi} \frac{1}{(j+x)} + O(1).$$

Hence, putting $j = K - k$ in (2.3), we have

$$\begin{aligned} g_{p,K}(x) &= i \left(\frac{K}{p}\right) \sum_{|j| < p/2} \left(\frac{K-j}{p}\right) \frac{1}{\zeta_p^{j+x} - 1} \\ &= \frac{p}{2\pi} \left(\frac{K}{p}\right) \sum_{|j| < J} \left(\frac{K-j}{p}\right) \frac{1}{j+x} \\ &\quad + i \left(\frac{K}{p}\right) \sum_{J \leq |j| < p/2} \left(\frac{K-j}{p}\right) \frac{1}{\zeta_p^j - 1} + O\left(J + \frac{p}{J}\right). \end{aligned}$$

We now show that the mean-square of the second term above is small, which proves the proposition. By Lemma 2,

$$\begin{aligned}
 & \sum_{K=1}^p \left| \sum_{J \leq |j| < p/2} \left(\frac{K-j}{p} \right) \frac{1}{\zeta_p^j - 1} \right|^2 \\
 &= \sum_{J \leq |j_1|, |j_2| < p/2} \frac{1}{(\zeta_p^{j_1} - 1)(\zeta_p^{-j_2} - 1)} \sum_{K=1}^p \left(\frac{K-j}{p} \right) \left(\frac{K-j_2}{p} \right) \\
 &= p \sum_{J \leq |j| < p/2} \frac{1}{|\zeta_p^j - 1|^2} - \left| \sum_{J \leq |j| < p/2} \frac{1}{\zeta_p^j - 1} \right|^2 \\
 &\ll p \sum_{J \leq |j| < p/2} \left(\frac{p}{j} \right)^2 + \left(\sum_{J \leq |j| < p/2} \frac{p}{j} \right)^2 \ll \frac{p^3}{J} + p^2 \log^2 p.
 \end{aligned}$$

This proves the proposition. \square

By Proposition 4 we know that for almost all K with $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$ the minimum value of $\frac{2\pi}{p} g_{p,K}(x)$ equals the minimum of $\left(\frac{K}{p}\right) \sum_{|j| < J} \left(\frac{K-j}{p}\right) \frac{1}{j+x} + O(J^{-\frac{1}{3}})$. For such K the minimum value of $g_{p,K}(x)$ is non-positive if and only if the minimum of $\left(\frac{K}{p}\right) \sum_{|j| < J} \left(\frac{K-j}{p}\right) \frac{1}{j+x}$ is non-positive, unless

$$(5.1) \quad \left(\frac{K}{p}\right) \sum_{|j| < J} \left(\frac{K-j}{p}\right) \frac{1}{j+x} \ll \frac{1}{J^{\frac{1}{3}}}.$$

Now choose $J = [\log p/10]$. Given any choice of $\delta_j \in \{-1, 1\}$, $0 < |j| < J$ with $\delta_0 = 1$, and $\delta_{-1} = -1$, by Lemma 3 there are $\sim p/2^{2J-2}$ values of K with $\left(\frac{K}{p}\right) \left(\frac{K-j}{p}\right) = \delta_j$ for each j . Therefore (5.1) fails, for almost all K , by Proposition 3. Appealing now to Theorem 2 we have proved that for $\sim \kappa_1 p/2$ values of K with $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$, the minimum of $g_{p,K}(x)$ is < 0 . For such K , $f_p(z)$ has two zeros on the arc from ζ_p^K to ζ_p^{K+1} , so that the total number of such zeros is $\sim \kappa_1 p$. Theorems 1 and $1\frac{1}{2}$ follow.

6. Pseudo-Gauss Sums: Proof of the first part of Theorem 4.

In this section, we wish to study the distribution of $f_p(\zeta_p^{K+1/2})$. By (2.3) and Proposition 4 we have (if $(\sqrt{p} >)J \rightarrow \infty$ as $p \rightarrow \infty$) for almost

all $1 \leq K \leq p-1$,

$$(6.1) \quad f_p(\zeta_p^{K+\frac{1}{2}}) = \frac{if_p(\zeta_p)}{\pi} \left(\sum_{|j|<J} \left(\frac{K-j}{p} \right) \frac{1}{j+\frac{1}{2}} + O\left(\frac{1}{J^{1/3}}\right) \right) \\ = \eta \frac{\sqrt{p}}{\pi} \left(\sum_{|j|<J} \left(\frac{K-j}{p} \right) \frac{1}{j+\frac{1}{2}} + O\left(\frac{1}{J^{1/3}}\right) \right),$$

where $\eta = \pm 1$ or $\pm i$ is fixed. Thus, by Lemma 3, we have that for any fixed real number ρ

$$\lim_{p \rightarrow \infty} \frac{1}{p} \# \left\{ K : 1 \leq K \leq p \text{ and } H_p \left(\frac{K+\frac{1}{2}}{p} \right) < \rho \sqrt{p} \right\}$$

exists and equals

$$(6.2) \quad \lim_{J \rightarrow \infty} \text{Prob} \left(\sum_{|j|<J} \frac{\delta_j}{j+\frac{1}{2}} < \pi \rho : \delta \in \Delta_{0,J} \right)$$

(using the notation $\Delta_{J,K}$ of Section 4). One may obtain an expression for this probability as follows: Recall that

$$\int_0^\infty \frac{\sin y}{y} dy = \frac{\pi}{2},$$

and so for any $k \neq 0$

$$\frac{2}{\pi} \int_0^\infty \frac{\sin(kx)}{x} dx = \text{sgn}(k) \frac{2}{\pi} \int_0^\infty \frac{\sin(|k|x)}{x} dx \\ = \text{sgn}(k) \frac{2}{\pi} \int_0^\infty \frac{\sin y}{y} dy = \text{sgn}(k),$$

where $\text{sgn}(k)$ is the sign of k ($= 1$ if $k > 0$ and -1 if $k < 0$). Hence the probability (6.2) equals

$$\frac{1}{2^{2J-1}} \sum_{\delta \in \Delta_{0,J}} \left(\frac{1}{2} - \frac{1}{\pi} \int_0^\infty \sin \left(\left(\sum_{|j|<J} \frac{\delta_j}{j+\frac{1}{2}} - \pi \rho \right) x \right) \frac{dx}{x} \right) \\ = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{1}{2^{2J-1}} \sum_{\delta \in \Delta_{0,J}} \left(\frac{e^{ix \left(\sum_{|j|<J} \frac{\delta_j}{j+\frac{1}{2}} - \pi \rho \right)}}{2i} - \frac{e^{-ix \left(\sum_{|j|<J} \frac{\delta_j}{j+\frac{1}{2}} - \pi \rho \right)}}{2i} \right) \frac{dx}{x} \\ = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \prod_{|j|<J} \left(\frac{e^{\frac{ix}{j+\frac{1}{2}}} + e^{-\frac{ix}{j+\frac{1}{2}}}}{2} \right) \left(\frac{e^{-ix\pi\rho} - e^{ix\pi\rho}}{2i} \right) \frac{dx}{x} \\ = \frac{1}{2} + \frac{1}{\pi} \int_{x=0}^\infty \sin(\rho\pi x) \prod_{|j|<J} \cos \left(\frac{2x}{2j+1} \right) \frac{dx}{x}.$$

Letting $J \rightarrow \infty$, we get

$$c_\rho = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \sin(\rho\pi x) C(x) \frac{dx}{x} \quad \text{where} \quad C(x) := \prod_{\substack{n \geq 1 \\ n \text{ odd}}} \cos^2\left(\frac{2x}{n}\right),$$

and thus Theorem 4 is proved. Note that this integral does converge: For any $x > 0$ we have

$$C(x) \ll \frac{1}{2^{\frac{3x}{\pi}}}$$

since this estimate is trivial for $x \leq 1$, and otherwise we note that $|\cos(2x/n)| < 1/2$ if $3x/\pi < n < 6x/\pi$. Thus the part of the integral with $x \geq 1$ is easily bounded. Since $\sin(\rho\pi x) \ll \rho\pi x$, the portion of the integral from 0 to 1 is also easily bounded.

Remark 1. — We use the above to study the multiplicative average size of $f_p(\zeta_p^{k+1/2})$. Due to the symmetry of c_ρ we have that

$$\frac{1}{p-1} \log \left(\prod_{k=1}^{p-1} \frac{f_p(\zeta_p^{k+1/2})}{\sqrt{p}} \right) = 2 \int_0^\infty \log \rho \, d\left(c_\rho - \frac{1}{2}\right).$$

Using our expression for c_ρ one can show that this is

$$= \gamma + \log \pi - \int_0^1 \frac{C(x) - 1}{x} dx - \int_1^\infty \frac{C(x)}{x} dx.$$

All of these integrals converge, though we do not know their exact values.

Remark 2. — The expansion given in (6.1) for f_p , and the general technique involved, is very similar to that used by Montgomery [5] in showing that

$$\text{i) } |f_p(z)| \ll \sqrt{p} \log p \text{ for all } |z| = 1.$$

ii) If p is sufficiently large then there exists some value of z with $|z| = 1$ for which $|f_p(z)| > \frac{2}{\pi} \sqrt{p} \log \log p$.

Indeed to prove a result like that in (ii) we note that we may select each δ_j equal to the sign of j for $|j| < J = \varepsilon \log p$. By Lemma 3 there are many such K and we proceed as before with the expansion in (6.1), but now taking a little more care over the set of excluded K .

Remark 3. — Fix $t \in (0, 1)$. By the argument above, we have, for

any fixed real number ρ ,

$$\begin{aligned} c_{\rho,t} &:= \lim_{p \rightarrow \infty} \frac{1}{p} \# \left\{ K : 1 \leq K \leq p \text{ and } H_p \left(\frac{K+t}{p} \right) < \rho \sqrt{p} \right\} \\ &= \lim_{J \rightarrow \infty} \text{Prob} \left(\delta \in \Delta_{0,J} : \sum_{|j| < J} \frac{\delta_j}{j+t} < \frac{\pi \rho}{\sin(\pi t)} \right) \\ &= \frac{1}{2} + \frac{1}{\pi} \int_{x=0}^{\infty} \sin \left(\frac{\rho \pi x}{\sin(\pi t)} \right) \prod_{j \in \mathbb{Z}} \cos \left(\frac{x}{j+t} \right) \frac{dx}{x}. \end{aligned}$$

Remark 4. — We can also use these techniques to investigate the distribution of values of $H_p(t)$ at $t = a/(p-1)$ for $1 \leq a \leq p-1$. We note that if $K \sim \alpha p$ then $\zeta_{p-1}^K = \zeta_p^{K+\alpha} \{1 + o(1/p)\}$. Therefore we can get an expression similar to (6.1) for almost all $F_p(\zeta_{p-1}^K)$, but now with $\sum_{|j| < J} \left(\frac{K-j}{p} \right) \frac{1}{j+\alpha}$ replacing the sum in (6.1), and multiplying the whole expression through by $\sin(\alpha\pi)$. Thus the density of those K , for which $H_p\left(\frac{K}{p-1}\right) \leq \rho\sqrt{p}$, is

$$\frac{1}{2} + \frac{1}{\pi} \int_{\alpha=0}^1 \int_{x=0}^{\infty} \sin \left(\frac{\rho \pi x}{\sin(\alpha\pi)} \right) \prod_{m \in \mathbb{Z}} \cos \left(\frac{x}{m+\alpha} \right) \frac{dx}{x} d\alpha.$$

We cannot see how to obtain a simpler expression.

It is not hard to modify this technique to determine the distribution of values of the Fekete polynomial (or, in fact, $H_p(t)$) at any “reasonably” distributed set of values.

7. The distribution of $g(1/2)$ for $g \in \mathcal{F}_J$ as $J \rightarrow \infty$.

We now look at the limiting distribution of $g(1/2) - 4$ for $g \in \mathcal{F}_J$ as $J \rightarrow \infty$. Define, for $N \geq 1$,

$$S_N(\underline{\delta}) = \sum_{|j+1/2| > N} \frac{\delta_j}{j + \frac{1}{2}},$$

where each $\delta_j = 1$ or -1 with probability $1/2$. We will prove that the distribution function of $S_1(\underline{\delta})$ decays double exponentially.

THEOREM 5. — *As $x \rightarrow \infty$, we have*

$$\text{Prob}(|S_1(\underline{\delta})| > x) = \exp(-e^{x/2+O(1)}).$$

Proof of the second part of Theorem 4. — Note that

$$\text{Prob}(S_1(\underline{\delta}) > x) = \text{Prob}(S_1(\underline{\delta}) < -x) = \exp(-e^{x/2+O(1)}),$$

by symmetry. Taking $x = \pi\rho$, the result follows from (6.2). \square

To prove Theorem 5 we study the $2k$ -th moment of $S_N(\underline{\delta})$, call it $M_N(k)$, that is, the expectation of $S_N(\underline{\delta})^{2k}$. For example

$$M_N(1) = \sum_{|j+1/2|>N} \frac{1}{(j+\frac{1}{2})^2}.$$

Our aim is to determine the asymptotic behaviour of $M_1(k)$ for large k .

PROPOSITION 5. — For large k ,

$$M_1(k) = (2 \log k - 2 \log \log k + O(1))^{2k}.$$

Proof. — To establish the lower bound, consider $\underline{\delta}$ such that $\delta_j = 1$ for all $1 \leq |j+1/2| \leq k/\log k$; and such that $S_{k/\log k}(\underline{\delta}) > 0$. The probability of this happening is $\asymp 1/2^{2k/\log k}$, and $S_1(\underline{\delta}) \geq 2 \log k - 2 \log \log k + O(1)$ for such $\underline{\delta}$. Hence

$$\begin{aligned} M_1(k) &\gg \frac{1}{2^{2k/\log k}} (2 \log k - 2 \log \log k + O(1))^{2k} \\ &= (2 \log k - 2 \log \log k + O(1))^{2k}. \end{aligned}$$

Now

$$M_N(k) = \sum_{j_1, j_2, \dots, j_{2k}} \mathbb{E} \left(\frac{\delta_{j_1}}{j_1 + \frac{1}{2}} \frac{\delta_{j_2}}{j_2 + \frac{1}{2}} \dots \frac{\delta_{j_{2k}}}{j_{2k} + \frac{1}{2}} \right),$$

where \mathbb{E} stands for the expectation. Observe that a summand above is non-zero only if each value of j appears an even number of times amongst j_1, j_2, \dots, j_{2k} . In particular $j_\ell = j_1$ for some $\ell > 1$, and then $\mathbb{E}(\prod_{1 \leq i \leq 2k} \delta_{j_i}) = \mathbb{E}(\prod_{1 \leq i \leq 2k, i \neq 1, \ell} \delta_{j_i})$. Summing over all $2k-1$ possibilities for ℓ in the above, we deduce that

$$(7.1) \quad M_N(k) \leq (2k-1) \sum_{|j+1/2|>N} \frac{1}{(j+\frac{1}{2})^2} M_N(k-1),$$

for all $k \geq 1$ and all $N \geq 1$. Iterating this inequality, we obtain

$$\begin{aligned} (7.2) \quad M_N(k) &\leq (2k-1) \cdot (2k-3) \cdots 3 \cdot 1 \cdot \left(\sum_{|j+1/2|>N} \frac{1}{(j+\frac{1}{2})^2} \right)^k \\ &\leq \frac{(2k)!}{k! 2^k} \left(\frac{2}{N - \frac{1}{2}} \right)^k = \frac{(2k)!}{k! (N - \frac{1}{2})^k}. \end{aligned}$$

Now

$$|S_1(\underline{\delta}) - S_N(\underline{\delta})| \leq 2\lambda_N, \text{ where } \lambda_N := \sum_{N \geq j+1/2 \geq 1} \frac{1}{j + \frac{1}{2}} = \log N + O(1).$$

Evidently the odd moments of $S_N(\underline{\delta})$ are zero. Therefore, by the binomial theorem and (7.2),

$$\begin{aligned} M_1(k) &= \sum_{j=0}^k \binom{2k}{2j} M_N(j) \mathbb{E}(|S_1(\underline{\delta}) - S_N(\underline{\delta})|^{2k-2j}) \\ &\leq \sum_{j=0}^k \binom{2k}{2j} \frac{(2j)!}{j!(N - \frac{1}{2})^j} (2\lambda_N)^{2k-2j} \\ &\leq (2\lambda_N)^{2k} \sum_{j=0}^k \frac{1}{j!} \left(\frac{k^2}{(N - \frac{1}{2})\lambda_N^2} \right)^j \leq (2\lambda_N)^{2k} \exp\left(\frac{k^2}{(N - \frac{1}{2})\lambda_N^2} \right). \end{aligned}$$

Taking $N = k/\log k$ we obtain the upper bound of the proposition. \square

Proof of Theorem 5. — Take $k = c_1 x e^{x/2} + O(1)$ for some $c_1 > 0$, and then $\text{Prob}(|S_1(\underline{\delta})| > x) \leq x^{-2k} M_1(k) \ll \exp(-c_2 e^{x/2})$ for some constant $c_2 > 0$, if c_1 is sufficiently small, by Proposition 5.

The lower bound is more involved. Select integer k so that $2 \log k - 2 \log \log k$ is as close as possible to x . The contribution to $M_1(k)$ of those $\underline{\delta}$ with $|S_1(\underline{\delta})| < x - c_3$ is $\leq (x - c_3)^{2k} \leq M_1(k)/4$ if c_3 is sufficiently large. The contribution to $M_1(k)$ of those $\underline{\delta}$ with $|S_1(\underline{\delta})| > x + c_3$ is $\leq \int_{t > x+c_3} \text{Prob}(|S_1(\underline{\delta})| > t) t^{2k} dt \ll \int_{t > x+c_3} \exp(-c_2 e^{t/2}) t^{2k} dt \leq M_1(k)/4$ if c_3 is sufficiently large, using the upper bound from the paragraph above. Thus $M_1(k)/2 \leq \text{Prob}(x - c_3 \leq |S_1(\underline{\delta})| \leq x + c_3) (x + c_3)^k$ which implies that $\text{Prob}(|S_1(\underline{\delta})| \geq x - c_3) \geq M_1(k)/2(x + c_3)^k \gg \exp(-c_4 e^{x/2})$ for some constant $c_4 > 0$, by Proposition 5. Replacing $x - c_3$ by x gives the lower bound and thus our result. \square

Remark. — We follow up on Remark 3 of Section 6. The arguments above (Theorem 5 and Proposition 5) hold just as well with “1/2” replaced by any fixed $t \in (0, 1)$. Thus $1 - c_{\rho,t}$ and $c_{-\rho,t} = \exp(-\exp(\pi\rho/2 \sin(\pi t) + O(1)))$ for $\rho > 0$.

8. Bounds on κ_1 .

Applying the method of Section 6, we note that for any real λ ,

$$(8.1) \quad \begin{aligned} \pi_\lambda &:= \lim_{J \rightarrow \infty} \text{Prob} \left\{ g \in \mathcal{F}_J : g\left(\frac{1}{2}\right) < 4\lambda \right\} \\ &= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \sin((1-\lambda)x) \prod_{\substack{n \geq 3 \\ n \text{ odd}}} \cos^2\left(\frac{x}{2n}\right) \frac{dx}{x}. \end{aligned}$$

We can use this to obtain numerical bounds on κ_1 using the following result.

PROPOSITION 6. — We have $\pi_{.013496...} \geq \kappa_1 \geq \pi_0$.

Using Simpson's rule to compute the integrals in (8.1) we obtain $.000813 > \pi_{.013496...} \geq \kappa_1 \geq \pi_0 > .000668$, from which we deduce the bounds on κ_0 in the introduction.

Proof. — Again selecting x_0 so that $g(x_0)$ is minimal, we have, by definition, that

$$\kappa_1 = \lim_{J \rightarrow \infty} \text{Prob}\{g \in \mathcal{F}_J : g(x_0) \leq 0\}.$$

Since $g(x_0) \leq g(1/2)$ we deduce the lower bound on κ_1 above.

To get the upper bound, write $x_0 = 1/2 + \nu$ so that $|\nu| < 1/2$. If $g(x_0) \leq 0$ then

$$\begin{aligned} g\left(\frac{1}{2}\right) &\leq g\left(\frac{1}{2}\right) - g(x_0) \\ &= 4 - \frac{1}{x_0} - \frac{1}{1-x_0} + \sum_{\substack{|j| < J \\ j \neq 0, -1}} \frac{\delta_j(x_0 - \frac{1}{2})}{(j + \frac{1}{2})(j + x_0)} \\ &\leq -\frac{4\nu^2}{\frac{1}{4} - \nu^2} + \sum_{j=1}^{\infty} \frac{|\nu|}{(j + \frac{1}{2})(j + \frac{1}{2} + \nu)} + \sum_{j=-\infty}^{-2} \frac{|\nu|}{(j + \frac{1}{2})(j + \frac{1}{2} + \nu)} \\ &= -\frac{4\nu^2}{\frac{1}{4} - \nu^2} + \sum_{j=1}^{\infty} \frac{2|\nu|}{(j + \frac{1}{2})^2 - \nu^2} = -\frac{(2|\nu| + 4\nu^2)}{\frac{1}{4} - \nu^2} + \pi \tan(\pi|\nu|). \end{aligned}$$

Using Maple to compute the \max_ν , we obtain

$$g\left(\frac{1}{2}\right) \leq \max_{|\nu| \leq \frac{1}{2}} \left(\pi \tan(\pi|\nu|) - \frac{(2|\nu| + 4\nu^2)}{\frac{1}{4} - \nu^2} \right) = 0.053986\dots,$$

the maximum being attained at $\nu = \pm 0.057052\dots$ \square

Remark. — One can refine the above to get better bounds for κ_1 . First note that $g(x) = 1/x + 1/(1-x)$ is the only element in \mathcal{F}_1 , and in this case $x_0 = 1/2$; thus “ $1/2$ ” appears in the definition of π_λ . More generally, let J be some positive integer. For each $\gamma \in \mathcal{F}_J$ select χ_0 so that $\gamma(\chi_0)$ is minimal. We again have $g(x_0) \leq g(\chi_0)$, so if $g(\chi_0) \leq 0$ then $g(x_0) \leq 0$. On the other hand, if $g(x_0) \leq 0$ then we can again get an explicit upper bound on $g(\chi_0)$ and proceed as above. This can be used to give another proof that κ_1 exists.

9. Zeros off the unit circle.

Proof of Theorem 3. — Theorem 3 holds trivially if there is a zero of $f_p(t)$ on the unit circle in the arc from ζ_p^K to ζ_p^{K+1} . Thus we shall henceforth assume that there is no such zero. Let $h(x) := H_p((K+x)/p)/H_p(K/p)$, so that $|h(x)| = |f_p(\zeta_p^{K+x})/\sqrt{p}|$, and $h(x)$ is a continuous real-valued function. Now the hypothesis implies that $h(y) < \epsilon$ for some $y \in (0, 1)$ (in fact, $t = \zeta_p^{K+y}$), while our assumption above implies that $h(x) \neq 0$ for all $x \in (0, 1)$. By (2.3) we have, uniformly for $|x| \leq 2/3$,

$$\begin{aligned} (9.1) \quad h(x) &= \frac{\sin(\pi x)}{p} \left(\frac{1}{\sin(\pi x/p)} + \left(\frac{K}{p}\right) \sum_{1 \leq |K-k| < p/2} \frac{(k/p)}{\sin(\pi(x+K-k)/p)} \right) \\ &= 1 - (C + O(1))x, \quad \text{where } C := -(K/p) \sum_{1 \leq |K-k| < p/2} \frac{(k/p)}{K-k}. \end{aligned}$$

So if $h(y) < \epsilon$ for some sufficiently small y then $h(2y) = 2h(y) - 1 + O(y) < 0$, contradicting our assumption. Therefore we may assume that $y \gg 1$, and also $1 - y \gg 1$ by the symmetric argument. Thus $g_{p,K}(y) \ll \sqrt{p}|f_p(t)|/\sin(\pi y) \ll \epsilon p$ by (2.3), so that

$$g_{p,K}(x_0) \leq g_{p,K}(y) \ll \epsilon p$$

where x_0 is defined as in Section 3.

Let $x_1 = x_0 - \epsilon^{1/2}$, and $x_2 = x_0 + \epsilon^{1/2}$, and then $\alpha_j = \zeta_p^{x_j}$ for $j = 1, 2$. Let $R = 1 - \epsilon^{1/3}/p$. We shall consider the variation in argument of

$$G(z) := i \left(\frac{K}{p}\right) \frac{p}{f_p(\zeta_p)} \frac{f_p(z)}{z^p - 1} = i \left(\frac{K}{p}\right) \sum_{|K-k| < p/2} \left(\frac{k}{p}\right) \frac{1}{z \zeta_p^{-k} - 1},$$

as z goes around (in the anti-clockwise direction) the box bounded by the four curves, \mathcal{C}_1 , the arc of the unit circle from α_1 to α_2 , then \mathcal{C}_2 , the straight line segment from α_2 to $R\alpha_2$, then \mathcal{C}_3 , the arc of the circle of radius R , from $R\alpha_2$ to $R\alpha_1$, then finally \mathcal{C}_4 , the straight line segment from $R\alpha_1$ back to α_1 .

We know that $G(z)$ is real valued and positive on the arc \mathcal{C}_1 . We shall show that $G(z)$ has positive imaginary part on \mathcal{C}_2 , that $G(z)$ has negative real part on \mathcal{C}_3 , and that $G(z)$ has negative imaginary part on \mathcal{C}_4 . This shows that the change in argument of $G(z)$ is 2π as we go around our box, so that there is exactly one zero in our box. This implies a little more than Theorem 3.

To estimate $H(r, x) := G(r\zeta_p^{(K+x)/p})$ when $R \leq r \leq 1$, for a value of $x \in [x_1, x_2]$, we calculate the Taylor series expansion around $r = 1$, which is

$$H(r, x) = g_{p,K}(x) - \frac{(1-r)^2}{2r} \left(\frac{p}{2\pi}\right)^2 g''_{p,K}(x) + i \frac{1-r^2}{2r} \frac{p}{2\pi} g'_{p,K}(x) + O\left(\frac{(1-r)^3}{r} p^4\right).$$

From the proof of Proposition 2 we have, since x is bounded away from 0 and 1,

$$g_{p,K}(x) = g_{p,K}(x_0) + O((x-x_0)^2 p), \quad g'_{p,K}(x) \asymp (x-x_0)p \text{ and } g''_{p,K}(x) \asymp p.$$

Therefore

$$\operatorname{Im}(G(z)) = \operatorname{Im}(H(r, x)) \asymp \epsilon^{1/2} p^2 (1-r) + O((1-r)\epsilon^{2/3} p^2) > 0 \quad \text{on } \mathcal{C}_2,$$

$$\operatorname{Im}(G(z)) = \operatorname{Im}(H(r, x)) \asymp -\epsilon^{1/2} p^2 (1-r) + O((1-r)\epsilon^{2/3} p^2) < 0 \quad \text{on } \mathcal{C}_4,$$

$$\operatorname{Re}(G(z)) = \operatorname{Re}(H(r, x)) \asymp -\epsilon^{2/3} p + O(\epsilon p) < 0 \quad \text{on } \mathcal{C}_3,$$

as required. \square

Remark. — By (9.1) we see that

$$\max_{|z|=1} |f_p(z)| \asymp \sqrt{p} \max_{K \in \mathbb{Z}} \sum_{j \neq 0} \frac{1}{j} \left(\frac{K+j}{p} \right).$$

This again allows us to recover the results of Montgomery [5], as in Remark 2 of Section 6.

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