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Zeros of Derivatives of Riemann's Xi-Function on the Critical Line. II

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Explicit lower bounds for the proportion of zeros of the derivatives of Riemann's xi-function which are on the critical line and simple are given. These lead to upper bounds for the proportion of zeros of the Riemann zeta-function with given multiplicity.

1. INTRODUCTION

Let

$$H(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$$

and $\xi(s) = H(s)\zeta(s)$, where ζ is Riemann's zeta-function. In a recent paper [1] we have shown that the Riemann Hypothesis implies that all the zeros of all the derivatives $\xi^{(m)}(s)$ are on the line $\sigma = \frac{1}{2}$ and that if α_m denotes the proportion of zeros of $\xi^{(m)}(s)$ on $\sigma = \frac{1}{2}$, then (unconditionally) $\alpha_m = 1 + O(m^{-2})$ as $m \to \infty$. However, we were not able to show that the zeros detected on $\sigma = \frac{1}{2}$ are simple (see [2]). By means of a new identity for $\xi^{(m)}(s)$ we now have

THEOREM. *Let β_m denote the proportion of zeros of $\xi^{(m)}(s)$ which are on $\sigma = \frac{1}{2}$ and are simple. Let $\phi_m(x) = (1-x)(1-2x)^m$, and let $F_m(R) = 2\Phi\Lambda \coth\Lambda + \frac{1}{2}$, where $\Phi = \int_0^1 e^{2Rx}\phi_m(x)^2\,dx$, $\Phi' = \int_0^1 e^{2Rx}\phi_m'(x)^2\,dx$, and $\Lambda^2 = (\Phi' - R - R^2\Phi)/(4\Phi)$ with $\Lambda \geqslant 0$. Then*

$$\beta_m \geqslant 1 - \frac{\log F_m(R)}{R}$$

for any $R \geqslant 0$.

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Our bounds for β_m are not quite as good as those for α_m of [1] but we can prove

COROLLARY 1. *With β_m as above we have $\beta_0 > 0.3485$, $\beta_1 > 0.7869$, $\beta_2 > 0.9314$, $\beta_3 > 0.9666$, $\beta_4 > 0.9799$, and $\beta_5 > 0.9863$. Further $\beta_m = 1 + O(m^{-2})$ as $m \to \infty$.*

A zero of $\zeta(s)$ of multiplicity $m+1$ is a zero of $\xi(s)$ of the same multiplicity, hence a double zero of $\xi^{(m)}(s)$. Therefore we have

COROLLARY 2. *Let δ_m be the proportion of zeros of $\zeta(s)$ which have multiplicity $\geqslant m$. Then $\delta_m \ll m^{-2}$.*

2. THE IDENTITY

To derive the new identity for $\xi^{(m)}(s)$ we need only the functional equation $\xi^{(m)}(1-s) = (-1)^m \xi^{(m)}(s)$ and the recursion relation for binomial coefficients $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$. We employ the notation $F(s) = (H'/H)(s)$ and $H_n(s) = H^{(n)}(s)/H(s) - F(s)^n$. (It follows from Lemma 1c of [1] that $H_n(s)$ is comparatively small.) Then

$$\begin{aligned}
\xi^{(m+1)}(s) &= \sum_{k=0}^{m+1} \binom{m+1}{k} \zeta^{(k)}(s) H^{(m+1-k)}(s) \\
&= \sum_{k=0}^{m} \binom{m}{k} \zeta^{(k)}(s) H^{(m+1-k)}(s) + \sum_{k=0}^{m} \binom{m}{k} \zeta^{(k+1)}(s) H^{(m-k)}(s) \\
&= H(s) \sum_{k=0}^{m} \binom{m}{k} \zeta^{(k)}(s)(F(s)^{m+1-k} + H_{m+1-k}(s)) + \frac{d^m}{ds^m}(H(s)\zeta'(s)).
\end{aligned}$$

Also

$$\begin{aligned}
H(s) &\sum_{k=0}^{m} \binom{m}{k} \zeta^{(k)}(s) F(s)^{m+1-k} \\
&= F(s) \left[\xi^{(m)}(s) - H(s) \sum_{k=0}^{m} \binom{m}{k} \zeta^{(k)}(s) H_{m-k}(s)\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
\xi^{(m+1)}(s) = F(s)\,\xi^{(m)}(s) &+ \frac{d^m}{ds^m}(H(s)\zeta'(s)) \\
&+ H(s) \sum_{k=0}^{m} \binom{m}{k} \zeta^{(k)}(s) \mathscr{H}_{m,k}(s), \qquad (1)
\end{aligned}$$

where

$$\mathscr{H}_{m,k}(s) = H_{m+1-k}(s) - F(s)\, H_{m-k}(s).$$

We replace s by $1-s$ in (1), multiply through by $(-1)^{m+1}$ and use the functional equation to eliminate $\xi^{(m+1)}(s)$ from the two expressions. Thus

$$-\xi^{(m)}(s) = [G(s) + (-1)^m\, G(1-s)]/[F(s) + F(1-s)], \tag{2}$$

where

$$G(s) = \frac{d^m}{ds^m}(H(s)\,\zeta'(s)) + H(s) \sum_{k=0}^{m} \binom{m}{k} \zeta^{(k)}(s)\, \mathscr{H}_{m,k}(s).$$

Now we add $2\xi^{(m)}(s) = \xi^{(m)}(s) + (-1)^m\, \xi^{(m)}(1-s)$ to (2) to obtain the desired identity

$$\xi^{(m)}(s) = G_m(s) + (-1)^m\, G_m(1-s), \tag{3}$$

where

$$G_m(s) = \xi^{(m)}(s) + \frac{(d^m/ds^m)(H(s)\,\zeta'(s))}{F(s)+F(1-s)} + \frac{H(s)\sum_{k=0}^{m}\binom{m}{k}\zeta^{(k)}(s)\,\mathscr{H}_{m,k}(s)}{F(s)+F(1-s)}. \tag{4}$$

This identity expresses $\xi^{(m)}(\frac{1}{2}+it)$ as a sum (m even) or a difference (m odd) of complex conjugates. Hence $\xi^{(m)}(\frac{1}{2}+it) = 0$ precisely when

(a) $\arg G_m(\frac{1}{2}+it) \equiv (m+1)/2\,\pi \bmod \pi$, or

(b) $G_m(\frac{1}{2}+it) = 0$.

Suppose that $(1/2)+it_0$ is a zero of $G_m(s)$ of multiplicity n. Then by (3) it is a zero of $\xi^{(m)}(s)$ of multiplicity at least n. From (1) and (4) we see that

$$\xi^{(m+1)}(s) = (F(s) + F(1-s))\, G_m(s) - F(1-s)\, \xi^{(m)}(s).$$

Hence $(1/2)+it_0$ is a zero of $\xi^{(m+1)}(s)$ of multiplicity at least n and so a zero of $\xi^{(m)}(s)$ of multiplicity at least $n+1$. (In [1] we did not have this last result.) It remains only to approximate G_m by a more tractable function and then follow [1, 2].

3. Simplification of $G_m(s)$

We write

$$G_m(s) = \frac{d^m}{ds^m}\left(H(s)\left(\zeta(s) + \frac{\zeta'(s)}{F(s)+F(1-s)}\right)\right) + R_m(s),$$

where

$$R_m(s) = \frac{H(s)\sum_{k=0}^{m}\binom{m}{k}\zeta^{(k)}(s)\mathscr{H}_{m,k}(s)}{F(s)+F(1-s)}$$
$$- \sum_{k=0}^{m-1}\binom{m}{k}\frac{d^k}{ds^k}(H(s)\,\zeta'(s))\frac{d^{m-k}}{ds^{m-k}}\left(\frac{1}{F(s)+F(1-s)}\right).$$

It follows easily from [1, Lemma 1] that if T is sufficiently large, then

$$\frac{R_m(s)}{H(s)} \ll_m T^{-(1/2)+\varepsilon}$$

for $t > T$, $0 < \sigma < \log T$, and any $\varepsilon > 0$. The Riemann–Siegel formula is

$$\zeta(s) = f_1(s) + \chi(s) f_2(s),$$

where $\chi(s) = H(1-s)/H(s)$ and f_1 and f_2 are certain entire functions (see [1, Lemma 3]). Therefore

$$\zeta'(s) = f_1'(s) + \chi(s)\,f_2'(s) + f_2(s)\,\chi'(s)$$
$$= f_1'(s) + \chi(s)\,f_2'(s) - \chi(s)[F(s)+F(1-s)]\,f_2(s)$$

from which we obtain

$$H(s)\left(\zeta(s) + \frac{\zeta'(s)}{F(s)+F(1-s)}\right) = H(s)\left(f_1(s) + \frac{f_1'(s)}{F(s)+F(1-s)}\right)$$
$$+ \frac{H(1-s) f_2'(s)}{F(s)+F(1-s)}.$$

If $F(s)+F(1-s)$ is replaced by $L = \log T/2\pi$, then the mth derivative with respect to s of the above expression is precisely $Q_m(s)$ of [1, Eq. (1) with $\phi(x) = 1-x$]. Since $F(s)+F(1-s) = \log(t/2\pi) + O(t^{-1})$ and $\log(t/2\pi) = L + O(L^{-10})$ for $T \leqslant t \leqslant T + TL^{-10}$ we can make this replacement with a tolerable error. Then the analysis on $G_m(s)$ proceeds exactly as the analysis on $Q_m(s)$ in [1, Sects. 4 – 6] and we obtain the Theorem of [1] with $\phi(x) = 1-x$. Moreover, the zeros obtained are simple by virtue of the argument in [2].

The numerical results of Corollary 1 follow from the Theorem by choosing $R = 1.3$, 0.9, 0.8, 0.7, 0.7, and 0.8 for $m = 0, 1, 2, 3, 4$, and 5, respectively. The second assertion in the Corollary 1 is proved exactly as in [1, Sect. 7].

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