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A Note on Some Positivity Conditions Related to Zeta and L-Functions

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1 Introduction

The theory of Hilbert spaces of entire functions was developed by Louis de Branges [1] in the late 1950s and early 1960s. It is a generalization of the part of Fourier analysis involving Fourier transforms and the Plancherel formula. In [2] de Branges proposed an approach to the generalized Riemann hypothesis, that is, the hypothesis that not only the Riemann zeta function $\zeta(s)$ but also all the Dirichlet L-functions $L(s,\chi)$ with χ primitive have their nontrivial zeros lying on the critical line $\Re s = 1/2$ (see [5]). In [2] de Branges mentioned that his approach to the generalized Riemann hypothesis using Hilbert spaces of entire functions is related to the Lax-Phillips theory of scattering [6]. In [6, Section 7, Appendix 2] Lax and Phillips explained the difficulty of approaching the Riemann hypothesis by using the scattering theory. In this note, we indicate the difficulty of approaching the Riemann hypothesis by using de Branges's positivity conditions (see [2], [3], and [4]). In fact, we give examples showing that de Branges's positivity conditions, which imply the generalized Riemann hypothesis, are not satisfied by defining functions of reproducing kernel Hilbert spaces associated with the Riemann zeta function $\zeta(s)$ and the Dirichlet L-function $L(s,\chi_4)$.

2 Reproducing kernel Hilbert spaces

We first outline an important part of de Branges's approach to the Riemann hypothesis.

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Let $E(z)$ be an entire function satisfying $|E(\bar{z})| < |E(z)|$ for z in the upper half-plane. A Hilbert space of entire functions $\mathcal{H}(E)$ is the set of all entire functions $F(z)$ such that $F(z)/E(z)$ is square integrable on the real axis and such that

$$\left|F(z)\right|^2 \leqslant \|F\|^2_{\mathcal{H}(E)} K(z,z) \tag{2.1}$$

for all complex z, where the inner product of the space is given by

$$\langle F(z), G(z)\rangle_{\mathcal{H}(E)} = \int_{-\infty}^{\infty} \frac{F(x)\bar{G}(x)}{|E(x)|^2}\,dx$$

for all elements $F, G \in \mathcal{H}(E)$ and where

$$K(w,z) = \frac{E(z)\bar{E}(w) - \bar{E}(\bar{z})E(\bar{w})}{2\pi i(\bar{w}-z)}$$

is the reproducing kernel function of the space $\mathcal{H}(E)$. That is, the identity

$$F(w) = \langle F(z), K(w,z)\rangle_{\mathcal{H}(E)} \tag{2.2}$$

holds for every complex w and for every element $F \in \mathcal{H}(E)$. The identity (2.2) is obtained by using Cauchy's integration formula in the upper half-plane (cf. [1]), and the condition (2.1) is made so that Cauchy's formula applies to all functions in the space $\mathcal{H}(E)$.

The following two theorems are essentially due to de Branges (cf. [2] and [3]).

Theorem 1. Let $E(z)$ be an entire function having no real zeros such that $|E(\bar{z})| < |E(z)|$ for $\Im z > 0$, such that $\bar{E}(\bar{z}) = \epsilon E(z-i)$ for a constant ϵ of absolute value one, and such that $|E(x+iy)|$ is a strictly increasing function of $y > 0$ for each fixed real x. If $\Re\langle F(z), F(z+i)\rangle_{\mathcal{H}(E)} \geqslant 0$ for every element $F(z) \in \mathcal{H}(E)$ with $F(z+i) \in \mathcal{H}(E)$, then the zeros of $E(z)$ lie on the line $\Im z = -1/2$, and $\Re\{\bar{E}'(w)E(w+i)/2\pi i\} \geqslant 0$ when w is a zero of $E(z)$. □

Proof. Let w be a zero of $E(z)$. Since $\bar{E}(\bar{z}) = \epsilon E(z-i)$ with $|\epsilon| = 1$, we have

$$\bar{E}(w+i)K(w, z+i) = -\bar{E}(w-i)K(w+i, z) \tag{2.3}$$

for all complex z. Since $E(z)$ has no real zeros and since $|E(\bar{z})| < |E(z)|$ for z in the upper half-plane, $E(w+i)$ and $E(w-i)$ are nonzero. It follows that $K(w,z)$ is a nonzero element of $\mathcal{H}(E)$ such that $K(w, z+i)$ belongs to the space. Assume that $F(z)$ is an element in $\mathcal{H}(E)$ such that $F(z+i)$ belongs to the space. Then, by (2.2) and (2.3), we have

$$\langle F(z+i), K(w,z)\rangle_{\mathcal{H}(E)} + \langle F(z), K(w, z+i)\rangle_{\mathcal{H}(E)} = \frac{E(w+i) - E(w-i)}{E(w+i)} F(w+i). \tag{2.4}$$

Define a new scalar product $\langle \cdot, \cdot \rangle$ by

$$\langle F(z), G(z) \rangle = \langle F(z+i), G(z) \rangle_{\mathcal{H}(E)} + \langle F(z), G(z+i) \rangle_{\mathcal{H}(E)}$$

for all $F, G \in \mathcal{H}(E)$ such that $F(z+i), G(z+i) \in \mathcal{H}(E)$. Since, by assumption, $\Re\langle F(z+i), F(z)\rangle_{\mathcal{H}(E)} \geqslant 0$ for every element $F(z) \in \mathcal{H}(E)$ such that $F(z+i) \in \mathcal{H}(E)$, we have $\langle F(z), F(z)\rangle \geqslant 0$. Then, by (2.4) and the Schwarz inequality, we have

$$\begin{aligned} \left| \frac{E(w+i) - E(w-i)}{E(w+i)} F(w+i) \right|^2 &= \left| \langle F(z), K(w,z) \rangle \right|^2 \\ &\leqslant \langle F(z), F(z) \rangle \langle K(w,z), K(w,z) \rangle \\ &= 4\Re K(w, w+i) \Re \langle F(z+i), F(z) \rangle_{\mathcal{H}(E)}. \end{aligned} \tag{2.5}$$

If $K(w, w+i) \neq 0$, then we must have $\Im w = -1/2$ because, otherwise, we have $K(w, w+i) = 0$ by the functional identity $\bar{E}(\bar{z}) = \epsilon E(z-i)$.

Next, we assume that $K(w, w+i) = 0$. Then $F(z) = K(w, z)$ is an element of $\mathcal{H}(E)$ such that $F(w+i) = 0$ and $F(z+i) \in \mathcal{H}(E)$. If $F(z)$ is a nonzero element of $\mathcal{H}(E)$ having zero at a point z_0, it is easy to see by definition that $F(z)/(z-z_0)$ belongs to $\mathcal{H}(E)$. Since $F(z)$ is an entire function, by using Taylor's expansion of $F(z)$ at the point z_0, we see that $F(z)/(z-z_0)^n$ does not vanish at z_0 for some positive integer n. By the repeated process of dividing out the factor $z - z_0$ from $F(z)$, we see that $F(z)/(z-z_0)^n$ belongs to $\mathcal{H}(E)$. If $F(z+i) \in \mathcal{H}(E)$, we also see that $F(z+i)/(z+i-z_0)^n \in \mathcal{H}(E)$. Therefore, there exists a nonzero element $F(z) \in \mathcal{H}(E)$ such that $F(w+i) \neq 0$ and $F(z+i) \in \mathcal{H}(E)$. Hence, if $K(w, w+i) = 0$, then, by (2.5), we have

$$E(w+i) - E(w-i) = 0.$$

Since $\bar{E}(\bar{w}) = \epsilon E(w-i)$, we have $|E(w+i)| = |E(\bar{w})|$. Note that $\Re(w+i) = \Re(\bar{w})$. Since $|E(x+iy)|$ $(0, \infty)$, we must have $\Im(w+i) = \Im(\bar{w})$, and hence $w + i = \bar{w}$. Therefore, we have $\Im w = -1/2$.

We have $K(w, w+i) = \bar{E}'(w)E(w+i)/2\pi i$. Since $F(z) = K(w, z)$ is an element of $\mathcal{H}(E)$ such that $F(z+i) \in \mathcal{H}(E)$, we have

$$\Re K(w, w+i) = \Re \langle F(z+i), F(z) \rangle_{\mathcal{H}(E)} \geqslant 0;$$

that is, $\Re\{\bar{E}'(w)E(w+i)/2\pi i\} \geqslant 0$ when w is a zero of $E(z)$.

This completes the proof of the theorem. ■

Let $W(z)$ be a function analytic having no zeros in the upper half-plane. Then a Hilbert space of analytic functions $\mathcal{F}(W)$ is the set of all analytic functions $F(z)$ in the upper half-plane such that $F(z)/W(z)$ can be written as a quotient of bounded analytic functions in the upper half-plane, has square integrable boundary values on the real axis, and satisfies the inequality

$$\log\left|\frac{F(x+iy)}{W(x+iy)}\right| \leqslant \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\log\left|\frac{F(t)}{W(t)}\right| dt}{(t-x)^2+y^2}$$

for $y > 0$. The inner product of $\mathcal{F}(W)$ is given by

$$\langle F(z), G(z)\rangle_{\mathcal{F}(W)} = \int_{-\infty}^{\infty}\frac{F(x)\bar{G}(x)}{|W(x)|^2}\,dx$$

for all $F, G \in \mathcal{F}(W)$. The reproducing kernel function of $\mathcal{F}(W)$ is given by the expression

$$K(w,z) = \frac{W(z)\bar{W}(w)}{2\pi i(\bar{w}-z)};$$

that is, for every complex w in the upper half-plane, we have

$$F(w) = \langle F(z), K(w,z)\rangle_{\mathcal{F}(W)} \tag{2.6}$$

for every element $F \in \mathcal{F}(W)$. The identity (2.6) is obtained by using Cauchy's integration formula in the upper half-plane (cf. [1]).

Theorem 2. Let $W(z)$ be a function analytic having no zeros in the upper half-plane. Let T be a linear transformation of $\mathcal{F}(W)$ into itself that takes $K(w,z)$ into $K(w+i,z)$ for all complex w with $\Im w > 0$. Assume that

$$\Re\langle F(z), TF(z)\rangle_{\mathcal{F}(W)} \geqslant 0$$

for all $F \in \mathcal{F}(W)$. Then $W(z)$ has an analytic extension to the half-plane $\Im z > -1/2$, and $W(z)/W(z+i)$ has a nonnegative real part in this half-plane. □

Proof. Let $w_1, \ldots, w_r$ be points in the upper half-plane, and let $c_1, \ldots, c_r$ be complex numbers. If $F(z) = \sum_{\alpha=1}^{r} c_\alpha K(w_\alpha, z)$, then $TF(z) = \sum_{\beta=1}^{r} c_\beta K(w_\beta + i, z)$. By assumption, we have

$$\sum_{\alpha,\beta=1}^{r} c_\alpha \bar{c}_\beta\left[K(w_\alpha, w_\beta + i) + K(w_\alpha + i, w_\beta)\right] = 2\Re\langle F(z), TF(z)\rangle_{\mathcal{F}(W)} \geqslant 0;$$

that is, the expression $K(w+i,z)+K(w,z+i)$ is positive definite for w, z in the upper half-plane. This implies that $\Re\{W(z)/W(z+i)\} \geqslant 0$ for z in the upper half-plane. Let

$$B(z) = \frac{W(z)-W(z+i)}{W(z)+W(z+i)}.$$

Then $B(z)$ is analytic and bounded by 1 in the upper half-plane. The positive definiteness of $K(w+i,z)+K(w,z+i)$ implies the positive definiteness of the expression

$$\frac{1-B(z)\bar{B}(w)}{2\pi i(\bar{w}-z-i)}$$

for w, z in the upper half-plane.

Let $\mathcal{H}$ be the Hilbert space of analytic functions in the half-plane $\Im z > -1/2$, which has the expression $L(w,z) = 1/2\pi i(\bar{w}-z-i)$ as its reproducing kernel function. The norm of an element F in the space $\mathcal{H}$ is given by

$$\left\langle F(z), G(z)\right\rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} F\left(x-\frac{i}{2}\right)\bar{G}\left(x-\frac{i}{2}\right)dx.$$

Let P be a transformation of $\mathcal{H}$ into itself that takes $L(w,z)$ into $\bar{B}(w)L(w,z)$. The positive definiteness of the expression

$$\frac{1-B(z)\bar{B}(w)}{2\pi i(\bar{w}-z-i)}$$

implies $\langle PF(z), PF(z)\rangle_{\mathcal{H}} \leqslant \langle F(z), F(z)\rangle_{\mathcal{H}}$ for all elements $F \in \mathcal{H}$ which are linear combinations of functions $L(w,z)$ with $\Im w > 0$. Since $L(w,z)$ is the reproducing kernel function of $\mathcal{H}$, if $F \in \mathcal{H}$ is orthogonal to all elements $L(w,z)$ with $\Im w > 0$, then

$$F(w) = \left\langle F(z), L(w,z)\right\rangle_{\mathcal{H}} = 0$$

for $\Im w > 0$. Since $F(z)$ is analytic for $\Im z > -1/2$, we must have $F \equiv 0$. Therefore, the set of elements $L(w,z)$ with $\Im w > 0$ is dense in $\mathcal{H}$. It follows that $\langle PF(z), PF(z)\rangle_{\mathcal{H}} \leqslant \langle F(z), F(z)\rangle_{\mathcal{H}}$ for all elements $F \in \mathcal{H}$. Thus, P is a bounded linear transformation of the Hilbert space $\mathcal{H}$ into itself, and, therefore, the adjoint P^* of P exists.

Let α be a complex number with $\Im\alpha > -1/2$, and let $F(z) = L(\alpha, z)$. Then $F \in \mathcal{H}$, and hence $P^*F(z) \in \mathcal{H}$. It follows that

$$2\pi i(\bar{\alpha}-w-i)\left\langle P^*F(z), L(w,z)\right\rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} P^*F\left(x-\frac{i}{2}\right)\frac{\bar{\alpha}-w-i}{x-\frac{i}{2}-w}\,dx$$

is an analytic function of w for $\Im w > -1/2$. Since

$$2\pi i(\bar{\alpha} - w - i)\langle P^* F(z), L(w,z)\rangle_{\mathcal{H}} = 2\pi i(\bar{\alpha} - w - i)\langle F(z), PL(w,z)\rangle_{\mathcal{H}} = B(w)$$

for w in the upper half-plane, $B(z)$ has an analytic extension to the half-plane $\Im z > -1/2$.

If $F \in \mathcal{H}$ and $\Im w > 0$, we have

$$B(w)F(w) = \langle P^* F(z), L(w,z)\rangle_{\mathcal{H}}. \tag{2.7}$$

Since both sides of (2.7) are analytic functions of w for $\Im w > -1/2$, the identity (2.7) remains true for all complex w with $\Im w > -1/2$ by analytic continuation. Since $\langle PF(z), PF(z)\rangle_{\mathcal{H}} \leqslant \langle F(z), F(z)\rangle_{\mathcal{H}}$ for all $F \in \mathcal{H}$, we have

$$\begin{aligned} \left|B(w)F(w)\right|^2 &= \left|\langle F(z), PL(w,z)\rangle_{\mathcal{H}}\right|^2 \\ &\leqslant \langle F(z), F(z)\rangle_{\mathcal{H}} \langle PL(w,z), PL(w,z)\rangle_{\mathcal{H}} \\ &\leqslant \langle F(z), F(z)\rangle_{\mathcal{H}} \langle L(w,z), L(w,z)\rangle_{\mathcal{H}} \\ &= \langle F(z), F(z)\rangle_{\mathcal{H}} L(w,w) \end{aligned} \tag{2.8}$$

for $\Im w > -1/2$ and for all $F \in \mathcal{H}$. In particular, if $F(z) = L(w,z)$, then $\langle F(z), F(z)\rangle_{\mathcal{H}} = F(w)$, and hence (2.8) becomes $|B(w)F(w)|^2 \leqslant |F(w)|^2$ for $\Im w > -1/2$; that is, $|B(w)| \leqslant 1$ for $\Im w > -1/2$.

Therefore, we have proved that $B(z)$ is analytic and bounded by 1 for $\Im z > -1/2$. It follows that $W(z)/W(z+i)$ is analytic and has a nonnegative real part in the half-plane $\Im z > -1/2$.

This completes the proof of the theorem. ■

3 Hilbert spaces associated with $\zeta(s)$ and $L(s, \chi_4)$

3.1 The Riemann zeta function

The Riemann zeta function $\zeta(s)$ is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re s > 1$. Let $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $\xi(s)$ is an entire function and satisfies the functional identity $\xi(s) = \xi(1-s)$. It is well known (see [5]) that we have the infinite product formula

$$\xi(s) = \prod \left(1 - \frac{s}{\rho}\right),$$

where the product is taken over all nontrivial zeros ρ of $\zeta(s)$ with ρ and $1-\rho$ being paired together for the convergence of the product.

Let $E(z) = \xi(1-iz)$. Then the Riemann hypothesis is that the zeros of $E(z)$ lie on the line $\Im z = -1/2$ and the functional identity $\xi(s) = \xi(1-s)$ can be written as $\bar{E}(\bar{z}) = E(z-i)$. If ρ is a nontrivial zero of $\zeta(s)$, then $0 < \Re\rho < 1$, a result proved independently by Hadamard and de la Vallée Poussin in 1896. Since

$$\left|E(z)\right|^2 = \prod \left|1 - \frac{iz}{\rho}\right|^2 = \prod \frac{(\Re\rho + y)^2 + (\Im\rho - x)^2}{|\rho|^2}$$

for $z = x + iy$, we see that $|E(x-iy)| < |E(x+iy)|$ for $y > 0$, and $|E(x+iy)|$ is a strictly increasing function of y on $(0, \infty)$ for each fixed real x.

In view of Theorem 1, it is natural to ask whether the Hilbert space of entire functions $\mathcal{H}(E)$ satisfies the condition that

$$\Re\left\langle F(z), F(z+i)\right\rangle_{\mathcal{H}(E)} \geqslant 0 \tag{3.1}$$

for every element $F(z)$ of $\mathcal{H}(E)$ such that $F(z+i) \in \mathcal{H}(E)$, because the nontrivial zeros of the Riemann zeta function $\zeta(s)$ would then lie on the critical line $\Re s = 1/2$ under this condition.

We give an example showing that condition (3.1) is unfortunately not true. Let $\rho = 1/2 + i111.0295355431696745\cdots$ be the 34th zero of the Riemann zeta function in the upper half-plane. By using *Mathematica*,[1] we compute that

$$-\Re\left\{\xi'(\rho)\xi(1+\rho)\right\} = -5.389100507182945\cdots \times 10^{-69} < 0. \tag{3.2}$$

Write $\rho = 1 - iw$. Then $E(w) = 0$ and $\bar{E}'(w)E(w+i)/i = -\xi'(\rho)\xi(1+\rho)$. Thus, (3.2) becomes

$$\Re\left\{\frac{\bar{E}'(w)E(w+i)}{2\pi i}\right\} < 0.$$

Therefore, by Theorem 1, we see that the Hilbert space of entire functions $\mathcal{H}(E)$ with $E(z) = \xi(1-iz)$ does not satisfy the condition (3.1).

Next, let $W(z) = 1/\xi(1-iz)$. Then $W(z)$ is analytic in the upper half-plane, is continuous, and has no zeros in the closed upper half-plane. If the Hilbert space of analytic functions $\mathcal{F}(W)$ satisfies the condition that

$$\Re\left\langle F(z), TF(z)\right\rangle_{\mathcal{F}(W)} \geqslant 0 \tag{3.3}$$

[1] Mathematica, 4.0, Wolfram Research, Champaign, IL, 1997.

for all $F \in \mathcal{F}(W)$, where T is the linear transformation of $\mathcal{F}(W)$ into itself that takes $K(w,z) = W(z)\bar{W}(w)/2\pi i(\bar{w} - z)$ into $K(w+i,z)$ for all complex w with $\Im w > 0$, then the function $W(z)$ would have an analytic extension to the half-plane $\Im z > -1/2$ by Theorem 2; that is, the Riemann zeta function $\zeta(s)$ would have no zeros for $\Re s > 1/2$.

We give an example showing that the space $\mathcal{F}(W)$ does not satisfy the condition (3.3). By using *Mathematica*, we compute that

$$\Re\left\{\frac{\xi(1+i282)}{\xi(2+i282)}\right\} = -0.000131957 < 0. \tag{3.4}$$

Let $w = -282$. Then $\Im w = 0 > -1/2$. Since $W(w) = 1/\xi(1 + i282)$ and $W(w + i) = 1/\xi(2 + i282)$, by (3.4), we have

$$\Re\left\{\frac{W(w)}{W(w+i)}\right\} < 0.$$

Therefore, by Theorem 2, we see that the space $\mathcal{F}(W)$ with $W(z) = 1/\xi(1 - iz)$ does not satisfy the condition (3.3).

3.2 The Dirichlet L-function $L(s, \chi_4)$

Note that χ_4 is the real primitive Dirichlet character $(\bmod 4)$, which is given by

$$\chi_4(n) = \begin{cases} (-1)^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

The Dirichlet L-function $L(s, \chi_4)$ is given by

$$L(s,\chi_4) = \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s}$$

for $\Re s > 0$. Let $\xi(s,\chi_4) = (4/\pi)^{s/2}\Gamma((1+s)/2)L(s,\chi_4)$. Then $\xi(s,\chi_4)$ is an entire function and satisfies the functional identity

$$\xi(1-s,\chi_4) = \epsilon(\chi_4)\xi(s,\chi_4), \tag{3.5}$$

where $\epsilon(\chi_4)$ is a constant of absolute value 1 (see [5]). Since χ_4 is a real character, by the argument of [5, Section 12], we have the infinite product formula

$$\xi(s,\chi_4) = \frac{\sqrt{\pi}}{2} \prod \left(1 - \frac{s}{\rho}\right), \tag{3.6}$$

where the product is taken over all nontrivial zeros of $L(s, \chi_4)$ with ρ and $1-\rho$ being put together.

Let $E_{\chi_4}(z) = \xi(1 - iz, \chi_4)$. Then the functional identity (3.5) can be written as

$$\bar{E}_{\chi_4}(\bar{z}) = \bar{\epsilon}(\chi_4) E_{\chi_4}(z - i).$$

By the product formula (3.6), we find that $|E_{\chi_4}(x + iy)|$ is a strictly increasing function of $y > 0$ for each fixed real x. Since the nontrivial zeros of $L(s, \chi_4)$ lie in the strip $0 < \Re s < 1$ (see [5, Section 14]), we have $|E_{\chi_4}(\bar{z})| < |E_{\chi_4}(z)|$ for $\Im z > 0$.

In view of Theorem 1, it is natural to ask whether the Hilbert space of entire functions $\mathcal{H}(E_{\chi_4})$ satisfies the condition that

$$\Re \langle F(z), F(z+i) \rangle_{\mathcal{H}(E_{\chi_4})} \geqslant 0 \tag{3.7}$$

for every element $F(z)$ of $\mathcal{H}(E_{\chi_4})$ such that $F(z+i) \in \mathcal{H}(E_{\chi_4})$, because the nontrivial zeros of the Dirichlet L-function $L(s, \chi_4)$ would then lie on the critical line $\Re s = 1/2$ under this condition.

We give an example showing that condition (3.7) is unfortunately not true. Let $\rho = 1/2 + i67.636920863546068398054 9\cdots$ be a zero of $L(s, \chi_4)$. By using *Mathematica*, we compute that

$$-\Re\left\{\xi'(\rho, \chi_4)\xi(1+\rho, \chi_4)\right\} = -2.310349004993483456 \cdots \times 10^{-45} < 0.$$

Write $\rho = 1 - iw$. Then $E_{\chi_4}(w) = 0$ and $E'_{\chi_4}(w) E_{\chi_4}(w + i)/i = -\xi'(\rho, \chi_4)\xi(1+\rho, \chi_4)$. Thus, the above inequality becomes

$$\Re\left\{\frac{E'_{\chi_4}(w)\bar{E}_{\chi_4}(w+i)}{2\pi i}\right\} < 0.$$

Therefore, by Theorem 1, we see that the Hilbert space of entire functions $\mathcal{H}(E_{\chi_4})$ with $E_{\chi_4}(z) = \xi(1 - iz, \chi_4)$ does not satisfy the condition (3.7).

Next, let $W_{\chi_4}(z) = 1/\xi(1 - iz, \chi_4)$. Then $W_{\chi_4}(z)$ is analytic in the upper half-plane, is continuous, and has no zeros in the closed upper half-plane. If the space $\mathcal{F}(W_{\chi_4})$ satisfies the condition that

$$\Re \langle F(z), TF(z) \rangle_{\mathcal{F}(W_{\chi_4})} \geqslant 0 \tag{3.8}$$

for all $F \in \mathcal{F}(W_{\chi_4})$, where T is the linear transformation of $\mathcal{F}(W_{\chi_4})$ into itself that takes

$K(w,z) = W_{\chi_4}(z)\bar{W}_{\chi_4}(w)/2\pi i(\bar{w} - z)$ into $K(w+i,z)$ for all complex w with $\Im w > 0$, then the function $W_{\chi_4}(z)$ would have an analytic extension to the half-plane $\Im z > -1/2$ by Theorem 2; that is, the Dirichlet L-function $L(s,\chi_4)$ would have no zeros for $\Re s > 1/2$.

We give an example showing that the space $\mathcal{F}(W_{\chi_4})$ does not satisfy the condition (3.8). By using *Mathematica*, we compute that

$$\Re\left\{\frac{\xi(1+i8714.2,\chi_4)}{\xi(2+i8714.2,\chi_4)}\right\} = -0.000422340607 < 0.$$

Let $w = -8714.2$. Then $\Im w = 0 > -1/2$. Since $W_{\chi_4}(w) = 1/\xi(1+i8714.2,\chi_4)$ and $W_{\chi_4}(w+i) = 1/\xi(2+i8714.2,\chi_4)$, we have

$$\Re\left\{\frac{W_{\chi_4}(w)}{W_{\chi_4}(w+i)}\right\} < 0.$$

Therefore, by Theorem 2, we see that the space $\mathcal{F}(W_{\chi_4})$ with $W_{\chi_4}(z) = 1/\xi(1-iz,\chi_4)$ does not satisfy the condition (3.8).

4 Conclusion

We have seen in Section 3 the difficulty of approaching the generalized Riemann hypothesis by using de Branges-type positivity conditions for reproducing kernel Hilbert spaces. It is possible that these positivity conditions are too strong for Hilbert spaces of entire functions associated with the Riemann zeta function and the Dirichlet L-functions.

Remark. After he looked at the manuscript of this paper, Peter Sarnak gave a proof for the statement that the space $\mathcal{F}(W)$ does not satisfy the condition (3.3), where $W(z) = 1/\xi(1-iz)$, and his argument involves no numerical calculations. For the convenience of readers, we sketch his proof here. Let $F(s) = \xi(s)/\xi(s+1)$. Assume that all logarithmic functions are defined by continuation from the point $s = 2$ at which their arguments are set to be zero. Then we have

$$\Im\{\log F(s)\} = \Im\{\log \zeta(s)\} + O(1)$$

for $\Re s > 1/2$. Since the set of values of $\log\zeta(s)$, $1/2 < \Re s < 2$, is dense in the complex plane (see [7, Chapter XI]), a complex number s_0 exists with $\Re s_0 > 1/2$ such that $\pi/2 < \Im\{\log F(s_0)\} < \pi$. Let $z_0 = i(s_0 - 1)$. Then $\Im z_0 > -1/2$ and $\Re\{W(z_0)/W(z_0+i)\} < 0$, and hence, by Theorem 2, the space $\mathcal{F}(W)$ does not satisfy the condition (3.3). Let r be any positive integer. For any Dirichlet character χ modulo r, let

$$\xi(s,\chi) = \left(\frac{\pi}{r}\right)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

where $L(s,\chi)$ is the Dirichlet L-function and where $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$. By using a similar argument, Sarnak also proved that the space $\mathcal{F}(W_\chi)$ does not satisfy the condition (3.8) where $W_\chi(z) = 1/\xi(1 - iz, \chi)$.

Appendix

Here is a list of *Mathematica* instructions, which can be used to verify our calculations.

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xi[s_]:=s*(s-1)*Pi^(-s/2)*Gamma[s/2]*Zeta[s]
f[ro_]:=Re[-xi'[ro]*xi[ro+1]]
g[t_]:=Re[xi[1+I*t]/xi[2+I*t]]
ro1=1/2+I*111.029535543169674524656
Zeta[ro1]
f[ro1]
g[282.]
Plot[g[t],{t,281.95,282.15}]
xi4[s_]:=(4*Pi)^(-s/2)*Gamma[(s+1)/2]*(Zeta[s,1/4]-Zeta[s,3/4])
f4[ro_]:=Re[-xi4'[ro]*xi4[ro+1]]
g4[t_]:=Re[xi4[1+I*t]/xi4[2+I*t]]
ro2=1/2+I*67.6369208635460683980549
Zeta[ro2,1/4]-Zeta[ro2,3/4]
f4[ro2]
g4[8714.2]
Plot[g4[t],{t,8714.1,8714.4}]
```

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