

HIGH MOMENTS OF THE RIEMANN ZETA-FUNCTION

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Abstract

In 1918 G. Hardy and J. Littlewood proved an asymptotic estimate for the Second moment of the modulus of the Riemann zeta-function on the segment $[1/2, 1/2+iT]$ in the complex plane, as T tends to infinity. In 1926 Ingham proved an asymptotic estimate for the fourth moment. However, since Ingham's result, nobody has proved an asymptotic formula for any higher moment. Recently J. Conrey and A. Ghosh conjectured a formula for the sixth moment. We develop a new heuristic method to conjecture the asymptotic size of both the sixth and eighth moments. Our estimate for the sixth moment agrees with and strongly supports, in a sense made clear in the paper, the one conjectured by Conrey and Ghosh. Moreover, both our sixth and eighth moment estimates agree with those conjectured recently by J. Keating and N. Snaitch based on modeling the zeta-function by characteristic polynomials of random matrices from the Gaussian unitary ensemble. Our method uses a conjectural form of the approximate functional equation for the zeta-function, a conjecture on the behavior of additive divisor sums, and D. Goldston and S. Gonek's mean value theorem for long Dirichlet polynomials. We also consider the question of the maximal order of the zeta-function on the critical line.

0. Introduction

One of the most important goals of number theorists in the past century has been to determine the moments of the Riemann zeta-function on the critical line. These are important because they can be used to estimate the maximal order of the zeta-function on the critical line and because of their applicability to the study of the distribution of prime numbers, often through zero-density estimates, and to divisor problems.

The most significant early results were obtained by Hardy and Littlewood [HL] in 1918 and by A. Ingham [I] in 1926. Hardy and Littlewood proved that

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$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim T \log T \quad (1)$$

as $T \rightarrow \infty$, and Ingham showed that

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \sim \frac{1}{2\pi^2} T \log^4 T. \quad (2)$$

No analogous formula has yet been proved for any higher moment, and it seems unlikely that any will be in the near future. In fact, the problem is so intractable that, until a few years ago, no one was able even to produce a plausible guess for the asymptotic main term. Recently, however, Conrey and Ghosh [CG2] found a special argument in the case of the sixth power moment that led them to conjecture that

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^6 dt \sim \frac{42}{9!} \prod_p \left(\left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right) T \log^9 T. \quad (3)$$

The object of this paper is to describe a new heuristic approach that leads to a conjecture for the asymptotic main term of

$$I_k(T) = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt$$

when k equals three and four. The resulting formula for the sixth power moment is identical to the conjecture of Conrey and Ghosh above and lends additional strong support to it in a sense to be described below. For the eighth power moment, we obtain the following new conjecture.

CONJECTURE 1

As $T \rightarrow \infty$,

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^8 dt \sim \frac{24024}{16!} \prod_p \left(\left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right) \right) T \log^{16} T.$$

We also discuss the size of higher moments of the zeta-function and its maximal order in the critical strip.

While writing this paper, the authors learned that Keating and Snaith [KS] have made a high moments conjecture based on a completely different approach. Instead of the attack through approximate functional equations, mean value theorems, and additive divisor sums employed here, they prove a general result on moments of random matrices whose eigenvalues have a GUE (Gaussian unitary ensemble) distribution. If the zeta-function is modeled by the determinant of such a matrix (and there are

reasons to believe it is), then the moments they calculate apply to the zeta-function as well. It is remarkable that our conjecture and theirs, which we state later, agree for the sixth and eighth moments, and it suggests that both are likely to be right.

We begin by outlining the main ideas behind our approach, starting with a brief discussion of approximate functional equations.

For $s = \sigma + it$ and $\sigma > 1$, $\zeta^k(s)$ has the Dirichlet series expansion

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s},$$

where $d_k(n)$ is the k th divisor function, which is multiplicative and defined at prime powers by $d_k(p^j) = \binom{k+j-1}{j}$. The series does not converge when $\sigma \leq 1$, but we can nevertheless approximate $\zeta^k(s)$ in this region by a sum of two Dirichlet polynomials. This is called an approximate functional equation, and its prototype is

$$\zeta(s)^k = \mathbb{D}_{k,N}(s) + \chi(s)^k \mathbb{D}_{k,M}(1-s) + \mathbb{E}_k(s), \tag{4}$$

where

$$\mathbb{D}_{k,N}(s) = \sum_{n=1}^N \frac{d_k(n)}{n^s},$$

$\mathbb{E}_k(s)$ is an error term, $MN = (t/(2\pi))^k$, and

$$\chi(s) = (\pi)^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)}$$

is the factor from the functional equation for the zeta-function, namely,

$$\zeta(s) = \chi(s)\zeta(1-s).$$

Note that from the last equation it follows that $\chi(s)$ satisfies

$$\chi(s)\chi(1-s) = 1.$$

Taking $s = 1/2 + it$ in (4), integrating the square of the modulus of both sides, and assuming that $\mathbb{E}_k(1/2 + it)$ is sufficiently small, we obtain

$$\begin{aligned} \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt &\sim \int_T^{2T} \left| \mathbb{D}_{k,N}\left(\frac{1}{2} + it\right) \right|^2 dt + \int_T^{2T} \left| \mathbb{D}_{k,M}\left(\frac{1}{2} + it\right) \right|^2 dt \\ &\quad + 2\Re \int_T^{2T} \chi\left(\frac{1}{2} - it\right)^k \mathbb{D}_{k,N}\left(\frac{1}{2} + it\right) \mathbb{D}_{k,M}\left(\frac{1}{2} + it\right) dt. \end{aligned} \tag{5}$$

Now

$$\chi\left(\frac{1}{2} - it\right) = \exp\left(it \log \frac{t}{2\pi e}\right) \left(1 + O\left(\frac{1}{t}\right)\right)$$

as $t \rightarrow \infty$, so we find that

$$\chi\left(\frac{1}{2} - it\right)^k (mn)^{-it} = \exp\left(it \log \frac{(t/(2\pi e))^k}{mn}\right) \left(1 + O\left(\frac{1}{t}\right)\right).$$

This has a stationary phase at $t = 2\pi(mn)^{1/k}$, which is generally outside the interval of integration. This suggests that when $MN \leq T^{k-\epsilon}$, the third integral on the right-hand side of (5) is smaller than the larger of the first two. (That it is no larger can be seen from the Cauchy-Schwarz inequality.) An extrapolation from this to $MN = T^k$ seems reasonable, and one can probably also replace t in the condition $MN = (t/(2\pi))^k$ by T when t is large. One can also show that if the Lindelöf hypothesis is true and if $M = 0$ and $N \gg T^k$, then $\zeta(1/2 + it)^k$ is well approximable in mean square by $\sum_{n=1}^{\infty} d_k(n) e^{-n/N} n^{-s}$. This suggests that $\zeta(1/2 + it)^k$ should also be well approximable by $\mathbb{D}_{k,N}(1/2 + it)$ for the same M and N . Thus, we expect the following to hold.

CONJECTURE 2

For every positive integer k , we have

$$\int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \sim \int_T^{2T} \left| \mathbb{D}_{k,N}\left(\frac{1}{2} + it\right) \right|^2 dt + \int_T^{2T} \left| \mathbb{D}_{k,M}\left(\frac{1}{2} + it\right) \right|^2 dt, \quad (6)$$

where

$$MN = \left(\frac{T}{2\pi}\right)^k \quad \text{with } M, N \geq \frac{1}{2}, \quad \text{or} \quad N \gg \left(\frac{T}{2\pi}\right)^k \quad \text{if } M = 0. \quad (7)$$

By classical methods, one can prove that Conjecture 2 holds when $k = 1$ and also when $k = 2$, provided that $\max(M, N) \ll T$. When $k \geq 3$, however, the known bounds for $\mathbb{E}_k(s)$ in (4) are too large to give (6), and it is also difficult to show that the third term in (5) really is smaller than the other two. Nevertheless, it may be possible to overcome these problems (when $k = 3$ or 4) by appealing to a more complicated form of the approximate functional equation first developed by A. Good [Go] for $\zeta(s)$ and for the L -functions attached to cusp forms (which are analogous to $\zeta(s)^2$). We hope to return to this question in a future article.

Our problem now reduces to determining an asymptotic estimate for the mean square of the Dirichlet polynomial $\mathbb{D}_{k,N}(1/2 + it)$. The standard tool for this is the classical mean value theorem for Dirichlet polynomials which, in the refined version

due to H. Montgomery and R. Vaughan [MV], asserts that

$$\int_T^{2T} \left| \sum_{n=1}^N a(n)n^{it} \right|^2 dt = \sum_{n=1}^N |a(n)|^2 (T + O(n)). \tag{8}$$

Using this, we see that

$$\int_T^{2T} \left| \mathbb{D}_{k,N} \left(\frac{1}{2} + it \right) \right|^2 dt = \sum_{n \leq N} \frac{d_k(n)^2}{n} (T + O(n)).$$

Now it is well known that

$$\sum_{n \leq N} d_k(n)^2 \sim \frac{a_k}{\Gamma(k^2)} N \log^{k^2-1} N$$

and

$$\sum_{n \leq N} \frac{d_k(n)^2}{n} \sim \frac{a_k}{\Gamma(k^2 + 1)} \log^{k^2} N, \tag{9}$$

where

$$a_k = \prod_p \left(\left(1 - \frac{1}{p} \right)^{k^2} \sum_{r=0}^{\infty} \frac{d_k^2(p^r)}{p^r} \right). \tag{10}$$

Thus, we deduce that

$$\int_T^{2T} \left| \mathbb{D}_{k,N} \left(\frac{1}{2} + it \right) \right|^2 dt \sim \frac{a_k}{\Gamma(k^2 + 1)} T \log^{k^2} N,$$

for $N \ll T$, say. Inserting this into (6), and assuming that M and N satisfy (7) and are each $\ll T$, we arrive at the previously known estimates for $I_1(T)$ and $I_2(T)$ in (1) and (2).

The problem we encounter for higher moments is that at least one of M and N must be significantly larger than T because of (7), but then the O -terms dominate the right-hand side of (8), and we lose the asymptotic formula. To get around this, we appeal to recent work of Goldston and Gonek [GG] on mean values of “long” Dirichlet polynomials, which allows one to evaluate

$$\int_T^{2T} \left| \sum_{n=1}^N \frac{a(n)}{n^{1/2+it}} \right|^2 dt, \tag{11}$$

provided one has a good handle on the coefficient sum

$$A(x) = \sum_{n \leq x} a(n)$$

and on the coefficient-correlation sums

$$A(x, h) = \sum_{n \leq x} a(n)a(n + h).$$

Specifically, one needs formulae of the type

$$A(x) = m(x) + E(x)$$

and

$$A(x, h) = m(x, h) + E(x, h)$$

in which $m(x)$ and $m(x, h)$ are differentiable with respect to x , $E(x) \ll x^\theta$, and $E(x, h) \ll x^\phi$, uniformly for $h \ll x^\eta$ with $0 < \theta, \phi, \eta < 1$. One can then obtain an asymptotic formula for (11) for $N \ll T^{\min(1/\theta, 1/\phi, 1/(1-\eta))-\epsilon}$ and any $\epsilon > 0$. If, in addition, one knows something about averages of the error terms $E(x, h)$ with respect to h , then N can be taken even larger. For example (cf. [GG]), taking $a(n) = d_k(n)$ and $k = 1$, we see that we can choose θ and ϕ to be ϵ and η to be $1 - \epsilon$ for any small positive ϵ , which means that N can be an arbitrarily large power of T . When $k = 2$, a result of D. Heath-Brown [HB1] allows us to take $\phi = \eta = 5/6$, and this permits us to take N up to $T^{6/5-\epsilon}$. (A better result can probably be obtained by using more recent tools, such as those of [DFI], combined with an averaging over h .) However, what we actually expect in this case is that $\phi = 1/2 + \epsilon$ and $\eta \geq 1/2 - \epsilon$, and if this is the case, then our method leads to Ingham’s formula (2) for all $N \ll T^{2-\epsilon}$.

For $k \geq 3$, unfortunately, the estimates we require for the additive divisor sums

$$D_k(x, h) = \sum_{n \leq x} d_k(n) d_k(n + h)$$

have never been proved. In fact, even an asymptotic formula for

$$\sum_{n \leq x} d_3(n) d_3(n + 1)$$

is not known. Still, a precise formula for the main term of $D_k(x, h)$ can be conjectured in several ways, the easiest being the so-called δ -method of W. Duke, J. Friedlander, and H. Iwaniec [DFI]. In Section 1 we describe how their method, together with a guess as to how the error term behaves, suggests the following conjecture.

CONJECTURE 3

Let $D_k(x, h) = \sum_{n \leq x} d_k(n) d_k(n + h)$. Then we have

$$D_k(x, h) = m_k(x, h) + O(x^{1/2+\epsilon}) \tag{12}$$

uniformly for $1 \leq h \leq x^{1/2}$, where $m_k(x, h)$ is a smooth function of x . The derivative of $m_k(x, h)$ is given by

$$m'_k(x, h) = \sum_{d|h} \frac{f_k(x, d)}{d}, \tag{13}$$

where

$$f_k(x, d) = \sum_{q=1}^{\infty} \frac{\mu(q)}{q^2} P_k(x, qd)^2, \tag{14}$$

$$P_k(x, q) = \frac{1}{2\pi i} \int_{|s|=1/8} \zeta^k(s+1) G_k(s+1, q) \left(\frac{x}{q}\right)^s ds, \tag{15}$$

$$G_k(s, q) = \sum_{d|q} \frac{\mu(d)}{\phi(d)} d^s \sum_{e|d} \frac{\mu(e)}{e^s} g_k\left(s, \frac{qe}{d}\right), \tag{16}$$

and, if $q = \prod_p p^\alpha$,

$$g_k(s, q) = \prod_{p|q} \left((1 - p^{-s})^k \sum_{j=0}^{\infty} \frac{d_k(p^{j+\alpha})}{p^{js}} \right). \tag{17}$$

Furthermore, for $d \leq x$ we have

$$f_k(x, d) \ll d_{k-1}^2(d) \log^{2k-2} x. \tag{18}$$

One can actually prove the last assertion, and we do so below. A similar conjecture for $m_k(x, h)$ appears in [Iv], but its form is less appropriate for our purposes.

Using Conjecture 3 together with [GG, Theorem 1], we are led to the following conjecture.

CONJECTURE 4

Let $N = T^{1+\eta}$ with $0 \leq \eta \leq 1$. Then

$$\int_T^{2T} \left| \mathbb{D}_{k,N} \left(\frac{1}{2} + it \right) \right|^2 dt \sim w_k(\eta) \frac{a_k}{\Gamma(k^2 + 1)} T L^{k^2},$$

where a_k is given by (10), and

$$w_k(\eta) = (1 + \eta)^{k^2} \left(1 - \sum_{n=0}^{k^2-1} \binom{k^2}{n+1} \gamma_k(n) (1 - (1 + \eta)^{-(n+1)}) \right),$$

where

$$\gamma_k(n) = (-1)^n \sum_{i=0}^k \sum_{j=0}^k \binom{k}{i} \binom{k}{j} \binom{n-1}{i-1, j-1, n-i-j+1}$$

for $n \geq 1$, and

$$\gamma_k(0) = k.$$

With more work we could *prove* that Conjecture 3 implies the mean value formula of Conjecture 4 for $0 \leq \eta \leq 1 - \epsilon$. However, we would still have to extrapolate to the full

interval $0 \leq \eta \leq 1$. Originally we expected the error terms in Conjecture 3 to exhibit considerable cancellation when summed over h ranges of length up to $x^{1-\epsilon}$, just as we believe they do when the coefficients $d_k(n)$ are replaced by $\Lambda(n)$ or $\mu(n)$. We were surprised to find, however, that this does not seem to be the case. For example, when $k = 2$, it appears that the error term in (12) actually leads to a main term in (2) if we use (6) with $N \gg T^{2+\epsilon}$. Thus, in the formula

$$w_2(\eta) = 1 + 4\eta - 6\eta^2 + 4\eta^3 - \eta^4,$$

if we take $\eta > 1$, we see that $w_2(\eta) > 2$. It follows that (6), (7), and Conjecture 3 cannot all be true if $\eta > 1$, since they contradict Ingham’s result (2). We believe (6) and (7) are correct, so our conclusion is that Conjecture 3 fails when $\eta > 1$. Since we have accounted for all of the obvious main terms, we are forced to conclude that somehow the error terms in (12) accumulate to deliver a new “mysterious” main term.

The restriction on the size of N in Conjecture 4 means that M and N must satisfy $\max(M, N) \ll T^2$ in (6), and, in light of (7), this forces k to be less than or equal to 4. When $k = 3$, for example, we may use (6) with N satisfying $T \ll N \ll T^2$. Then $M = T^3/N$ satisfies the same bound and we show that, for any choice of N in this range, Conjecture 4 leads to

$$\begin{aligned} \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^6 dt &\sim \int_T^{2T} \left| \mathbb{D}_{3,N}\left(\frac{1}{2} + it\right) \right|^2 dt + \int_T^{2T} \left| \mathbb{D}_{3,T^3/N}\left(\frac{1}{2} + it\right) \right|^2 dt \\ &\sim 42 \frac{a_3}{9!} T \log^9 T. \end{aligned}$$

Adding this up for T replaced by $T/2, T/4, \dots$, we obtain

$$I_3(T) \sim 42 \frac{a_3}{9!} T \log^9 T,$$

which is (3). The persistence of this estimate throughout the range $T \ll N \ll T^2$ gives very strong independent confirmation of the sixth power moment conjecture of Conrey and Ghosh. When $k = 4$, we are forced to take $N = M = T^2$, and in this case Conjecture 4 leads to

$$\int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^8 dt \sim 2 \int_T^{2T} \left| \mathbb{D}_{4,N}\left(\frac{1}{2} + it\right) \right|^2 dt \sim 24024 \frac{a_4}{16!} T \log^{16} T,$$

so that

$$I_4(T) \sim 24024 \frac{a_4}{16!} T \log^{16} T.$$

The form of these results suggests that there exists a constant g_k such that

$$I_k(T) \sim g_k \frac{a_k}{\Gamma(1 + k^2)} T \log^{k^2} T. \tag{19}$$

With this notation, Hardy and Littlewood’s result asserts that $g_1 = 1$, Ingham’s that

$g_2 = 2$, the conjecture of Conrey and Ghosh that $g_3 = 42$, and Conjecture 1 that $g_4 = 24024$. Since we do not know whether g_k exists in general, it is convenient to define

$$g_k(T) = \left(\frac{a_k}{\Gamma(1+k^2)} T \log^{k^2} T \right)^{-1} I_k(T),$$

so that

$$g_k = \lim_{T \rightarrow \infty} g_k(T),$$

provided the limit exists.

There have been numerous papers devoted to the estimation of $g_k(T)$. For example, writing $g_k(T) \geq C$ to mean that $g_k(T) \geq (1 + o(1))C$, Conrey and Ghosh [CG3] showed unconditionally that $g_3(T) \geq 10.13$ as $T \rightarrow \infty$. K. Soundararajan [S] increased the bound to 20.26 and later (unpublished) to 24.59. Subject to the truth of the Lindelöf hypothesis, Conrey and Ghosh [CG3] also obtained the lower asymptotic bounds $g_4(T) \geq 205$, $g_5(T) \geq 3242$, and $g_6(T) \geq 28130$. Their method uses the auxiliary means

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \mathbb{D}_{k,N}\left(\frac{1}{2} + it\right)^2 dt \quad (20)$$

and

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \zeta\left(\frac{1}{2} + it\right)^k \mathbb{D}_{k,N}\left(\frac{1}{2} - it\right) dt \quad (21)$$

for $N = T^\theta$ with $0 < \theta < 1/2$. Under the assumption that the Lindelöf hypothesis is true and that these formulae also hold when θ tends to 1, Conrey and Ghosh deduced the stronger lower bounds $g_3(T) \geq 38.76$, $g_4(T) \geq 21528$, $g_5(T) \geq 48438800$, and

$$g_k(T) \geq \left(\frac{ek}{2}\right)^{2k-2} \quad (22)$$

as $k \rightarrow \infty$. Their conjecture that $g_3 = 42$ was based on similar ideas.

The function $g_k(T)$ has also been studied for nonintegral values of k by K. Ramachandra, R. Balasubramanian and Ramachandra, Heath-Brown, Conrey and Ghosh, Gonek, and Soundararajan, among others. Here we summarize just a few of the results. Ramachandra [R] proved that $g_k(T) \approx 1$ for $k = 1/2$. Heath-Brown [HB2] extended this to $k = 1/n$ for n any positive integer. Conrey and Ghosh [CG1] showed that the Riemann hypothesis (RH) implies $g_k(T) \geq 1$ for all $k > 0$, and Soundararajan [S] improved this for all $k \geq 2$ by showing that $g_k(T) \geq 2$. Gonek [G] proved that, on RH, $g_k(T) \geq 1$ for $-1/2 < k < 0$. Conrey and Ghosh [CG3] also gave conjectural improvements in the lower bound for the interval $1 < k < 2$, assuming their conjecture that $\theta = 1$ is permissible in (20) and (21).

One rationale for studying $g_k(T)$ for nonintegral k is that if g_k exists and is

meromorphic as a function of k , then it can be identified from its values for small real k . In fact, the other components in the conjectural formula (19) for $I_k(T)$ are known to be entire functions of order 2. This is clearly the case for $\log^{k^2} T$ and $1/\Gamma(1+k^2)$, and it was proved for a_k by Conrey and Ghosh [CG3]. It would therefore be interesting to know how g_k behaves as $k \rightarrow \infty$. The conjectural result (22) suggests that g_k grows at least like a function of order 1. We see below that the function $w_k(\eta)$ in Conjecture 4 is $\geq (1 + \eta)^{k^2}$, so that

$$\int_T^{2T} \left| \mathbb{D}_{k,T^2} \left(\frac{1}{2} + it \right) \right|^2 dt \geq 2^{k^2} \frac{a_k}{\Gamma(1+k^2)} T \log^{k^2} T.$$

Using this in (6) and dropping the second term, which is positive and probably much larger than the first, we deduce that $g_k(T) \geq 2^{k^2}$ as $k \rightarrow \infty$ through the integers. Thus, Conjecture 4 implies that if g_k exists, it grows at least as fast as a function of order 2. Probably it grows no faster than this. To see why, take $M = N$ in (6) to obtain

$$\int_T^{2T} \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2k} dt \sim 2 \int_T^{2T} \left| \mathbb{D}_{k,N} \left(\frac{1}{2} + it \right) \right|^2 dt$$

with $N = T^{k/2}$. According to (9), the contribution to this of the “diagonal” terms is

$$2T \sum_{n \leq N} \frac{d_k(n)^2}{n} \sim \frac{2a_k}{\Gamma(k^2 + 1)} T \log^{k^2} N.$$

But we expect this to be larger than the entire mean value, as it is when $N \ll T$ by Montgomery and Vaughan’s mean value theorem and when $T \ll N \ll T^2$ by Conjecture 4. This reasoning suggests that $g_k \leq 2(k/2)^{k^2}$. Thus, we believe that

$$2^{k^2} \leq g_k \leq 2 \left(\frac{k}{2} \right)^{k^2}.$$

This is consistent with the conjecture of Keating and Snaith referred to above, which is that

$$g_k = \Gamma(1+k^2) \lim_{N \rightarrow \infty} N^{-k^2} \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}.$$

By Stirling’s formula, we easily see that this implies

$$g_k = \left(\frac{k}{4e^{1/2}} \right)^{k^2(1+o(1))} \tag{23}$$

as $k \rightarrow \infty$.

With conjectural estimates for g_k in hand, we can approach the question of the maximal order of the zeta-function. Define

$$m_T = \max_{0 \leq t \leq T} \left| \zeta \left(\frac{1}{2} + it \right) \right|,$$

and for convenience write

$$L = \log T.$$

On RH, it is known that

$$m_T \ll \exp \left(C_u \frac{L}{\log L} \right) \quad (24)$$

for some positive constant C_u . On the other hand, it follows from work of Montgomery [M] that if RH is true, then

$$m_T \gg \exp \left(C_l \sqrt{\frac{L}{\log L}} \right) \quad (25)$$

with $C_l = 1/20$. Subsequently, Balasubramanian and Ramachandra [BR] eliminated the need for RH in the lower bound, and Balasubramanian [B] increased the constant to $C_l = 0.5305 \dots$ (The constant is quoted as “3/4” in his paper, but Soundararajan has pointed out that there is an error in the computation of $\max D(\ell)$ there; it seems to be larger by a factor of $\sqrt{2}$ than it should be.) The wide disparity between the upper and lower bounds here appears in several other problems as well, and for the same reasons. For example, on RH it is known that $S(T) = (1/\pi) \arg \zeta(1/2 + iT)$ satisfies

$$S(T) \ll \frac{L}{\log L}, \quad (26)$$

and also (see [M]) that there exists a sequence of values of $T \rightarrow \infty$ such that

$$S(T) \gg \sqrt{\frac{L}{\log L}}.$$

On the 1-line, the disparity appears as a factor of 2. Namely, RH implies that

$$|\zeta(1 + it)| \leq 2e^\gamma \log \log t \quad (27)$$

as $t \rightarrow \infty$, while unconditionally there exists a sequence of $t \rightarrow \infty$ for which

$$\zeta(1 + it) \geq e^\gamma \log \log t.$$

The q -analogue of this asserts that if the generalized Riemann hypothesis is true, then

$$|L(1, \chi)| \leq 2e^\gamma \log \log q \quad (28)$$

for every primitive character $\chi \pmod{q}$, whereas unconditionally there is a sequence of $q \rightarrow \infty$ such that

$$L(1, \chi_q) \geq e^\gamma \log \log q,$$

with χ_q a quadratic, primitive character \pmod{q} . (See D. Shanks [Sh] for a discussion of his extensive numerical work on this question.)

We can obtain lower bounds for m_T directly from lower bounds for $I_k(T)$ by observing that

$$m_T \geq \left(\frac{1}{T} \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2k} dt \right)^{1/2k}.$$

To estimate the right-hand side of this, we require an estimate for a_k in addition to our conjectural estimates for g_k . To this end we prove the following proposition.

PROPOSITION

Let a_k be defined as above. Then we have

$$\log a_k = -k^2 \log(2e^\gamma \log k) + o(k^2)$$

as $k \rightarrow \infty$.

With more work and assuming RH, we could obtain the much more precise result

$$\begin{aligned} \log a_k = & -k^2 \left(\log \log k + \log(2e^\gamma) + \log \left(1 + \frac{\log 2}{\log k} \right) \right) \\ & + 8k^2 \left(\int_1^\infty \frac{\log(J_0(iw))}{w^3 \log(4k^2/w^2)} dw + \int_0^1 \frac{\log(J_0(iw)) - w^2/4}{w^3 \log(4k^2/w^2)} dw \right) \\ & + O(k^{1+\epsilon}), \end{aligned}$$

where J_0 is the Bessel function of the first kind of order zero. The integrals can then be expanded further to give an asymptotic expansion in decreasing powers of $\log k$.

We now calculate a lower bound for m_T assuming a lower bound for g_k of the form $(Ak + B)^{k^2}$. By Stirling’s formula, we have

$$m_T \geq \left(\frac{g_k a_k L^{k^2}}{\Gamma(1 + k^2)} \right)^{1/2k} \geq \left(\frac{(Ak + B)L e^{1-\gamma}}{2k^2 \log k} \right)^{k/2}. \tag{29}$$

It is not difficult to see that when $A = 0$, the right-hand side is maximized (as a function of k) essentially by taking $k^2 \log k = (B/2e^{1+\gamma})L$. Then $\log k \sim (1/2) \log L$, and we have

$$m_T \geq e^k \geq \exp \left(\sqrt{\frac{BL}{e^{1+\gamma} \log L}} \right).$$

Note that if $B = 2$, then $(B/e^{1+\gamma})^{1/2} = 0.64 \dots$, and if $B = 1$, it equals $0.45 \dots$. Of course, this assumes we have uniformity in k out to \sqrt{L} . On the other hand, if $B = 0$ in (29), so that g_k has the form suggested by Keating and Snaith, then we find that the maximum is attained when $k \log k$ is near $(A/2e^{1/2+\gamma})L$. This implies that $\log k \sim \log L$ and that

$$m_T \geq \exp\left(\frac{AL}{4e^\gamma \log L}\right).$$

Thus, if g_k grows as suggested by (23) and if (29) holds uniformly for $k \ll L$, then (24) is closer to the truth than (25). Similarly, (26) and (27) would reflect the true order of $S(T)$ and $|\zeta(1 + it)|$, respectively. Analogously, this suggests that (28) reflects the correct maximal size of $|L(1, \chi)|$. This is at odds with the usual view in these questions which is that the true order is most likely to be near the lower bounds. See the forthcoming paper of A. Granville and Soundararajan [GS], where one may find similar computations and an asymptotic evaluation of $\sum_n d_k(n)^2/n^2$ which resembles the result of the above proposition.

1. The conjectural formula for $D_k(x, h)$

In this section we sketch the derivation of the form and properties of $m_k(x, h)$, the conjectural main term for

$$D_k(x, h) = \sum_{n \leq x} d_k(n) d_k(n + h).$$

We assume that $(a, q) = 1$ and that the main term for the sum $\sum_{n \leq x} d_k(n) e(an/q)$ can be written in the form $(1/q) \int_0^x P_k(y, q) dy$ independently of a . Then applying the δ -method of Duke, Friedlander, and Iwaniec [DFI] in exactly the same way they do in the case $k = 2$, but ignoring all error terms, immediately leads to

$$m'_k(x, h) = \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} P_k(x, q)^2, \tag{30}$$

where $c_q(h) = \sum_{d|q, d|h} d \mu(q/d)$ is Ramanujan’s sum. Substituting this expression for $c_q(h)$ into the right-hand side, changing the order of summation, which is justified by absolute convergence once we have established the bound for $P_k(x, q)$ in (38), and relabeling variables, we find that

$$m'_k(x, h) = \sum_{d|h} \frac{f_k(x, d)}{d},$$

where

$$f_k(x, d) = \sum_{q=1}^{\infty} \frac{\mu(q)}{q^2} P_k(x, qd)^2.$$

Next we need to determine an explicit expression for $P_k(x, q)$. To do this we consider the generating function

$$D_k\left(s, \frac{a}{q}\right) = \sum_{n=1}^{\infty} d_k(n) e\left(\frac{an}{q}\right) n^{-s} \tag{31}$$

with $(a, q) = 1$ and $\sigma > 1$. Now for any integer m we have

$$e\left(\frac{m}{q}\right) = \sum_{d|m, d|q} \phi\left(\frac{q}{d}\right)^{-1} \sum_{\chi \pmod{q/d}} \tau(\bar{\chi}) \chi\left(\frac{m}{d}\right),$$

where the inner sum is over all characters χ to the modulus q/d and where $\tau(\chi) = \sum_{b \pmod{q}} \chi(b)e(b/q)$ is Gauss's sum. Using this to replace the exponential in (31) and rearranging the resulting sums, we obtain

$$D_k\left(s, \frac{a}{q}\right) = q^{-s} \sum_{d|q} \phi(d)^{-1} d^s \sum_{\chi \pmod{q}} \chi(a)\tau(\bar{\chi}) \sum_{m=1}^{\infty} d_k\left(\frac{qm}{d}\right) \chi(m)m^{-s}.$$

Now if $r = \prod_p p^\alpha$, we see that

$$\begin{aligned} \sum_{m=1}^{\infty} d_k(rm)\chi(m)m^{-s} &= \prod_{p|r} \left(\frac{\sum_{j=0}^{\infty} d_k(p^{j+\alpha})\chi(p^j)p^{-js}}{\sum_{j=0}^{\infty} d_k(p^j)\chi(p^j)p^{-js}} \right) \\ &\quad \times \prod_p \left(\sum_{j=0}^{\infty} d_k(p^j)\chi(p^j)p^{-js} \right) \\ &= g_k(s, r, \chi)L^k(s, \chi), \end{aligned} \tag{32}$$

say. Thus, for $\sigma > 1$ we have

$$D_k\left(s, \frac{a}{q}\right) = q^{-s} \sum_{d|q} \phi(d)^{-1} d^s \sum_{\chi \pmod{q}} \chi(a)\tau(\bar{\chi}) g_k\left(s, \frac{q}{d}, \chi\right) L^k(s, \chi).$$

This provides a meromorphic continuation of $D_k(s, a/q)$ to the whole complex plane and shows that its only possible pole in $\sigma > 0$ occurs at $s = 1$ and is due to the principal character $\chi_d^{(0)} \pmod{d}$ for each d dividing q . Thus, the singular part of $D_k(s, a/q)$ is the same as that of

$$q^{-s} \sum_{d|q} \phi(d)^{-1} d^s \chi_d^{(0)}(a)\tau\left(\chi_d^{(0)}\right) \sum_{m=1}^{\infty} d_k\left(\frac{qm}{d}\right) \chi_d^{(0)}(m)m^{-s}.$$

Now

$$\tau\left(\chi_d^{(0)}\right) = \sum_{\substack{b \pmod{d} \\ (b,d)=1}} e\left(\frac{b}{d}\right) = c_d(1) = \mu(d),$$

and $\chi_d^{(0)}(m) = \sum_{e|m, e|d} \mu(e)$, so we find that this equals

$$q^{-s} \sum_{d|q} \frac{\mu(d)}{\phi(d)} d^s \sum_{e|d} \mu(e)e^{-s} \sum_{n=1}^{\infty} d_k\left(\frac{qen}{d}\right) n^{-s}. \tag{33}$$

We can use (32) to express the sum over n here in terms of the zeta-function. Taking χ

equal to $\chi_1^{(0)}$, the principal character (mod 1), and writing $g_k(s, r) = g_k(s, r, \chi_1^{(0)})$, we deduce from (32) that

$$\sum_{n=1}^{\infty} d_k(rn)n^{-s} = g_k(s, r)\zeta^k(s),$$

where

$$g_k(s, r) = \prod_{p|r} \left((1 - p^{-s})^k \sum_{j=0}^{\infty} d_k(p^{j+\alpha})p^{-js} \right).$$

Inserting this into (33), we find that the singular part of $D_k(s, a/q)$ is identical to that of

$$q^{-s} \zeta^k(s) G_k(s, q),$$

where

$$G_k(s, q) = \sum_{d|q} \frac{\mu(d)}{\phi(d)} d^s \sum_{e|d} \mu(e) e^{-s} g_k\left(s, \frac{qe}{d}\right),$$

and we note that this is independent of a . From this and Perron's formula, we see that $(1/q) \int_0^x P_k(y, q) dy$, the main term for $\sum_{n \leq x} d_k(n) e(an/q)$, should be given by

$$\frac{1}{2\pi i} \int_{|s-1|=1/8} \zeta(s)^k G_k(s, q) \frac{(x/q)^s}{s} ds.$$

Thus, differentiating with respect to x , we find that

$$P_k(x, q) = \frac{1}{2\pi i} \int_{|s-1|=1/8} \zeta(s)^k G_k(s, q) \left(\frac{x}{q}\right)^{s-1} ds. \tag{34}$$

On changing s to $s + 1$, we obtain (15).

It only remains to prove (18). However, before doing this we derive a formula that we feel is interesting in its own right, even though we do not require it here. Define

$$\mathcal{D}_k(s, h) = \sum_{n=1}^{\infty} d_k(n) d_k(n + h) n^{-s}.$$

Then

$$m_k(x, h) = \frac{1}{2\pi i} \int_{|s-1|=1/8} \mathcal{D}_k(s, h) \frac{x^s}{s} ds,$$

so we see that

$$m'_k(x, h) = \text{res}_{s=1} \mathcal{D}_k(s, h) x^{s-1}.$$

On the other hand, since $D_k(s, 1/q)$ and $q^{-s} \zeta^k(s) G_k(s, q)$ have the same singular part at $s = 1$, from (34) we have that

$$\frac{1}{q} P_k(x, q) = \operatorname{res}_{s=1} D_k\left(s, \frac{1}{q}\right) x^{s-1}.$$

It therefore follows from (30) that

$$\operatorname{res}_{s=1} \mathcal{D}_k(s, h) x^{s-1} = \sum_{q=1}^{\infty} c_q(h) \left(\operatorname{res}_{s=1} D_k\left(s, \frac{1}{q}\right) x^{s-1} \right)^2.$$

This is the formula referred to above.

We now proceed to the proof of (18). From (16) and (17), we see that $g_k(s, 1) = G_k(s, 1) = 1$ and that, for $\alpha \geq 1$,

$$G_k(s, p^\alpha) = \left(1 - \frac{1}{p}\right)^{-1} \left(g_k(s, p^\alpha) - p^{s-1} g_k(s, p^{\alpha-1})\right).$$

Using this with (17) and the easily proven identity

$$d_k(p^\beta) = d_{k-1}(p^\beta) + d_k(p^{\beta-1}), \tag{35}$$

we obtain

$$G_k(s, p^\alpha) = \left(1 - \frac{1}{p}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^k \sum_{j=0}^{\infty} p^{-js} \left(d_{k-1}(p^{\alpha+j}) + (1 - p^{s-1}) d_k(p^{\alpha+j-1})\right). \tag{36}$$

It is also not difficult to show, for example by induction on β , that

$$d_l(p^{\alpha+\beta}) \leq d_l(p^\alpha) d_l(p^\beta). \tag{37}$$

Therefore, we have that

$$\begin{aligned} |G_k(s, p^\alpha)| \leq & \left(1 - \frac{1}{p}\right)^{-1} \left|1 - \frac{1}{p^s}\right|^k \left(d_{k-1}(p^\alpha) (1 - p^{-\sigma})^{-k+1} \right. \\ & \left. + d_k(p^{\alpha-1}) (1 - p^{-\sigma})^{-k} |1 - p^{s-1}|\right). \end{aligned}$$

Now we assume that $p^\alpha \parallel q$, and we restrict s to the circle $|s - 1| = A_1/(\log qx)$. Then there exist positive constants A_2 and A_3 such that

$$\left(1 - \frac{1}{p}\right) e^{-(A_2 \log p)/(p \log qx)} \leq \left|1 - \frac{1}{p^s}\right| \leq \left(1 - \frac{1}{p}\right) e^{(A_2 \log p)/(p \log qx)}$$

and

$$|1 - p^{s-1}| \leq \frac{A_3 \log p}{\log qx}.$$

Hence, for some positive constant A_4 we have

$$|G_k(s, p^\alpha)| \leq e^{(A_4 k \log p)/(p \log qx)} \left(d_{k-1}(p^\alpha) + d_k(p^{\alpha-1}) \left(1 - \frac{1}{p} \right)^{-1} \frac{A_3 \log p}{\log qx} \right).$$

We now use the simple formula

$$d_k(p^{\alpha-1}) = \frac{\alpha}{k-1} d_{k-1}(p^\alpha),$$

and we find that

$$\begin{aligned} |G_k(s, p^\alpha)| &\leq d_{k-1}(p^\alpha) e^{(A_4 k \log p)/(p \log qx)} \left(1 + \frac{A_5 \alpha \log p}{k \log qx} \right) \\ &\leq d_{k-1}(p^\alpha) e^{(B_k \log p^\alpha)/(\log qx)}, \end{aligned}$$

where B_k depends only on k . It follows that

$$|G_k(s, q)| \leq d_{k-1}(q) e^{(B_k \log q)/(\log qx)} \ll_k d_{k-1}(q).$$

Since $\zeta(s) \ll |s-1|^{-1}$ near $s=1$, if we use this in (34) and shrink the path of integration to the circle $|s-1| = A_1/(\log qx)$, we deduce that

$$P_k(x, q) \ll d_k(q) \log^{k-1} qx. \tag{38}$$

Finally, we use this bound in (14), separate the resulting divisor functions by means of (37), and recall that $d \leq x$ to obtain

$$f_k(x, d) \ll d_{k-1}^2(d) \log^{2k-2} x,$$

which is (18).

2. The generating function $F_k(x, z)$

In applying Conjecture 3 in Section 3, it turns out that what we actually require is the behavior of the generating function

$$F_k(x, z) = \sum_{d=1}^{\infty} \frac{f_k(dx, d)}{d^{z+1}}$$

near $z=0$. By (14) and (15), we see that

$$F_k(x, z) = \frac{1}{(2\pi i)^2} \iint_{\substack{|s|=1/8 \\ |w|=1/8}} \zeta^k(s+1) \zeta^k(w+1) \mathcal{H}_k(z, s, w) x^{s+w} ds dw, \tag{39}$$

where

$$\mathcal{H}_k(z, s, w) = \sum_{d=1}^{\infty} \frac{1}{d^{1+z}} \sum_{q=1}^{\infty} \frac{\mu(q) G_k(s+1, dq) G_k(w+1, dq)}{q^{2+s+w}}.$$

If we define

$$\begin{aligned} h_k(p^\alpha) &= h_k(s, w, p^\alpha) \\ &= G_k(s+1, p^\alpha) G_k(w+1, p^\alpha) \end{aligned} \tag{40}$$

$$- G_k(s+, p^{\alpha+1})G_k(w + 1, p^{\alpha+1})p^{-2-s-w},$$

then from the multiplicativity of $G_k(s + 1, q)$ we see that the sum over q in the definition of $\mathcal{H}_k(z, s, w)$ equals

$$\prod_{p||d} \left(\frac{h_k(p^\alpha)}{h_k(p)} \right) \prod_p h_k(p).$$

The first product defines a multiplicative function of d , so we find that

$$\begin{aligned} \mathcal{H}_k(z, s, w) &= \left(\prod_p h_k(p) \right) \sum_{d=1}^\infty \frac{1}{d^{1+z}} \prod_{p||d} \left(\frac{h_k(p^\alpha)}{h_k(p)} \right) \\ &= \left(\prod_p h_k(p) \right) \prod_p \left(\sum_{\alpha=0}^\infty \frac{h_k(p^\alpha)/h_k(p)}{p^{\alpha(1+z)}} \right) \\ &= \prod_p \left(\sum_{\alpha=0}^\infty \frac{h_k(p^\alpha)}{p^{\alpha(1+z)}} \right). \end{aligned} \tag{41}$$

Now $G_k(s + 1, 1) = 1$, and, for $\alpha \geq 1$,

$$G_k(s + 1, p^\alpha) = d_{k-1}(p^\alpha) + d_k(p^{\alpha-1})(1 - p^s) + O_k(p^{-1-\sigma}) + O_k(p^{-1})$$

by (36). Thus, writing $\Re s = \sigma$, $\Re w = u$, and $\Re z = x$, we see that

$$\begin{aligned} h_k(1) &= 1 + O_k(p^{-2+|\sigma|+|u|}), \\ h_k(p^\alpha) &= \left(d_{k-1}(p^\alpha) + d_k(p^{\alpha-1})(1 - p^s) \right) \left(d_{k-1}(p^\alpha) + d_k(p^{\alpha-1})(1 - p^w) \right) \\ &\quad + O_k(p^{-1+|\sigma|+|u|}), \end{aligned}$$

and

$$\sum_{\alpha=0}^\infty \frac{h_k(p^\alpha)}{p^{\alpha(1+z)}} = 1 + \frac{(k - p^s)(k - p^w)}{p^{1+z}} + O_k(p^{-2+|\sigma|+|u|+|x|}),$$

where the function bounded by the O -term in the last line is analytic in the region $|\sigma| + |u| + |x| < 2$. When we use this with (41), we obtain

$$\begin{aligned} \mathcal{H}_k(z, s, w) &= \prod_p \left(1 + \frac{k^2}{p^{1+z}} - \frac{k}{p^{1+z-s}} - \frac{k}{p^{1+z-w}} + \frac{1}{p^{1+z-s-w}} + \dots \right) \\ &= \zeta^{k^2}(1+z)\zeta^{-k}(1+z-s)\zeta^{-k}(1+z-w)\zeta(1+z-s-w)\mathcal{H}_k^*(z, s, w), \end{aligned}$$

where

$$\mathcal{H}_k^*(z, s, w) = \prod_p \left(\left(1 - \frac{1}{p^{1+z}} \right)^{k^2} \left(1 - \frac{1}{p^{1+z-s}} \right)^{-k} \left(1 - \frac{1}{p^{1+z-w}} \right)^{-k} \right)$$

$$\times \left(1 - \frac{1}{p^{1+z-s-w}}\right) \left(\sum_{\alpha=0}^{\infty} \frac{h_k(p^\alpha)}{p^{\alpha(1+z)}}\right)$$

is analytic for $|\sigma| + |u| + |x| < 1$. Combining this with (39), we find that

$$F_k(x, z) = \frac{1}{(2\pi i)^2} \iint_{\substack{|s|=1/8 \\ |w|=1/8}} \frac{\zeta(s+1)^k \zeta(w+1)^k \zeta(z+1)^{k^2} \zeta(z+1-s-w)x^{s+w}}{\zeta(z+1-s)^k \zeta(z+1-w)^k} \mathcal{H}_k^*(z, s, w) ds dw. \tag{42}$$

Finally, we evaluate $\mathcal{H}_k^*(0, 0, 0)$, as this is also required in Section 3. By (40) and the definition of $\mathcal{H}_k^*(z, s, w)$, we have

$$\mathcal{H}_k^*(0, 0, 0) = \prod_p \left(\left(1 - \frac{1}{p}\right)^{(k-1)^2} \sum_{\alpha=0}^{\infty} \frac{(G_k^2(1, p^\alpha) - G_k^2(1, p^{\alpha+1})p^{-2})}{p^\alpha} \right). \tag{43}$$

By (36), $G_k(1, p^\alpha) = (1 - 1/p)^{k-1} \sum_{j=0}^{\infty} d_{k-1}(p^{\alpha+j})p^{-j}$. Hence, factoring the numerator in the sum over α as a difference of two squares, we obtain

$$\begin{aligned} &G_k^2(1, p^\alpha) - G_k^2(1, p^{\alpha+1})p^{-2} \\ &= \left(1 - \frac{1}{p}\right)^{2k-2} d_{k-1}(p^\alpha) \left(d_{k-1}(p^\alpha) + 2 \sum_{j=0}^{\infty} \frac{d_{k-1}(p^{\alpha+j+1})}{p^{j+1}} \right). \end{aligned}$$

We therefore see that the typical factor in (43) equals

$$\left(1 - \frac{1}{p}\right)^{k^2-1} \left(\sum_{\alpha=0}^{\infty} \frac{d_{k-1}^2(p^\alpha)}{p^\alpha} + 2 \sum_{\alpha=1}^{\infty} \frac{d_{k-1}(p^\alpha)}{p^\alpha} \left(\sum_{l=0}^{\alpha-1} d_{k-1}(p^l) \right) \right).$$

Since $\sum_{q|n} d_{k-1}(q) = d_k(n)$, the sum over l equals $d_k(p^{\alpha-1})$, so this is

$$\left(1 - \frac{1}{p}\right)^{k^2-1} \left(1 + \sum_{\alpha=1}^{\infty} \frac{(d_{k-1}(p^\alpha) + d_k(p^{\alpha-1}))^2 - d_k^2(p^{\alpha-1})}{p^\alpha} \right).$$

We now use (35) to see that this is

$$\left(1 - \frac{1}{p}\right)^{k^2-1} \left(\sum_{\alpha=0}^{\infty} \frac{d_k^2(p^\alpha)}{p^\alpha} - \sum_{\alpha=1}^{\infty} \frac{d_k^2(p^{\alpha-1})}{p^\alpha} \right) = \left(1 - \frac{1}{p}\right)^{k^2} \left(\sum_{\alpha=0}^{\infty} \frac{d_k^2(p^\alpha)}{p^\alpha} \right).$$

Inserting this into (43) and comparing with (10), we deduce that

$$\mathcal{H}_k^*(0, 0, 0) = a_k.$$

3. Conjecture 4

A precise version of the mean value formula we require to derive the formula in

Conjecture 4 is given in [GG]. However, we adopt a simpler heuristic approach that leads to the same formula more quickly.

Recall that

$$\mathbb{D}_{k,N}(s) = \sum_{n=1}^N \frac{d_k(n)}{n^s}$$

and that we are to estimate

$$\mathcal{I}(T) = \mathcal{I}_{k,N}(T) = \int_T^{2T} \left| \mathbb{D}_{k,N}\left(\frac{1}{2} + it\right) \right|^2 dt.$$

The reason we integrate from T to $2T$ rather than from zero to T is that for t near zero the integrand is very large, on the order of $N^{1/2}$, so the mean square for small t is about N . Since N can be as large as T^2 , this would dominate, and so obscure, the behavior of the mean value away from the real axis. We let

$$\mathcal{J}(T) = \mathcal{J}_{k,N}(T) = \frac{1}{2} \int_{-T}^T \left| \mathbb{D}_{k,N}\left(\frac{1}{2} + it\right) \right|^2 dt, \tag{44}$$

and then we obtain our estimate for $\mathcal{I}(T)$ via the formula

$$\mathcal{I}(T) = \mathcal{J}(2T) - \mathcal{J}(T). \tag{45}$$

Squaring out and integrating term-by-term in (44), we find that

$$\mathcal{J}(T) = T \sum_{n=1}^N \frac{d_k(n)^2}{n} + \sum_{m \neq n} \frac{d_k(m) d_k(n)}{(mn)^{1/2}} \frac{\sin(T \log(m/n))}{\log(m/n)} = \mathcal{J}_d(T) + \mathcal{J}_o(T), \tag{46}$$

say. Using the symmetry in m and n and writing $m = n + h$, we find that

$$\mathcal{J}_o = 2 \sum_{n=1}^N \sum_{h=1}^{N-n} \frac{d_k(n) d_k(n+h)}{n \sqrt{1+h/n}} \frac{\sin(T \log(1+h/n))}{\log(1+h/n)}.$$

Now we make several approximations that are justified in [GG] for the weighted version of this formula. Namely, we replace $\log(1+h/n)$ by h/n , $\sqrt{1+h/n}$ by 1, and $d_k(n) d_k(n+h)$ by $m'_k(x, h)$ from Conjecture 3. Finally, the sum over n can be replaced by an integral, and the sum over h can be extended to ∞ . This leads to

$$\mathcal{J}_o \sim 2 \int_0^N \sum_{h=1}^{\infty} \frac{m'_k(x, h)}{h} \sin\left(\frac{Th}{x}\right) dx.$$

By (13), the right-hand side equals

$$\begin{aligned}
 2 \int_0^N \sum_{h=1}^{\infty} \frac{1}{h} \sum_{d|h} \frac{f_k(x, d)}{d} \sin\left(\frac{Th}{x}\right) dx &= 2 \int_0^N \sum_{d=1}^{\infty} \frac{f_k(x, d)}{d^2} \sum_{h=1}^{\infty} \frac{\sin(Th d/x)}{h} dx \\
 &= -2\pi \int_0^N \sum_{d=1}^{\infty} \frac{f_k(x, d)}{d^2} \left(\left\{ \frac{Td}{2\pi x} \right\} - \frac{1}{2} \right) dx,
 \end{aligned}$$

where

$$\{x\} = \begin{cases} x - [x] & \text{if } x \text{ is not an integer,} \\ \frac{1}{2} & \text{if } x \text{ is an integer.} \end{cases}$$

The “ $-1/2$ ” term leads to a large contribution if N is large, but it is independent of T and so disappears when we take the difference $\mathcal{J}(2T) - \mathcal{J}(T) = \mathcal{I}(T)$. Thus, we may express \mathcal{I}_o as

$$\mathcal{I}_o(T) = \mathcal{I}_{o,1}(T) + \mathcal{I}_{o,2}, \tag{47}$$

where $\mathcal{I}_{o,1}(T)$ is the part with “ $\{\}$ ” and where $\mathcal{I}_{o,2}$ is the part with “ $-1/2$ ”.

We now make the change of variable $y = Td/(2\pi x)$ in $\mathcal{I}_{o,1}$, and we find that

$$\begin{aligned}
 \mathcal{I}_{o,1}(T) &= -2\pi \int_0^N \sum_{d=1}^{\infty} \frac{f_k(x, d)}{d^2} \left\{ \frac{Td}{2\pi x} \right\} dx \\
 &= -T \sum_{d=1}^{\infty} \frac{1}{d} \int_{(Td)/(2\pi N)}^{\infty} \frac{f_k(Td/2\pi y, d)}{y^2} \{y\} dy.
 \end{aligned}$$

We split the sum over d into $d \leq 2\pi N/T$ and $d > 2\pi N/T$. From our estimate for f_k in (18), we find that the contribution to $\mathcal{I}_{o,1}$ from the upper range of d is

$$\ll T \sum_{N/T \ll d} \frac{\tau_{k-1}(d)^2}{d} \frac{N}{Td} L^{2k-2} \ll TL^{(k-1)^2-1+2k-2} = TL^{k^2-2}.$$

In the lower range of d , we split the integral over y into two ranges: $Td/2\pi N \leq y < 1$ and $y \geq 1$. The contribution from the upper range of y is

$$\ll T \sum_{d \ll N/T} \frac{\tau_{k-1}(d)^2}{d} L^{2k-2} \frac{N}{dT} \ll TL^{k^2-1}.$$

Hence, since $\{y\} = y$ for $0 < y < 1$, we see that

$$\begin{aligned}
 \mathcal{I}_{o,1}(T) &= -T \sum_{d \leq N/T} \frac{1}{d} \int_{(Td)/(2\pi N)}^1 \frac{f_k(Td/2\pi y, d)}{y} dy + O(TL^{k^2-1}) \\
 &= -T \int_{T/(2\pi N)}^1 \sum_{d \leq yN/T} \frac{f_k(Td/2\pi y, d)}{d} \frac{dy}{y} + O(TL^{k^2-1}).
 \end{aligned}$$

We now use the expression in (42) for the generating function $F_k(x, z)$ of

$f_k(dx, d)$. In doing this we retain only the first term in the Laurent expansion at zero of the various factors in the integrand, and we find that

$$\mathcal{J}_{o,1}(T) \sim -a_k T \int_{T/N}^1 \frac{1}{(2\pi i)^3} \iiint_{\substack{|s|=1/8 \\ |w|=1/8 \\ |z|=1/2}} \frac{(T/y)^{s+w-z} N^z (z-s)^k (z-w)^k}{z^{k^2+1} (z-s-w) s^k w^k} ds dw dz \frac{dy}{y}.$$

To evaluate this we make the substitutions $s \rightarrow sz$ and $w \rightarrow wz$, and we carry out the integration in z . We then find that

$$\begin{aligned} \mathcal{J}_{o,1}(T) \sim & -\frac{a_k T}{\Gamma(k^2)(2\pi i)^2} \int_{T/N} \iint_{\substack{|s|=1/16 \\ |w|=1/16}} \frac{((s+w-1)\log(T/y) + \log N)^{k^2-1} (1-s)^k (1-w)^k}{(1-s-w)s^k w^k} ds dw \frac{dy}{y}. \end{aligned} \tag{48}$$

Next we write

$$N = T^{1+\eta},$$

where $0 \leq \eta \leq 1$, and we make the substitution

$$y = T^{\alpha(1+\eta)-\eta}.$$

Then $y = 1$ corresponds to $\alpha = 1 - 1/(1 + \eta)$, and $y = T/N$ corresponds to $\alpha = 0$. Also,

$$(s + w - 1) \log \frac{T}{y} + \log N = \log N (1 + (s + w - 1)(1 - \alpha))$$

and

$$\frac{dy}{y} = \log N d\alpha.$$

Thus, the asymptotic expression for (48) becomes

$$\mathcal{J}_{o,1}(T) \sim \frac{a_k}{\Gamma(k^2)} (1 + \eta)^{k^2} T L^{k^2} \mathcal{M}_k, \tag{49}$$

where

$$\begin{aligned} \mathcal{M}_k = & -\frac{1}{(2\pi i)^2} \int_0^{1-1/(1+\eta)} \iint_{\substack{|s|=1/16 \\ |w|=1/16}} \frac{(1 + (s + w - 1)(1 - \alpha))^{k^2-1} (1-s)^k (1-w)^k}{(1-s-w)s^k w^k} ds dw d\alpha. \end{aligned}$$

We observe for future reference that $\mathcal{M}_k \rightarrow 0$ as $k \rightarrow \infty$. In fact, since

$$1 + (s + w - 1)(1 - \alpha) = \alpha + (s + w)(1 - \alpha),$$

we find that

$$|\mathcal{M}_k| \leq \left(\frac{1-1}{\eta+1} + \frac{1}{8}\right)^{k^2-1} 15^{2k} \ll \left(\frac{4}{5}\right)^{k^2}$$

as $k \rightarrow \infty$.

Next, we expand $(1 + (s + w - 1)(1 - \alpha))^{k^2-1}$ into powers of $(s + w - 1)$, and we find that

$$(1 - (1 - s - w)(1 - \alpha))^{k^2-1} = \sum_{n=0}^{k^2-1} \binom{k^2-1}{n} (-1)^n (1 - s - w)^n (1 - \alpha)^n.$$

Thus,

$$\mathcal{M}_k = - \int_0^{1-1/(1+\eta)} \sum_{n=0}^{k^2-1} (-1)^n \gamma_k(n) (1 - \alpha)^n d\alpha,$$

where

$$\gamma_k(n) = \frac{1}{(2\pi i)^2} \iint_{\substack{|s|=1/16 \\ |w|=1/16}} \frac{(1 - s - w)^{n-1} (1 - s)^k (1 - w)^k}{s^k w^k} ds dw.$$

Now

$$\begin{aligned} \gamma_k(0) &= \frac{1}{(2\pi i)^2} \iint_{\substack{|s|=1/16 \\ |w|=1/16}} \frac{(1 - s)^k (1 - w)^k}{(1 - s - w) s^k w^k} ds dw \\ &= \sum_{i=0}^k \sum_{j=0}^k \binom{k}{i} \binom{k}{j} (-1)^{i+j} \frac{1}{(2\pi i)^2} \iint_{\substack{|s|=1/16 \\ |w|=1/16}} \frac{ds dw}{(1 - s - w) s^i w^j} ds dw \\ &= \sum_{i=0}^k \sum_{j=0}^k \sum_{m=0}^{\infty} \binom{k}{i} \binom{k}{j} (-1)^{i+j} \frac{1}{(2\pi i)^2} \iint_{\substack{|s|=1/16 \\ |w|=1/16}} \frac{(s + w)^m ds dw}{s^i w^j} ds dw \\ &= \sum_{i=0}^k \sum_{j=0}^k \binom{k}{i} \binom{k}{j} \binom{i + j - 2}{i - 1} (-1)^{i+j}. \end{aligned}$$

Actually, this can be simplified to $\gamma_k(0) = k$, but this fact is not necessary to proceed.

In a similar manner we find that

$$\gamma_k(n) = (-1)^n \sum_{i=0}^k \sum_{j=0}^k \binom{k}{i} \binom{k}{j} \binom{n-1}{i-1, j-1, n-i-j+1}.$$

Finally, integrating with respect to α , we obtain

$$\mathcal{M}_k = - \sum_{n=0}^{k^2-1} (-1)^n \gamma_k(n) \frac{(1 - (\eta + 1)^{-(n+1)})}{n + 1}.$$

Combining this with (47) and (49), we obtain

$$\begin{aligned} \mathcal{J}_o(2T) - \mathcal{J}_o(T) &= \mathcal{J}_{o,1}(2T) - \mathcal{J}_{o,1}(T) \\ &\sim -\frac{a_k}{\Gamma(k^2)}(1 + \eta)^{k^2} TL^{k^2} \sum_{n=0}^{k^2-1} (-1)^n \gamma_k(n) \frac{(1 - (\eta + 1)^{-(n+1)})}{n + 1}. \end{aligned} \tag{50}$$

Also, from (9) we have

$$\mathcal{J}_d(2T) - \mathcal{J}_d(T) \sim \frac{a_k}{\Gamma(k^2 + 1)}(1 + \eta)^{k^2} L^{k^2}.$$

It now follows from (45), (46), (50), and this that

$$\mathcal{J}_{k,N} = \int_T^{2T} \left| \mathbb{D}_{k,N} \left(\frac{1}{2} + it \right) \right|^2 dt \sim w_k(\eta) \frac{a_k}{\Gamma(k^2 + 1)} TL^{k^2},$$

where

$$w_k(\eta) = (1 + \eta)^{k^2} \left(1 - \sum_{n=0}^{k^2-1} \binom{k^2}{n+1} \gamma_k(n) \frac{(1 - (1 + \eta)^{-(n+1)})}{n + 1} \right).$$

This is Conjecture 4.

4. The sixth and eighth power moment conjectures

We first note an alternative expression for a_k . Conrey and Ghosh [CG3] have shown that $a_k = a_{1-k}$. Thus, by (10),

$$a_k = \prod_p \left(\left(1 - \frac{1}{p} \right)^{(k-1)^2} \sum_{r=0}^{\infty} \frac{d_{1-k}^2(p^r)}{p^r} \right).$$

Since $d_k(p^r) = \binom{k+r-1}{r} = (-1)^r \binom{-k}{r}$, we see that $d_{1-k}(p^r) = (-1)^r \binom{k-1}{r}$, so we have that

$$a_k = \prod_p \left(\left(1 - \frac{1}{p} \right)^{(k-1)^2} \sum_{r=0}^{k-1} \frac{\binom{k-1}{r}}{p^r} \right). \tag{51}$$

This is the expression for a_k we have used in (3) and Conjecture 1.

By (6) we expect that

$$\int_T^{2T} \left| \zeta \left(\frac{1}{2} + it \right) \right|^6 dt \sim \int_T^{2T} \left(\left| \mathbb{D}_{3,T^{1+\eta}} \left(\frac{1}{2} + it \right) \right|^2 + \left| \mathbb{D}_{3,T^{2-\eta}} \left(\frac{1}{2} + it \right) \right|^2 \right) dt$$

for any η with $0 \leq \eta \leq 2$. Using Conjecture 4 and adding the results together for $T/2, T/4, \dots$, we obtain

$$I_3(T) = \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^6 dt \sim (w_3(\eta) + w_3(1 - \eta)) \frac{a_3}{9!} TL^9.$$

From the formula for $w_k(\eta)$ in the conjecture, we calculate that

$$w_3(\eta) = 1 + 9\eta + 36\eta^2 + 84\eta^3 + 126\eta^4 - 630\eta^5 + 588\eta^6 + 180\eta^7 - 9\eta^8 + 2\eta^9,$$

and it is not difficult to verify that

$$w_3(\eta) + w_3(1 - \eta) = 42$$

for all η . This identity provides compelling evidence for the sixth power moment conjecture of Conrey and Ghosh.

Similarly, we calculate that

$$\begin{aligned} w_4(\eta) = & 1 + 16\eta + 120\eta^2 + 560\eta^3 + 1820\eta^4 + 4368\eta^5 + 8008\eta^6 + 11440\eta^7 \\ & + 12870\eta^8 + 11440\eta^9 - 152152\eta^{10} + 179088\eta^{11} - 78260\eta^{12} \\ & + 14000\eta^{13} - 1320\eta^{14} + 16\eta^{15} - 3\eta^{16}. \end{aligned}$$

We then find that

$$w_4(1) = 12012,$$

and this leads to

$$I_4(T) \sim 24024 \frac{a_4}{16!} T L^{16},$$

which is Conjecture 1.

5. Proof of the proposition

To prove the proposition we work from the expression for a_k given in (51). Using this, we first prove an upper bound for a_k . Write

$$a_k = \Pi^- \Pi^+,$$

where Π^- is the part of the product over the primes less than or equal to $2k^2$ and where Π^+ is the part over the primes greater than $2k^2$. Then

$$\begin{aligned} \Pi^- &\leq \prod_{p \leq 2k^2} \left(1 - \frac{1}{p}\right)^{(k-1)^2} \left(1 + \frac{k-1}{p^{1/2}} + \frac{\binom{k-1}{2}}{p} + \frac{\binom{k-1}{3}}{p^{3/2}} + \dots\right)^2 \\ &\leq \left(\frac{e^{-\gamma}}{\log 2k^2} (1 + o(1))\right)^{(k-1)^2} \prod_{p \leq 2k^2} \left(1 + \frac{1}{p^{1/2}}\right)^{2(k-1)} \\ &\leq \left(\frac{e^{-\gamma}}{2 \log k}\right)^{k^2} e^{o(k^2)} \exp\left(\sum_{p \leq 2k^2} \frac{2(k-1)}{p^{1/2}}\right) \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{e^{-\gamma}}{2 \log k} \right)^{k^2} e^{o(k^2)} e^{O(k^2/\log k)} \\
&= \left(\frac{e^{-\gamma}}{2 \log k} \right)^{k^2} e^{o(k^2)}.
\end{aligned}$$

Next, it is easy to show that

$$\binom{k-1}{r}^2 \leq \binom{(k-1)^2}{r}$$

for $r = 0, 1, 2, \dots$ and $k = 1, 2, 3, \dots$, so we see that

$$\Pi^+ \leq \prod_{p > 2k^2} \left(1 - \frac{1}{p}\right)^{(k-1)^2} \left(1 + \frac{1}{p}\right)^{(k-1)^2} = \prod_{p > 2k^2} \left(1 - \frac{1}{p^2}\right)^{(k-1)^2} \leq 1.$$

Thus, we find that

$$a_k \leq \left(\frac{e^{-\gamma}}{2 \log k} \right)^{k^2} e^{o(k^2)}.$$

Now we deduce a lower bound. First we have

$$\Pi^- \geq \prod_{p \leq 2k^2} \left(1 - \frac{1}{p}\right)^{(k-1)^2} = \left(\frac{e^{-\gamma}}{\log 2k^2} (1 + o(1)) \right)^{(k-1)^2} = \left(\frac{e^{-\gamma}}{2 \log k} \right)^{k^2} e^{o(k^2)}.$$

Also, since $(1-x)^n \geq 1-nx$ for $0 < x < 1$, we have

$$\begin{aligned}
\Pi^+ &\geq \prod_{p > 2k^2} \left(1 - \frac{1}{p}\right)^{(k-1)^2} \left(1 + \frac{(k-1)^2}{p}\right) \\
&\geq \prod_{p > 2k^2} \left(1 - \frac{(k-1)^2}{p}\right) \left(1 + \frac{(k-1)^2}{p}\right) \\
&\geq \prod_{p > 2k^2} \left(1 - \frac{(k-1)^4}{p^2}\right) \\
&= \exp \left(\sum_{p > 2k^2} \log \left(1 - \frac{(k-1)^4}{p^2}\right) \right).
\end{aligned}$$

Since $\log(1-x) \geq -2x$ for $0 \leq x \leq 0.8$, and $(k-1)^4/p^2 \leq 0.8$ for $p > 2k^2$, this is greater than or equal to

$$\exp \left(-2 \sum_{p > 2k^2} \frac{(k-1)^4}{p^2} \right) \geq \exp \left(-O \left(\frac{k^4}{k^2 \log k} \right) \right) = e^{o(k^2)}.$$

Thus, we find that

$$a_k \geq \left(\frac{e^{-\gamma}}{2 \log k} \right)^{k^2} e^{o(k^2)}.$$

Since the upper and lower bounds are the same, the proposition follows. \square

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