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# On the trace of Hecke operators for Maass forms for congruence subgroups

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**Abstract.** Let $E_\lambda$ be a Hilbert space, whose elements are functions spanned by the eigenfunctions of the Laplace-Beltrami operator associated with an eigenvalue $\lambda > 0$. The norm of elements in this space is given by the Petersson inner product. In this paper, the trace of Hecke operators $T_n$ acting on the space $E_\lambda$ is computed for congruence subgroups $\Gamma_0(N)$ of square free level, which may be considered as the analogue of the Eichler-Selberg trace formula [11] for non-holomorphic cusp forms of weight zero.

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## 1 Introduction

Let $N$ be a positive integer greater than one. Denote by $\Gamma_0(N)$ the Hecke congruence subgroup of level $N$. The Laplace-Beltrami operator $\Delta$ on the upper half-plane $\mathscr{H}$ is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Let $D$ be the fundamental domain of $\Gamma_0(N)$. Eigenfunctions of the discrete spectrum of $\Delta$ are nonzero real-analytic solutions of the equation

$$\Delta\psi = \lambda\psi$$

such that $\psi(\gamma z) = \psi(z)$ for all $\gamma$ in $\Gamma_0(N)$ and such that

$$\int_D |\psi(z)|^2\, dz < \infty$$

where $dz$ represents the Poincaré measure of the upper half-plane.

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Let $\mathfrak{a}$ be a cusp of $\Gamma_0(N)$. Its stabilizer is denoted by $\Gamma_{\mathfrak{a}}$. An element $\sigma_{\mathfrak{a}} \in \mathrm{PSL}(2, \mathbb{R})$ exists such that $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. Let $f$ be a $\Gamma_0(N)$-invariant function. If $\mathfrak{a}$ is a cusp of $\Gamma_0(N)$, then $f(\sigma_{\mathfrak{a}}z)$ is $\Gamma_\infty$-invariant, and hence it admits a formal Fourier expansion

$$f(\sigma_{\mathfrak{a}}z) = \sum_{n \in \mathbb{Z}} c_{\mathfrak{a}n}(y)e^{2n\pi ix}.$$

A $\Gamma_0(N)$-invariant function is said to be a Maass cusp form if it is square-integrable and is an eigenvector of $\Delta$, such that the Fourier coefficient $c_{\mathfrak{a}0}(y) = 0$ for every cusp $\mathfrak{a}$ of $\Gamma_0(N)$. If $\psi$ is a cusp form associated with a positive discrete eigenvalue $\lambda$, then it has the Fourier expansion [5]

$$\psi(\sigma_{\mathfrak{a}}z) = \sqrt{y}\sum_{m \neq 0} \rho_{\mathfrak{a}}(m)K_{i\kappa}(2\pi|m|y)e^{2m\pi ix},$$

where $\kappa = \sqrt{\lambda - 1/4}$ and $K_\nu(y)$ is given by the formula §6.32, [18]

$$\begin{aligned} K_\nu(y) &= \frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{y^\nu\sqrt{\pi}}\int_0^\infty \frac{\cos(yt)}{(1+t^2)^{\nu+(1/2)}}\,dt \\ &= \frac{1}{2}\int_0^\infty \exp\left(-\frac{y}{2}\left(t+\frac{1}{t}\right)\right)\frac{dt}{t^{\nu+1}}. \end{aligned} \tag{1.1}$$

The complex numbers $\rho_{\mathfrak{a}}(m), m(\neq 0) \in \mathbb{Z}$, are called the Fourier coefficients of $\psi$ around the cusp $\mathfrak{a}$.

The Hecke operators $T_n, n = 1, 2, \ldots, (n, N) = 1$, which act in the space of automorphic functions with respect to $\Gamma_0(N)$, are defined by

$$(T_nf)(z) = \frac{1}{\sqrt{n}}\sum_{ad=n,\, 0\le b<d} f\left(\frac{az+b}{d}\right).$$

An orthonormal system of eigenfunctions of $\Delta$ exists [5] such that each of them is an eigenfunction of all the Hecke operators. We call these eigenfunctions Maass-Hecke eigenfunctions. Let $\lambda_j, j = 1, 2, \ldots$, be an enumeration in increasing order of all positive discrete eigenvalues of $\Delta$ for $\Gamma_0(N)$ with an eigenvalue of multiplicity $m$ appearing $m$ times, and let $\kappa_j = \sqrt{\lambda_j - 1/4}$. If $\psi_j(z)$ is a Maass-Hecke eigenfunction of $\Delta$ associated with the $j$th eigenvalue $\lambda_j$, then

$$(T_n\psi_j)(z) = \tau_j(n)\psi_j(z)$$

where $\rho_{j\infty}(m) = \rho_{j\infty}(d)\tau_j(n)$ if $m = dn$ with $n \geqslant 1$, $(n, dN) = 1$.

Let $E_\lambda$ be a Hilbert space of functions spanned by the eigenfunctions of $\Delta$ with a positive eigenvalue $\lambda$. The inner product of the space is given by

$$(1.2)\qquad \langle F(z), G(z)\rangle = \int_D F(z)\bar{G}(z)\,dz.$$

The analogue for Maass forms of Eichler-Selberg's trace formula [11], p. 85 for modular forms is obtained for the full modular group in [8]. In this paper, the trace $\operatorname{tr} T_n$ of Hecke operators acting on the space $E_\lambda$ is computed for congruence subgroups $\Gamma_0(N)$. Some of this computation is implicit in Hejhal [3].

Denote by $h_d$ the class number of indefinite rational quadratic forms with discriminant $d$. Define

$$(1.3)\qquad \varepsilon_d = \frac{v_0 + u_0\sqrt{d}}{2}$$

where the pair $(v_0, u_0)$ is the fundamental solution [9] of Pell's equation $v^2 - du^2 = 4$. Denote by $\Omega$ the set of all the positive integers $d$ such that $d \equiv 0$ or $1 \pmod 4$ and such that $d$ is not a square of an integer.

**Theorem 1.** *Let $N$ be a square free positive integer, and let $n$ be a positive integer with $(n, N) = 1$. Put*

$$L_n(s) = \sum_{m|N}\sum_{k|N} k^{1-2s}\frac{\mu((m,k))}{(m,k)}\sum_{d\in\Omega}\sum_{u}\left(\frac{d}{m}\right)\frac{h_d \ln \varepsilon_d}{(du^2)^s}$$

*for $\Re s > 1$, where the summation on $u$ is taken over all the positive integers $u$ such that $\sqrt{4n + dk^2u^2} \in \mathbb{Z}$. Then $L_n(s)$ is analytic for $\Re s > 1$ and can be extended by analytic continuation to the half-plane $\Re s > 0$ except for a possible pole at $s = 1/2$ and for possible simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j$, $j = 1, 2, \ldots$. For any eigenvalue $\lambda > 0$ of the Laplace-Beltrami operator for $\Gamma_0(N)$, we have*

$$\operatorname{tr} T_n = 2n^{i\kappa}\operatorname{Res}_{s=1/2+i\kappa} L_n(s),$$

*where $\kappa = \sqrt{\lambda - 1/4}$.*

The paper is organized as follows. In section 2, we recall the theory of Selberg's trace formula [11]. Elements of the set $\Gamma^*$, which is defined in section 2, can be divided into four types, the identity, hyperbolic, elliptic and parabolic elements. Next, in section 3 we compute contributions of the identity, elliptic, hyperbolic and parabolic elements to the trace formula. A technical part of this section (Lemma 3.3—Theorem 3.9) is to compute the contribution from hyperbolic elements whose fixed points are cusps of $\Gamma_0(N)$, and the result is given in Theorem 3.9. By using the Selberg trace formula and by considering the contributions to the trace formula of the identity, elliptic, parabolic elements and hyperbolic elements whose fixed points are cusps, we obtain the analyticity information about a series formed by contributions to the trace formula of hyperbolic elements whose fixed points are not cusps, and a precise statement is given in Theorem 3.12. In section 4, we compute explicitly the total contribution to the trace formula of hyperbolic elements whose fixed points are not cusps. The result is stated in Theorem 4.6. A technical part of this section is to relate the number of certain indefinite primitive quadratic forms, which are not equivalent under $\Gamma_0(N)$, to the class number of indefinite primitive quadratic forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, and the relation is given in Lemma 4.5. Finally, the main theorem of this paper follows from Theorem 3.12 and Theorem 4.6.

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## 2 The Selberg trace formula

Let $s$ be a complex number with $\Re s > 1$. Define

$$k(t) = \left(1 + \frac{t}{4}\right)^{-s}$$

and

$$k(z, z') = k\left(\frac{|z - z'|^2}{yy'}\right),$$

for $z = x + iy$ and $z' = x' + iy'$ in the upper half-plane. Then $k(mz, mz') = k(z, z')$ for every $2 \times 2$ matrix $m$ of determinant one with real entries. The kernel $k(z, z')$ is of (a)–(b) type in the sense of Selberg [11], p. 60. Let

$$g(u) = \int_w^\infty k(t) \frac{dt}{\sqrt{t - w}}$$

with $w = e^u + e^{-u} - 2$. Write

$$h(r) = \int_{-\infty}^\infty g(u) e^{iru}\, du.$$

Then

$$\text{(2.1)} \qquad g(u) = \sqrt{w} \int_0^1 \left(t + \frac{w}{4}\right)^{-s} t^{s-(3/2)} \frac{dt}{\sqrt{1-t}} = c\left(1 + \frac{w}{4}\right)^{(1/2)-s}$$

where $c = 2\sqrt{\pi}\Gamma\left(s - \frac{1}{2}\right)\Gamma^{-1}(s)$. Since

$$h(r) = c4^{s-(1/2)} \int_0^\infty \left(u + \frac{1}{u} + 2\right)^{(1/2)-s} u^{ir-1}\, du$$

$$= c4^{s-(1/2)} \int_1^\infty \left(u + \frac{1}{u} + 2\right)^{(1/2)-s} (u^{ir} + u^{-ir}) \frac{du}{u}$$

$$= c \frac{4^s(s-\frac{1}{2})}{(s-\frac{1}{2})^2 + r^2} + A(r,s),$$

where $A(r,s)$ is finite for $|\Im r| \leqslant 1/2$ and for $\Re s > 0$, we obtain that

$$\text{(2.2)} \qquad \lim_{s \to 1/2 + i\kappa} \left(s - \frac{1}{2} - i\kappa\right) h(r) = \begin{cases} 4^{1/2+i\kappa}\sqrt{\pi}\dfrac{\Gamma(i\kappa)}{\Gamma(1/2+i\kappa)}, & \text{for } r = \pm\kappa; \\ 0, & \text{for } r \neq \pm\kappa. \end{cases}$$

**Remark.** Atle Selberg told the second author that he used the function $k(t) = \left(1 + \frac{t}{4}\right)^{-s}$ in some of his unpublished works, which was convenient for computations (cf. Selberg [13]).

Let $n$ be a positive integer with $(n, N) = 1$. Define

$$\Gamma^* = \bigcup_{\substack{ad=n \\ 0 \leq b < d}} \frac{1}{\sqrt{n}} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \Gamma_0(N).$$

Since $(n, N) = 1$, we have that $T^{-1} \in \Gamma^*$ whenever $T \in \Gamma^*$. Every element of $\Gamma^*$ is represented uniquely in the form

$$\frac{1}{\sqrt{n}} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \gamma$$

with $ad = n$, $0 \leq b < d$ and $\gamma \in \Gamma_0(N)$. It follows that $\Gamma^*$ satisfies all the requirements given in [11], p. 69. Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{\nu(N)}$ with $\nu(N) = \sum_{w|N,\, w>0} \varphi\left(\left(w, \frac{N}{w}\right)\right)$ be a complete set of inequivalent cusps of $\Gamma_0(N)$, where $\varphi$ is Euler's function. We choose an element $\sigma_{\mathfrak{a}_i} \in \mathrm{PSL}(2, \mathbb{R})$ such that $\sigma_{\mathfrak{a}_i}\infty = \mathfrak{a}_i$ and $\sigma_{\mathfrak{a}_i}^{-1}\Gamma_{\mathfrak{a}_i}\sigma_{\mathfrak{a}_i} = \Gamma_\infty$ for $i = 1, 2, \ldots, \nu(N)$. The Eisenstein series $E_i(z, s)$ for the cusp $\mathfrak{a}_i$ is defined by

$$E_i(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}_i} \backslash \Gamma_0(N)} (\Im(\sigma_{\mathfrak{a}_i}^{-1}\gamma z))^s$$

for $\Re s > 1$ when $z$ is in the upper half-plane. Define

$$K(z,z') = \sum_{T\in\Gamma^*} k(z,Tz')$$

and

$$H(z,z') = \sum_{i=1}^{\nu(N)} \sum_{ad=n,0\le b<d} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)E_i\left(\frac{az+b}{d},\frac{1}{2}+ir\right)E_i\left(z',\frac{1}{2}-ir\right)dr.$$

Denote by $\mathrm{tr}_j T_n$ the trace of the Hecke operator $T_n$ acting on the space $E_{\lambda_j}$. It follows from (2.14) of [11], the argument of [7], pp. 96–98, Theorem 5.3.3 of [7], and the spectral decomposition formula (5.3.12) of [7] that

$$d(n)h\left(-\frac{i}{2}\right) + \sqrt{n}\sum_{j=1}^{\infty} h(\kappa_j)\,\mathrm{tr}_j\,T_n = \int_D \{K(z,z)-H(z,z)\}\,dz \tag{2.3}$$

for $\Re s > 1$, where $d(n)$ is the number of positive divisors of $n$.

## 3 Evaluation of components of the trace

For every element $T$ of $\Gamma^*$, denote by $\Gamma_T$ the set of all the elements of $\Gamma_0(N)$ which commute with $T$. Put $D_T = \Gamma_T\backslash\mathscr{H}$. The elements of $\Gamma^*$ can be divided into four types, of which the first consists of the identity element, while the others are respectively the hyperbolic, the elliptic and the parabolic elements. If $T$ is not a parabolic element, put

$$c(T) = \int_{D_T} k(z,Tz)\,dz.$$

### 3.1 The identity component

If $\Gamma^*$ contains the identity element $I$, then

$$c(I) = \int_{\Gamma_0(N)\backslash\mathscr{H}} dz.$$

### 3.2 Elliptic components

There are only a finite number of elliptic conjugacy classes.

**Lemma 3.1.** *Let $R$ be an elliptic element of $\Gamma^*$. Then*

$$c(R) = \frac{\pi}{2m\sin\theta}\int_0^{\infty}\frac{k(t)}{\sqrt{t+4\sin^2\theta}}\,dt,$$

*where m is the order of a primitive element of* $\Gamma_R$ *and where* $\theta$ *is defined by the formula* $\mathrm{trace}(R) = 2\cos\theta$.

*Proof.* Since $R$ is an elliptic element of $\Gamma^*$, an element $\sigma \in \mathrm{PSL}(2,\mathbb{R})$ exists such that

$$\sigma R \sigma^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \tilde{R}$$

for some real number $0 < \theta < \pi$. Denote by $(\sigma\Gamma_0(N)\sigma^{-1})_{\tilde{R}}$ the set of all the elements of $\sigma\Gamma_0(N)\sigma^{-1}$ which commute with $\tilde{R}$. We have

$$c(R) = \int_{D_{\tilde{R}}} k(z, \tilde{R}z)\, dz$$

where $D_{\tilde{R}} = (\sigma\Gamma_0(N)\sigma^{-1})_{\tilde{R}} \backslash \mathscr{H}$.

Let $\gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be an element of $\Gamma_0(N)$ which has the same fixed points as $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $(\alpha - \delta)c = \gamma(a-d)$ and $\beta c = \gamma b$. It follows that $\gamma$ commutes with $R$. By Proposition 1.16 of [14], a primitive elliptic element $\gamma_0$ of $\Gamma_0(N)$ exists such that $(\eta\Gamma\eta^{-1})_{\tilde{R}}$ is generated by $\eta\gamma_0\eta^{-1}$. Since $\eta\gamma_0\eta^{-1}$ commutes with $\tilde{R}$, it is of the form

$$\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

for some real number $\theta_0$. By Proposition 1.16 of [14], $\theta_0 = \pi/m$ for some positive integer $m$. It follows from the argument of [7], p. 99 that

$$c(R) = \frac{1}{m}\int_0^\infty \int_{-\infty}^\infty k\left(\frac{|z^2+1|^2}{y^2}\sin^2\theta\right) dz.$$

By the argument of [7], p. 100 we have

$$c(R) = \frac{\pi}{2m\sin\theta}\int_0^\infty \frac{k(t)}{\sqrt{t + 4\sin^2\theta}}\, dt. \quad \square$$

### 3.3 Hyperbolic components

Let $P$ be a hyperbolic element of $\Gamma^*$. Then an element $\rho$ exists in $\mathrm{SL}_2(\mathbb{R})$ such that

$$\rho P \rho^{-1} = \begin{pmatrix} \lambda_P & 0 \\ 0 & \lambda_P^{-1} \end{pmatrix} = \tilde{P}$$

with $\lambda_P > 1$. The number $\lambda_P^2$ is called the norm of $P$, and is denoted by $NP$. It follows that

$$c(P) = \int_{D_{\tilde{P}}} k(z, NPz)\, dz$$

where $D_{\tilde{P}} = (\rho\Gamma_0(N)\rho^{-1})_{\tilde{P}} \backslash \mathscr{H}$. Let $P_0$ be a primitive hyperbolic element of $\mathrm{SL}_2(\mathbb{Z})$, which generates the group of all elements of $\mathrm{SL}_2(\mathbb{Z})$ commutating with $P$. Then there exists a hyperbolic element $P_1 \in \Gamma_0(N)$, which generates $\Gamma_P$, such that $P_1$ is the smallest positive integer power of $P_0$ among all the generators of $\Gamma_P$ in $\Gamma_0(N)$. Detail discussion about the "primitive" hyperbolic element $P_1$ is given in the proof of Theorem 4.6.

**Theorem 3.2.** *Let $P$ be a hyperbolic element of $\Gamma^*$ such that $\Gamma_P \neq \{1_2\}$. If $P_1$ is a "primitive" hyperbolic element of $\Gamma_0(N)$ which generates the group $\Gamma_P$, then*

$$c(P) = \frac{\ln NP_1}{(NP)^{1/2} - (NP)^{-1/2}} g(\ln NP).$$

*Proof.* An argument similar to that made for the elliptic elements shows that every element of $\Gamma_0(N)$, which has the same fixed points as $P$, commutes with $P$. Because $\rho P_1 \rho^{-1}$ commutes with $\tilde{P}$, it is of the form

$$\begin{pmatrix} \lambda_{P_1} & 0 \\ 0 & \lambda_{P_1}^{-1} \end{pmatrix}$$

for some real number $\lambda_{P_1} > 1$. Then

$$c(P) = \int_1^{NP_1} \frac{dy}{y^2} \int_{-\infty}^{\infty} k\left( \frac{(NP-1)^2}{NP} \frac{|z|^2}{y^2} \right) dx.$$

The stated identity follows. $\square$

We next consider $c(P)$ for hyperbolic element $P \in \Gamma^*$ with $\Gamma_P = \{1_2\}$.

**Lemma 3.3.** *Let $P = \frac{1}{\sqrt{n}} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a hyperbolic element of $\Gamma^*$ such that $\Gamma_P = \{1_2\}$. Then fixed points of $P$ are cusps of $\Gamma_0(N)$. Moreover, $\Gamma_P = \{1_2\}$ if and only if $A + D = \frac{1}{2}\left( m + \frac{4n}{m} \right)$ for some divisor $m$ of $4n$ with $m \neq 2\sqrt{n}$ for $C \neq 0$, and if and only if $A \neq D$ for $C = 0$.*

*Proof.* Let $P$ a hyperbolic element of $\Gamma^*$ such that $\Gamma_P = \{1_2\}$. If the fixed points of $P$ are not rational numbers or infinity, then they are zeros of an irreducible polynomial

$ax^2 + bx + c$ with $a \equiv 0 \bmod (N)$. If $d = b^2 - 4ac$ and $(u, v)$ is a solution of Pell's equation $v^2 - du^2 = 4$, then

$$\begin{pmatrix} \dfrac{v - bu}{2} & -cu \\ au & \dfrac{v + bu}{2} \end{pmatrix}$$

belongs to $\Gamma_0(N)$ and has the same fixed points as $P$, and hence it commutes with $P$. This contradicts to $\Gamma_P = \{1_2\}$. Therefore, fixed points of $P$ are rational numbers or infinity. If $u/w$ is a fixed point of $P$, then

$$\begin{pmatrix} 1 + Nuw & -Nu^2 \\ Nw^2 & 1 - Nuw \end{pmatrix}$$

is a parabolic element of $\Gamma_0(N)$ and has $u/w$ as a fixed point. Hence, fixed points of $P$ are cusps of $\Gamma_0(N)$.

Conversely, if $T$ is an element of $\Gamma^*$ having two distinct fixed points with at least one of them being a rational number, then $\Gamma_T = \{1_2\}$. Otherwise, let $r_1, r_2$ be the fixed points of $T$, and let $\gamma \in \Gamma_0(N)$ $(\gamma \neq 1_2)$ with $\gamma T = T\gamma$. Then either $\gamma$ or $\gamma^2$ $(\neq 1_2)$, say $\gamma^2$, has $r_1, r_2$ as its fixed points, and hence it is a hyperbolic element of $\Gamma_0(N)$. Since rational points are cusps of $\Gamma_0(N)$, by Proposition 1.17 of [14] $\gamma^2$ is parabolic. This is a contradiction, and hence we have $\Gamma_T = \{1_2\}$. Thus, we have proved that $\Gamma_T = \{1_2\}$ for an element $T = \dfrac{1}{\sqrt{n}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^*$ if and only if $a + d = \dfrac{1}{2}\left(m + \dfrac{4n}{m}\right)$ for some divisor $m$ of $4n$ with $m \neq 2\sqrt{n}$ if $c \neq 0$ or $a \neq d$ if $c = 0$. $\square$

Every cusp of $\Gamma_0(N)$ is equivalent to one of the following inequivalent cusps

$$\text{(3.1)} \qquad \frac{u}{w} \text{ with } u, w > 0, \quad (u, w) = 1, \quad w | N.$$

Two such cusps $u/w$ and $u_1/w_1$ are $\Gamma_0(N)$-equivalent if and only if $w = w_1$ and $u \equiv u_1$ modulo $(w, N/w)$. Let $\mathfrak{a} = u/w$ be given as in (3.1). By (2.2) and (2.3) of Deshouillers and Iwaniec [1], we have

$$\text{(3.2)} \qquad \Gamma_{\mathfrak{a}} = \left\{ \begin{pmatrix} 1 + cu/w & -cu^2/w^2 \\ c & 1 - cu/w \end{pmatrix} : c \equiv 0 \ (\mathrm{mod}[w^2, N]) \right\}$$

and

$$\sigma_{\mathfrak{a}} \infty = \mathfrak{a} \quad \text{and} \quad \sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = \Gamma_\infty$$

where

$$\sigma_{\mathfrak{a}} = \begin{pmatrix} \mathfrak{a}\sqrt{[w^2, N]} & 0 \\ \sqrt{[w^2, N]} & 1/\mathfrak{a}\sqrt{[w^2, N]} \end{pmatrix}. \tag{3.3}$$

Let $P$ be a hyperbolic element of $\Gamma^*$ such that $\Gamma_P = \{1_2\}$. Assume that $\mathfrak{a}$ is a fixed point of $P$. Then $\infty$ is a fixed point of $\sigma_{\mathfrak{a}}^{-1} P \sigma_{\mathfrak{a}}$, and hence there exist positive numbers $a, d$ with $ad = n, a \neq d$ such that $P$ is of the form

$$P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \sigma_{\mathfrak{a}}^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} a - b[w^2, N]\dfrac{u}{w} & b[w^2, N]\dfrac{u^2}{w^2} \\ (a-d)\dfrac{w}{u} - b[w^2, N] & d + b[w^2, N]\dfrac{u}{w} \end{pmatrix}. \tag{3.4}$$

We claim that $a, d$ are integers. Since $P = \frac{1}{\sqrt{n}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^*$, we have $A, B, C, D \in \mathbb{Z}$, $AD - BC = n$, and $C \equiv 0 \bmod(N)$. By (3.4), we find that

$$a = D + C\frac{u}{w}, \quad d = D - B\frac{w}{u}, \quad b = B\frac{w^2}{u^2[w^2, N]} \tag{3.5}$$

and

$$A - D = C\frac{u}{w} - B\frac{w}{u}. \tag{3.6}$$

Since $(w, u) = 1$, we have $u|B$ by (3.6). It follows from (3.5) that $a, d$ are integers. Since $c(P)$ depends only on the conjugacy class $\{P\}$ represented by $P$, we can replace $P$ by $\gamma^{-1}P\gamma$ without changing the value of $c(P)$. Replacing $P$ by $\gamma^{-1}P\gamma$ for some element $\gamma \in \Gamma_{\mathfrak{a}}$, we can assume without loss of generality that $0 \leqslant b < |a - d|$ in (3.4).

**Lemma 3.4.** *Let* $\mathfrak{a} = u/w$ *be given as in* (3.1). *Then*

$$P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \sigma_{\mathfrak{a}}^{-1}$$

*with* $a, d \in \mathbb{Z}$, $ad = n$, $a \neq d$ *is a hyperbolic element of* $\Gamma^*$ *with* $\Gamma_P = \{1_2\}$ *and* $P(\mathfrak{a}) = \mathfrak{a}$ *if and only if* $(w, N/w)|(a - d)$ *with* $b$ *being chosen so that* $(a-d)\frac{w}{u} - b[w^2, N]$ *is divisible by* $N$.

*Proof.* Let $P$ be a hyperbolic element of $\Gamma^*$ with $\Gamma_P = \{1_2\}$ and $P(\mathfrak{a}) = \mathfrak{a}$. By (3.4), $(a-d)\frac{w}{u} - b[w^2, N]$ is divisible by $N$, and there exists an integer $k$ such that

$\frac{u}{w}N|a-d-kw$. This implies that $(w, N/w)|a-d$, which can also be seen directly from the identities $P(\mathfrak{a}) = \mathfrak{a}$ and $a+d = \frac{1}{2}\left(m+\frac{4n}{m}\right)$ for some divisor $m$ of $4n$.

Conversely, let $a, d \in \mathbb{Z}^+$, $ad = n$, $a \neq d$ and $(w, N/w)|a-d$. Write $a-d = l(w, N/w)$ for some integer $l$. Since $uN/w(w, N/w)$ and $w/(w, N/w)$ are coprime, there exist integers $\lambda$ and $\tau$ such that $\lambda\mathfrak{a}N + \tau w = (w, N/w)$, and hence $l\lambda\mathfrak{a}N = a - d - l\tau w$. This means that there exists an integer $k$ such that $\frac{u}{w}N|a-d-kw$. If we choose $A = a-kw$, $B = ku$, $C = \frac{w}{u}(a-d-kw)$ and $D = d+kw$, then $AD - BC = n$ and $C \equiv 0 \bmod(N)$. Let $P = \frac{1}{\sqrt{n}}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then $P$ can be expressed as in (3.4), $P(\mathfrak{a}) = \mathfrak{a}$, and $\Gamma_P = \{1_2\}$. Let $\gamma = C/(C, D)$ and $\delta = D/(C, D)$. There exist integers $\alpha, \beta$ such that $\alpha\delta - \beta\gamma = 1$. Since $(n, N) = 1$, we have $\gamma \equiv 0 \bmod(N)$. It follows that

$$P = \frac{1}{\sqrt{n}}\begin{pmatrix} \frac{n}{(C, D)} & B\alpha - A\beta \\ 0 & (C, D) \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

This implies that $P \in \Gamma^*$. Thus, we have proved that

$$P = \frac{1}{\sqrt{n}}\sigma_{\mathfrak{a}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}$$

with $a, d \in \mathbb{Z}$, $ad = n$, $a \neq d$ is an element of $\Gamma^*$ with $\Gamma_P = \{1_2\}$ and $P(u/w) = u/w$ if and only if there exists an integer $k$ such that $\frac{u}{w}N|a-d-kw$, that is, if and only if $(w, N/w)|(a-d)$ with $b$ being chosen so that $(a-d)\frac{w}{u} - b[w^2, N]$ is divisible by $N$. $\square$

**Lemma 3.5.** *Let* $\mathfrak{a} = u/w$ *be given as in* (3.1). *Let* $a, d \in \mathbb{Z}^+$, $ad = n, a \neq d$. *If* $(w, N/w)|a-d$, *then there are exactly* $|a-d|$ *number of* $\Gamma_0(N)$*-inequivalent hyperbolic elements* $P \in \Gamma^*$ *with* $P(\mathfrak{a}) = \mathfrak{a}$ *and* $\Gamma_P = \{1_2\}$, *which are of the form*

$$P = \frac{1}{\sqrt{n}}\sigma_{\mathfrak{a}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}$$

*with* $0 \leqslant b < |a-d|$.

*Proof.* For $a, d \in \mathbb{Z}$, $ad = n$, $a \neq d$, let

$$P = \frac{1}{\sqrt{n}}\sigma_{\mathfrak{a}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\sigma_{\mathfrak{a}}^{-1} \quad \text{and} \quad P' = \frac{1}{\sqrt{n}}\sigma_{\mathfrak{a}}\begin{pmatrix} a & b' \\ 0 & d \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}$$

be two elements of $\Gamma^*$ with $0 \leqslant b, b' < |a-d|$ such that $\Gamma_P = \{1_2\}$, $P(\mathfrak{a}) = \mathfrak{a}$ and $\Gamma_{P'} = \{1_2\}$, $P'(\mathfrak{a}) = \mathfrak{a}$. We claim that $b' = b + l$ for some integer $l$. In fact, for $P$ there exists an integer $k$ such that $\frac{u}{w}N | a - d - kw$, and for $P'$ there exists an integer $k'$ such that $\frac{u}{w}N | a - d - k'w$. Then we have $\frac{u}{w}N | (k'-k)w$, that is,

$$k' = k + \frac{luN}{w(w, N/w)}$$

for some integer $l$. By using (3.4), we find that

$$b' = b + \frac{lwN}{[w^2, N](w, N/w)} = b + l.$$

The claim then follows. Conversely, given an integer $l$ with $0 \leqslant b + l < |a-d|$, let $b' = b + l$ and

$$k' = k + \frac{luN}{w(w, N/w)}.$$

Put $A' = a - k'w$, $B' = k'u$, $C' = \frac{w}{u}(a - d - k'w)$ and $D' = d + k'w$. Then

$$P' = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}} \begin{pmatrix} a & b' \\ 0 & d \end{pmatrix} \sigma_{\mathfrak{a}}^{-1}$$

is a hyperbolic element of $\Gamma^*$, $\Gamma_{P'} = \{1_2\}$, and $P'(\mathfrak{a}) = \mathfrak{a}$. Moreover, if $b' \neq b$ modulo $|a-d|$, then $P'$ and $P$ are not $\Gamma_0(N)$-equivalent. Otherwise, an element $\gamma \in \Gamma_0(N)$ exists such that $P'\gamma = \gamma P$. Put $\gamma = \begin{pmatrix} * & * \\ C & * \end{pmatrix}$ and $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} = \begin{pmatrix} \alpha & \beta \\ \eta & \delta \end{pmatrix}$. By using the relation $P'\gamma = \gamma P$, we obtain that $(a-d)\beta = \alpha b - \delta b'$ and $\eta = 0$. Since $\gamma \in \Gamma_0(N)$, by using $\eta = 0$ and the relation $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ we find that $\alpha$ and $\delta$ are integers, and hence $\alpha = \delta = 1$. Furthermore, we obtain that $\beta = -C/[w^2, N]$. Since $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$, we have $(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}})(\infty) = \infty$, that is, $\gamma(\mathfrak{a}) = \mathfrak{a}$. Hence $\gamma \in \Gamma_{\mathfrak{a}}$, which is given by (3.2). This implies that $\beta$ is an integer. It follows that $(a-d) | (b - b')$. This is a contradiction. Therefore, for fixed $a, d \in \mathbb{Z}^+$, $ad = n$, $a \neq d$, there are exactly $|a-d|$ $\Gamma_0(N)$-inequivalent hyperbolic elements $P$ in $\Gamma^*$ with $P(u/w) = u/w$ and $\Gamma_P = \{1_2\}$, which are of the form

$$P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \sigma_{\mathfrak{a}}^{-1}$$

with $0 \leqslant b < |a-d|$. $\square$

**Lemma 3.6.** *Let* $\mathfrak{a} = u/w$, *and let*

$$P = \frac{1}{\sqrt{n}}\sigma_{\mathfrak{a}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\sigma_{\mathfrak{a}}^{-1} \quad \textit{and} \quad P' = \frac{1}{\sqrt{n}}\sigma_{\mathfrak{a}}\begin{pmatrix} d & b' \\ 0 & a \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}$$

*be two hyperbolic elements of* $\Gamma^*$ *with* $P(\mathfrak{a}) = \mathfrak{a}$, $P'(\mathfrak{a}) = \mathfrak{a}$, $\Gamma_P = \{1_2\}$ *and* $\Gamma_{P'} = \{1_2\}$. *Then* $P$ *is* $\Gamma_0(N)$*-conjugate to* $P'$ *for some number* $b'$ *if and only if the two fixed points of* $P$ *are* $\Gamma_0(N)$*-conjugate. That is,* $P$ *is* $\Gamma_0(N)$*-conjugate to* $P'$ *for some number* $b'$ *if and only if*

$$\left(\frac{(d-a)\frac{w}{u} + b[w^2, N]}{l}, N\right) = w,$$

*where* $l$ *is the greatest common divisor of* $b[w^2, N]\frac{u}{w}$ *and* $(d-a)\frac{w}{u} + b[w^2, N]$.

*Proof.* We first assume that the two fixed points of $P$ are $\Gamma_0(N)$-conjugate. That is, there exists an element $T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$, which is not the identity, such that $\mathfrak{a}, T(\mathfrak{a})$ are the two fixed points of $P$. Let $P' = T^{-1}PT$. Then $P'(\mathfrak{a}) = \mathfrak{a}$. Since $\Gamma_P = \{1_2\}$, we have $\Gamma_{P'} = \{1_2\}$. We claim that $P'$ is of the form given in the statement of the lemma.

We can write $P'$ in the form

$$P' = \frac{1}{\sqrt{n}}\sigma_{\mathfrak{a}}\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}$$

with $a', d' \in \mathbb{Z}^+$ and $a'd' = n$. Replacing $T$ by $T\gamma$ for some element $\gamma \in \Gamma_{\mathfrak{a}}$, we assume that $0 \leqslant b' < |a' - d'|$. Since $P' = T^{-1}PT$, we have

$$(\sigma_{\mathfrak{a}}^{-1}T\sigma_{\mathfrak{a}})\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(\sigma_{\mathfrak{a}}^{-1}T\sigma_{\mathfrak{a}}).$$

In general, from the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $AD - BC \neq 0$, we obtain that $(d - a')C = 0, (d - d')D = b'C$ and $(a' - a)A = bC$. If $C = 0$, we have $a' = a$ and $d' = d$, and hence $P'$ is of the form

$$P' = \frac{1}{\sqrt{n}}\sigma_{\mathfrak{a}}\begin{pmatrix} a & b' \\ 0 & d \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}.$$

Since $P'$ is $\Gamma_0(N)$-conjugate to $P$ and $0 \leqslant b' < |a-d|$, we must have $P' = P$ by the proof of Lemma 3.5. That is, $T \in \Gamma_P$, and hence $T = 1_2$. A contradiction is then derived. Therefore, we must have $C \neq 0$. It follows that $a' = d$. Since $ad = n$ and $a'd' = n$, we have $d' = a$, and our claim follows.

Conversely, suppose that there exists an element $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(N)$ such that $T^{-1}PT = P'$, that is,

$$\sigma_{\mathfrak{a}}^{-1}T^{-1}\sigma_{\mathfrak{a}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} d & b' \\ 0 & a \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}T^{-1}\sigma_{\mathfrak{a}}. \tag{3.7}$$

If we write

$$\sigma_{\mathfrak{a}}^{-1}T^{-1}\sigma_{\mathfrak{a}} = \begin{pmatrix} D - B\dfrac{w}{u} & \dfrac{-Bw^2}{[w^2, N]u^2} \\ \dfrac{u}{w}[w^2, N]\Big(A + B\dfrac{w}{u} - C\dfrac{u}{w} - D\Big) & A + B\dfrac{w}{u} \end{pmatrix},$$

then (3.7) is equivalent to the system of equations

$$\begin{cases} (a-b)\Big(A + B\dfrac{w}{u}\Big) = b[w^2, N]\dfrac{u}{w}\Big(A + B\dfrac{w}{u} - C\dfrac{u}{w} - D\Big), \\ b\Big(D - B\dfrac{w}{u}\Big) = b'\Big(A + B\dfrac{w}{u}\Big). \end{cases}$$

Write this system of equations as

$$\begin{cases} ((a-d)\dfrac{w}{u} - b[w^2, N])u\Big(A + B\dfrac{w}{u}\Big) = b[w^2, N]\dfrac{u}{w}(-Cu - Dw), \\ b\Big(D - B\dfrac{w}{u}\Big) = b'\Big(A + B\dfrac{w}{u}\Big). \end{cases} \tag{3.8}$$

The first equation of (3.8) can be written as

$$T(\mathfrak{a}) = \frac{\mathfrak{a}}{1 - \dfrac{a-d}{b[w^2, N]\mathfrak{a}}},$$

which is the second fixed point of $P$. That is, the two fixed points of $P$ are $\Gamma_0(N)$-conjugate. Since $\mathfrak{a} = \dfrac{u}{w}$ and

$$\frac{b[w^2, N]u/w}{(d-a)\dfrac{w}{u} + b[w^2, N]}$$

are the two fixed points of $P$, the last statement of the lemma follows from Lemma 3.6 of Deshouillers and Iwaniec [1]. $\square$

**Lemma 3.7.** *Let* $a, d \in \mathbb{Z}^+$, $a \neq d$, $ad = n$, *and let* $\mathfrak{a} = u/w$ *be given as in* (3.1) *with* $(w, N/w) | (a - d)$. *Assume that* $\mathfrak{a}$ *and* $\mathfrak{a}' = u_1/w_1$ *are the two distinct fixed point of a hyperbolic element* $P \in \Gamma^*$. *Then* $(w', N/w') | (a - d)$ *for* $w' = (w_1, N)$. *In other words, if* $P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \sigma_{\mathfrak{a}}^{-1}$ *is a hyperbolic element of* $\Gamma^*$ *with* $P(\mathfrak{a}') = \mathfrak{a}'$, *then it can also be written in the form*

$$P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}'} \begin{pmatrix} d & b' \\ 0 & a \end{pmatrix} \sigma_{\mathfrak{a}'}^{-1}$$

*for some number* $b'$.

*Proof.* It follows from the argument made in the paragraph preceding Lemma 3.4 that there exist positive integers $a', d'$ with $a'd' = n$ such that

$$P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}'} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \sigma_{\mathfrak{a}'}^{-1}$$

for some number $b'$. Then we have

$$\sigma_{\mathfrak{a}}^{-1} \sigma_{\mathfrak{a}'} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \sigma_{\mathfrak{a}}^{-1} \sigma_{\mathfrak{a}'}. \tag{3.9}$$

Write $\sigma_{\mathfrak{a}}^{-1} \sigma_{\mathfrak{a}'} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. It follows from (3.9) that $(d - a')C = 0, (a' - a)A = bC$ and $(d - d')D = b'C$. Since $\mathfrak{a} \neq \mathfrak{a}'$, by using (3.3) we find that $C \neq 0$, and hence we have $a' = d$. It follows that $d' = a$, that is, $P$ can also be written in the form stated in the lemma. Then by the argument made in the first paragraph of the proof of Lemma 3.4, we have $(w', N/w') | (a - d)$. $\square$

For a large positive number $Y$, define

$$D_Y = \{ z \in D : \Im \sigma_{\mathfrak{a}_i}^{-1} z < Y, i = 1, 2, \ldots, h \}.$$

Let

$$(D_P)_Y = \bigcup_{\gamma \in \Gamma_0(N)} \gamma D_Y.$$

Write

$$c(P)_Y = \int_{(D_P)_Y} k(z, Pz)\, dz.$$

**Lemma 3.8.** *Let* $\mathfrak{a} = u/w$ *be given as in* (3.1), *and let* $P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \sigma_{\mathfrak{a}}^{-1}$ *with* $a, d \in \mathbb{Z}^+$, $ad = n$, $a \neq d$ *be a hyperbolic element of* $\Gamma^*$ *with* $\Gamma_P = \{1_2\}$ *and* $P(\mathfrak{a}) = \mathfrak{a}$. *Then*

$$c(P)_Y = \frac{\sqrt{n} \ln\{(a-d)^2 [w^2, N] Y/2\rho\}}{a-d} g\left(\ln \frac{a}{d}\right) + \int_1^\infty k\left(\frac{(a-d)^2}{n} t\right) \frac{\ln t}{\sqrt{t-1}} \, dt + o(1)$$

*where* $o(1) \to 0$ *as* $Y \to \infty$ *and* $\rho = C^2/2[C^2/l^2, N]Y$ *with* $C = (a-d)\frac{w}{u} - b[w^2, N]$ *and with* $l$ *being given as in Lemma 3.6.*

*Proof.* Let

$$\gamma = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

be an element of $\mathrm{SL}_2(\mathbb{R})$. Then linear fractional transformation, which takes every complex $z$ in the upper half-plane into $\gamma(z)$, maps the horizontal line $\Im z = Y$ into a circle of radius $\frac{1}{2q^2 Y}$ with center at $\frac{p}{q} + \frac{i}{2q^2 Y}$. Let

$$\mu = \begin{pmatrix} 1 & \frac{b}{a-d} \\ 0 & 1 \end{pmatrix}.$$

Then

$$P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}} \mu^{-1} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mu \sigma_{\mathfrak{a}}^{-1}.$$

Note that

$$\mu \sigma_{\mathfrak{a}}^{-1} = \begin{pmatrix} \frac{w/u}{\sqrt{[w^2, N]}} - \frac{b\sqrt{[w^2, N]}}{a-d} & \frac{bu\sqrt{[w^2, N]}}{(a-d)w} \\ -\sqrt{[w^2, N]} & \frac{u}{w}\sqrt{[w^2, N]} \end{pmatrix}.$$

Since

$$P = \frac{1}{\sqrt{n}} \sigma_{\mathfrak{a}} \mu^{-1} \begin{pmatrix} 1/y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix} \mu \sigma_{\mathfrak{a}}^{-1}$$

for any $y \neq 0$, by choosing $y = (a-d)\sqrt{[w^2, N]}$ we obtain that

$$c(P)_Y = \int_{\mu_{\mathfrak{a}}\{(D_P)_Y\}} k\left(z, \frac{a}{d} z\right) dz$$

where

$$\mu_{\mathfrak{a}} = \begin{pmatrix} (a-d)\dfrac{w}{u} - b[w^2, N] & b[w^2, N]\dfrac{u}{w} \\ \dfrac{-1}{a-d} & \dfrac{u}{(a-d)w} \end{pmatrix}.$$

The linear transformation $z \to \sigma_{\mathfrak{a}} z$ maps the half-plane $\Im z > Y$ into a disk $D_{\mathfrak{a}}$ of radius $\dfrac{1}{2[w^2, N]Y}$ with center at $\mathfrak{a} + \dfrac{i}{2[w^2, N]Y}$. Then the transformation $z \to \mu_{\mathfrak{a}}(z)$ maps the disk $D_{\mathfrak{a}}$ into the half-plane $\Im z > (a-d)^2[w^2, N]Y$. Let

$$\mathfrak{p} = \frac{-b[w^2, N]u/w}{(a-d)\dfrac{w}{u} - b[w^2, N]}.$$

If

$$\sigma_{\mathfrak{p}} = \begin{pmatrix} \mathfrak{p}\sqrt{[C^2/l^2, N]} & 0 \\ \sqrt{[C^2/l^2, N]} & 1/\mathfrak{p}\sqrt{[C^2/l^2, N]} \end{pmatrix},$$

then $\sigma_{\mathfrak{p}}\infty = \mathfrak{p}$ and $\sigma_{\mathfrak{p}}^{-1} \Gamma_{\mathfrak{p}} \sigma_{\mathfrak{p}} = \Gamma_\infty$. By the definition of $D_Y$, the image of the half-plane $\Im z > Y$ under the linear transformation $z \to \sigma_{\mathfrak{p}} z$ is not contained in $(D_P)_Y$. Since the linear transformation $z \to (\mu_{\mathfrak{a}} \sigma_{\mathfrak{p}})(z)$ maps the half-plane $\Im z > Y$ into a disk $D_\rho$ of radius $\rho$ centered at $i\rho$, where $\rho = C^2/2[C^2/l^2, N]Y$, the disk $D_\rho$ is not contained in $\mu_{\mathfrak{a}}\{(D_P)_Y\}$. It follows that

$$\begin{aligned} c(P)_Y &= \int_0^\pi d\theta \int_{2\rho \sin\theta}^{(a-d)^2[w^2, N]Y/\sin\theta} k\left(\frac{(a-d)^2}{n \sin^2\theta}\right) \frac{dr}{r \sin^2\theta} + o(1) \\ &= \int_1^\infty k\left(\frac{(a-d)^2}{n} t\right) \frac{\ln\{(a-d)^2[w^2, N]Yt/2\rho\}}{\sqrt{t-1}}\, dt + o(1) \end{aligned}$$

where $o(1)$ has a limit zero when $Y \to \infty$, and hence we have

$$c(P)_Y = \frac{\sqrt{n}\ln\{(a-d)^2[w^2, N]Y/2\rho\}}{a-d} g\left(\ln\frac{a}{d}\right) + \int_1^\infty k\left(\frac{(a-d)^2}{n}t\right)\frac{\ln t}{\sqrt{t-1}}\,dt + o(1).$$

This completes the proof of the lemma. $\square$

In the rest of the paper, we shall indicate explicitly when we assume that $N$ is square free.

**Theorem 3.9.** *Let $N$ be a square free positive integer. Then we have*

$$\sum_{\{P\},\, \Gamma_P=\{1_2\}} c(P)_Y$$

$$= \nu(N)\sqrt{n} \sum_{ad=n,\, d>0,\, a\neq d} g\left(\ln\frac{a}{d}\right)\ln Y$$

$$+ \frac{\sqrt{n}}{2}\sum_{w|N,\, w>0}\ \sum_{ad=n,\, d>0,\, a\neq d}\ \sum_b \frac{\ln\left\{(a-d)^2 wN\left[\frac{C^2}{l}, N\right]\Big/ C^2\right\}}{a-d} g\left(\ln\frac{a}{d}\right)$$

$$+ \sum_{ad=n,\, d>0,\, a\neq d} \frac{1}{2}\nu(N)|a-d| \int_1^\infty k\left(\frac{(a-d)^2}{n}t\right)\frac{\ln t}{\sqrt{t-1}}\,dt + o(1)$$

*with $C = (a-d)w - bwN$, where $l = (C, bN)$ and $\sum_b$ is over the $|a-d|$ numbers such that $C, bN/w \in \mathbb{Z}, N|C$ and $0 \leqslant b < |a-d|$.*

*Proof.* Let $P$ and $P'$ be hyperbolic elements in $\Gamma^*$ such that $\Gamma_P = \{1_2\}$ and $\Gamma_{P'} = \{1_2\}$. Assume that $P'$ and $P$ are $\Gamma_0(N)$-conjugate, that is, $T^{-1}P'T = P$ for some element $T \in \Gamma_0(N)$. If $\mathfrak{a}_1, \mathfrak{a}_2$ are the two fixed points of $P$, then $T(\mathfrak{a}_1), T(\mathfrak{a}_2)$ are the two fixed points of $P'$. In other words, if at least one of the two fixed points of $P$ is not $\Gamma_0(N)$-conjugate to a fixed point of $P'$, then $\{P'\}$ and $\{P\}$ represent different $\Gamma_0(N)$-conjugacy classes.

Denote $P(a, d; b; w) = \frac{1}{\sqrt{n}}\sigma_{1/w}\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\sigma_{1/w}^{-1}$ with $\sigma_{1/w}$ being given as in (3.3). It follows from Lemma 3.4, Lemma 3.5, Lemma 3.6 and Lemma 3.7 that

$$\sum_{w|N,\, w>0}\ \sum_{ad=n,\, d>0,\, a\neq d}\ \sum_{b} c(P(a,d;b;w))_Y = 2 \sum_{\{P\},\, \Gamma_P=\{1_2\}} c(P)_Y.$$

The stated identity then follows from Lemma 3.8. □

### 3.4 Parabolic components

Let $S$ be a parabolic element of $\Gamma^*$. An argument similar to that made for the elliptic elements shows that every element of $\Gamma_0(N)$ which has the same fixed point as $S$ commutes with $S$. If $\mathfrak{a} = u/w$ is the fixed point of $S$, then $\Gamma_S = \Gamma_{\mathfrak{a}}$, and hence we have $\sigma_{\mathfrak{a}}^{-1}\Gamma_S\sigma_{\mathfrak{a}} = \Gamma_\infty$, where $\sigma_{\mathfrak{a}}$ is given as in (3.3). Since $\sigma_{\mathfrak{a}}^{-1}S\sigma_{\mathfrak{a}}$ commutes with every element of $\Gamma_\infty$, we have

$$\sigma_{\mathfrak{a}}^{-1}S\sigma_{\mathfrak{a}} = \frac{1}{\sqrt{n}}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

for some real numbers $a, b$ with $a^2 = n$. An argument similar to that made in the paragraph following (3.4) shows that $a$ is an integer. This implies that $\Gamma^*$ has parabolic elements only if $n$ is the square of an integer. Furthermore, by (3.4) we see that elements of the form

$$S = \sigma_{\mathfrak{a}}\begin{pmatrix} 1 & b/\sqrt{n} \\ 0 & 1 \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}, \quad 0 \neq b \in \mathbb{Z}$$

constitute a complete set of representatives for the conjugacy classes of parabolic elements of $\Gamma^*$ having $\mathfrak{a}$ as its fixed point. It follows that

$$\sum_{\{S\}} \int_{D_Y} k(z, Sz)\, dz = \int_0^Y \int_0^1 \sum_{0\neq b\in\mathbb{Z}} k\left(z, z + \frac{b}{\sqrt{n}}\right) dz + o(1),$$

where the summation on $\{S\}$ is taken over all parabolic classes represented by parabolic elements whose fixed point is $\mathfrak{a} = u/w$ and where $o(1)$ tends to zero as $Y \to \infty$. Define $\delta_n$ to be one if $n$ is the square of an integer and to be zero otherwise.

**Theorem 3.10.** *Put*

$$c(\infty)_Y = \delta_n \nu(N) \int_0^Y \int_0^1 \sum_{0\neq b\in\mathbb{Z}} k\left(z, z + \frac{b}{\sqrt{n}}\right) dz - \int_{D_Y} H(z,z)\, dz.$$

*Then*

$$\frac{c(\infty)_Y}{\sqrt{n}} = \delta_n v(N) g(0) \ln \frac{\sqrt{\mathrm{n}}}{2} - v(N) \sum_{ad=n,\, d>0,\, a\neq d} g\left(\ln \frac{a}{d}\right) \ln Y$$

$$-\frac{\delta_n v(N)}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1+ir)\, dr + \frac{1}{4} h(0) \left\{ \delta_n v(N) + d(n) \sum_{i=1}^{v(N)} \varphi_{ii}\left(\frac{1}{2}\right) \right\}$$

$$+ \sum_{i,j=1}^{v(N)} \sum_{ad=n,\, d>0} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \left(\frac{a}{d}\right)^{ir} \varphi'_{ij}\left(\frac{1}{2}+ir\right) \varphi_{ij}\left(\frac{1}{2}-ir\right) dr + o(1)$$

*where* $o(1)$ *tends to zero as* $Y \to \infty$ *and* $\varphi_{ii}(1/2) \neq \infty$ (*cf. Selberg* [13]).

*Proof.* By the argument of [7], pp. 102–106 we have

$$\frac{1}{\sqrt{n}} \int_0^Y \int_0^1 \sum_{0\neq b\in\mathbb{Z}} k\left(z, z+\frac{b}{\sqrt{n}}\right) dz$$

$$= g(0) \ln(\sqrt{n} Y) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1+ir)\, dr - g(0) \ln 2 + \frac{1}{4} h(0) + o(1).$$

Let

$$\varphi_{ij,m}(s) = \sum_c \frac{1}{|c|^{2s}} \left( \sum_d e(md/c) \right)$$

where summations are taken over $c > 0, d$ modulo $c$ with $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}_i}^{-1} \Gamma_0(N) \sigma_{\mathfrak{a}_j}$. Then we have

$$E_i(\sigma_{\mathfrak{a}_j} z, s) = \delta_{ij} y^s + \varphi_{ij}(s) y^{1-s}$$

$$+ \frac{2\pi^s \sqrt{y}}{\Gamma(s)} \sum_{m\neq 0} |m|^{s-(1/2)} K_{s-(1/2)}(2|m|\pi y) \varphi_{ij,m}(s) e(mx)$$

where

$$\varphi_{ij}(s) = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} \varphi_{ij,0}(s).$$

By (1.1) and the Maass-Selberg relation (Theorem 2.3.1 of [7]), we obtain that

$$\int_{D_Y} E_i\left(\frac{az+b}{d}, s\right) E_i(z, \bar{s})\, dza$$

$$= \sum_{j=1}^{v(N)} \left\{ \frac{\delta_{ij}(a/d)^s Y^{s+\bar{s}-1} - \varphi_{ij}(s)\varphi_{ij}(\bar{s})(a/d)^{1-s} Y^{1-s-\bar{s}}}{s+\bar{s}-1} \right.$$

$$\left. + \delta_{ij} \frac{(a/d)^s \varphi_{ij}(\bar{s}) Y^{s-\bar{s}} - \varphi_{ij}(s)(a/d)^{1-s} Y^{\bar{s}-s}}{s-\bar{s}} \right\} + o(1)$$

for nonreal $s$ with $\Re s > 1/2$, where $o(1)$ tends to zero as $Y \to \infty$. Hence we have

$$\int_{D_Y} \left( \sum_{ad=n,\, 0 \leqslant b<d} E_i\left(\frac{az+b}{d}, s\right) \right) E_i(z, \bar{s})\, dz$$

$$= \left\{ \frac{Y^{s+\bar{s}-1} - \sum_{j=1}^{v(N)} \varphi_{ij}(s)\varphi_{ij}(\bar{s}) Y^{1-s-\bar{s}}}{s+\bar{s}-1} + \frac{\varphi_{ii}(\bar{s}) Y^{s-\bar{s}} - \varphi_{ii}(s) Y^{\bar{s}-s}}{s-\bar{s}} \right\}$$

$$\times \left\{ \sum_{ad=n,\, d>0} a^s d^{1-s} \right\} + o(1).$$

By partial integration, we obtain

$$h(r) = \frac{1}{r^4} \int_0^\infty g^{(4)}(\ln u) u^{ir-1}\, du$$

for nonzero $r$. Then it follows that

(3.10)

$$\lim_{S \to (1/2)^+} \int_{-\infty}^{\infty} h(r) \left\{ \int_{D_Y} \left( \sum_{ad=n,\, 0 \leqslant b<d} E_i\left(\frac{az+b}{d}, S+ir\right) \right) E_i(z, S-ir)\, dz \right\} dr$$

$$= \sqrt{n} \sum_{ad=n,\, d>0} \left\{ 4\pi g\left(\ln \frac{a}{d}\right) \ln Y - \sum_{j=1}^{v(N)} \int_{-\infty}^{\infty} h(r) \left(\frac{a}{d}\right)^{ir} \varphi_{ij}'\left(\frac{1}{2}+ir\right) \varphi_{ij}\left(\frac{1}{2}-ir\right) dr \right.$$

$$\left. + \int_{-\infty}^{\infty} h(r) \left(\frac{a}{d}\right)^{ir} \frac{\varphi_{ii}(\frac{1}{2}-ir) Y^{2ir}}{ir}\, dr \right\} + o(1).$$

By the Riemann-Lebesgue theorem (cf. §1.8 of [16]), we have

$$\lim_{Y\to\infty}\int_{-\infty}^{\infty} h(r)\left(\frac{a}{d}\right)^{ir}\frac{\varphi_{ii}(\frac{1}{2}-ir)\,Y^{2ir}}{ir}\,dr \tag{3.11}$$

$$= \lim_{Y\to\infty}\int_{-\infty}^{\infty} h(r)\Re\left(\varphi_{ii}\left(\frac{1}{2}-ir\right)\right)\frac{\sin\{r\ln(aY^2/d)\}}{r}\,dr = \pi h(0)\varphi_{ii}\left(\frac{1}{2}\right).$$

By (3.10) and (3.11), we have

$$\frac{1}{\sqrt{n}}\int_{D_Y} H(z,z)\,dz$$

$$= \sum_{i=1}^{v(N)}\sum_{ad=n,\,d>0}\left\{g\left(\ln\frac{a}{d}\right)\ln Y + \frac{1}{4}h(0)\varphi_{ii}\left(\frac{1}{2}\right)\right.$$

$$\left. - \sum_{j=1}^{v(N)}\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\left(\frac{a}{d}\right)^{ir}\varphi_{ij}'\left(\frac{1}{2}+ir\right)\varphi_{ij}\left(\frac{1}{2}-ir\right)dr\right\} + o(1).$$

The stated identity then follows. $\square$

Let $N$ be a square free positive integer. It follows from Theorem 3.9 and Theorem 3.10 that

$$\lim_{Y\to\infty}\left(c(\infty)_Y + \sum_{\{P\},\,\Gamma_P=\{1_2\}} c(P)_Y\right) \tag{3.12}$$

$$= \sqrt{n}\delta_n v(N)g(0)\ln\frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{4}h(0)\left(\delta_n v(N) + d(n)\sum_{i=1}^{v(N)}\varphi_{ii}\left(\frac{1}{2}\right)\right)$$

$$+ \frac{\sqrt{n}}{2}\sum_{w|N,\,w>0}\ \sum_{ad=n,\,d>0,\,a\neq d}\ \sum_{b}\frac{\ln\left\{(a-d)^2wN\left[\frac{C^2}{l},N\right]/C^2\right\}}{a-d}g\left(\ln\frac{a}{d}\right)$$

$$- \frac{\delta_n v(N)\sqrt{n}}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr$$

$$+ \sum_{i,j=1}^{v(N)}\sum_{ad=n,\,d>0}\frac{\sqrt{n}}{4\pi}\int_{-\infty}^{\infty} h(r)\left(\frac{a}{d}\right)^{ir}\varphi_{ij}'\left(\frac{1}{2}+ir\right)\varphi_{ij}\left(\frac{1}{2}-ir\right)dr$$

$$+ \sum_{ad=n,\,d>0,\,a\neq d}\frac{1}{2}v(N)|a-d|\int_{1}^{\infty} k\left(\frac{(a-d)^2}{n}t\right)\frac{\ln t}{\sqrt{t-1}}\,dt$$

with $C = (a-d)w - bwN$, where $l = (C, bN)$ and the summation on $b$ is taken over all numbers $b$ such that $C, bN/w \in \mathbb{Z}, N|C$ and $0 \leqslant b < |a-d|$. Note that there are exactly $|a-d|$ number of such numbers $b$ by Lemma 3.5.

Denote by $c(\infty)$ the right side of the identity (3.12). We conclude that the trace formula (2.3) can be written as

$$(3.13) \quad d(n)h\left(-\frac{i}{2}\right) + \sqrt{n}\sum_{j=1}^{\infty} h(\kappa_j)\,\mathrm{tr}_j T_n$$
$$= c(I) + \sum_{\{R\}} c(R) + \sum_{\{P\},\,\Gamma_P \neq \{1_2\}} c(P) + c(\infty)$$

for $\Re s > 1$, where the summations on the right side of the identity are taken over the conjugacy classes.

**Lemma 3.11 (Hejhal [3]).** (cf. Proposition 13.6 of Iwaniec [6]) *Let $N$ be a square free positive integer. Then, for any pair of cusps $\mathfrak{a}_i = 1/w_i$, $\mathfrak{a}_j = 1/w_j$ with $N = v_i w_i = v_j w_j$, we have*

$$\varphi_{ij}(s) = \varphi(s) p_{ij}(s)$$

*where*

$$\varphi(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}$$

*and*

$$p_{ij}(s) = \varphi((v_i, v_j)(w_i, w_j)) \prod_{p|N} (p^{2s}-1)^{-1} \prod_{p|(w_i,v_j)(w_j,v_i)} (p^s - p^{1-s}).$$

**Theorem 3.12.** *Let $N$ be square free, and put $c(P) = \int_{\Gamma_P\backslash\mathscr{H}} k(z, Pz)\,dz$ for hyperbolic elements $P \in \Gamma^*$. Then the series*

$$\sum_{\{P\},\,\Gamma_P \neq \{1_2\}} c(P)$$

*represents an analytic function in the half-plane $\Re s > 0$ except for a possible pole at $s = 1/2$ and for possible simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j$, $j = 1, 2, \ldots$.*

*Proof.* We have

$$g^{(4)}(\log u) = A(s)u^{(1/2)-s} + O_s(u^{-(1/2)}),$$

where $A(s)$ is an analytic function of $s$ for $\Re s > 0$ and where $O_s(u^{-(1/2)})$ means that, for every complex number $s$ with $\Re s > 0$, there exists a finite constant $B(s)$ depending only on $s$ such that

$$|O_s(u^{-(1/2)})| \leqslant B(s)u^{-(1/2)}.$$

Moreover, for every fixed value of $u$, the term $O_s(u^{-(1/2)})$ also represents an analytic function of $s$ for $\Re s > 0$. Since

$$h(r) = \frac{1}{r^4}\int_0^\infty g^{(4)}(\ln u)u^{ir-1}\,du$$

for nonzero $r$, we have

$$\begin{aligned} h(r) &= \frac{1}{r^4}\int_1^\infty g^{(4)}(\ln u)(u^{ir}+u^{-ir})\frac{du}{u} \\ &= \frac{A(s)}{r^4}\int_1^\infty u^{-(1/2)-s}(u^{ir}+u^{-ir})\,du + O_s\left(\frac{1}{r^4}\int_1^\infty u^{-1-\varepsilon}\,du\right) \\ &= \frac{A(s)}{r^4}\left(\frac{1}{s-\frac{1}{2}-ir}+\frac{1}{s-\frac{1}{2}+ir}\right) + O_s(r^{-4}) \end{aligned}$$

for $\Re s > 1$ and for nonzero $r$ with $|\Im r| < \frac{1}{2} - \varepsilon$. By analytic continuation, we obtain that

$$h(r) = \frac{2A(s)(s-\frac{1}{2})}{r^4[(s-\frac{1}{2})^2+r^2]} + O_s(r^{-4}) \tag{3.14}$$

for $\Re s > 0$ and for nonzero $r$ with $|\Im r| < \frac{1}{2} - \varepsilon$. It follows from results of [17] that the left side of (3.13) is an analytic function of $s$ for $\Re s > 0$ except for simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j,\ j = 1, 2, \ldots$. Then the right side of (3.13) can be interpreted as an analytic function of $s$ in the same region by analytic continuation.

Since $k(t) = (1 + t/4)^{-s}$, by Lemma 3.1 we have that $c(R)$ is analytic for $\Re s > 0$ except for a simple pole at $s = 1/2$. There are only a finite number of elliptic conjugacy classes $\{R\}$. The term $c(I)$ is a constant.

Since $g(0) = 2\sqrt{\pi}\Gamma(s-\frac{1}{2})\Gamma(s)^{-1}$,

$$h(0) = 2\sqrt{\pi}4^s\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\int_1^\infty\left(u+\frac{1}{u}+2\right)^{(1/2)-s}\frac{du}{u}$$

and

$$g\left(\ln \frac{a}{d}\right) = 2\sqrt{\pi}4^{s-(1/2)} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \left(\frac{a}{d} + \frac{d}{a} + 2\right)^{(1/2)-s},$$

the sum of first three terms on the right side of the identity (3.12) is analytic for $\Re s > 0$ except for a pole at $s = 1/2$.

Since $k(t) = (1 + t/4)^{-s}$, the sixth term on the right side of the identity (3.12) is analytic for $\Re s > 0$ except for a pole at $s = 1/2$.

By Stirling's formula the identity

$$\text{(3.15)} \quad \frac{\Gamma'(z)}{\Gamma(z)} = \ln z + O(1)$$

holds uniformly when $|\arg z| \le \pi - \delta$ for a small positive number $\delta$. It follows from (3.14) and (3.15) that the fourth term on the right side of the identity (3.12) is analytic for $\Re s > 0$ except for a possible pole at $s = 1/2$.

By the functional identity of the Riemann zeta function $\zeta(s)$, we have $|\varphi(s)| = 1$ for $\Re s = 1/2$. This implies that

$$\text{(3.16)} \quad \varphi'_{ij}(s)\varphi_{ij}(\bar{s}) = \frac{\varphi'(s)}{\varphi(s)} |p_{ij}(s)|^2 + p'_{ij}(s)p_{ij}(\bar{s})$$

for $\Re s = 1/2$ by Lemma 3.11. The identity

$$\text{(3.17)} \quad \frac{\varphi'(s)}{\varphi(s)} = 2\ln\pi - \frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(1-s)}{\Gamma(1-s)} - 2\frac{\zeta'(2s)}{\zeta(2s)} - 2\frac{\zeta'(2-2s)}{\zeta(2-2s)}$$

holds for $\Re s = 1/2$. It follows from (3.14)–(3.17) and the formula for $p_{ij}(s)$ given in Lemma 3.11 that the fifth term on the right side of the identity (3.12) is analytic for $\Re s > 0$ except for a possible pole at $s = 1/2$.

Therefore, by (3.13) we have proved that the series

$$\sum_{\{P\},\, \Gamma_P \neq \{1_2\}} c(P)$$

represents an analytic function of $s$ in the half-plane $\Re s > 0$ except for a possible pole at $s = 1/2$ and for possible simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j$, $j = 1, 2, \ldots$. □

## 4 Proof of the Main Theorem

**Lemma 4.1.** *Let $\lambda > 0$ be an eigenvalue of $\Delta$ for $\Gamma_0(N)$ with $N$ square free. Assume that $(n, N) = 1$. Put $\tau = 1/2 + i\kappa$ with $\kappa = \sqrt{\lambda - 1/4}$. Then we have*

$$4^{\tau}\sqrt{\pi n}\,\frac{\Gamma(i\kappa)}{\Gamma(\tau)}\,\mathrm{tr}\,T_n=\lim_{s\to\tau}(s-\tau)\sum_{\{P\},\,\Gamma_P\neq\{1_2\}}c(P)$$

*where the right side is defined as in Theorem 3.12.*

*Proof.* By (2.2), (3.14) and results of [17], we have

$$\lim_{s\to\tau}(s-\tau)\sum_{j=1,\,\kappa_j\neq\kappa}^{\infty}h(\kappa_j)\,\mathrm{tr}_j\,T_n=0.$$

By the proof of Theorem 3.12, we have

$$\lim_{s\to\tau}(s-\tau)\left(d(n)h\left(-\frac{i}{2}\right)-c(I)-c(\infty)-\sum_{\{R\}}c(R)\right)=0.$$

Then the stated identity follows from (2.2), (3.13) and Theorem 3.12. □

A quadratic form $ax^2+bxy+cy^2$, which is denoted by $[a,b,c]$, is said to be primitive if $(a,b,c)=1$ and $b^2-4ac=d\in\Omega$. Two quadratic forms $[a,b,c]$ and $[a',b',c']$ are equivalent if an element $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ exists such that

$$\begin{pmatrix}a' & b'/2\\ b'/2 & c'\end{pmatrix}=\gamma^t\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}\gamma,$$

where $\gamma^t$ is the transpose of $\gamma$. This relation partitions the quadratic forms into equivalence classes, and two such forms from the same class have the same discriminant. The number of classes $h_d$ of primitive indefinite quadratic forms of a given discriminant $d$ is finite, and is called the class number of indefinite quadratic forms.

**Remark.** Siegel [15] proved that

$$\lim_{d\to\infty}\frac{\ln(h_d\ln\varepsilon_d)}{\ln d}=\frac{1}{2}. \tag{4.1}$$

**Lemma 4.2.** *$P$ is a hyperbolic element of $\Gamma^*$ with $\Gamma_P\neq\{1_2\}$ if and only if there exists a primitive indefinite quadratic form $[a,b,c]$ of discriminant $d$ such that*

$$P=\frac{1}{\sqrt{n}}\begin{pmatrix}\dfrac{v-bNu(a,N)^{-1}}{2} & -cNu(a,N)^{-1}\\ aNu(a,N)^{-1} & \dfrac{v+bNu(a,N)^{-1}}{2}\end{pmatrix}$$

*with*

$$v^2 - \{dN^2(a,N)^{-2}\}u^2 = 4n.$$

*If $\lambda_P$ is an eigenvalue of $P$, then*

$$\lambda_P - \frac{1}{\lambda_P} = \pm \frac{Nu}{(a,N)} \frac{\sqrt{d}}{\sqrt{n}}.$$

*Let*

$$P_0 = \begin{pmatrix} \frac{v_0 - bu_0}{2} & -cu_0 \\ au_0 & \frac{v_0 + bu_0}{2} \end{pmatrix},$$

*where the pair $(v_0, u_0)$ is the fundamental solution of Pell's equation $v^2 - du^2 = 4$. Then $P$ is $\Gamma_0(N)$-conjugate to a hyperbolic element $P' \in \Gamma^*$ with $\Gamma_{P'} = \{1_2\}$ if and only if $P_0$ is $\Gamma_0(N)$-conjugate to $P'_0$, where $P'_0$ is associated with $P'$ similarly as $P_0$ is associated with $P$.*

*Proof.* Let

$$P = \frac{1}{\sqrt{n}} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a hyperbolic element of $\Gamma^*$ such that $\Gamma_P \neq \{1_2\}$. Then fixed points $r_1, r_2$ of $P$ are not rational numbers by Lemma 3.3, which satisfy the equation $Cr^2 + (D - A)r - B = 0$. This implies that $\Gamma_P$ is the subgroup of elements in $\Gamma_0(N)$ having $r_1, r_2$ as fixed points. Let $a = C/\mu$, $b = (D - A)/\mu$ and $c = -B/\mu$, where $\mu = (C, D - A, -B)$. Then $[a, b, c]$ is a primitive quadratic form with $r_1, r_2$ being the roots of the equation $ar^2 + br + c = 0$. By Sarnak [9], the subgroup of elements in $\mathrm{SL}_2(\mathbb{Z})$ having $r_1, r_2$ as fixed points consists of matrices of the form

$$\begin{pmatrix} \frac{v - bu}{2} & -cu \\ au & \frac{v + bu}{2} \end{pmatrix}$$

with $v^2 - du^2 = 4$ where $d = b^2 - 4ac$, and it is generated by the primitive hyperbolic element

$$P_0 = \begin{pmatrix} \frac{v_0 - bu_0}{2} & -cu_0 \\ au_0 & \frac{v_0 + bu_0}{2} \end{pmatrix}$$

where the pair $(v_0, u_0)$ is the fundamental solution of Pell's equation $v^2 - du^2 = 4$.

Since $P$ and $P_0$ have the same fixed points, we have $A = D - bC/a$ and $B = -cC/a$. Since $P$ belongs to $\Gamma^*$ and $AD - BC = n$, $C$ satisfies

$$\begin{cases} aD^2 - bDC + cC^2 = na \\ a|C, N|C. \end{cases} \tag{4.2}$$

Let $\lambda_P$ be an eigenvalue of $P$. Then it is a solution of the equation $\lambda^2 - \frac{A+D}{\sqrt{n}}\lambda + 1 = 0$. By using $A = D - bC/a$ and $B = -cC/a$, we obtain that

$$\lambda_P - \frac{1}{\lambda_P} = \pm\frac{C\sqrt{d}}{a\sqrt{n}} \tag{4.3}$$

and

$$\lambda_P + \frac{1}{\lambda_P} = \frac{1}{\sqrt{n}}\left(2D - \frac{b}{a}C\right). \tag{4.4}$$

Conversely, let a pair $(C, D)$ be a solution of the equation (4.2). Define $A = D - bC/a$ and $B = -cC/a$. Then the matrix

$$P = \frac{1}{\sqrt{n}}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has the same fixed points as $P_0$, and eigenvalues of $P$ satisfies (4.3) and (4.4). We have the decomposition

$$P = \frac{1}{\sqrt{n}}\begin{pmatrix} \frac{n}{(D,C)} & -\frac{c}{a}\alpha C - \left(D - \frac{b}{a}C\right)\beta \\ 0 & (D,C) \end{pmatrix} T \cdot T^{-1}\begin{pmatrix} \alpha & \beta \\ C/(D,C) & D/(D,C) \end{pmatrix},$$

where $\alpha$ and $\beta$ are integers such that $\alpha D - \beta C = (D, C)$ and where $T = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ is chosen so that

$$\begin{pmatrix} n/(D,C) & * \\ 0 & (D,C) \end{pmatrix} T = \begin{pmatrix} n/(D,C) & -s \\ 0 & (D,C) \end{pmatrix}$$

with $0 \le s < n/(D, C)$. The first equation of (4.2) can be written as

$$D^2 - bD\frac{C}{a} + cC\frac{C}{a} = n.$$

Since $(n, N) = 1$ and $a|C$, we have $(N, (D, C)) = 1$. Hence $\begin{pmatrix} \alpha & \beta \\ C/(D, C) & D/(D, C) \end{pmatrix}$ is an element of $\Gamma_0(N)$. Therefore, $P$ is an hyperbolic element of $\Gamma^*$ with $\Gamma_P \neq \{1_2\}$ by Lemma 3.3.

Next, let $v = 2D - b\dfrac{C}{a}$ and $u = \dfrac{C}{a}$. Then the equation (4.2) becomes $v^2 - du^2 = 4n$ with $N|au$. Since $N|au$, this equation can be written as

$$v^2 - \frac{dN^2}{(a, N)^2}u^2 = 4n. \tag{4.5}$$

Moreover, we have

$$\frac{1}{\sqrt{n}}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{\sqrt{n}}\begin{pmatrix} \dfrac{v - bNu(a, N)^{-1}}{2} & -cNu(a, N)^{-1} \\ aNu(a, N)^{-1} & \dfrac{v + bNu(a, N)^{-1}}{2} \end{pmatrix}. \tag{4.6}$$

Let $P_0'$ be the primitive hyperbolic element of $\mathrm{SL}_2(\mathbb{Z})$ corresponding to $[a', b', c']$. Since the identity

$$\begin{pmatrix} v_0/2 & 0 \\ 0 & v_0/2 \end{pmatrix} + u_0\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\gamma^t\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\gamma = \gamma^{-1}P_0\gamma$$

holds for every element $\gamma \in \Gamma_0(N)$, two forms $[a, b, c]$ and $[a', b', c']$ of the same discriminant are equivalent in $\Gamma_0(N)$ if and only if an element $\gamma = \Gamma_0(N)$ exists such that $\gamma^{-1}P_0\gamma = P_0'$. $\square$

**Lemma 4.3.** *Let* $[a, b, c]$ *be a primitive indefinite quadratic form of discriminant* $d$, *and let*

$$P = \frac{1}{\sqrt{n}}\begin{pmatrix} \dfrac{v - bNu(a, N)^{-1}}{2} & -cNu(a, N)^{-1} \\ aNu(a, N)^{-1} & \dfrac{v + bNu(a, N)^{-1}}{2} \end{pmatrix}$$

*with* $v^2 - \{dN^2(a, N)^{-2}\}u^2 = 4n$ *be a hyperbolic element of* $\Gamma^*$ *with* $\Gamma_P \neq \{1_2\}$. *Let* $(v_1, u_1)$ *with* $v_1, u_1 > 0$ *be the fundamental solution of Pell's equation* $v^2 - d_1u^2 = 4$, *where* $d_1 = dN^2/(a, N)^2$. *Then* $\Gamma_P$ *is generated by the hyperbolic element*

$$P_1 = \begin{pmatrix} \frac{1}{2}\left(v_1 - b\frac{N^2u_1}{(a,N)^2}\right) & -c\frac{N^2u_1}{(a,N)^2} \\ a\frac{N^2u_1}{(a,N)^2} & \frac{1}{2}\left(v_1 + b\frac{N^2u_1}{(a,N)^2}\right) \end{pmatrix}$$

*of* $\Gamma_0$ (N).

*Proof.* Let $r_1$ and $r_2$ be the fixed points of $P$. Then they satisfy $ar^2 + br + c = 0$, and $\Gamma_P$ is the subgroup of elements in $\Gamma_0(N)$ having $r_1, r_2$ as fixed points. By Sarnak [9], the subgroup of elements in $\mathrm{SL}_2(\mathbb{Z})$ having $r_1, r_2$ as fixed points consists of matrices of the form

$$\begin{pmatrix} \frac{v - bu}{2} & -cu \\ au & \frac{v + bu}{2} \end{pmatrix}$$

with $v^2 - du^2 = 4$ where $d = b^2 - 4ac$, and it is generated by the primitive hyperbolic element

$$P_0 = \begin{pmatrix} \frac{v_0 - bu_0}{2} & -cu_0 \\ au_0 & \frac{v_0 + bu_0}{2} \end{pmatrix}$$

where the pair $(v_0, u_0)$ is the fundamental solution of Pell's equation $v^2 - du^2 = 4$.

Since $\Gamma_P$ is cyclic, a solution $v_1, u_1' > 0$ of Pell's equation $v^2 - du^2 = 4$ exists such that $\Gamma_P$ is generated by

$$P_1 = \begin{pmatrix} \frac{v_1 - bu_1'}{2} & -cu_1' \\ au_1' & \frac{v_1 + bu_1'}{2} \end{pmatrix}$$

and such that $P_1$ is the smallest positive integer power of $P_0$ among all powers of $P_0$ belonging to $\Gamma_0(N)$. Note that the eigenvalues of $P_1$ are

$$\frac{v_1 \pm \sqrt{d}u_1'}{2}.$$

Since $\Gamma_P$ is generated by $P_1$, $(v_1, u_1')$ is the minimal solution of the equation $v^2 - du^2 = 4$ with $N | au_1'$ in the sense that $(v_1 + \sqrt{d}u_1')/2$ is of the smallest value among all such solutions. Since $N | au_1'$, we have $\frac{N}{(a,N)} | u_1'$. Write

$$u_1' = \frac{Nu_1}{(a,N)}.$$

Then the pair $(v_1, u_1)$ with $v_1, u_1 > 0$ must be the fundamental solution of the Pell equation $v^2 - d_1u^2 = 4$, where $d_1 = dN^2(a,N)^{-2}$. The stated result then follows. □

Two quadratic forms $[a,b,c]$ and $[a',b',c']$ are equivalent in $\Gamma_0(N)$ if an element $\gamma \in \Gamma_0(N)$ exists such that

$$\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma.$$

This relation partitions the quadratic forms into equivalence classes, and two such forms from the same class have the same discriminant. The number of such classes of a given discriminant $d$ is finite, and is denoted by $H_d$.

**Lemma 4.4.** *Let $[a_j, b_j, c_j]$, $j = 1, 2, \ldots, H_d$, be a set of representatives for classes of primitive indefinite quadratic forms of discriminant d, which are not equivalent under $\Gamma_0(N)$. Then we have*

$$\sum_{\{P\}:P\in\Gamma^*,\Gamma_P\neq\{1_2\}} c(P)$$

$$= 4\sqrt{\pi n}\frac{\Gamma(s-\frac{1}{2})}{N\Gamma(s)} \sum_{d\in\Omega} \sum_{j=1}^{H_d} \sum_u \frac{(a_j,N)\ln\varepsilon_{d_1}}{u\sqrt{d}} \left(1 + \frac{d(Nu)^2}{4n(a_j,N)^2}\right)^{(1/2)-s}$$

*for $\Re s > 1$, where $d_1 = dN^2(a_j,N)^{-2}$ and where the summation on u is taken over all the positive integers u such that $4n + dN^2(a_j,N)^{-2}u^2$ is the square of an integer.*

*Proof.* It follows from (2.1) and Theorem 3.2 that

$$\sum_{\{P\}:P\in\Gamma^*,\Gamma_P\neq\{1_2\}} c(P)$$

$$= 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{\{P\}} \frac{\ln NP_1}{\lambda_P - 1/\lambda_P} \left(1 + \frac{(\lambda_P - 1/\lambda_P)^2}{4}\right)^{(1/2)-s}$$

for $\Re s > 1$, where $\lambda_P > 1$ is an eigenvalue of $P$ and $P_1$ is given in Lemma 4.3. Let $P$ be associated with a primitive indefinite quadratic form $[a_j, b_j, c_j]$ as in Lemma 4.2. Then by Lemma 4.2, we have

$$\lambda_P - \frac{1}{\lambda_P} = \frac{Nu}{(a_j,N)} \frac{\sqrt{d}}{\sqrt{n}}.$$

By Lemma 4.3, we have

$$\sqrt{NP_1} = \frac{v_1 + \sqrt{d_1}u_1}{2} = \varepsilon_{d_1}.$$

If $P'$ is a hyperbolic of $\Gamma^*$ with $\Gamma_{P'} \neq \{1_2\}$, and is associated with a primitive indefinite quadratic form $[a_{j'}, b_{j'}, c_{j'}]$ as in Lemma 4.2, then $P$ and $P'$ are $\Gamma_0(N)$-conjugate if and only if $[a_j, b_j, c_j]$ and $[a_{j'}, b_{j'}, c_{j'}]$ are $\Gamma_0(N)$-conjugate by the last statement of Lemma 4.2. Next, let $T$ be a hyperbolic of $\Gamma^*$ with $\Gamma_T \neq \{1_2\}$. Assume that $T$ is associated with a primitive indefinite quadratic form $[a, b, c]$ as in Lemma 4.2. If the discriminant of $[a, b, c]$ is not equal to the discriminant of $[a_j, b_j, c_j]$ which is associated with $P$, then $P$ and $T$ are not $\Gamma_0(N)$-conjugate. The stated identity then follows. □

**Lemma 4.5.** *Let $N$ be square free, and let $k$ be a divisor of $N$. Then the number of indefinite primitive quadratic forms $[a, b, c]$ with $(a, N) = k$ of discriminant $d$, which are not equivalent under $\Gamma_0(N)$, is equal to*

$$h_{d_1} \prod_{p|k} \left(1 + \left(\frac{d}{p}\right)\right)$$

*where $d_1 = dN^2/k^2$.*

*Proof.* Let $[a, b, c]$ and $[a', b', c']$ be two indefinite primitive quadratic forms of discriminant $d$ with $(a, N) = k = (a', N)$. If they are equivalent under $\Gamma_0(N)$, then an element $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$ exists such that

$$\frac{b'}{2} = \alpha\left(a\beta + \frac{b}{2}\delta\right) + \gamma\left(\frac{b}{2}\beta + c\delta\right).$$

This implies that $b' \equiv b \pmod{2k}$. In particular, if $b' \not\equiv b \pmod{2k}$, then $[a, b, c]$ and $[a', b', c']$ are not $\Gamma_0(N)$-equivalent.

Assume that $\varrho$ is an integer with $1 \leqslant \varrho \leqslant 2k$. Denote by $\Lambda_{k,d,\varrho}$ the set of representatives of indefinite primitive quadratic forms $[a, b, c]$ of discriminant $d$ with $(a, N) = k$ and $b \equiv \varrho \pmod{2k}$, which are not equivalent under $\Gamma_0(N)$. Let $\mathscr{L}^0_{N,d_1,\varrho N/k}$ be the set of representatives of indefinite quadratic forms $[aN, b, c]$ of discriminant $d_1$ with $(a, b, c) = 1$, $(N, b, c) = N/k$ and $b \equiv \varrho N/k \pmod{2N}$, which are not equivalent under $\Gamma_0(N)$.

A map from $\Lambda_{k,d,\varrho}$ to $\mathscr{L}^0_{N,d_1,\varrho N/k}$ is defined by $T : [a, b, c] \to [aN/k, bN/k, cN/k]$. We claim that $T$ is bijective. By the definition of $\Lambda_{k,d,\varrho}$, we see that $T$ is injective. Conversely, if $[a_1 N, b_1, c_1]$ is an element of $\mathscr{L}^0_{N,d_1,\varrho N/k}$, then we have $(a_1, b_1, c_1) = 1$, $(N, b_1, c_1) = N/k$, $b_1 \equiv \varrho N/k \pmod{2N}$ and $b_1^2 - 4Na_1c_1 = d_1$. Let $[a_0, b_0, c_0]$ be an element of $\Lambda_{k,d,\varrho}$. Then

$$b_1^2 = (b_0 N/k)^2 + 4lb_0 N \frac{N}{k} + 4l^2 N^2$$

for some integer $l$. Since

$$d_1 = (b_0 N/k)^2 - 4\frac{a_0}{k} N \cdot c_0 \frac{N}{k},$$

we have

$$(b_0 N/k)^2 - 4\frac{a_0}{k} N \cdot c_0 \frac{N}{k} = d_1 = (b_0 N/k)^2 + 4lb_0 N \frac{N}{k} + 4l^2 N^2 - 4Na_1 c_1.$$

That is, the identity

$$(4.7) \qquad -\frac{a_0}{k} \cdot c_0 \frac{N}{k} = lb_0 \frac{N}{k} + l^2 N - a_1 c_1$$

holds for some integer $l$. Note that $(a_0, N) = k$. Since $(N, b_1, c_1) = N/k$ and $(a_1, b_1, c_1) = 1$, we have $(a_1, N/k) = 1$, and hence it follows from (4.7) that $N/k | c_1$. Let $a = a_1 k$, $b = b_1 k/N$ and $c = c_1 k/N$. Then we have $(a, N) = k$, $b \equiv \varrho \pmod{2k}$ and $d = b^2 - 4ac$. We claim that $(a, b, c) = 1$, that is, $(a_1 k, b_1 k/N, c_1 k/N) = 1$. Since $(a_1, b_1, c_1) = 1$, it is enough to show that $(k, b_1, c_1) = 1$. Since $(N, b_1, c_1) = N/k$, we must have $(k, b_1, c_1) = 1$, and therefore we have $(a, b, c) = 1$. Thus, $[a, b, c]$ is an element of $\Lambda_{k,d,\varrho}$, and $T$ maps it into the element $[a_1 N, b_1, c_1]$ of $\mathscr{L}^0_{N,d_1,\varrho N/k}$. Therefore, $T$ is surjective. Thus, we have proved that $T$ is a bijection if the set $\Lambda_{k,d,\varrho}$ is not empty. By Proposition, p. 505 of Gross, Kohnen and Zagier [2], the number of elements contained in $\mathscr{L}^0_{N,d_1,\varrho N/k}$ is $h_{d_1}$, and hence the set $\Lambda_{k,d,\varrho}$ contains $h_{d_1}$ elements if it is not empty.

If $\varrho_1 \not\equiv \varrho_2 \pmod{2k}$, then the set of $\Gamma_0(N)$-equivalence classes represented by elements in $\Lambda_{k,d,\varrho_1}$ is disjoint from the set of $\Gamma_0(N)$-equivalence classes represented by elements in $\Lambda_{k,d,\varrho_2}$. Now, we want to count the number of non-empty sets $\Lambda_{k,d,\varrho}$. That is, we want to count the number of solutions $\varrho$ of the equation

$$(4.8) \qquad \varrho^2 \equiv d \pmod{4k}, \quad 1 \leqslant \varrho \leqslant 2k.$$

If $(d, k) = 1$, by Theorem 3.4 of Chapter 12, Hua [4] the number of solution of the equation (4.8) is equal to

$$\prod_{p|k} \left(1 + \left(\frac{d}{p}\right)\right).$$

Next, we consider the case when there exists a prime number $q$ satisfying $q|k$ and $q^2|d$. Then we have $q|\varrho$, and the equation (4.8) can be written as

$$\left(\frac{\varrho}{q}\right)^2 \equiv \frac{d}{q^2} \left(\mathrm{mod}\, \frac{4k}{q}\right), \quad 1 \leqslant \frac{\varrho}{q} \leqslant \frac{2k}{q}. \tag{4.9}$$

Dividing out $\varrho, d$ and $k$ by all such prime numbers $q$ as in (4.9), we can reduce the second case to the first case when $(d,k) = 1$. Then, by using properties of the Legendre symbol we obtain that the number of solution of the equation (4.8) in the second case is still equal to

$$\prod_{p|k}\left(1 + \left(\frac{d}{p}\right)\right).$$

Finally, we consider the case when there exists a prime number $q$ satisfying $q|k, q|d$ and $q^2 \nmid d$. We have again $q|\varrho$. The equation (4.8) can be written as

$$q\left(\frac{\varrho}{q}\right)^2 \equiv \frac{d}{q} \left(\mathrm{mod}\, \frac{4k}{q}\right), \quad 1 \leqslant \frac{\varrho}{q} \leqslant \frac{2k}{q}. \tag{4.10}$$

We can assume that $q \neq 2$. Otherwise, if $q = 2$ then we must have $q^2|d$. Then we have $(q, 4k/q) = 1$, and hence a number $x_q$ exists such that $qx_q \equiv 1 \ (\mathrm{mod}\, 4k/q)$. Then the equation (4.10) can be written as

$$\left(\frac{\varrho}{q}\right)^2 \equiv x_q \frac{d}{q} \left(\mathrm{mod}\, \frac{4k}{q}\right), \quad 1 \leqslant \frac{\varrho}{q} \leqslant \frac{2k}{q}. \tag{4.11}$$

Note that we have

$$\left(\frac{x_q d/q}{p}\right) = \left(\frac{q}{p}\right)\left(\frac{d/q}{p}\right) = \left(\frac{d}{p}\right) \tag{4.12}$$

for any prime number $p|(k/q)$. Dividing out all such primes $q$ as in (4.11) and using (4.12), we obtain that the number of solution of the equation (4.8) in the final case is equal to

$$\prod_{p|k}\left(1 + \left(\frac{d}{p}\right)\right).$$

This completes the proof of the lemma. $\square$

By Lemma 4.5, we get the following corollary.

**Corollary.** *Let $H_d$ be the number of indefinite primitive quadratic forms of discriminant d, which are not equivalent under $\Gamma_0(N)$. Then we have*

$$H_d = \sum_{k|N} h_{d(N/k)^2} \prod_{p|k}\left(1 + \left(\frac{d}{p}\right)\right).$$

**Theorem 4.6.** *Let N be a square free positive integer with $(n, N) = 1$. Then we have*

$$\sum_{\{P\}:P\in\Gamma^*,\Gamma_P\neq\{1_2\}} c(P)$$

$$= 4\sqrt{\pi n}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{m|N} \sum_{k|N} \frac{\mu((m,k))}{(m,k)} \sum_{d\in\Omega} \sum_{u} \left(\frac{d}{m}\right)$$

$$\times \frac{h_d \ln \varepsilon_d}{u\sqrt{d}} \left(1 + \frac{dk^2u^2}{4n}\right)^{(1/2)-s}$$

*for $\Re s > 1$, where the summation on u is taken over all the positive integers u such that $\sqrt{4n + dk^2u^2} \in \mathbb{Z}$.*

*Proof.* Let $[a_j, b_j, c_j]$, $j = 1, 2, \ldots, H_d$, be a set of representatives for classes of primitive indefinite quadratic forms of discriminant $d$, which are not equivalent under $\Gamma_0(N)$. By Lemma 4.4, we have

$$\sum_{\{P\}:P\in\Gamma^*,\Gamma_P\neq\{1_2\}} c(P) = 4\sqrt{\pi n}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{d\in\Omega} \sum_{j=1}^{H_d} \sum_{u} \frac{\ln \varepsilon_{d_1}}{u\sqrt{d_1}} \left(1 + \frac{d_1u^2}{4n}\right)^{(1/2)-s}$$

for $\Re s > 1$, where $d_1 = dN^2(a_j, N)^{-2}$ and where the summation on $u$ is taken over all the positive integers $u$ such that $\sqrt{4n + d_1u^2} \in \mathbb{Z}$. Denote $k = N/(a_j, N)$. Then $k|N$. By using Lemma 4.5, we can write the above identity as

$$\sum_{\{P\}:P\in\Gamma^*,\Gamma_P\neq\{1_2\}} c(P) \tag{4.13}$$

$$= 4\sqrt{\pi n}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{k|N} \sum_{d\in\Omega} \sum_{u} \prod_{p|(N/k)} \left(1 + \left(\frac{d}{p}\right)\right) \frac{h_{d_1} \ln \varepsilon_{d_1}}{u\sqrt{d_1}}$$

$$\times \left(1 + \frac{d_1u^2}{4n}\right)^{(1/2)-s}$$

for $\Re s > 1$, where $d_1 = dk^2$. By using Dirichlet's class number formula

$$h_{d_1} \ln \varepsilon_{d_1} = \sqrt{d_1}L(1, \chi_{d_1})$$

and by using the identity (See Theorem 11.2 of Chapter 12, Hua [4])

$$L(1,\chi_{d_1}) = L(1,\chi_d)\prod_{p|k}\left(1-\left(\frac{d}{p}\right)p^{-1}\right),$$

we can write (4.13) as

$$\sum_{\{P\}:P\in\Gamma^*,\Gamma_P\neq\{1_2\}} c(P)$$

$$= 4\sqrt{\pi n}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{k|N}\sum_{d\in\Omega}\sum_{u}\prod_{p|(N/k)}\left(1+\left(\frac{d}{p}\right)\right)$$

$$\times\prod_{p|k}\left(1-\left(\frac{d}{p}\right)p^{-1}\right)\frac{h_d\ln\varepsilon_d}{u\sqrt{d}}\left(1+\frac{dk^2u^2}{4n}\right)^{(1/2)-s}$$

for $\Re s>1$, where the summation on $u$ is taken over all the positive integers $u$ such that $\sqrt{4n+dk^2u^2}\in\mathbb{Z}$. Since

$$\prod_{p|(N/k)}\left(1+\left(\frac{d}{p}\right)\right)\prod_{p|k}\left(1-\left(\frac{d}{p}\right)p^{-1}\right) = \sum_{m|N}\left(\frac{d}{m}\right)\frac{\mu((m,k))}{(m,k)},$$

we have

$$\sum_{\{P\}:P\in\Gamma^*,\Gamma_P\neq\{1_2\}} c(P) = 4\sqrt{\pi n}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{m|N}\sum_{k|N}\frac{\mu((m,k))}{(m,k)}\sum_{d\in\Omega}\sum_{u}\left(\frac{d}{m}\right)$$
$$\times\frac{h_d\ln\varepsilon_d}{u\sqrt{d}}\left(1+\frac{dk^2u^2}{4n}\right)^{(1/2)-s}. \tag{4.14}$$

Next, we want to show that

$$\sum_{d\in\Omega}\sum_{u}\left(\frac{d}{m}\right)\frac{h_d\ln\varepsilon_d}{u\sqrt{d}}\left(1+\frac{dk^2u^2}{4n}\right)^{(1/2)-s}$$

is absolutely convergent for $\sigma=\Re s>1$. Since

$$\left|\sum_{d\in\Omega}\sum_{u}\left(\frac{d}{m}\right)\frac{h_d\ln\varepsilon_d}{u\sqrt{d}}\left(1+\frac{dk^2u^2}{4n}\right)^{(1/2)-s}\right| \leqslant \sum_{d\in\Omega}\sum_{u}\frac{h_d\ln\varepsilon_d}{u\sqrt{d}}\left(1+\frac{dk^2u^2}{4n}\right)^{(1/2)-\sigma}. \tag{4.15}$$

It is proved in Li [8] that the right side of (4.15) is convergent for $\sigma > 1$, and hence, the right side of the stated identity is absolutely convergent for $\Re s > 1$.

This completes the proof of the theorem. $\square$

*Proof of Theorem 1.* It is proved at the end of Li [8] that

$$\sum_{d \in \Omega, u} \left| \frac{h_d \ln \varepsilon_d}{\sqrt{d}u} \left(1 + \frac{du^2}{4n}\right)^{(1/2)-\sigma} - \frac{h_d \ln \varepsilon_d}{(du^2)^{\sigma}} \right| \ll \sum_{d \in \Omega, u} \frac{(du^2)^{(1/2)+\varepsilon-1-\sigma}}{u^{1+2\varepsilon}} < \infty$$

for $\sigma > 0$. Then it follows from Theorem 4.6 that

$$\begin{aligned} (4.16) \qquad & \lim_{s \to 1/2+i\kappa} \left(s - \frac{1}{2} - i\kappa\right) \sum_{\{P\}, \Gamma_P \neq \{1_2\}} c(P) \\ &= \frac{(4\pi)^{1/2}\Gamma(i\kappa)}{(4n)^{-1/2-i\kappa}\Gamma(1/2+i\kappa)} \sum_{m|N} \sum_{k|N} k^{-2i\kappa} \prod_{p|k} (1 - p^{-1}) \\ &\quad \times \lim_{s \to 1/2+i\kappa} \left(s - \frac{1}{2} - i\kappa\right) \sum_{d \in \Omega} \sum_{u} \left(\frac{d}{m}\right) \frac{h_d \ln \varepsilon_d}{(du^2)^s}. \end{aligned}$$

Theorem 3.12 shows that the function on the right side of (4.16) represents an analytic function in the half-plane $\Re s > 0$ except for a possible pole at $s = 1/2$ and for possible simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j,\ j = 1, 2, \ldots$. The stated identity then follows from Lemma 4.1.

This completes the proof of the theorem.

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