ON THE FREQUENCY OF VANISHING OF QUADRATIC TWISTS OF MODULAR L-FUNCTIONS

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Abstract. We present theoretical and numerical evidence for a random matrix theoretical approach to a conjecture about vanishings of quadratic twists of certain L-functions.

In this paper we[1] present some evidence that methods from random matrix theory can give insight into the frequency of vanishing for quadratic twists of modular L-functions. The central question is the following: given a holomorphic newform f with integral coefficients and associated L-function $L_f(s)$, for how many fundamental discriminants d with $|d| \le x$, does $L_f(s,\chi_d)$, the L-function twisted by the real, primitive, Dirichlet character associated with the discriminant d, vanish at the center of the critical strip to order at least 2?

This question is of particular interest in the case that the L-function is associated with an elliptic curve, in light of the conjecture of Birch and Swinnerton-Dyer. This case corresponds to weight $k = 2$. We will focus on this case for most of the paper, though we do make some remarks about higher weights (see (26) and below).

Suppose that E/Q is an elliptic curve with associated L-function

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a_n^*}{n^s} \tag{1}$$

for $\Re s > 1$. Then, as a consequence of the Taniyama-Shimura conjecture, recently solved by Wiles, Taylor, ([W], [TW]), and Breuil, Conrad, and Diamond, L_E is entire and satisfies a functional equation

$$\left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L_E(s) = \Phi_E(s) = w_E \Phi(1-s) \tag{2}$$

where N is the conductor of E and $w_E = \pm 1$ is called the sign of the functional equation. Note that we have normalized the coefficients a_n^* so that the functional equation of L_E relates the values at s and $1-s$. The numbers a_n^* satisfy $|a_n^*| \le d(n)$, where $d(n)$ is the number of positive divisors of n, and $a_n = \sqrt{n} a_n^*$ is an integer (related to the numbers of points on E mod p for primes p which divide n). Let

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d represent a fundamental discriminant and let χ_d be the associated quadratic character (i.e. $\chi_d(n) = \left(\frac{d}{n}\right)$, the Kronecker symbol). We assume, for simplicity, that $(d, N) = 1$. Then the twisted L-function is

$$L_E(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a_n^* \chi_d(n)}{n^s}. \tag{3}$$

This L-function has functional equation

$$\left(\frac{2\pi}{|d|\sqrt{N}}\right)^{-s} \Gamma(s) L_E(s, \chi_d) = \Phi_E(s, \chi_d) = w_E \chi_d(-N) \Phi(1-s, \chi_d); \tag{4}$$

it is actually the L-function of another elliptic curve, namely the quadratic twist of E by d.

The conjecture of Birch and Swinnerton-Dyer predicts that the order of vanishing of the L-function of an elliptic curve at the central critical point $s = 1/2$ is the same as the rank of the Mordell-Weil group of the elliptic curve. Thus it would be of interest to find an asymptotic formula for

$$V_E(x) := \sum_{\substack{|d| \le x \\ w_E \chi_d(-N)=1 \\ L_E(1/2,\chi_d)=0}} 1 \tag{5}$$

since this would potentially be counting how often the twists of a given elliptic curve have rank at least 2. (Note that we have restricted the sum to twists for which the sign of the functional equation is $+1$; these L-functions vanish to order at least 2 because of the symmetry implied by the functional equation.) Goldfeld [G] has predicted that $V_E(x) = o(x)$. More specifically, he predicts that asymptotically, $1/2$ of all twists will have rank 0 and $1/2$ of all twists will have rank 1; consequently ranks 2 and higher should be infrequent. We will give a more precise conjecture about the frequency of twists with ranks at least 2.

Sarnak has predicted that $V_E(x)$ should be about $x^{3/4}$. His reasoning has to do with the formulas of Waldspurger [Wa], Shimura [Sh], and Kohnen and Zagier [KZ] which relate the value of $L_E(1/2, \chi_d)$ to the Fourier coefficient of a half-integral weight modular form. Roughly,

$$L_E(1/2, \chi_d) = \kappa_E c_E(|d|)^2/\sqrt{d} \tag{6}$$

where κ_E depends only on E and where the integers $c_E(|d|)$ are the Fourier coefficients of a half-integral weight form. The Ramanujan conjecture for these coefficients predicts that $c_E(|d|) \ll |d|^{1/4+\epsilon}$ for every $\epsilon > 0$. If $c_E(d)$ takes on each integer value up to $|d|^{1/4}$ about the same number of times for $|d| \le x$, then it should take the value 0 about $x^{3/4}$ times.

Using random matrix theory, we would like to give a conjecture of the form

$$V_E(x) \sim b_E x^{3/4} (\log x)^{e_E} \tag{7}$$

for certain constants b_E and e_E. The basic idea is to regard the family

$$\mathcal{F}_{E^+} = \{L_E(s, \chi_d) : w_E \chi_d(-N) = +1\} \tag{8}$$

as an orthogonal family, in the sense of the families introduced by Katz and Sarnak ([KS1], [KS2]). More specifically, this family conjecturally has symmetry type O^+. Thus, for example, we believe that the statistics of the low lying zeros of the L-functions in *this family* will match the statistics of eigenvalues near 1 of the matrices in $SO(2N)$.

The point of departure for our conjectures is the work of Keating and Snaith [KeSn1] and [KeSn2] (see also [BH] and [CF]) which indicates that the moments

$$M_E(T,k) = \frac{1}{T^*} \sum_{\substack{|d|\le T \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} L_E(1/2,\chi_d)^k$$

(with $T^* = \sum_{\substack{|d|\le T \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} 1$) apparently behave like the moments of the characteristic polynomials of matrices in $SO(2N)$ where N is of size $\log T$. Precisely, they conjecture that

$$M_E(T,k) \sim g_k(O^+) a_k(E) (\log T)^{k(k-1)/2} \tag{9}$$

where

$$g_k(O^+) = 2^{k(k+1)/2} \prod_{\ell=1}^{k-1} \frac{\ell!}{2\ell!}$$

for integer k and

(10)

$$a_k(E) = \prod_p \left(1-\frac{1}{p}\right)^{k(k-1)/2} \left(\frac{\left(1-\frac{a_p}{p}+\frac{1}{p}\right)^{-k} + \left(1+\frac{a_p}{p}+\frac{1}{p}\right)^{-k}}{2} \frac{p}{p+1} + \frac{1}{p+1} \right).$$

This conjecture arises from arithmetical considerations together with the fact from random matrix theory that the moments of the characteristic polynomials of matrices in SO($2N$), evaluated at the point 1, averaged over the group can be explicitly evaluated. Thus,

$$\begin{aligned} M_O(N,s) &= \int_{SO(2N)} |\det(U-I)|^s \, dU \\ &= 2^{2Ns} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(s+j-1/2)}{\Gamma(j-1/2)\Gamma(s+j+N-1)} \end{aligned} \tag{11}$$

where dU is the Haar measure for SO($2N$). The connection with g_k is that

$$M_O(N,k) \sim g_k(O^+) N^{k(k-1)/2} \tag{12}$$

as $N\to\infty$. Note that the formula for $g_k(O^+)$ can be extended to all real k by

$$g_k(O^+) = 2^{k^2/2} \frac{G(1+k)\sqrt{\Gamma(1+2k)}}{\sqrt{G(1+2k)\Gamma(1+k)}} \tag{13}$$

where G is Barnes' Double Gamma function.

Continuing to follow Keating and Snaith ([KeSn2], equations (74) - (81)), we observe that knowledge of all the complex moments of characteristic polynomials, evaluated at 1, in $SO(2N)$ gives complete information about the density function for the distribution of values of the characteristic polynomials at this point. Specifically, the latter is the Mellin transform of the former:

$$P_O(N,x) = \frac{1}{2\pi i x}\int_{(c)} M_O(N,s) x^{-s} ds \tag{14}$$

where (c) denotes the vertical line path from $c - i\infty$ to $c + i\infty$; this formula is valid for all real x. Note that $P_O(N,x)\ dx$ gives the probability that $\det(U - I) = x$ for an element U of $SO(2N)$.

For small positive x, the pole of $M_O(N,s)$ at $s = -1/2$ determines the dominating behavior of $P_O(N,x)$. In fact, we see that

$$\begin{aligned} P_O(N,x) \sim & x^{-1/2} 2^{-N} \Gamma(N)^{-1} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(j)}{\Gamma(j-1/2)\Gamma(j+N-3/2)} \\ := & x^{-1/2} h(N) \end{aligned} \tag{15}$$

as $x \to 0^+$. As $N \to \infty$,

$$h(N) \sim 2^{-7/8} G(1/2) \pi^{-1/4} N^{3/8}. \tag{16}$$

An interpretation of the above is that the probability that an element of SO($2N$) has a characteristic polynomial whose value at 1 is X or smaller is

$$\sim \int_0^X x^{-1/2} h(N)\ dx = 2X^{1/2} h(N). \tag{17}$$

We apply this reasoning to the values of $L_E(1/2, \chi_d)$. In particular, by (6) the fact that the $c_E(|d|)$ are integers implies that these values are discretized. If, for example, it is known that

$$L_E(1/2, \chi_d) < \kappa_E/\sqrt{|d|}, \tag{18}$$

then it follows that $L_E(1/2, \chi_d) = 0$. Similarly, if

$$\kappa_E/\sqrt{|d|} \le L_E(1/2, \chi_d) < 4\kappa_E/\sqrt{|d|}, \tag{19}$$

then it must be the case that $L_E(1/2, \chi_d) = \kappa_E/\sqrt{|d|}$.

We assume now that the distribution of values of $L_E(1/2, \chi_d)$ will behave like the values of the determinants of random orthogonal matrices with some suitable restrictions and use this assumption to conjecture results about the frequency of vanishing of $L_E(1/2, \chi_d)$. The restrictions we have in mind are of an arithmetical nature. First of all, we want to include the arithmetical factor $a_k(E)$. We expect that

$$M_E(T,s) \sim a_s(E) M_O(N,s) \tag{20}$$

with $N \sim \log T$. Thus, using

$$P_E(T,x) = \frac{1}{2\pi i x}\int_{(c)} M_E(T,s)x^{-s}ds \tag{21}$$

together with (20), we reiterate the conjecture of [KeSn2] (equation (81)) that

(22)

$$\begin{aligned} P_E(T,x) &\sim a_{-1/2}(E)x^{-1/2}2^{-N}\Gamma(N)^{-1}\prod_{j=1}^{N}\frac{\Gamma(N+j-1)\Gamma(j)}{\Gamma(j-1/2)\Gamma(j+N-3/2)} \\ &= a_{-1/2}x^{-1/2}h(N) \end{aligned}$$

should approximate for small x the probability density function for values of $L_E(1/2,\chi_d)$. Of course, this formula cannot be too accurate as we have already remarked that the values of $L_E(1/2,\chi_d)$ are discretized. The precise nature of this discretization is somewhat involved; it involves the constant κ_E for which we have explicit formulas, but it also involves the coefficients $c_E(|d|)$ whose arithmetic nature is difficult to describe. Simplistically, we would like to use the integral of (22), as in (17), and (18) to predict that

$$\begin{aligned} &\#\{|d| \le T : L_E(1/2,\chi_d) = 0, L_E(s,\chi_d) \in \mathcal{F}_{E^+}\} \\ &\sim \frac{8}{3}\sqrt{\kappa_E}a_{-1/2}T^*/T^{1/4}h(N) \end{aligned} \tag{23}$$

with $N \sim \log T$. However, the $c(|d|)$ are divisible by some predetermined powers of 2 which change this discretization. For example, in the case of the congruent number curve $E_{32} : y^2 = x^3 - x$, the number $c(|d|)$ is divisible by $\tau(d)$ for squarefree d where $\tau(d)$ is the number of divisors of d. Thus, if

$$L_E(1/2,\chi_d) < \kappa_{E_{32}}\tau(d)^2/\sqrt{|d|}, \tag{24}$$

then $L_{E_{32}}(1/2,\chi_d) = 0$. If we introduce the factor $\tau(d)$ into (23) it will raise the expected frequency of vanishing by a factor of about $\log T$ giving a total order of magnitude $T^{3/4}(\log T)^{11/8}$ for the frequency of vanishing. We expect this to be the correct order of magnitude but are not able to say yet what constant we expect.

If we restricted to prime twists ($|d| = p$) then the extra powers of 2 are not so significant. This leads to

Conjecture 1. *Let E be an elliptic curve defined over Q. Then there is a constant $c_E > 0$ such that*

$$\sum_{\substack{p \le T \\ L_E(1/2,\chi_p)=0 \\ L_E(s,\chi_p)\in\mathcal{F}_{E^+}}} 1 \sim c_E T^{3/4}(\log T)^{-5/8}. \tag{25}$$

We will return to a discussion of the value of c_E in another paper.

We remark that our arguments apply equally well to newforms f of weight 4 with integral Fourier coefficients and even functional equation. Here we have a different discretization and expect that

$$\sum_{\substack{p \le T \\ L_f(1/2,\chi_p)=0}} 1 \sim c_f T^{1/4}(\log T)^{-5/8}. \tag{26}$$

Note in particular that the exponent on T is now $1/4$. For a newform of weight 6 or higher, we expect that there will be at most a finite number of twists that vanish. We have some numerical evidence to support these conjectures.

For the remainder of this paper we would like to discuss a numerical experiment which allows us to skirt the delicate issue of the arithmetic nature of the $c(|d|)$. For a prime p we consider the ratios

$$R_p(T) = \Big(\sum_{\substack{|d|\le T \\ \chi_d(p)=1 \\ L_E(1/2,\chi_d)=0 \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} 1\Big) / \Big(\sum_{\substack{|d|\le T \\ \chi_d(p)=-1 \\ L_E(1/2,\chi_d)=0 \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} 1\Big). \tag{27}$$

By considering this ratio, the powers of T, of $\log T$, and the constants intrinsic to the curve E should all cancel out.

More generally, let

$$Q_p(k) = \lim_{T\to\infty} \sum_{\substack{|d|\le T \\ \chi_d(p)=1 \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} L_E(1/2,\chi_d)^k / \sum_{\substack{|d|\le T \\ \chi_d(p)=-1 \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} L_E(1/2,\chi_d)^k \tag{28}$$

assuming that this limit exists. What is a reasonable conjecture for $Q_p(k)$?

Using standard techniques from analytic number theory (see [I]), we can evaluate $Q_p(1)$. Based on the analysis involved in such an evaluation, we expect that

$$Q_p(k) = \frac{(p+1+a_p)^k}{(p+1-a_p)^k} \tag{29}$$

where a_p is the p-th Fourier coefficient of the modular form associated with E. The heuristics are as follows. Consider either sum that appears in (28)

$$\begin{aligned} \sum_{\substack{|d|\le T \\ \chi_d(p)=\pm1 \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} L_E(1/2,\chi_d)^k &= \sum_{\substack{|d|\le T \\ \chi_d(p)=\pm1 \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} \left(\sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n} \right)^k \\ &= \sum_{\substack{|d|\le T \\ \chi_d(p)=\pm1 \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} \sum_{n=1}^{\infty} \frac{b_n \chi_d(n)}{n} \end{aligned} \tag{30}$$

where

$$b_n = \sum_{n=n_1 n_2\cdots n_k} a_{n_1} a_{n_2} \cdots a_{n_k},$$

the sum being over all ways of writing n as a product of k factors. Summing over d, the main contribution to (30) comes from those n's that are of the form $p^r m^2$, i.e. a power of p times a perfect square (since, unless $(d,n)>1$, these always have $\chi_d(n) = \chi_d(p)^r$ while, for other n's, we get cancellation as we sum over d). So, the main contribution to (30) is roughly

$$\sum_{\substack{|d|\le T \\ \chi_d(p)=\pm1 \\ L_E(s,\chi_d)\in\mathcal{F}_{E+}}} \sum_{p^r m^2} \frac{b_{p^r m^2} \chi_d(p)^r}{p^r m^2}$$

But, $b_{uv} = b_u b_v$ when $(u,v)=1$, so the inner sum above equals

$$\sum_{(m,p)=1} \frac{b_{m^2}}{m^2} \sum_{r=0}^{\infty} \frac{b_{p^r}\chi_d^r(p)}{p^r}.$$

Now,

$$\sum_{r=0}^{\infty} \frac{b_{p^r}\chi_d^r(p)}{p^r} = \left(\sum_{r=0}^{\infty} \frac{a_{p^r}\chi_d^r(p)}{p^r} \right)^k. \tag{31}$$

Further, one can write an explicit formula for a_{p^r} in terms of a_p and p. Assume that E has good reduction mod p. Writing the corresponding Euler factor $(1 - a_p p^{-s} + p^{1-2s}) = (1-\alpha p^{-s})(1-\beta p^{-s})$, with $\alpha + \beta = a_p$ and $\alpha\beta = p$, we have, using partial fractions, $a_{p^r} = (\alpha^{r+1} - \beta^{r+1})/(\alpha - \beta)$. Substituting this into (31), and summing the geometric series we get

$$\left(\sum_{r=0}^{\infty} \frac{a_{p^r}\chi_d^r(p)}{p^r} \right)^k = \frac{p^{2k}}{(p+1-\chi_d(p)a_p)^k}.$$

Hence, the only difference in the numerator and denominator of (28), asymptotically, is a factor of

$$\frac{(p+1+a_p)^k}{(p+1-a_p)^k}.$$

Thus, by the random matrix theory considerations above (as in (15)), taking $k = -1/2$ leads to

Conjecture 2. *With $R_p(T)$ defined as above, and E having good reduction mod p,*

$$R_p = \lim_{T\to\infty} R_p(T) = \sqrt{\frac{p+1-a_p}{p+1+a_p}}.$$

Note that the number N_p of points on the elliptic curve E over the finite field F_p of p elements can be computed as

$$N_p = p + 1 - a_p$$

so that the above ratio is the square-root of the ratio of the number of points on E/F_p to the number of points on E^χ/F_p for any character χ with $\chi(p) = -1$.

We conclude with some numerical evidence to support these conjectures. We consider three elliptic curves which we call E_{11}, E_{19}, and E_{32}. These are associated to the unique newforms of weight 2 and levels 11, 19, and 32 respectively. The values of $L(s,\chi_d)$ were evaluated by computing the $c(|d|)$'s of the corresponding weight 3/2 forms. For the level 11 and 19 curves, we only computed these for $d < 0$ and d odd. For the level 32 curve, we computed these for all odd d. The relevant form for the level 32 case is described in [Ko]. The forms for the level 11

p	conjectured R_p for E_{11}	data for E_{11}	conjectured R_p for E_{19}	data for E_{19}	conjectured R_p for E_{32}	data for E_{32}
3	1.2909944	1.2774873	1.7320508	1.7018241	1	0.99925886
5	0.84515425	0.84938811	0.57735027	0.57825622	1.4142136	1.4113424
7	1.2909944	1.288618	1.1338934	1.134852	1	1.0003445
11		0	0.77459667	0.76491219	1	1.0001457
13	0.74535599	0.73266305	1.3416408	1.3632977	0.63245553	0.61626177
17	1.118034	1.1282072	1.183216	1.196637	0.89442719	0.88962298
19	1	1.000864		0	1	1.0006726
23	1.0425721	1.0470095	1	0.99857962	1	1.0000812
29	1	0.99769402	0.81649658	0.80174375	1.4142136	1.4615854
31	0.80064077	0.78332934	1.1338934	1.143379	1	1.0008405
37	0.92393644	0.91867671	0.9486833	0.94311279	1.0540926	1.0603105
41	1.2126781	1.2400086	1.1547005	1.1683113	0.78446454	0.76494748
43	1.1470787	1.1642671	1.0229915	1.0229106	1	1.0006774
47	0.84515425	0.82819492	1.0645813	1.0708874	1	0.99951502
53	1.118034	1.1332312	0.79772404	0.77715638	0.76696499	0.74137107
59	0.91986621	0.91329134	1.1055416	1.1196252	1	0.99969828
61	0.82199494	0.79865031	1.0162612	1.0199932	1.1766968	1.1996892
67	1.1088319	1.1216776	1.0606602	1.0705574	1	1.0002831
71	1.0425721	1.0497774	0.91986621	0.90939741	1	0.99992715
73	0.94733093	0.94345043	1.099525	1.1110782	1.0846523	1.0950853
79	1.1338934	1.1562237	0.90453403	0.8922209	1	0.99882039
83	1.0741723	1.0854551	0.8660254	0.84732408	1	0.99979996
89	0.84515425	0.82410673	0.87447463	0.85750248	0.89442719	0.88154899
97	1.0741723	1.0877289	0.92144268	0.90867892	0.8304548	0.80811684
101	0.98058068	0.97846254	0.94280904	0.93032086	1.0198039	1.0229108
103	1.1677484	1.1976448	0.87333376	0.855721	1	1.0004009
107	0.84515425	0.82186438	1.183216	1.2153554	1	1.0009282
109	0.91287093	0.89933354	1.1577675	1.1844329	0.94686415	0.94015124
113	0.92393644	0.9146531	0.9486833	0.93966595	1.1313708	1.1534106
127	0.93933644	0.93052596	0.98449518	0.98005032	1	0.99904006
131	1.1470787	1.171545	1.1208971	1.1413931	1	0.99916309
137	1.052079	1.0603352	1.0219806	1.0285831	1.1744404	1.2066518
139	0.93094934	0.91532106	1.0975994	1.1176423	1	1.0000469
149	1.069045	1.0833831	0.86855395	0.84844439	0.91064169	0.89706709

Table 1. A table in support of Conjecture 2, comparing R_p v.s. $R_p(T)$ for the three elliptic curves E_{11}, E_{19}, E_{32} (T equal to 333605031, 263273979, 930584451 respectively). More of this data, for $p < 2000$, is depicted in the figures below. The 0 entries for $p = 11$ and $p = 19$ are explained by the fact that we are restricting ourselves to twists with even functional equation, $w_E \chi_d(-N) = 1$. Hence for E_{11} and E_{19}, we are only looking at twists with $\chi_d(11) = \chi_d(19) = -1$.

and 19 cases can be computed according to [Gr] and were given to us by Fernando Rodriguez-Villegas.

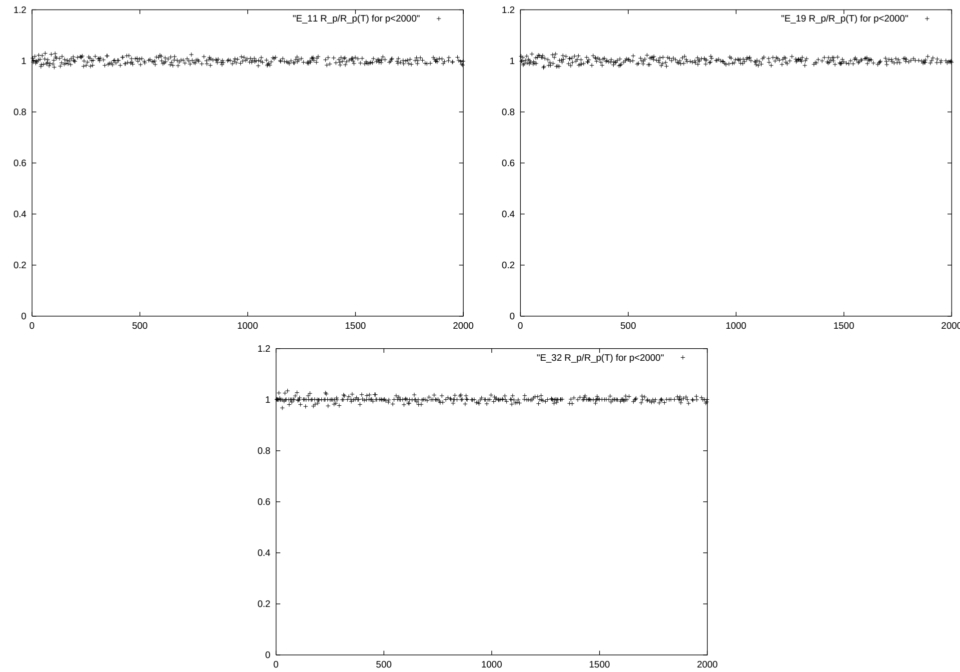

FIGURE 1. Pictures depicting $R_p/R_p(T)$, for $p < 2000$, T as in Table 1.

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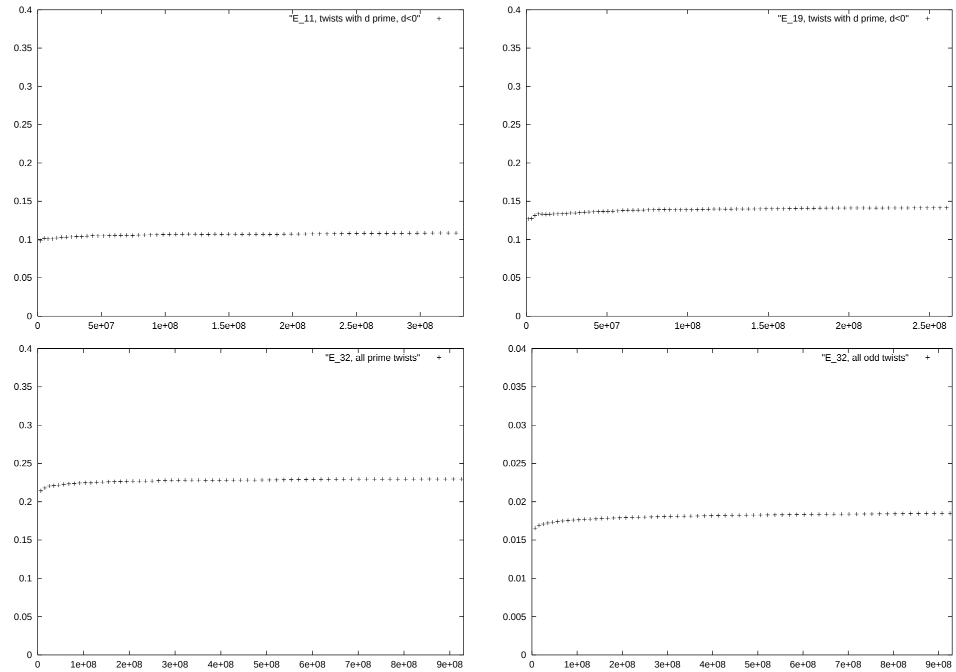

FIGURE 2. Figures in support of Conjecture 1. These depict the l.h.s. of (25) divided by $T^{3/4}(\log T)^{-5/8}$. For the level 11 and 19 curves, we only looked at twists with $d < 0$, d prime, even functional equation. We also depict the l.h.s of (23) divided by $T^{3/4}(\log T)^{11/8}$ for the level 32 curve and odd d. While the pictures are reasonably flat, $\log(T)$ is almost constant for most of the interval in question. The flatness we are observing reflects the main dependence on $T^{3/4}$.

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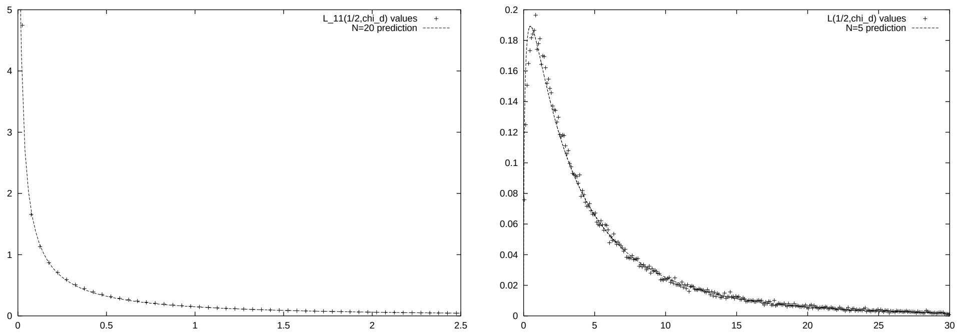

FIGURE 3. The first picture depicts the value distribution of $L_{E_{11}}(1/2, \chi_d)$, for prime $|d|$, $-788299808 < d < 0$, even functional equation, compared to $P_O(N, x)$, with $N = 20$. For contrast, we depict, in the second picture, the value distribution of $L(1/2, \chi_d)$ (Dirichlet L-functions) for all fundamental $800000 < |d| < 1000000$. Here, the Katz-Sarnak philosophy predicts a Unitary Symplectic family, and so we compare the data against $P_{USp}(N, x)$, $N = 5$. In these pictures, we have renormalized the L values so as to have the same means as $P_O(20, x)$, and $P_{USp}(5, x)$ respectively, and have not incorporated the a_k values into the pictures.

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