Commun. Math. Phys. 237, 365–395 (2003)
Digital Object Identifier (DOI) 10.1007/s00220-003-0852-2

Communications in
**Mathematical
Physics**

Autocorrelation of Random Matrix Polynomials

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Received: 1 August 2002 / Accepted: 25 December 2002
Published online: 7 May 2003 –

Abstract: We calculate the autocorrelation functions (or shifted moments) of the characteristic polynomials of matrices drawn uniformly with respect to Haar measure from the groups $U(N)$, $O(2N)$ and $USp(2N)$. In each case the result can be expressed in three equivalent forms: as a determinant sum (and hence in terms of symmetric polynomials), as a combinatorial sum, and as a multiple contour integral. These formulae are analogous to those previously obtained for the Gaussian ensembles of Random Matrix Theory, but in this case are identities for any size of matrix, rather than large-matrix asymptotic approximations. They also mirror exactly the autocorrelation formulae conjectured to hold for L-functions in a companion paper. This then provides further evidence in support of the connection between Random Matrix Theory and the theory of L-functions.

1. Introduction

The conjectured connection between random matrices and number theory dates back to an exchange between H. L. Montgomery and F. J. Dyson [18] in which they discovered that the two-point correlation function of the zeros of the Riemann zeta function, studied by the former, is the same, in the appropriate limit, as the two-point correlation function of the eigenvalues of random matrices, calculated by the latter. Since then calculations of the three-point zero correlation function by Hejhal [12], the general n-point zero correlation functions by Rudnick and Sarnak [22] and Bogomolny and Keating [3, 4], the study of the low-lying zeros of families of L-functions by Katz and Sarnak [15], and extensive numerical computations [20, 21] have strengthened the connection.

In the past few years, following the work of Keating and Snaith [16, 17], Conrey and Farmer [7] and Hughes, Keating and O'Connell [13, 14], it has become clear that the leading order asymptotics of the mean values (or moments) of the Riemann zeta function and families of L-functions can be understood, again conjecturally, in terms of the corresponding value distribution of the characteristic polynomials of random matrices. In the random matrix case, the average is performed with respect to Haar measure for either the group of unitary ($U(N)$), orthogonal ($O(2N)$) or unitary symplectic ($USp(2N)$) matrices, depending on the symmetries of the family in question.

Our purpose here is to calculate the autocorrelation functions (sometimes called the shifted moments) for the characteristic polynomials of random matrices from the groups just listed. Specifically, let $\Lambda_M(s)$ represent the characteristic polynomial of a matrix M associated with an element of a compact group G, and let dM denote Haar measure on G. We calculate

$$\int_G \Lambda_M(s_1^{-1})\cdots\Lambda_M(s_m^{-1})\Lambda_{M^\dagger}(s_{m+1})\cdots\Lambda_{M^\dagger}(s_n)dM \tag{1.1}$$

when $G=U(N)$ (here $M^\dagger$ is the Hermitian conjugate of M), and

$$\int_G \Lambda_M(s_1^{-1})\cdots\Lambda_M(s_k^{-1})dM \tag{1.2}$$

when $G=O(2N)$ and $G=USp(2N)$. (The reason for having a different definition in the first case is related to symmetries in the eigenvalue spectra.) In each case the result will be presented in three equivalent forms: as a determinant sum, in the style of Basor and Forrester [2] (and hence in terms of symmetric polynomials); as a combinatorial sum; and as a contour integral, in the style of Brézin and Hikami [5].

Conjectures based on these random matrix results for the autocorrelation functions of L-functions are presented in a companion paper to this one [8]. We here prove the results stated there. The combination of the random matrix results derived here and the numerical evidence in favour of the conjectures for L-functions put forward in [8] add considerable weight to the idea that there are fundamental connections between the two subjects. In addition, the random matrix calculations carry an interest of their own in connection with work on Toeplitz matrices [2, 6] in the unitary case, and with the elegant dual pair method of Zirnbauer and Nonnenmacher [19] for all three of the above mentioned compact groups.

Similar calculations to those described here have been performed on ensembles of Hermitian matrices, first by Andreev and Simons [1] and then by Brézin and Hikami [5]. In those cases the analogous formulae are asymptotic approximations in the large-matrix limit. The expressions we obtain here are exact. Several stages of our work were inspired by [2] and [5]. We note that Fyodorov and Strahov [10, 11, 9] have recently extended the results of [1] and [5] for products as well as for ratios of shifted characteristic polynomials of Hermitian matrices.

This paper is divided into three main sections, one devoted to each of the three compact groups: $U(N)$, $O(2N)$ and $USp(2N)$. In each we briefly present a related conjecture for the autocorrelation functions for families of L-functions having the same unitary, orthogonal or symplectic symmetry. For more details on the number theoretical side, see [8].

2. Unitary Group: $U(N)$

As mentioned in the introduction, we will calculate the autocorrelation function

$$\int_{U(N)} \Lambda_M(s_1^{-1})\cdots\Lambda_M(s_m^{-1})\Lambda_{M^\dagger}(s_{m+1})\cdots\Lambda_{M^\dagger}(s_n)dM, \tag{2.1}$$

where dM denotes Haar measure. The characteristic polynomial, which in this case we will define as

$$\Lambda_M(s) = \det(I - Ms) = \prod_{n=1}^{N}(1 - e^{i\theta_n} s), \tag{2.2}$$

where $e^{i\theta_n}$ are the eigenvalues of M, obeys the functional equation

$$\Lambda_M(s) = (-1)^N \det M s^N \Lambda_{M^\dagger}(1/s), \tag{2.3}$$

where $MM^\dagger = I$.

We will actually examine

$$\begin{aligned} I_{m,n}(U(N), w) &\equiv I_{m,n}(U(N); w_1, \ldots, w_m; w_{m+1}, \ldots, w_n) \\ &= \prod_{k=m+1}^{n} w_k^N \int_{U(N)} \prod_{i=m+1}^{n} \Lambda_M(w_i^{-1}) \prod_{j=1}^{m} \Lambda_{M^\dagger}(w_j) dM, \end{aligned} \tag{2.4}$$

for which it transpires that the result is simply interpreted through the work of Nonnenmacher and Zirnbauer [19] as a character of the group $U(n)$. This is related to the correlation function (2.1) via

$$\begin{aligned} &\int_{U(N)} \Lambda_M(s_1^{-1}) \cdots \Lambda_M(s_m^{-1}) \Lambda_{M^\dagger}(s_{m+1}) \cdots \Lambda_{M^\dagger}(s_n) dM \\ &= \left(\prod_{i=1}^{m} s_i^{-N}\right) I_{n-m,n}(U(N); s_{m+1}, \ldots, s_n; s_1, \ldots, s_m). \end{aligned} \tag{2.5}$$

Our initial approach in this case (up to (2.9)) is identical to [2]. We present this part of the calculation in full, because it will be generalized in the subsequent sections to the cases of $O(2N)$ and $USp(2N)$.

Using the expression for Haar measure in terms of the eigenvalues of M [23],

$$\begin{aligned} I_{m,n}(U(N), w) = \frac{1}{N!(2\pi)^N} &\int_0^{2\pi} \cdots \int_0^{2\pi} \\ &\times \left[\prod_{p=1}^{N}\left[\left(\prod_{r=1}^{m}(1 - e^{-i\theta_p} w_r)\right)\left(\prod_{j=m+1}^{n}(w_j - e^{i\theta_p})\right)\right]\right] \\ &\times \prod_{1\le \ell < q \le N} |e^{i\theta_q} - e^{i\theta_\ell}|^2 d\theta_1 \cdots d\theta_N. \end{aligned} \tag{2.6}$$

Let Δ denote the Vandermonde determinant

$$\Delta(x_1, \ldots, x_n) \equiv \prod_{1\le j<k\le n} (x_k - x_j) = \det\left[x_j^{k-1}\right]_{1\le j,k\le n}. \tag{2.7}$$

The object is to create in the integrand in (2.6) a Vandermonde determinant in the variables $w_1, \ldots, w_n, e^{i\theta_1}, \ldots, e^{i\theta_N}$. To this end we introduce an extra factor $\prod_{1\le \ell<m\le k}(w_m - w_\ell)$, and, making use of the symmetry of the rest of the integrand,

replace $\prod_{1\le\ell<q\le N}|e^{i\theta_q}-e^{i\theta_\ell}|^2$ by $(N!\prod_{j=1}^N e^{-i(j-1)\theta_j})\prod_{1\le\ell<q\le N}(e^{i\theta_q}-e^{i\theta_\ell})$ in the integral. This gives

$$\begin{aligned}
&I_{m,n}(U(N),w)\\
&= \frac{(-1)^{(n-m)N}}{(2\pi)^N\prod_{1\le\ell<q\le n}(w_q-w_\ell)}\int_0^{2\pi}\cdots\int_0^{2\pi}\left[\prod_{p=1}^N e^{-im\theta_p}\right]\\
&\quad\times\left[\prod_{p=1}^N\left(\prod_{r=1}^m(e^{i\theta_p}-w_r)\right)\left(\prod_{j=m+1}^n(e^{i\theta_p}-w_j)\right)\right]\left(\prod_{1\le\ell<q\le n}(w_q-w_\ell)\right)\\
&\quad\times\left(\prod_{1\le\ell<q\le N}(e^{i\theta_q}-e^{i\theta_\ell})\right)\left(\prod_{j=1}^N e^{-i(j-1)\theta_j}\right)d\theta_1\cdots d\theta_N\\
&= \frac{(-1)^{(n-m)N}}{(2\pi)^N\prod_{1\le\ell<q\le n}(w_q-w_\ell)}\int_0^{2\pi}\cdots\int_0^{2\pi}\left[\prod_{j=1}^N e^{-i(m+j-1)\theta_j}\right]\\
&\quad\times\begin{vmatrix}
1 & w_1 & w_1^2 & \cdots & w_1^{N+n-1}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & w_n & w_n^2 & \cdots & w_n^{N+n-1}\\
1 & e^{i\theta_1} & e^{2i\theta_1} & \dots & e^{i(N+n-1)\theta_1}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & e^{i\theta_N} & e^{2i\theta_N} & \dots & e^{i(N+n-1)\theta_N}
\end{vmatrix} d\theta_1\cdots d\theta_N.
\end{aligned}\tag{2.8}$$

If the factor $e^{-i(m+j-1)\theta_j}$ and the integration over θ_j are pulled into the row of the determinant which contains only θ_j, then the integration in the final N rows of the determinant results in zeros throughout these rows, with the exception of a diagonal line of ones running from column $m+1$ in row $n+1$ to column $m+N$ in row $n+N$. Thus we are left with the representation of I as a determinant:

$$\begin{aligned}
&I_{m,n}(U(N),w)\\
&= \frac{1}{\prod_{1\le\ell<q\le n}(w_q-w_\ell)}\begin{vmatrix}
1 & w_1 & w_1^2 & \cdots & w_1^{m-1} & w_1^{N+m} & w_1^{N+m+1} & \cdots & w_1^{N+n-1}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
1 & w_n & w_n^2 & \cdots & w_n^{m-1} & w_n^{N+m} & w_n^{N+m+1} & \cdots & w_n^{N+n-1}
\end{vmatrix}.
\end{aligned}\tag{2.9}$$

This result first appears in the work of Basor and Forrester [2].

The notation can be simplified by recalling that the general form of a Schur polynomial associated with the partition $\mu=(\mu_1,\mu_2,\dots,\mu_n)$ (where the μ_j are integers and

$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$) is

$$S_\mu(x_1, \ldots, x_n) = \frac{\begin{vmatrix} x_1^{\mu_1+n-1} & x_1^{\mu_2+n-2} & x_1^{\mu_3+n-3} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1+n-1} & x_2^{\mu_2+n-2} & x_2^{\mu_3+n-3} & \cdots & x_2^{\mu_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n^{\mu_1+n-1} & x_n^{\mu_2+n-2} & x_n^{\mu_3+n-3} & \cdots & x_n^{\mu_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & x_1^{n-3} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & x_2^{n-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n^{n-1} & x_n^{n-2} & x_n^{n-3} & \cdots & 1 \end{vmatrix}}. \tag{2.10}$$

Thus,

$$I_{m,n}(U(N), w) = S_{\lambda^{(n-m)}}(w_1, \ldots, w_n), \tag{2.11}$$

where $\lambda^{(n-m)} = (N, N, \ldots, N)$, with $(n-m)$ N's. This is, as predicted from the approach of Zirnbauer and Nonnenmacher [19] using Lie theory and dual pairs, a character of an irreducible representation of the group $U(n)$ when $w_1, \ldots, w_n$ lie on the unit circle.

We concentrate now on the determinant

$$\begin{aligned} D_{N,m,n}&(w_1, \ldots, w_n) \\ &\equiv \begin{vmatrix} 1 & w_1 & w_1^2 & \cdots & w_1^{m-1} & w_1^{N+m} & \cdots & w_1^{N+n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_n & w_n^2 & \cdots & w_n^{m-1} & w_n^{N+m} & \cdots & w_n^{N+n-1} \end{vmatrix} \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) w_{\sigma(1)}^0 w_{\sigma(2)}^1 w_{\sigma(3)}^2 \cdots w_{\sigma(m)}^{m-1} w_{\sigma(m+1)}^{N+m} w_{\sigma(m+2)}^{N+m+1} \cdots w_{\sigma(n)}^{N+n-1}, \end{aligned} \tag{2.12}$$

where the sum is over S_n, all permutations of $\{1, 2, \ldots, n\}$. We break up the sum over all permutations into subsets. Let Ξ_m be the set of the $\binom{n}{m}$ permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(n)$,

$$\begin{aligned} D_{N,m,n}(w_1, \ldots, w_n) = \sum_{\sigma \in \Xi_m} \operatorname{sgn}(\sigma) &\left(\sum_\rho \operatorname{sgn}(\rho) w_{\rho(1)}^0 w_{\rho(2)}^1 \cdots w_{\rho(m)}^{m-1} \right) \\ &\times \left(\sum_\delta \operatorname{sgn}(\delta) w_{\delta(1)}^0 w_{\delta(2)}^1 \cdots w_{\delta(n-m)}^{n-m-1} \right) \\ &\times (w_{\sigma(m+1)} w_{\sigma(m+2)} \cdots w_{\sigma(n)})^{N+m}, \end{aligned} \tag{2.13}$$

where ρ is a permutation taking $\sigma(1), \sigma(2), \ldots, \sigma(m)$ to $\rho(1), \rho(2), \ldots, \rho(m)$ and δ is a permutation taking $\sigma(m+1), \sigma(m+2), \ldots, \sigma(n)$ to $\delta(1), \delta(2), \ldots, \delta(n-m)$.

Finally, using the definition of the Vandermonde determinant from (2.7),

$$D_{N,m,n}(w_1, \ldots, w_n) = \sum_{\sigma \in \Xi_m} \operatorname{sgn}(\sigma) \left[\prod_{1 \leq \ell < j \leq m} (w_{\sigma(j)} - w_{\sigma(\ell)}) \right]$$

$$\times \left[\prod_{m+1 \le p < q \le n} (w_{\sigma(q)} - w_{\sigma(p)}) \right] \times (w_{\sigma(m+1)} w_{\sigma(m+2)} \cdots w_{\sigma(n)})^{N+m}. \tag{2.14}$$

So,

$$I_{m,n}(U(N), w) = \sum_{\sigma \in \Xi_m} \frac{(w_{\sigma(m+1)} w_{\sigma(m+2)} \cdots w_{\sigma(n)})^{N+m}}{\prod_{\substack{1 \le \ell \le m \\ m+1 \le q \le n}} (w_{\sigma(q)} - w_{\sigma(\ell)})}. \tag{2.15}$$

In (2.14) each factor $(w_i - w_j)$ is ordered such that $i > j$. In the denominator of (2.15) we wish the ordering to be such that the first w in each pair is chosen from $w_{\sigma(m+1)}, \dots, w_{\sigma(n)}$. The sign required to accomplish this reordering cancels exactly with $\mathrm{sgn}(\sigma)$ in the numerator of (2.14). Thus we obtain an expression for I as a combinatorial sum:

$$I_{m,n}(U(N), w) = \sum_{\sigma \in \Xi_m} \frac{(w_{\sigma(m+1)} w_{\sigma(m+2)} \cdots w_{\sigma(n)})^{N}}{\prod_{\substack{1 \le \ell \le m \\ m+1 \le q \le n}} (1 - w_{\sigma(\ell)} w_{\sigma(q)}^{-1})}. \tag{2.16}$$

We now use [8]

Lemma 2.1. *If*

$$G(a_1, \dots, a_m; b_1, \dots, b_{n-m}) = F(a_1, \dots, a_m; b_1, \dots, b_{n-m}) \prod_{i=1}^{m} \prod_{j=1}^{n-m} f(a_i - b_j),$$

where F is regular near $(0, \dots, 0)$ and $f(x) = \frac{1}{x} + c_0 + c_1 x + \cdots$, then

$$\sum_{\sigma \in \Xi_m} G(u_{\sigma(1)}, \dots, u_{\sigma(m)}; u_{\sigma(m+1)}, \dots, u_{\sigma(n)}) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n m!(n-m)!} \oint \cdots \oint G(z_1, \dots, z_m; z_{m+1}, \dots, z_n) \times \frac{\Delta(z_1, \dots, z_m, z_{m+1}, \dots, z_n)^2}{\prod_{i=1}^n \prod_{j=1}^n (z_i - u_j)} dz_1 \cdots dz_n,$$

where Ξ_m is the set of the $\binom{n}{m}$ permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(n)$ and the contour integrals enclose the variables u_j,

which allows us to write the sum (2.16) as a contour integral:

$$I_{m,n}(U(N); e^{-\alpha_1}, e^{-\alpha_2}, \dots, e^{-\alpha_m}; e^{-\alpha_{m+1}}, \dots, e^{-\alpha_n}) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n m!(n-m)!} \oint \cdots \oint e^{-N(z_{m+1} + z_{m+2} + \cdots + z_n)} \prod_{\substack{1 \le \ell \le m \\ m+1 \le q \le n}} (1 - e^{z_q - z_\ell})^{-1} \times \frac{\Delta(z_1, \dots, z_m, z_{m+1}, \dots, z_n)^2}{\prod_{i=1}^n \prod_{j=1}^n (z_i - \alpha_j)} dz_1 \cdots dz_n. \tag{2.17}$$

Brézin and Hikami arrive at an integral of a very similar form for the autocorrelation functions of characteristic polynomials of random Hermitian matrices in the limit of large matrix size N [5]. Note that in our case the result is an identity for any N.

2.1. Comparison with the Riemann Zeta Function. The main motivation for the calculations presented above is to understand the autocorrelation function and moments of the Riemann zeta function. The Riemann zeta function is defined for Re$s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and has a continuation to a meromorphic function on the complex plane with a single, simple pole at $s = 1$. As described in detail in [8], for the autocorrelation functions of $\zeta(s)$ we have the following:

Conjecture 2.2.

$$\int_0^T \zeta\left(\tfrac{1}{2} + \alpha_1 + it\right) \cdots \zeta\left(\tfrac{1}{2} + \alpha_k + it\right) \zeta\left(\tfrac{1}{2} - \alpha_{k+1} - it\right) \cdots \zeta\left(\tfrac{1}{2} - \alpha_{2k} - it\right) dt$$
$$= \int_0^T W_k(t; \alpha_1, \ldots, \alpha_k; \alpha_{k+1}, \ldots, \alpha_{2k}) \left(1 + O(t^{-\frac{1}{2}+\epsilon})\right) dt,$$

where

$$\begin{aligned} &W_k(t; \alpha_1, \ldots, \alpha_k; \alpha_{k+1}, \ldots, \alpha_{2k}) \\ &\quad = e^{\frac{1}{2}\log\frac{t}{2\pi}(-\alpha_1-\alpha_2-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})} \\ &\quad\quad \times \sum_{\sigma\in\Xi} e^{\frac{1}{2}\log\frac{t}{2\pi}(\alpha_{\sigma(1)}+\alpha_{\sigma(2)}+\cdots+\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}-\cdots-\alpha_{\sigma(2k)})} \\ &\quad\quad \times A_k(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(2k)}) \prod_{\substack{1\le\ell\le k \\ k+1\le m\le 2k}} \zeta(1 + \alpha_{\sigma(\ell)} - \alpha_{\sigma(m)}), \end{aligned} \tag{2.18}$$

and Ξ *is the set of the* $\binom{2k}{k}$ *permutations* $\sigma \in S_{2k}$ *such that* $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ *and* $\sigma(k+1) < \cdots < \sigma(2k)$*. Here* $A_k(u_1, \ldots, u_{2k}) \equiv A_k(u)$ *is an Euler product containing arithmetic information:*

$$A_k(u) = \prod_p \prod_{i=1}^k \prod_{j=1}^k \left(1 - \frac{1}{p^{1+u_i-u_{j+k}}}\right) \int_0^1 \prod_{j=1}^k \left(1 - \frac{e(\theta)}{p^{1/2+u_j}}\right)^{-1} \times \left(1 - \frac{e(-\theta)}{p^{1/2-u_{j+k}}}\right)^{-1} d\theta.$$

Note that by Lemma 2.1 we can also write

$$\begin{aligned} &W_k(t; \alpha_1, \ldots, \alpha_k; \alpha_{k+1}, \ldots, \alpha_{2k}) \\ &\quad = e^{\frac{1}{2}\log\frac{t}{2\pi}(-\alpha_1-\alpha_2-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})} \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \\ &\quad\quad \times \oint \cdots \oint \frac{A_k(z_1, \ldots, z_{2k}) \prod_{i=1}^k \prod_{j=1}^k \zeta(1 + z_i - z_{j+k}) \Delta(z_1, \ldots, z_{2k})^2}{\prod_{i=1}^{2k} \prod_{j=1}^{2k} (z_i - \alpha_j)} \\ &\quad\quad \times e^{\frac{1}{2}\log\frac{t}{2\pi}\sum_{j=1}^k z_j - z_{j+k}} \, dz_1 \ldots dz_{2k}. \end{aligned} \tag{2.19}$$

The Riemann zeta function satisfies a functional equation

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} \zeta(1 - s). \tag{2.20}$$

The Riemann Hypothesis is that the complex zeros of $\zeta(s)$ lie on the line $\mathrm{Re}s = 1/2$. The characteristic polynomial, on the other hand, obeys the functional equation (2.3) and its zeros lie on the unit circle, so in analogy with the autocorrelation functions of $\zeta(s)$, we let $s_j = \exp(\alpha_j)$ in (2.5). Now when α_i is purely imaginary, $e^{-\alpha_i}$ sits on the unit circle, in analogy with $1/2 + it + \alpha_i$ lying on the critical line when α_i is purely imaginary in the Riemann zeta case. We compare (2.18) with

$$\begin{aligned}\int_{U(N)} &\Lambda_M(e^{-\alpha_1})\cdots\Lambda_M(e^{-\alpha_k})\Lambda_{M^\dagger}(e^{\alpha_{k+1}})\cdots\Lambda_{M^\dagger}(e^{\alpha_{2k}})dM\\ &= e^{\frac{N}{2}(-\alpha_1-\alpha_2-\cdots-\alpha_k+\alpha_{k+1}+\cdots\alpha_{2k})}\\ &\quad\times\left(\sum_{\sigma\in\,\Xi} e^{\frac{N}{2}(\alpha_{\sigma(1)}+\alpha_{\sigma(2)}+\cdots+\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}-\cdots-\alpha_{\sigma(2k)})}\prod_{\substack{1\le\ell\le k\\ k+1\le m\le 2k}}(1-e^{\alpha_{\sigma(m)}-\alpha_{\sigma(\ell)}})^{-1}\right),\end{aligned} \tag{2.21}$$

which follows from (2.5). These two formulae clearly have a similar structure if we equate the density of the Riemann zeros and the density of the eigenvalues of M on the unit circle to obtain the relation $N = \log\frac{t}{2\pi}$. The random matrix expression is, not surprisingly, missing the arithmetical factor $A(\alpha_1, \ldots, \alpha_{2k})$; also, the function which provides the simple poles in each term of the sum is $\zeta(1+z)$ in the Riemann zeta case and $(1-e^{-z})^{-1}$ in the random matrix case.

3. Unitary Symplectic Group: $USp(2N)$

Now we turn to the group of symplectic unitary matrices, $USp(2N)$. These are $2N\times 2N$ matrices, M, with $MM^\dagger = 1$ and $M^tJM = J$, where $J = \begin{pmatrix} 0 & I_N\\ -I_N & 0\end{pmatrix}$ and I_N is the $N\times N$ identity matrix. For these matrices, the eigenvalues lie on the unit circle and come in complex conjugate pairs $e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, e^{-i\theta_2}, \ldots\ e^{i\theta_N}, e^{-i\theta_N}$. Thus we let the characteristic polynomial related to such a matrix take the form

$$\Lambda_M(s) = \det(I - Ms) = \prod_{n=1}^{N}(1-e^{i\theta_n}s)(1-e^{-i\theta_n}s). \tag{3.1}$$

The weighting in the average over $USp(2N)$ of the matrix with eigenphases $\pm\theta_1, \ldots,$ $\pm\theta_N$ is derived from Haar measure on the group, and can be manipulated into the form

$$N_{Sp}\frac{(-1)^{N(N-1)/2}}{4^{N^2}}\Delta(e^{i\theta_1},\ldots,e^{i\theta_N},e^{-i\theta_1},\ldots,e^{-i\theta_N})\prod_{k=1}^{N}(e^{i\theta_k}-e^{-i\theta_k}), \tag{3.2}$$

where $N_{Sp} = \frac{2^{2N^2-2N}}{\pi^N N!}$.

We define the autocorrelation function in this case to be

$$\begin{aligned}
I(USp(2N), w_1, \dots, w_k) &\equiv \int_{USp(2N)} \Lambda_M(w_1)\cdots\Lambda_M(w_k)dM \\
&= \frac{N_{Sp}(-1)^{N(N-1)/2}}{4^{N^2}} \int_0^{2\pi}\cdots\int_0^{2\pi} \Delta(e^{i\theta_1},\dots,e^{i\theta_N},e^{-i\theta_1},\dots,e^{-i\theta_N}) \\
&\quad \times \prod_{m=1}^{k}\prod_{n=1}^{N}(e^{i\theta_n}-w_m)(e^{-i\theta_n}-w_m)\prod_{j=1}^{N}(e^{i\theta_j}-e^{-i\theta_j})d\theta_1\cdots d\theta_N \\
&= \frac{N_{Sp}}{4^{N^2}}\frac{(-1)^{N(N-1)/2}}{\prod_{1\le i<j\le k}(w_j-w_i)}\int_0^{2\pi}\cdots\int_0^{2\pi} \\
&\quad \times \Delta(w_1,w_2,\dots,w_k,e^{i\theta_1},\dots,e^{i\theta_N},e^{-i\theta_1},\dots,e^{-i\theta_N}) \\
&\quad \times \prod_{j=1}^{N}(e^{i\theta_j}-e^{-i\theta_j})d\theta_1\cdots d\theta_N \\
&= \frac{N_{Sp}}{4^{N^2}}\frac{(-1)^{N(N-1)/2}}{\prod_{1\le i<j\le k}(w_j-w_i)}\int_0^{2\pi}\cdots\int_0^{2\pi} \\
&\quad \times \sum_{\sigma\in S_{k+2N}} \operatorname{sgn}(\sigma) w_1^{\sigma(1)-1}w_2^{\sigma(2)-1}\cdots w_k^{\sigma(k)-1}(e^{i\sigma(k+1)\theta_1}-e^{i(\sigma(k+1)-2)\theta_1}) \\
&\quad \times (e^{i\sigma(k+2)\theta_2}-e^{i(\sigma(k+2)-2)\theta_2})\cdots(e^{i\sigma(k+N)\theta_N}-e^{i(\sigma(k+N)-2)\theta_N}) \\
&\quad \times e^{-i(\sigma(k+N+1)-1)\theta_1}\cdots e^{-i(\sigma(k+2N)-1)\theta_N}d\theta_1\cdots d\theta_N. \qquad (3.3)
\end{aligned}$$

As we are integrating each θ_j from 0 to 2π, the term in the sum belonging to a given permutation σ is zero unless for every j, $\sigma(k+j) = \sigma(k+N+j)-1$ or $\sigma(k+j) = \sigma(k+N+j)+1$. Upon integration this places the condition $i_1 < i_2 < \cdots < i_k \in \{0,1,2,\dots,2N+k-1\}$, i_j is even if j is odd and i_j is odd if j is even, on the resulting sum over $k\times k$ determinants:

$$I(USp(2N), w_1,\dots,w_k) = \frac{1}{\prod_{1\le i<j\le k}(w_j-w_i)} \sum_{\substack{0\le i_1<i_2<\cdots<i_k\le 2N+k-1 \\ i_j\equiv j-1 \bmod 2}} \begin{vmatrix} w_1^{i_1} & w_1^{i_2} & \cdots & w_1^{i_k} \\ w_2^{i_1} & w_2^{i_2} & \cdots & w_2^{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ w_k^{i_1} & w_k^{i_2} & \cdots & w_k^{i_k} \end{vmatrix}. \qquad (3.4)$$

Note that this can also be written in terms of Schur functions (see (2.10)),

$$I(USp(2N), w_1,\dots,w_k) = \sum_{\lambda \text{ even}} S_\lambda(w_1,\dots,w_k), \qquad (3.5)$$

where the sum is over partitions $\lambda = (\lambda_1,\dots,\lambda_k)$ with all parts λ_j even and $2N \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k \ge 0$.

Examination of examples when k is small leads to the guess that in general,

$$I(USp(2N), w_1,\dots,w_k) = w_1^N\cdots w_k^N\left[\sum_{\epsilon_j\in\{-1,1\}}\left(\prod_{j=1}^{k} w_j^{\epsilon_j N}\right)\prod_{1\le i\le j\le k}\left(1-w_i^{-\epsilon_i}w_j^{-\epsilon_j}\right)^{-1}\right], \qquad (3.6)$$

and we will now prove this to be true.

Before embarking on the proof of (3.6) we note that letting $w_j^N = e^{b_j}$ and taking N large,

$$\begin{aligned} &I(USp(2N), e^{b_1/N}, \dots, e^{b_k/N}) \\ &\approx e^{b_1} \cdots e^{b_k} \left[\sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^{k} e^{\epsilon_j b_j} \right) \prod_{1 \le i \le j \le k} \left(\frac{\epsilon_i b_i}{N} + \frac{\epsilon_j b_j}{N} \right)^{-1} \right] \\ &= N^{\frac{k^2+k}{2}} e^{b_1} \cdots e^{b_k} \left[\sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^{k} e^{\epsilon_j b_j} \right) \prod_{1 \le i \le j \le k} (\epsilon_i b_i + \epsilon_j b_j)^{-1} \right]. \end{aligned} \tag{3.7}$$

The sum here has just the same structure as Brézin and Hikami's results for the large N asymptotics of Hermitian ensembles [5], showing that when distances are measured in terms of the mean level spacing of the eigenvalues then, as expected, in the large N limit averages over the compact groups and the Hermitian ensembles are equivalent.

To prove (3.6), we need two identities. The first is

Identity 3.1.

$$\sum_{j=1}^{n} \Delta(w_1, \dots, w_n)\big|_{w_j=0} \prod_{m=1}^{n} (1 - w_j w_m) = \left(1 - w_1^2 \cdots w_n^2\right) \Delta(w_1, \dots, w_n).$$

This is a special case, with $f(w) = \prod_{m=1}^{n}(1 - w_m w)$, of the following lemma:

Lemma 3.2. *Given a polynomial function of order n, $f(w) = c_0 + c_1 w + \cdots + c_n w^n$, we have the relation*

$$\begin{aligned} &\sum_{j=1}^{n} \Delta(w_1, \dots, w_n)\big|_{w_j=0} f(w_j) \\ &\quad = (c_0 + (-1)^{n-1} c_n w_1 \cdots w_n) \Delta(w_1, \dots, w_n). \end{aligned}$$

To prove Lemma 3.2 we notice first of all that we can write the left side of the relation as a determinant:

$$\sum_{j=1}^{n} \Delta(w_1, \dots, w_n)\big|_{w_j=0} f(w_j) = \begin{vmatrix} f(w_1) & w_1 & w_1^2 & \cdots & w_1^{n-1} \\ f(w_2) & w_2 & w_2^2 & \cdots & w_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(w_n) & w_n & w_n^2 & \cdots & w_n^{n-1} \end{vmatrix}. \tag{3.8}$$

However, since $f(w)$ is a polynomial of order n, in the first column of the above determinant, all the terms in f with coefficients $c_1, \dots, c_{n-1}$ can be cancelled by column manipulations, leaving just

$$\begin{aligned} \sum_{j=1}^{n} \Delta(w_1, \dots, w_n)\big|_{w_j=0} f(w_j) &= \begin{vmatrix} c_0 + c_n w_1^n & w_1 & \cdots & w_1^{n-1} \\ c_0 + c_n w_2^n & w_2 & \cdots & w_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_0 + c_n w_n^n & w_n & \cdots & w_n^{n-1} \end{vmatrix} \\ &= (c_0 + (-1)^{n-1} c_n w_1 \cdots w_n) \Delta(w_1, \dots, w_n). \end{aligned} \tag{3.9}$$

The second identity is

Identity 3.3.

$$\sum_{C\cup D=[n],C\cap D=\emptyset} (-1)^{S(C,D)} \prod_{\alpha\in C} w_\alpha^{n-1} \prod_{\substack{i,j\in C\\ i<j}} (w_j - w_i) \prod_{\substack{i,j\in D\\ i<j}} (w_j - w_i) \times \prod_{\substack{\alpha\in C\\ \beta\in D}} (1 - w_\alpha w_\beta) = 0,$$

where the left-hand side is a polynomial in the variables $w_1, \dots, w_n$ and the notation is $[n] = \{1, 2, \dots, n\}$, $|C|$ is the number of elements in C, $W(C, D) = \sum_{\substack{m\in C,\, n\in D\\ m>n}} 1$ and $S(C, D) = |C||D| + |C|(|C| + 1)/2 + W(C, D)$.

We prove this by showing that when $r = n - 1$, with the same notation as above,

$$\begin{aligned} &F_n(w_1, \dots, w_n; x; r) \\ &= \sum_{C\cup D=[n],C\cap D=\emptyset} (-1)^{S(C,D)} \prod_{\alpha\in C} w_\alpha^r \quad \Delta(C)\Delta(D) \\ &\quad \times \prod_{\substack{\alpha\in C\\ \beta\in D}} (x^2 - w_\alpha w_\beta) \quad x^{|D|^2+(r-n)|D|} \end{aligned} \tag{3.10}$$

is identically zero. Here $\Delta(C) = \prod_{\substack{i,j\in C\\ i<j}} (w_j - w_i)$. We proceed by showing that the polynomial $F_n(w_1, \dots, w_n; x; n - 1)$, which is of order $n(n - 1)$ in x, has at least $n(n - 1) + 1$ roots, implying that it is identically zero. Since the left-hand side of the equation in Identity 3.3 is merely the instance of $F_n(w_1, \dots, w_n; x; n-1)$ when $x = 1$, this proves Identity 3.3.

First of all we note that $F_n(w_1, \dots, w_n; x; n - 1)$ is zero when x is zero. Only the terms with $|D| = 0, 1$ contribute in this case, due to the factor $x^{|D|^2-|D|}$. Thus we are looking at

$$\begin{aligned} &(-1)^{n(n+1)/2}\Delta(w_1, \dots, w_n) \prod_{i=1}^{n} w_i^{n-1} \\ &\quad + (-1)^{n(n+1)/2-1} \sum_{j=1}^{n} \Delta(w_1, \dots, w_n)\big|_{w_j=0} \prod_{i=1}^{n} w_i^{n-1}, \end{aligned} \tag{3.11}$$

which we can see is zero by a simple application of Lemma 3.2 (with $f(w) = (-1)^{n(n+1)/2} \prod_{i=1}^{n} w_i^{n-1}$).

Next we prove that (3.10) is zero for certain values of the integer $r \le n - 1$ when $x^2 = w_a w_b$, with $a \ne b = 1, 2, \dots, n$. This yields $n(n - 1)$ other zeros (assuming none of the w_j are zero) and proves that (3.10) is identically zero for $r = n - 1$. We start with $F_n(w_1, \dots, w_n; \sqrt{w_a w_b}; r)$. We note immediately that in the sum over C and D, any term in which a and b do not occur both in C or both in D is zero. Thus,

$$\begin{aligned}
&F_n(w_1, \dots, w_n; \sqrt{w_a w_b}; r) \\
&\quad = \sum_{A\cup B=[n]_{a,b},\ A\cap B=\emptyset} (-1)^{S(A,B\cup\{a,b\})} \prod_{\alpha\in A} w_\alpha^r \quad \Delta(A)\Delta(B\cup\{a,b\}) \\
&\qquad \times \prod_{\substack{\alpha\in A \\ \beta\in B\cup\{a,b\}}} (w_a w_b - w_\alpha w_\beta) \quad (\sqrt{w_a w_b})^{|B\cup\{a,b\}|^2+(r-n)|B\cup\{a,b\}|} \\
&\qquad + \sum_{A\cup B=[n]_{a,b},\ A\cap B=\emptyset} (-1)^{S(A\cup\{a,b\},B)} \prod_{\alpha\in A\cup\{a,b\}} w_\alpha^r \quad \Delta(A\cup\{a,b\})\Delta(B) \\
&\qquad \times \prod_{\substack{\alpha\in A\cup\{a,b\} \\ \beta\in B}} (w_a w_b - w_\alpha w_\beta) \quad (\sqrt{w_a w_b})^{|B|^2+(r-n)|B|},
\end{aligned} \tag{3.12}$$

where $[n]_{a,b}$ is the set of elements $\{1, 2, \dots, n\}$ with a and b removed. However, after some manipulations we can write both the sum in (3.12) containing A and $B\cup\{a,b\}$ and the sum containing $A\cup\{a,b\}$ and B in terms of just A and B. To this end, we note that $|B\cup\{a,b\}| = |B|+2$, $S(A\cup\{a,b\}, B) = (S(A,B)+s_b+1) \mod 2$ and $S(A, B\cup\{a,b\}) = (S(A,B)+s_a) \mod 2$, where (assuming $a > b$) s_A is $\sum_{i\in A,\ a>i>b} 1$ and s_B is $\sum_{i\in B,\ a>i>b} 1$. Hence

$$\begin{aligned}
&F_n(w_1, \dots, w_n; \sqrt{w_a w_b}; r) \\
&\quad = \sum_{A\cup B=[n]_{a,b},\ A\cap B=\emptyset} (-1)^{S(A,B)+s_A+s_B} \prod_{\alpha\in A} w_\alpha^r \quad \Delta(A)\Delta(B) \\
&\qquad \times \left(\prod_{\beta\in B}(w_a - w_\beta)(w_b - w_\beta)\right)(w_a - w_b) \prod_{\substack{\alpha\in A \\ \beta\in B}}(w_a w_b - w_\alpha w_\beta) \\
&\qquad \times \prod_{\alpha\in A}(w_b - w_\alpha)(w_a - w_\alpha) \quad (\sqrt{w_a w_b})^{|B|^2+(r-n)|B|} \\
&\qquad \times (w_a w_b)^{|A|}(w_a w_b)^{2|B|+(r-n)+2} \\
&\qquad + \sum_{A\cup B=[n]_{a,b},\ A\cap B=\emptyset} (-1)^{S(A,B)+s_A+s_B+1} \prod_{\alpha\in A} w_\alpha^r \quad \Delta(A)\Delta(B) \\
&\qquad \times \left(\prod_{\alpha\in A}(w_a - w_\alpha)(w_b - w_\alpha)\right)(w_a - w_b) \prod_{\substack{\alpha\in A \\ \beta\in B}}(w_a w_b - w_\alpha w_\beta) \\
&\qquad \times \prod_{\beta\in B}(w_b - w_\beta)(w_a - w_\beta) \quad (\sqrt{w_a w_b})^{|B|^2+(r-n)|B|}(w_a w_b)^r (w_a w_b)^{|B|}.
\end{aligned} \tag{3.13}$$

Since $|A|+|B|+2 = n$, we see that the two sums above cancel each other exactly, term by term. Thus we have that

$$F_n(w_1, \dots, w_n; \sqrt{w_a w_b}; r) = 0. \tag{3.14}$$

If $n-r$ is odd, it immediately follows that $F_n(w_1, \dots, w_n; -\sqrt{w_a w_b}; r) = 0$ also, as (3.10) will be even in x. This proves Identity 3.3.

Since the proof of $F_n(w_1, \dots, w_n; \sqrt{w_a w_b}; r) = 0$ involved cancellation in (3.10) only amongst terms in which $|C|$ has the same parity, we can restrict the sum over C to sets of even cardinality or sets of odd cardinality. Note then that if $r = n-2$, we can write a further identity (which will be of use in Sect. 4.2)

Identity 3.4.

$$\sum_{\substack{C\cup D=[n], C\cap D=\emptyset \\ |C| \text{ even}}} (-1)^{S(C,D)} \prod_{\alpha\in C} w_\alpha^{n-2} \quad \Delta(C)\Delta(D) \prod_{\substack{\alpha\in C \\ \beta\in D}} (x^2 - w_\alpha w_\beta) \ \ x^{|D|^2-2|D|+1} = 0,$$

because in the same manner as above, we would see that the left side of the expression is zero when $x = +\sqrt{w_a w_b}$, $a \neq b = 1, 2, \dots, n$. Note that an extra factor of x has been included in each term to ensure that the expression on the left of Identity 3.4 is a polynomial in x; that is, there are no terms with negative exponents on x. To deal with $x = -\sqrt{w_a w_b}$, we note that if n is odd, the expression is an even polynomial in x, and if n is even, then $x^{|D|^2+2|D|+1}$ is always an odd power of x. Thus the expression is zero when $x = \pm\sqrt{w_a w_b}$, $a \neq b = 1, 2, \dots, n$ (this means we have $n(n-1)$ zeros), and the polynomial in x is of order $n(n-1) - n + 1$. Thus if $n \geq 2$, it is everywhere zero and Identity 3.4 is true.

We are now in a position to return to the proof of (3.6). We need to prove that this is identical to (3.4). We will now prove that

$$\begin{aligned} \frac{1}{\Delta(w_1, \dots, w_k)} \sum_{\substack{0\le i_1<i_2<\dots<i_k\le n \\ i_j \equiv j-1 mod 2}} \begin{vmatrix} w_1^{i_1} & w_1^{i_2} & \cdots & w_1^{i_k} \\ w_2^{i_1} & w_2^{i_2} & \cdots & w_2^{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ w_k^{i_1} & w_k^{i_2} & \cdots & w_k^{i_k} \end{vmatrix} \\ = w_1^{(n-k+1)/2} \cdots w_k^{(n-k+1)/2} \sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{m=1}^{k} w_m^{\epsilon_m (n-k+1)/2} \right) \\ \times \left(\prod_{1\le m\le q\le k} (1 - w_m^{-\epsilon_m} w_q^{-\epsilon_q})^{-1} \right). \end{aligned} \tag{3.15}$$

As a first step it is convenient to add to the notation already introduced to help simplify the equations. If A and B are sets of positive integers, then we let $w_A = \prod_{m\in A} w_m$. Further, we define

$$E(A) = \prod_{\substack{m\le n \\ m,n\in A}} (1 - w_m w_n) \tag{3.16}$$

as well as

$$E(A, B) = \prod_{\substack{m\in A \\ n\in B}} (1 - w_m w_n), \tag{3.17}$$

and, as previously,

$$\Delta(A) = \prod_{\substack{m<n \\ m,n\in A}} (w_n - w_m) \tag{3.18}$$

and

$$D(A,B) = \prod_{\substack{m\in A\\ n\in B}} (w_n - w_m). \tag{3.19}$$

Armed with this notation, the right side of (3.15) can be written, where A is the set of indices j for which $\epsilon_j = +1$, as

$$\begin{aligned}
&w_1^{(n-k+1)/2} \cdots w_k^{(n-k+1)/2} \sum_{A\cup B=[k],\ A\cap B=\emptyset} w_A^{(n-k+1)/2} w_B^{-(n-k+1)/2}\\
&\qquad \times \left(\prod_{\substack{m,q\in A\\ m\le q}} \frac{1}{1-\frac{1}{w_m w_q}}\right)\left(\prod_{\substack{m,q\in B\\ m\le q}} \frac{1}{1-w_m w_q}\right)\left(\prod_{\substack{m\in A\\ q\in B}} \frac{1}{1-w_q/w_m}\right)\\
&= \sum_{A\cup B=[k],\ A\cap B=\emptyset} \frac{w_A^{n-k+1}}{E(A)E(B)D(A,B)}\left(\prod_{\substack{m,q\in A\\ m\le q}} -w_m w_q\right)\left(\prod_{\substack{m\in A\\ q\in B}} -w_m\right).
\end{aligned} \tag{3.20}$$

A straightforward manipulation gives

$$\begin{aligned}
\prod_{\substack{m,n\in A\\ m\le n}} -w_m w_n &= w_A^{|A|+1}(-1)^{|A|(|A|+1)/2},\\
\prod_{\substack{m\in A\\ n\in B}} -w_m &= w_A^{|B|}(-1)^{|A||B|} \text{ and}\\
E(A\cup B) &= E(A)E(B)E(A,B) \text{ and similarly}\\
\Delta(A\cup B) &= \Delta(A)\Delta(B)D(A,B)(-1)^{W(A,B)},
\end{aligned} \tag{3.21}$$

so we arrive at a re-expression of (3.15):

$$\begin{aligned}
&\sum_{\substack{0\le i_1<i_2<\cdots<i_k\le n\\ i_j\equiv j-1 mod 2}} \begin{vmatrix} w_1^{i_1} & w_1^{i_2} & \cdots & w_1^{i_k}\\ w_2^{i_1} & w_2^{i_2} & \cdots & w_2^{i_k}\\ \vdots & \vdots & \ddots & \vdots\\ w_k^{i_1} & w_k^{i_2} & \cdots & w_k^{i_k} \end{vmatrix}\\
&\qquad = \frac{1}{E([k])} \sum_{A\cup B=[k],\ A\cap B=\emptyset} (-1)^{S(A,B)} w_A^{n+2} E(A,B)\Delta(A)\Delta(B).
\end{aligned} \tag{3.22}$$

We prove this by induction on k. We see that when $k=1$

$$\sum_{\substack{0\le i_1\le n\\ i_1 \text{ even}}} w_1^{i_1} = \frac{1-w_1^{n+2}}{1-w_1^2}, \tag{3.23}$$

which clearly satisfies (3.22). We now show that if (3.22) holds with k replaced by $k-1$, then it holds for k as well. We start by expanding the determinant in (3.22) with respect to the last column so that the left side becomes

$$\sum_{\substack{k-1\le i_k\le n\\ i_k\equiv k-1 \bmod 2}} \sum_{j=1}^{k} w_j^{i_k}(-1)^{k-j} \sum_{\substack{0\le i_1<i_2<\cdots<i_{k-1}\le i_k-1\\ i_r\equiv r-1 \bmod 2}} \det\left(w_m^{i_q}\right)_{\substack{1\le m\le k,\, m\ne j\\ 1\le q\le k-1}}. \tag{3.24}$$

By the induction hypothesis, this is

$$\sum_{\substack{k-1\le i_k\le n\\ i_k\equiv k-1 \bmod 2}} \sum_{j=1}^{k} w_j^{i_k}(-1)^{k-j}\frac{1}{E([k]_j)} \sum_{\substack{A\cup B=[k]_j\\ A\cap B=\emptyset}} (-1)^{S(A,B)} w_A^{i_k+1} E(A,B)\Delta(A)\Delta(B), \tag{3.25}$$

where $A_j = A - \{j\}$.

If we redefine A to include j, and switch the order of the sum over j and the sum over the sets A and B in (3.25), we obtain

$$\sum_{\substack{k-1\le i_k\le n\\ i_k\equiv k-1 \bmod 2}} \sum_{\substack{A\cup B=[k]\\ A\cap B=\emptyset}} \sum_{j\in A} \frac{(-1)^{k-j} w_j^{i_k}}{E([k]_j)} (-1)^{S(A_j,B)} w_{A_j}^{i_k+1} E(A_j,B)\Delta(A_j)\Delta(B). \tag{3.26}$$

Applying the definition of E, it is straightforward to show that for $A\cup B=[k]$, $A\cap B=\emptyset$ and $j\in A$, then $E(A_j,B)/E([k]_j) = E(A,B)E(\{j\},A)/E([k])$. This leads us to

$$\sum_{\substack{k-1\le i_k\le n\\ i_k\equiv k-1 \bmod 2}} \frac{1}{E([k])} \sum_{\substack{A\cup B=[k]\\ A\cap B=\emptyset}} w_A^{i_k} E(A,B)\Delta(B) \times \sum_{j\in A} (-1)^{k-j}(-1)^{S(A_j,B)} w_{A_j} E(\{j\},A)\Delta(A_j). \tag{3.27}$$

In this notation, a simple generalization of Identity 3.1 can be written as

$$\sum_{j\in A} (-1)^{W(\{j\},A)} w_{A_j} E(\{j\},A)\Delta(A_j) = \Delta(A)(1-w_A^2), \tag{3.28}$$

and this combined with $(-1)^{S(A_j,B)+k-j} = (-1)^{W(\{j\},A)}(-1)^{S(A,B)+1}$, where $A\cup B = [k]$, $A\cap B=\emptyset$ and $j\in A$, gives us

$$\sum_{\substack{k-1\le i_k\le n\\ i_k\equiv k-1 \bmod 2}} \frac{1}{E([k])} \sum_{A\cup B=[k],A\cap B=\emptyset} (-1)^{S(A,B)+1}\Delta(B)E(A,B)w_A^{i_k}\Delta(A)(1-w_A^2). \tag{3.29}$$

Summing over i_k, this yields

$$\frac{1}{E([k])} \sum_{A\cup B=[k],A\cap B=\emptyset} (-1)^{S(A,B)}\Delta(A)\Delta(B)E(A,B)\left[w_A^{n+2} - w_A^{k-1}\right]. \tag{3.30}$$

Applying Identity 3.3, we see that the terms resulting from w_A^{k-1} in the square brackets above all cancel out, leaving us with

$$\frac{1}{E([k])} \sum_{A\cup B=[k], A\cap B=\emptyset} \Delta(A)\Delta(B)E(A,B)w_A^{n+2}(-1)^{S(A,B)}, \tag{3.31}$$

which proves (3.22) and so proves (3.6).

We also have the following lemma [8]

Lemma 3.5. *If F is a symmetric function of k variables, regular near $(0,\dots,0)$, and $f(x)$ has a simple pole of residue 1 at $x=0$ and is otherwise analytic in a neighbourhood of $x=0$, and either*

$$G(a_1,\dots,a_k) = F(a_1,\dots,a_k) \prod_{1\le i\le j\le k} f(a_i+a_j), \tag{3.32}$$

or

$$G(a_1,\dots,a_k) = F(a_1,\dots,a_k) \prod_{1\le i< j\le k} f(a_i+a_j), \tag{3.33}$$

then when $\pm\alpha_i \pm \alpha_j$ are contained in the region of analyticity of $f(x)$,

$$\sum_{\epsilon_j\in\{-1,1\}} G(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k) = \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k}\frac{2^k}{k!}\oint\cdots\oint G(z_1,\dots,z_k) \times \frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{i=1}^k\prod_{j=1}^k (z_i-\alpha_j)(z_i+\alpha_j)} dz_1\cdots dz_k, \tag{3.34}$$

and

$$\sum_{\epsilon_j\in\{-1,1\}} \left(\prod_{j=1}^k \epsilon_j\right) G(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k) = \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k}\frac{2^k}{k!}\oint\cdots\oint G(z_1,\dots,z_k) \times \frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^k \alpha_j}{\prod_{i=1}^k\prod_{j=1}^k (z_i-\alpha_j)(z_i+\alpha_j)} dz_1\cdots dz_k, \tag{3.35}$$

where the contour of integration encircles the $\pm\alpha$'s.

With the help of Lemma 3.5 we can write

$$I(USp(2N), e^{-\alpha_1},\dots,e^{-\alpha_k}) = \frac{(-1)^{k(k-1)/2}2^k}{(2\pi i)^k k!} e^{-N\sum_{j=1}^k \alpha_j} \oint\cdots\oint \prod_{1\le \ell\le m\le k} (1-e^{-z_m-z_\ell})^{-1} \times \frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{i=1}^k\prod_{j=1}^k (z_j-\alpha_i)(z_j+\alpha_i)} e^{N\sum_{j=1}^k z_j} dz_1\cdots dz_k. \tag{3.36}$$

3.1. Comparison with L-functions. Note that $\Lambda_M(s) = \prod_{n=1}^{N}(1 - e^{i\theta_n}s)(1 - e^{-i\theta_n}s)$ satisfies the functional equation $\Lambda_M(s) = s^{2N}\overline{\Lambda_M}(1/s)$. However, we can instead define

$$\mathcal{Z}_M(s) = s^{-N}\Lambda_M(s), \tag{3.37}$$

which satisfies $\mathcal{Z}_M(s) = \overline{\mathcal{Z}_M}(1/s)$, where $\overline{\mathcal{Z}_M}(z) = \overline{\mathcal{Z}_M(\overline{z})}$ and $\overline{z}$ denotes the complex conjugate of z .

In [8] we conjecture the form of autocorrelation functions of L-functions averaged over the family comprised of $L(s, \chi_d)$, with d a fundamental discriminant and $\chi_d(n) = (\frac{d}{n})$, where here the family is ordered by the conductor d. In that paper the conjecture is formulated in terms of a "Z-function" closely related to the L-function but satisfying the functional equation

$$Z_L(s) = \overline{Z_L}(1 - s). \tag{3.38}$$

This is analogous to the random matrix function $\mathcal{Z}_M(s)$ and its functional equation, because the transformation from s to $1 - s$ in the number theory case reflects round the symmetry point of the zeros of the L-function in the same manner as the transformation from s to $1/s$ in the random matrix theory case reflects around the symmetry point of the eigenvalues.

The family of L-functions just defined is said to show symplectic symmetry [15, 21] in as much as the statistics of the zeros around the symmetry point are those of the eigenvalues of random matrices from $USp(2N)$.

The conjecture stated in [8] is then

Conjecture 3.6. *Suppose $g(u)$ is a suitable weight function. Then, if $\mathcal{F}$ is the family of real Dirichlet L-functions with fundamental discriminants $d < 0$ (the sum over these fundamental discriminants is indicated by $\sum^*$) we have*

$$\sum_{L\in\mathcal{F}} Z_L(\tfrac{1}{2} + \alpha_1)\cdots Z_L(\tfrac{1}{2} + \alpha_k)g(|d|)$$
$$= \sideset{}{^*}\sum_{d<0} Q_k\left(\alpha, \log\tfrac{|d|}{2\pi}\right) g(|d|)\left(1 + O(|d|^{-\frac{1}{2}+\epsilon})\right), \tag{3.39}$$

in which

$$Q_k(\alpha, x) = \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}$$
$$\times \oint\cdots\oint \frac{G_-(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2)^2\prod_{j=1}^{k} z_j}{\prod_{\ell=1}^{k}\prod_{j=1}^{k}(z_j - \alpha_\ell)(z_j + \alpha_\ell)} e^{\frac{x}{2}\sum_{j=1}^{k} z_j}\, dz_1 \ldots dz_k, \tag{3.40}$$

where the path of integration encloses the $\pm\alpha$'s. Here

$$G_-(z_1, \ldots, z_k) = A_k(z_1, \ldots, z_k)\prod_{j=1}^{k}\left(\frac{\Gamma(\frac{3}{4} + \frac{z_j}{2})2^{z_j}}{\Gamma(\frac{3}{4} - \frac{z_j}{2})}\right)^{\frac{1}{2}}\prod_{1\le i\le j\le k}\zeta(1 + z_i + z_j), \tag{3.41}$$

and A_k is the Euler product, which is absolutely convergent for $|\Re z_j| < 1/2$, for $j = 1, \dots, k$, defined by

$$A_k(z_1, \dots, z_k) = \prod_p \prod_{1 \le i \le j \le k} \left(1 - \frac{1}{p^{1+z_i+z_j}}\right) \times \left(\frac{1}{2}\left(\prod_{j=1}^{k}\left(1 - \frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1} + \prod_{j=1}^{k}\left(1 + \frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\right) + \frac{1}{p}\right)\left(1 + \frac{1}{p}\right)^{-1}. \tag{3.42}$$

There is a similar conjecture for the analogous sum over positive fundamental discriminants. For this conjecture G_- is replaced by G_+, where

$$G_+(z_1, \dots, z_k) = A_k(z_1, \dots, z_k) \prod_{j=1}^{k} \left(\frac{\Gamma(\frac{1}{4} + \frac{z_j}{2})2^{z_j}}{\Gamma(\frac{1}{4} - \frac{z_j}{2})}\right)^{\frac{1}{2}} \prod_{1 \le i \le j \le k} \zeta(1 + z_i + z_j), \tag{3.43}$$

and A_k is as before.

When comparing $\int_{USp(2N)} \mathcal{Z}_M(e^{-\alpha_1}) \cdots \mathcal{Z}_M(e^{-\alpha_k})dM$, which is very closely related to (3.36), with the autocorrelation function (3.40) in Conjecture 3.6, we note that equating the density of zeros gives an equivalence $N = \frac{1}{2}\log\frac{|d|}{2\pi}$. Then we see immediately that the structure of the k-fold integrals is very similar. The role of $\prod_{1 \le \ell \le m \le k}(1 - e^{-z_m - z_\ell})^{-1}$ in (3.36) is played by $G_\pm(z_1, \dots, z_k)$ in the L-function case. Note that in both cases this factor produces poles when $z_m = -z_\ell$, for $1 \le \ell \le m \le k$. Extra arithmetic information is in evidence in the A_k factor in $G_\pm$ which, of course, does not feature in the random matrix result. Again, the underlying similarity between the two formulae lends support to the number theoretical conjecture and illustrates the strong connection between L-functions and random matrix theory.

4. Orthogonal Group: $O(2N)$

We now turn our attention to the group $O(2N)$ of $2N \times 2N$ orthogonal matrices. This group divides into two halves: the group $SO(2N)$ of matrices from $O(2N)$ with determinant +1, and $O^-(2N)$ which is comprised of the matrices with determinant -1. We will examine these two components separately.

4.1. $O^-(2N)$. We are considering orthogonal $2N \times 2N$ matrices with determinant -1. These matrices have eigenvalues at $1, -1, e^{i\theta_1}, e^{-i\theta_1}, \dots, e^{i\theta_{N-1}}, e^{-i\theta_{N-1}}$. The measure may be expressed in the form

$$\frac{(-1)^{((N-1)^2-(N-1))/2}}{(N-1)!\,\pi^{N-1}2^{2(N-1)}} \Delta(e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N}) \times \prod_{j=1}^{N-1}(e^{i\theta_j} - e^{-i\theta_j})d\theta_1 \cdots d\theta_{N-1}. \tag{4.1}$$

The characteristic polynomial for one of these matrices can be defined as

$$\Lambda_M(s) = \det(I - Ms) = (1-s)(1+s)\prod_{n=1}^{N-1}(1-e^{i\theta_n}s)(1-e^{-i\theta_n}s). \tag{4.2}$$

The autocorrelation function is then

$$\begin{aligned} I(O^-(2N), w_1, \ldots, w_k) &\equiv \int_{O^-(2N)} (-1)^k \Lambda_M(w_1)\cdots\Lambda_M(w_k)dM \\ &= \left(\frac{(-1)^{((N-1)^2-(N-1))/2}}{(N-1)!\,\pi^{N-1}2^{2(N-1)}} \int_0^{2\pi} \cdots \right. \\ &\quad \times \int_0^{2\pi} d\theta_1 \cdots d\theta_{N-1} \Delta(e^{i\theta_1}, \ldots, e^{i\theta_N}, e^{-i\theta_1}, \ldots, e^{-i\theta_N}) \\ &\quad \left. \times \prod_{m=1}^{k}\prod_{n=1}^{N-1}(e^{i\theta_n} - w_m)(e^{-i\theta_n} - w_m)\prod_{j=1}^{N-1}(e^{i\theta_j} - e^{-i\theta_j}) \right) \times \prod_{m=1}^{k}(w_m^2 - 1). \end{aligned} \tag{4.3}$$

Following exactly the calculation in the previous section for the group $USp(2N)$,

$$\begin{aligned} I(O^-(2N), w_1, \ldots, w_k) \\ = \frac{\prod_{m=1}^{k}(w_m^2-1)}{\prod_{1\le i<j\le k}(w_j - w_i)} \left[\sum_{\substack{0\le i_1<i_2<\cdots<i_k\le 2N+k-1 \\ i_j \equiv j-1 \bmod 2}} \begin{vmatrix} w_1^{i_1} & \cdots & w_1^{i_k} \\ \vdots & \ddots & \vdots \\ w_k^{i_1} & \cdots & w_k^{i_k} \end{vmatrix} \right]. \end{aligned} \tag{4.4}$$

This then leads to

$$\begin{aligned} I(O^-(2N), w_1, \ldots, w_k) = \left(\prod_{m=1}^{k}(w_m^2-1)\right) \times w_1^{N-1}\cdots w_k^{N-1} \\ \times \left[\sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^{k} w_j^{\epsilon_j(N-1)}\right) \prod_{1\le i\le j\le k}(1 - w_i^{-\epsilon_i}w_j^{-\epsilon_j})^{-1} \right]. \end{aligned} \tag{4.5}$$

However, in terms where $\epsilon_m = 1$, we have a factor

$$\frac{w_m^2-1}{1-w_m^{-2\epsilon_m}} = \frac{w_m^2-1}{1-w_m^{-2}} = w_m^2 = \epsilon_m w_m \times w_m^{\epsilon_m}, \tag{4.6}$$

and if $\epsilon_m = -1$, then

$$\frac{w_m^2-1}{1-w_m^{-2\epsilon_m}} = \frac{w_m^2-1}{1-w_m^{2}} = -1 = \epsilon_m w_m \times w_m^{\epsilon_m}. \tag{4.7}$$

Therefore,

$$I(O^-(2N), w_1, \dots, w_k) = w_1^N \cdots w_k^N \left[\sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^k \epsilon_j w_j^{\epsilon_j N} \right) \prod_{1 \le i < j \le k} (1 - w_i^{-\epsilon_i} w_j^{-\epsilon_j})^{-1} \right], \tag{4.8}$$

and

$$I(O^-(2N), e^{\alpha_1}, \dots, e^{\alpha_k}) = \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} e^{N \sum_{j=1}^k \alpha_j} \oint \cdots \oint \prod_{1 \le \ell \le m \le k} (1 - e^{-z_m - z_\ell})^{-1} \times \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k \alpha_j}{\prod_{i=1}^k \prod_{j=1}^k (z_j - \alpha_i)(z_j + \alpha_i)} e^{N \sum_{j=1}^k z_j} dz_1 \cdots dz_k, \tag{4.9}$$

using Lemma 3.5.

4.2. $SO(2N)$. We now consider the group of $2N \times 2N$ orthogonal matrices which have positive determinant. The eigenvalues of such matrices come in complex conjugate pairs $e^{i\theta_1}, e^{-i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_N}$. The measure is

$$\frac{(-1)^{N(N-1)/2} 2^{-2N+1}}{\pi^N N!} \Delta(e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N}) \times \prod_{k=1}^N (e^{-i\theta_k} - e^{i\theta_k})^{-1} d\theta_1 \cdots d\theta_N. \tag{4.10}$$

The characteristic polynomial for these matrices is

$$\Lambda_M(s) = \det(I - Ms) = \prod_{n=1}^N (1 - e^{i\theta_n} s)(1 - e^{-i\theta_n} s), \tag{4.11}$$

so the autocorrelation function which we wish to evaluate is

$$\begin{aligned} I(SO(2N), w_1, \dots, w_k) &\equiv \int_{SO(2N)} \Lambda_M(w_1) \cdots \Lambda_M(w_k) dM \\ &= \frac{(-1)^{N(N-1)/2} 2^{-2N+1}}{\pi^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \Delta(e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N}) \\ &\quad \times \prod_{m=1}^k \prod_{n=1}^N (e^{i\theta_n} - w_m)(e^{-i\theta_n} - w_m) \left[\prod_{n=1}^N (e^{-i\theta_n} - e^{i\theta_n}) \right]^{-1} d\theta_1 \cdots d\theta_N \end{aligned}$$

$$= \frac{(-1)^{N(N-1)/2} 2^{-2N+1}}{\pi^N N!} \frac{1}{\prod_{1\le i<j\le k}(w_j - w_i)} \int_0^{2\pi} \cdots \int_0^{2\pi} \left[\prod_{n=1}^{N} (e^{-i\theta_n} - e^{i\theta_n}) \right]^{-1}$$
$$\times \Delta(w_1, \ldots, w_k, e^{i\theta_1}, \ldots, e^{i\theta_N}, e^{-i\theta_1}, \ldots, e^{-i\theta_N}) d\theta_1 \cdots d\theta_N$$
$$= \frac{2^{-2N+1}}{\pi^N N!} \frac{1}{\prod_{1\le i<j\le k}(w_j - w_i)} \int_0^{2\pi} \cdots \int_0^{2\pi} \left[\prod_{n=1}^{N} (e^{-i\theta_n} - e^{i\theta_n}) \right]^{-1}$$
$$\times \sum_{\sigma \in S_{2N+k}} \operatorname{sgn}\sigma \; w_1^{\sigma(1)-1} w_2^{\sigma(2)-1} \cdots w_k^{\sigma(k)-1} e^{i[\sigma(k+1)-1]\theta_1} e^{-i[\sigma(k+2)-1]\theta_1} \cdots$$
$$e^{i[\sigma(k+2N-1)-1]\theta_N} e^{-i[\sigma(k+2N)-1]\theta_N} d\theta_1 \cdots d\theta_N, \qquad (4.12)$$

where in the final line we have the determinant expansion of $\Delta(w_1, \ldots, w_k, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_N}, e^{-i\theta_N}) = (-1)^{N(N-1)/2} \Delta(w_1, \ldots, w_k, e^{i\theta_1}, \ldots, e^{i\theta_N}, e^{-i\theta_1}, \ldots, e^{-i\theta_N})$ expressed in terms of the permutations of $\{1, 2, \ldots, 2N + k\}$.

The sum over $\sigma \in S_{2N+k}$ in (4.12) can be broken up and written as follows:

$$\sum_{\delta \in D} \operatorname{sgn}(\delta) \left(\sum_{\alpha \in A} \operatorname{sgn}(\alpha) w_1^{\alpha(1)-1} \cdots w_k^{\alpha(k)-1} \right)$$
$$\times \left(\sum_{\beta \in B} \operatorname{sgn}(\beta) \prod_{j=1}^{N} (e^{i\theta_j(\beta(2j-1)-\beta(2j))} - e^{-i\theta_j(\beta(2j-1)-\beta(2j))}) \right), \qquad (4.13)$$

where $D \subset S_{2N+k}$ is the set of permutations such that $\delta(1) < \cdots < \delta(k)$ and $\delta(k+1) < \cdots < \delta(k+2N)$, A is the set of all permutations of $\delta(1), \ldots, \delta(k)$, and B is the set of permutations of $\delta(k+1), \ldots, \delta(k+2N)$ such that $\beta(1) < \beta(2)$,$\beta(3) < \beta(4), \ldots, \beta(2N-1) < \beta(2N)$.

Using $x^N - y^N = (x-y)(x^{N-1} + x^{N-2}y + \cdots + xy^{N-2} + y^{N-1})$, we see that the product over j in (4.13) contains a factor $\prod_{n=1}^{N}(e^{-i\theta_n} - e^{i\theta_n})$ which cancels with the identical factor in (4.12). Since the integral in (4.12) integrates to zero unless the integrand is independent of all θ_j, and since in our case $x = e^{-i\theta}$ and $y = e^{i\theta}$, we obtain zero for any term in the sum over β unless $\beta(2j) - \beta(2j-1)$ is an odd number for all $j = 1, 2, \ldots, N$, in which case (4.12) reduces to

$$\frac{2^{-N+1}}{N!} \frac{1}{\prod_{1\le i<j\le k}(w_j - w_i)} \sum_{\delta \in D} \operatorname{sgn}(\delta) \left(\sum_{\alpha \in A} \operatorname{sgn}(\alpha) w_1^{\alpha(1)-1} \cdots w_k^{\alpha(k)-1} \right)$$
$$\times \left(\sum_{\substack{\beta \in B \\ \beta(2j)-\beta(2j-1)=\text{odd}}} \operatorname{sgn}(\beta) \right). \qquad (4.14)$$

To perform the remaining sum over β, recall that the permutation β rearranges $\delta(k+1), \ldots, \delta(2N+k)$ (which are arranged in ascending order), and note that the sum over β in (4.14) will contain zero terms unless N of $\delta(k+1), \ldots, \delta(2N+k)$ are even and N are odd. In particular, one of $\beta(2j-1)$ and $\beta(2j)$ must be even and one must be odd for each $j = 1, \ldots, N$. To perform the β sum, we essentially need to count (with signs) all the ways to pair up each even number with an odd number. It can be seen that if in the original ascending order $\delta(k+1),\dots,\delta(2N+k)$ even and odd numbers alternate, then the sum over β is given by $N!$ times the $N\times N$ determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{vmatrix} = 2^{N-1}, \tag{4.15}$$

where the determinant accounts (with sign) for the pairing of each even number with an odd number, while the $N!$ accounts for the further permutation of the N pairs. The same reasoning produces an $N\times N$ determinant which is zero when the arrangement $\delta(k+1),\dots,\delta(2N+k)$ contains two consecutive even or two consecutive odd numbers. Noting that sgnδ in (4.13) is always +1 for δ such that even and odd numbers alternate in $\delta(k+1),\dots,\delta(2N+k)$, we arrive at

$$I(SO(2N),w_1,\dots,w_k) = \frac{1}{\prod_{1\le i<j\le k}(w_j-w_i)} \sum_{i_1,\dots,i_k} \begin{vmatrix} w_1^{i_1} & \cdots & w_1^{i_k} \\ \vdots & \ddots & \vdots \\ w_k^{i_1} & \cdots & w_k^{i_k} \end{vmatrix}, \tag{4.16}$$

where the conditions on $i_1,\dots,i_k$ are that $i_j \in \{0,1,\dots,2N+k-1\}$, $i_1 < i_2 < \cdots < i_k$ and

$$\begin{aligned} &k \text{ even} \begin{cases} i_1=0, i_2=i_3-1, i_4=i_5-1,\dots,i_{k-2}=i_{k-1}-1, i_k=2N+k-1 \\ \text{or} \\ i_1=i_2-1, i_3=i_4-1,\dots,i_{k-1}=i_k-1, \end{cases} \\ &k \text{ odd} \begin{cases} i_1=0, i_2=i_3-1, i_4=i_5-1,\dots,i_{k-1}=i_k-1 \\ \text{or} \\ i_1=i_2-1, i_3=i_4-1,\dots,i_{k-2}=i_{k-1}-1, i_k=2N+k-1. \end{cases} \end{aligned} \tag{4.17}$$

Once more, this is a sum over Schur functions,

$$\begin{aligned} &I(SO(2N),w_1,\dots,w_k) \\ &\quad= \sum_{\substack{\lambda' \text{ odd} \\ k\ge\lambda'_1\ge\cdots\ge\lambda'_{2N}>0}} S_\lambda(w_1,\dots,w_k) + \sum_{\substack{\lambda' \text{ even} \\ k\ge\lambda'_1\ge\cdots\ge\lambda'_{2N}\ge0}} S_\lambda(w_1,\dots,w_k), \end{aligned} \tag{4.18}$$

where the sum is over partitions $\lambda'=(\lambda'_1,\dots,\lambda'_{2N})$, with no part greater than k, and λ' is the conjugate partition to λ. Note that the condition on the odd λ' implies that the partition has exactly $2N$ non-zero parts, whereas the sum over even partitions only requires that λ' has no more than $2N$ non-zero parts.

We now show that (4.16) may be expressed in the form

$$\begin{aligned} &I(SO(2N),w_1,\dots,w_k) \\ &\quad= w_1^N\cdots w_k^N \left[\sum_{\epsilon_j\in\{-1,1\}} \left(\prod_{j=1}^k w_j^{N\epsilon_j}\right) \prod_{1\le i<j\le k} (1-w_i^{-\epsilon_i}w_j^{-\epsilon_j})^{-1} \right]. \end{aligned} \tag{4.19}$$

In order to prove (4.19) we must first prove an identity very similar in form to Identity 3.1 in the symplectic symmetry section. This is

Identity 4.1.

$$\sum_{j=1}^{n} w_j^2 \Delta(w_1, \dots, w_n)\big|_{w_j=0} \prod_{m\neq j}(1 - w_m w_j)$$
$$= \begin{cases} w_1^2 \cdots w_n^2 \Delta(w_1, \dots, w_n) & \text{if } n \text{ odd} \\ (w_1^2 \cdots w_n^2 - w_1 \cdots w_n)\Delta(w_1, \dots, w_n) & \text{if } n \text{ even} \end{cases}.$$

To prove this we rewrite the factor w_j^2 as $1 - (1 - w_j^2)$. The contribution to Identity 4.1 from the $(1 - w_j^2)$ term, is just the right side of Identity 3.1; that is, $\Delta(w_1, \dots, w_n)$ $(1 - w_1^2 \cdots w_n^2)$. The remainder of the left side of Identity 4.1 we write as in (3.8), where $f(w_j) = \prod_{m\neq j}(1 - w_m w_j)$. Note that we cannot immediately apply Lemma 3.2 because $f(x)$ is not symmetric amongst the w's. However, if we write $f(w_j) = \prod_{m\neq j}(1 - w_m w_j) = \sum_{i=0}^{n-1} a_i w_j^i$ and $g(w) = \prod_{m=1}^{n}(1 - w_m w) = \sum_{i=0}^{n} b_i w^i$, then the identity $g(w_j) = f(w_j)(1 - w_j^2)$ produces the recurrence relation $b_0 = a_0 = 1$, $b_i = a_i - w_j a_{i-1}$ (for $i = 1, \dots, n-1$) and $b_n = -w_j a_{n-1}$. This allows us to write

$$\sum_{i=0}^{n-1} a_i w_j^i = \begin{cases} \sum_{i=0}^{(n-1)/2} \sum_{q=0}^{i} w_j^{2i-q} b_q + \sum_{i=(n+1)/2}^{n-1} \sum_{q=0}^{n-i-1} -w_j^{2i-n+q} b_{n-q} & n \text{ odd} \\ \sum_{i=0}^{n/2} \sum_{q=0}^{i} w_j^{2i-q} b_q + \sum_{i=(n+2)/2}^{n-1} \sum_{q=0}^{n-i-1} -w_j^{2i-n+q} b_{n-q} & n \text{ even} \end{cases}. \tag{4.20}$$

Since the b's are symmetric functions of the w's and in (3.8) we have columns containing powers of the w's from 1 to $n - 1$, the only terms in the expression for $f(w_j) = \sum_{i=0}^{n-1} a_i w_j^i$ on the right side of (4.20) above which *cannot* be cancelled by adding or subtracting one of these columns multiplied by a symmetric function of the w's are $a_0 = 1$ and, in the case that n is even, w_j^n. The determinant is then easily evaluated as $\Delta(w_1, \dots, w_n)$ if n is odd, and $\Delta(w_1, \dots, w_n) - w_1 \dots w_n \Delta(w_1, \dots, w_n)$ if n is even. This proves Identity 4.1.

Now we move on to determining the form of the autocorrelation functions in (4.19). When k is even in (4.19) we will write

$$I(SO(2N), w_1, \dots, w_{2k}) = I^M_{2N+2k-1}(w_1, \dots, w_{2k}) + I^E_{2N+2k-1}(w_1, \dots, w_{2k}), \tag{4.21}$$

where

$$I_n^M(w_1, \dots, w_{2k}) \equiv \frac{1}{\Delta(w_1, \dots, w_{2k})} \sum_{\substack{i_1 < \dots < i_{2k} \in \{0, \dots, n\} \\ i_{2j-1} = i_{2j} - 1, j = 1, \dots, k}} \begin{vmatrix} w_1^{i_1} & \cdots & w_1^{i_{2k}} \\ \vdots & \ddots & \vdots \\ w_{2k}^{i_1} & \cdots & w_{2k}^{i_{2k}} \end{vmatrix} \tag{4.22}$$

and

$$I_n^E(w_1, \dots, w_{2k}) \equiv \frac{1}{\Delta(w_1, \dots, w_{2k})} \sum_{\substack{i_1 < \dots < i_{2k} \in \{0, \dots, n\} \\ i_1 = 0, i_{2j} = i_{2j+1} - 1, j = 1, \dots, k-1, i_{2k} = n}} \begin{vmatrix} w_1^{i_1} & \cdots & w_1^{i_{2k}} \\ \vdots & \ddots & \vdots \\ w_{2k}^{i_1} & \cdots & w_{2k}^{i_{2k}} \end{vmatrix}. \tag{4.23}$$

Similarly, when k is odd in (4.19) we will write

$$I(SO(2N), w_1, \dots, w_{2k+1}) = I^R_{2N+2k}(w_1, \dots, w_{2k+1}) + I^L_{2N+2k}(w_1, \dots, w_{2k+1}), \tag{4.24}$$

where

$$I^R_n(w_1, \dots, w_{2k+1}) \equiv \frac{1}{\Delta(w_1, \dots, w_{2k+1})} \sum_{\substack{i_1 < \dots < i_{2k+1} \in \{0,\dots,n\} \\ i_{2j-1} = i_{2j}-1, j=1,\dots,k, i_{2k+1}=n}} \begin{vmatrix} w_1^{i_1} & \cdots & w_1^{i_{2k+1}} \\ \vdots & \ddots & \vdots \\ w_{2k+1}^{i_1} & \cdots & w_{2k+1}^{i_{2k+1}} \end{vmatrix} \tag{4.25}$$

and

$$I^L_n(w_1, \dots, w_{2k+1}) \equiv \frac{1}{\Delta(w_1, \dots, w_{2k+1})} \sum_{\substack{i_1 < \dots < i_{2k+1} \in \{0,\dots,n\} \\ i_1=0, i_{2j} = i_{2j+1}-1, j=1,\dots,k}} \begin{vmatrix} w_1^{i_1} & \cdots & w_1^{i_{2k+1}} \\ \vdots & \ddots & \vdots \\ w_{2k+1}^{i_1} & \cdots & w_{2k+1}^{i_{2k+1}} \end{vmatrix}. \tag{4.26}$$

We now prove the following identities

$$I^M_n(w_1, \dots, w_{2k}) = \frac{1}{\mathcal{E}([2k])\Delta([2k])} \sum_{\substack{A\cup B=[2k],\ A\cap B=\emptyset \\ |B| \text{ even}}} w_A^n E(A,B)\Delta(A)\Delta(B)(-1)^{S(A,B)-|A|}, \tag{4.27a}$$

$$I^E_n(w_1, \dots, w_{2k}) = \frac{1}{\mathcal{E}([2k])\Delta([2k])} \sum_{\substack{A\cup B=[2k],\ A\cap B=\emptyset \\ |B| \text{ odd}}} w_A^n E(A,B)\Delta(A)\Delta(B)(-1)^{S(A,B)-|A|}, \tag{4.27b}$$

$$I^R_n(w_1, \dots, w_{2k+1}) = \frac{1}{\mathcal{E}([2k+1])\Delta([2k+1])} \sum_{\substack{A\cup B=[2k+1],\ A\cap B=\emptyset \\ |B| \text{ even}}} w_A^n E(A,B)\Delta(A)\Delta(B)(-1)^{S(A,B)-|A|}, \tag{4.27c}$$

and

$$I^L_n(w_1, \dots, w_{2k+1}) = \frac{1}{\mathcal{E}([2k+1])\Delta([2k+1])} \sum_{\substack{A\cup B=[2k+1],\ A\cap B=\emptyset \\ |B| \text{ odd}}} w_A^n E(A,B)\Delta(A)\Delta(B)(-1)^{S(A,B)-|A|}. \tag{4.27d}$$

Here the only notation not already defined in Sect. 3 is

$$\mathcal{E}(A) = \prod_{\substack{m<n \\ m,n\in A}} (1 - w_m w_n). \tag{4.28}$$

For the case $k = 1$ it is easy to show that (4.27) holds. We now prove (4.27) for any k by induction. First we note that Identity 4.1 can be written as

$$\sum_{j\in A} w_j \Delta(A_j) E(\{j\}, A_j)(-1)^{W(A_j,\{j\})} = (-1)^{|A|-1}\Delta(A)\begin{cases}(w_A - 1) & \text{if } |A| \text{ even}\\ w_A & \text{if } |A| \text{ odd}\end{cases}, \tag{4.29}$$

where $A \subset \{1, 2, \ldots, m\} \equiv [m]$, A_j is the set of elements of A with j removed, and $|A|$ is the number of elements in the set A. Also, $w_A = \prod_{m\in A} w_m$ and $W(A, B) = \sum_{\substack{m\in A,\ n\in B\\ m>n}} 1$.

If we make use of Identity 3.4 with $x = 1$, then in the current notation this appears as

$$\sum_{\substack{A\cup B=[m], A\cap B=\emptyset\\ |A| \text{ even}}} (-1)^{S(A,B)} E(A, B)\Delta(A)\Delta(B) w_A^{m-2} = 0. \tag{4.30}$$

To prove the form of I^R in (4.27c), we need to show that

$$\begin{aligned}&\sum_{\substack{i_1<\cdots<i_{2k+1}\in\{0,\ldots,n\}\\ i_{2j-1}=i_{2j}-1, j=1,\ldots,k, i_{2k+1}=n}} \begin{vmatrix} w_1^{i_1} & \cdots & w_1^{i_{2k+1}}\\ \vdots & \ddots & \vdots\\ w_{2k+1}^{i_1} & \cdots & w_{2k+1}^{i_{2k+1}}\end{vmatrix}\\ &= \frac{1}{\mathcal{E}([2k+1])} \sum_{\substack{A\cup B=[2k+1],\ A\cap B=\emptyset\\ |B| \text{ even}}} w_A^n E(A, B)\Delta(A)\Delta(B)(-1)^{S(A,B)-|A|}.\end{aligned} \tag{4.31}$$

Using the definition of I^M (4.22), we see that the left side of the above is

$$\sum_{j=1}^{2k+1} (-1)^{j-1} w_j^n \Delta([2k+1]_j) I_{n-1}^M([2k+1]_j), \tag{4.32}$$

and then by induction using (4.27a), the line above equals

$$\begin{aligned}&\sum_{j=1}^{2k+1} (-1)^{j-1} w_j^n \frac{1}{\mathcal{E}([2k+1]_j)}\\ &\quad\times \sum_{\substack{F\cup B=[2k+1]_j,\ F\cap B=\emptyset\\ |B| \text{ even}}} w_F^{n-1} E(F, B)\Delta(F)\Delta(B)(-1)^{S(F,B)-|F|}.\end{aligned} \tag{4.33}$$

Now we define $A = F \cup \{j\}$ and then exchange the order of the two sums, to obtain

$$\sum_{\substack{A\cup B=[2k+1],\ A\cap B=\emptyset\\ |B| \text{ even}}} \sum_{j\in A} \frac{(-1)^{j-1} w_j^n}{\mathcal{E}([2k+1]_j)} (-1)^{S(A_j,B)-|A_j|} w_{A_j}^{n-1} E(A_j, B)\Delta(A_j)\Delta(B). \tag{4.34}$$

We note that $\frac{E(A_j,B)}{\mathcal{E}(A_j\cup B)} = \frac{E(A,B)E(\{j\},A_j)}{\mathcal{E}(A\cup B)}$ and that if $A \cup B = [m]$, then $(-1)^{j+S(A_j,B)-|A_j|} = (-1)^{S(A,B)-W(A_j,\{j\})}$ if m is odd, and $(-1)^{j+S(A_j,B)-|A_j|} = (-1)^{S(A,B)-W(A_j,\{j\})-1}$ if m is even. So we have

$$\sum_{\substack{A\cup B=[2k+1],\ A\cap B=\emptyset \\ |B| \text{ even}}} w_A^{n-1} \frac{E(A,B)}{\mathcal{E}([2k+1])}(-1)^{S(A,B)}\Delta(B) \times \sum_{j\in A}(-1)^{W(A_j,\{j\})-1} w_j E(\{j\},A_j)\Delta(A_j). \tag{4.35}$$

Using (4.29), we then find

$$\frac{1}{\mathcal{E}([2k+1])} \sum_{\substack{A\cup B=[2k+1],\ A\cap B=\emptyset \\ |B| \text{ even}}} w_A^n E(A,B)\Delta(A)\Delta(B)(-1)^{S(A,B)-|A|}, \tag{4.36}$$

which proves (4.31) and so confirms the form of I^R in (4.27c).

The expression for I^M is proved similarly, using induction and the form of I^R found in (4.27c). We need to show that

$$\sum_{\substack{i_1<\cdots<i_{2k}\in\{0,\dots,n\} \\ i_{2j-1}=i_{2j}-1,\, j=1,\dots,k}} \begin{vmatrix} w_1^{i_1} & \cdots & w_1^{i_{2k}} \\ \vdots & \ddots & \vdots \\ w_{2k}^{i_1} & \cdots & w_{2k}^{i_{2k}} \end{vmatrix} = \frac{1}{\mathcal{E}([2k])} \sum_{\substack{A\cup B=[2k], A\cap B=\emptyset \\ |B| \text{ even}}} w_A^n E(A,B)\Delta(A)\Delta(B)(-1)^{S(A,B)-|A|}. \tag{4.37}$$

The left side of this expression, written in terms of I^R, is

$$\sum_{i_{2k}=2k-1}^{n} \sum_{j=1}^{2k} (-1)^j w_j^{i_{2k}} \Delta([2k]_j) I^R_{i_{2k}-1}([2k]_j). \tag{4.38}$$

Now we proceed by induction and use (4.27c). Continuing exactly as we did in the case of I^R above, we find that (4.38) reduces to

$$\sum_{\substack{A\cup B=[2k], A\cap B=\emptyset \\ |B| \text{ even}}} \frac{E(A,B)}{\mathcal{E}([2k])}(-1)^{S(A,B)}(w_A^n - w_A^{2k-2})\Delta(A)\Delta(B). \tag{4.39}$$

Finally, by identity (4.30), we see that all the terms in which w_A appears with exponent $2k-2$ disappear, leaving us with

$$\sum_{\substack{A\cup B=[2k], A\cap B=\emptyset \\ |B| \text{ even}}} \frac{E(A,B)}{\mathcal{E}([2k])}(-1)^{S(A,B)} w_A^n \Delta(A)\Delta(B), \tag{4.40}$$

which proves (4.37) and so also (4.27a).

To prove (4.27b) for I^E and (4.27d) for I_L, we follow exactly the same procedure as above.

Finally, we show that the form of the expressions in (4.27) can be written as sums over $\epsilon_j \in \{-1, 1\}$ and so complete the proof of (4.19). Note that, using $D(A, B) = (-1)^{W(A,B)}\Delta([m])/(\Delta(A)\Delta(B))$ and $\mathcal{E}(A)\mathcal{E}(B) = \mathcal{E}([m])/E(A, B)$ (with $A \cup B = [m] \equiv \{1, \ldots, m\}$, $A \cap B = \emptyset$) and letting "parity" stand for *either* "even" or "odd",

$$\frac{1}{\mathcal{E}([m])\Delta([m])} \sum_{\substack{A\cup B=[m], A\cap B=\emptyset \\ |B| \text{ parity}}} w_A^n E(A, B)\Delta(A)\Delta(B)(-1)^{S(A,B)-|A|}$$
$$= \sum_{\substack{A\cup B=[m], A\cap B=\emptyset \\ |B| \text{ parity}}} w_A^{n-m+1} \frac{(-1)^{|A|(|A|-1)/2} w_A^{|A|-1}}{\mathcal{E}(A)} \frac{1}{\mathcal{E}(B)} \frac{(-1)^{|A||B|} w_A^{|B|}}{D(A, B)}$$

$$= w^{\frac{n-m+1}{2}} \sum_{\substack{A\cup B=[m], A\cap B=\emptyset \\ |B| \text{ parity}}} w_A^{\frac{n-m+1}{2}} w_B^{-\frac{n-m+1}{2}} \left(\prod_{\substack{p,q\in A \\ p<q}} \frac{1}{1 - \frac{1}{w_p w_q}} \right)$$
$$\times \left(\prod_{\substack{p,q\in B \\ p<q}} \frac{1}{1 - w_p w_q} \right) \left(\prod_{\substack{p\in A \\ q\in B}} \frac{1}{1 - \frac{w_q}{w_p}} \right)$$
$$= w^{\frac{n-m+1}{2}} \sum_{\substack{\epsilon_j\in\{1,-1\} \\ \prod_j \epsilon_j=(-1)^{\text{parity}}}} \left(\prod_{\ell=1}^{m} w_\ell^{\epsilon_\ell(\frac{n-m+1}{2})} \right) \left(\prod_{1\le q<\ell\le m} (1 - w_q^{-\epsilon_q} w_\ell^{-\epsilon_\ell})^{-1} \right). \tag{4.41}$$

Thus we end up with

$$I(SO(2N), w_1, \ldots, w_k)$$
$$= w_1^N \cdots w_k^N \left[\sum_{\epsilon_j\in\{1,-1\}} \left(\prod_{j=1}^{k} w_j^{N\epsilon_j} \right) \prod_{1\le i<j\le k} (1 - w_i^{-\epsilon_i} w_j^{-\epsilon_j})^{-1} \right], \tag{4.42}$$

which is exactly (4.19).

Using Lemma 3.5 we can express this as a multiple integral:

$$I(SO(2N), e^{\alpha_1}, \ldots, e^{\alpha_k})$$
$$= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} e^{N\sum_{j=1}^k \alpha_j} \oint \cdots \oint \prod_{1\le\ell<m\le k} (1 - e^{-z_m - z_\ell})^{-1}$$
$$\times \frac{\Delta(z_1^2, \ldots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_j - \alpha_i)(z_j + \alpha_i)} e^{N\sum_{j=1}^k z_j} dz_1 \cdots dz_k. \tag{4.43}$$

4.3. Comparison with L-functions. In [8] we give a conjecture for the autocorrelation functions of

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}, \tag{4.44}$$

near the critical point $s = 1/2$ averaged over $f \in H_k(N)$. Here we denote by $H_k(N)$ the set of primitive newforms $f \in S_k(\lambda_0(N))$ and the λ_f are the Fourier coefficients of the newform. For simplicity, we restrict attention to $k = 2$ and $N = q$, a prime. The zeros of this family near the critical point display orthogonal symmetry.

The L-function satisfies the functional equation

$$L_f(s) = \varepsilon_f X(s) L_f(1-s), \tag{4.45}$$

with $\varepsilon_f = -\sqrt{q}\lambda_f(q) = \pm 1$. If instead we define

$$Z_f(s) = X(s)^{-1/2} L_f(s), \tag{4.46}$$

then $Z_f(s)$ obeys the functional equation

$$Z_f(s) = \varepsilon_f Z_f(1-s). \tag{4.47}$$

After defining the "harmonic average"

$$\begin{aligned} &\sum_{f\in H_2(q)}^{h} Z_f(1/2+\alpha_1)\cdots Z_f(1/2+\alpha_k) \\ &\quad \equiv \sum_{f\in H_2(q)} Z_f(1/2+\alpha_1)\cdots Z_f(1/2+\alpha_k)/ < f, f >, \end{aligned} \tag{4.48}$$

we have the following three conjectures:

Conjecture 4.2.

$$\begin{aligned} &\sum_{f\in H_2^*(q)}^{h} Z_f(1/2+\alpha_1)\cdots Z_f(1/2+\alpha_k) \\ &= \sum_{\substack{\epsilon_j \in \{-1,+1\} \\ \prod_{j=1}^k \epsilon_j = 1}} \prod_{j=1}^{k} X(1/2+\epsilon_j\alpha_j)^{-1/2} \\ &\quad \times \prod_{1\le i<j\le k} \zeta(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j) A(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)\left(1+O(q^{-\frac{1}{2}+\epsilon})\right), \end{aligned}$$

Conjecture 4.3.

$$\sum_{\substack{f \in H_2^*(q) \\ f \text{ even}}}^{h} Z_f(1/2+\alpha_1)\cdots Z_f(1/2+\alpha_k)$$
$$= \frac{1}{2} \sum_{\epsilon_j \in \{-1,+1\}} \prod_{j=1}^{k} X(1/2+\epsilon_j\alpha_j)^{-1/2}$$
$$\times \prod_{1\le i<j\le k} \zeta(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j) A(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)\left(1+O(q^{-\frac{1}{2}+\epsilon})\right)$$

and

Conjecture 4.4.

$$\sum_{\substack{f \in H_2^*(q) \\ f \text{ odd}}}^{h} Z_f(1/2+\alpha_1)\cdots Z_f(1/2+\alpha_k)$$
$$= \frac{1}{2} \sum_{\epsilon_j \in \{-1,+1\}} \prod_{j=1}^{k} \epsilon_j X(1/2+\epsilon_j\alpha_j)^{-1/2}$$
$$\times \prod_{1\le i<j\le k} \zeta(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j) A(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)\left(1+O(q^{-\frac{1}{2}+\epsilon})\right),$$

where in all of the above

$$A(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k) = \prod_p \prod_{1\le i<j\le k}\left(1-\frac{1}{p^{1+\epsilon_i\alpha_i+\epsilon_j\alpha_j}}\right)$$
$$\times \frac{2}{\pi}\int_0^{\pi} \sin^2\theta \prod_{j=1}^{k} \frac{e^{i\theta}\left(1-\frac{e^{i\theta}}{p^{\frac{1}{2}+\epsilon_j\alpha_j}}\right)^{-1} - e^{-i\theta}\left(1-\frac{e^{-i\theta}}{p^{\frac{1}{2}+\epsilon_j\alpha_j}}\right)^{-1}}{e^{i\theta}-e^{-i\theta}}\, d\theta. \tag{4.49}$$

In the case of odd orthogonal symmetry, $\Lambda_M(s) = (1-s)(1+s)\prod_{n=1}^{N-1}(1-e^{i\theta_n}s)(1-e^{-i\theta_n}s)$ satisfies the functional equation

$$\Lambda_M(s) = -s^{2N}\overline{\Lambda_M}(\tfrac{1}{s}) \tag{4.50}$$

while

$$\mathcal{Z}_M(s) = -s^{-N}\Lambda_M(s) \tag{4.51}$$

satisfies

$$\mathcal{Z}_M(s) = -\overline{\mathcal{Z}_M}(\tfrac{1}{s}), \tag{4.52}$$

(the equivalent of (4.47)). The structure of

$$\int_{O^-(2N)} \mathcal{Z}_M(e^{-\alpha_1}) \cdots \mathcal{Z}_M(e^{-\alpha_k}) dM = \sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^{k} \epsilon_j e^{\epsilon_j N \alpha_j} \right) \prod_{1 \le i < j \le k} (1 - e^{-\epsilon_i \alpha_i - \epsilon_j \alpha_j})^{-1}. \quad (4.53)$$

parallels (in the manner described in Section 2.1) that of Conjecture 4.4.

Similarly, for even orthogonal symmetry, $\Lambda_M(s) = \prod_{n=1}^{N}(1 - e^{i\theta_n}s)(1 - e^{-i\theta_n}s)$ satisfies the functional equation

$$\Lambda_M(s) = s^{2N}\overline{\Lambda_M}(\tfrac{1}{s}) \quad (4.54)$$

while

$$\mathcal{Z}_M(s) = s^{-N}\Lambda_M(s) \quad (4.55)$$

satisfies

$$\mathcal{Z}_M(s) = \overline{\mathcal{Z}_M}(1/s), \quad (4.56)$$

and

$$\int_{SO(2N)} \mathcal{Z}_M(e^{-\alpha_1}) \cdots \mathcal{Z}_M(e^{-\alpha_k}) dM = \sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^{k} e^{\epsilon_j N \alpha_j} \right) \prod_{1 \le i < j \le k} (1 - e^{-\epsilon_i \alpha_i - \epsilon_j \alpha_j})^{-1} \quad (4.57)$$

has the same structure of Conjecture 4.3. Clearly, due to the cancellation caused by the extra ϵ_j factors in (4.53), the sum of (4.57) and (4.53) agrees with Conjecture 4.2 in the usual way.

Acknowledgements. We are grateful to Peter Forrester for introducing NCS to the method which was the starting point for this work. The research was partially supported by the American Institute of Mathematics and a Focused Research Group grant from the National Science Foundation. The last author was also supported by a Royal Society Dorothy Hodgkin Fellowship.

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Communicated by P. Sarnak