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INTEGRAL MOMENTS OF L-FUNCTIONS

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Abstract

We give a new heuristic for all of the main terms in the integral moments of various families of primitive L-functions. The results agree with previous conjectures for the leading order terms. Our conjectures also have an almost identical form to exact expressions for the corresponding moments of the characteristic polynomials of either unitary, orthogonal, or symplectic matrices, where the moments are defined by the appropriate group averages. This lends support to the idea that arithmetical L-functions have a spectral interpretation, and that their value distributions can be modeled using Random Matrix Theory. Numerical examples show good agreement with our conjectures.

Contents

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1. *Introduction and statement of results*

Random Matrix Theory (RMT) has recently become a fundamental tool for understanding L-functions. Montgomery [**38**] showed that the two-point correlations between the non-trivial zeros of the Riemann ζ-function, on the scale of the mean zero spacing, are similar to the corresponding correlations between the eigenvalues of random unitary matrices in the limit of large matrix size [**37**] and conjectured that these correlations are, in fact, identical to each other. There is extensive numerical evidence [**41**] in support of this conjecture. Rudnick and Sarnak [**45**] extended Montgomery's analysis to all n-point correlations, and to the zeros of other principal L-functions. Katz and Sarnak [**29**] introduced the idea of studying zero distributions within families of L-functions (see also [**42**, **44**]) and have conjectured that these coincide with the eigenvalue distributions of the classical compact groups. In this context symmetries of an L-function family determine the associated classical group. We shall here be concerned with the distribution of values taken by L-functions, either individually (that is, along the appropriate critical line), or with respect to averages over families. Specifically, we shall calculate the integral moments of these distributions.

Keating and Snaith [**30**] suggested that the value distribution of the Riemann ζ-function (or any other principal L-function) on its critical line is related to that of the characteristic polynomials of random unitary matrices. This led them to a general conjecture for the leading-order asymptotics of the moments of this distribution in the limit of large averaging range. Their conjecture agrees with a result of Hardy and Littlewood [**19**] for the second moment and a result of Ingham [**21**] for the fourth moment (see, for example [**48**]). It also agrees with conjectures, based on number-theoretical calculations, of Conrey and Ghosh [**11**] and Conrey and Gonek [**12**] for the sixth and eighth moments. General conjectures for the leading-order asymptotics of the moments of L-functions within families, based on random-matrix calculations for the characteristic polynomials of matrices from the orthogonal and unitary-symplectic groups, were developed by Conrey and Farmer [**8**] and Keating and Snaith [**31**]. These are also in agreement with what is known, and with previous conjectures.

Our purpose here is, for the integral moments of a family of primitive L-functions, to go beyond the leading order asymptotics previously investigated: we give conjectures for the full main terms. We propose a refined definition of 'conductor' of an L-function, which to leading order is the (logarithm of) the 'usual' conductor. We find that often, but not always, the mean values can be expressed as polynomials in the conductor. Importantly, our conjectures show a striking formal similarity with analogous expressions for the characteristic polynomials of random matrices. This provides a strong measure of the depth of the connection between L-functions and RMT. We also perform numerical calculations which show very good agreement with our conjectures. Non-primitive families can also be handled by our methods, but we do not treat those here.

The conjectures we develop here can also be obtained by techniques of multiple Dirichlet series, as described by Diaconu, Goldfeld and Hoffstein [**14**]. In their formulation, one considers Dirichlet series in several complex variables. The mean values we conjecture would then follow from a plausible conjecture about the polar divisors of the function. An interesting feature of their approach is that for higher moments it seems to predict lower order terms of the form cT^A with

$\frac{1}{2} < A < 1$, while in this paper we conjecture that our main terms are valid with an error of size $O(T^{1/2+\varepsilon})$. The cubic moment of quadratic Dirichlet L-functions is a specific case for which there is a conjectured lower order term [**49**] which possibly could be tested numerically.

There are many theorems dealing with moments of L-functions in particular families. The technique to prove these theorems usually involves invoking an approximate functional equation and averaging the coefficients of the L-function over the family. The averaging process behaves like a harmonic detection device. This harmonic detector usually presents itself as a formula with a relatively simple part and a somewhat more complicated part that is smaller in the first approximation. In the theorems in the literature it is often the case that the simple part of the harmonic detector is sufficiently good to determine the first or second moment of the family. The terms involved here are usually called the 'diagonal' terms. But invariably the more complicated version is needed to determine the asymptotics of the third or fourth moments; in these situations one has gone 'beyond the diagonal'. In at least one situation (the fourth moment of cusp form L-functions) it has been necessary to identify three separate stages of more subtle harmonic detection: the first featuring diagonal term contributions and the second and third featuring contributions to the main terms by two different types of off-diagonal terms. We believe that as one steps up the moments of a family then at every one or two steps a new type of off-diagonal contribution will emerge. The whole process is poorly understood; we only have glimpses of a mechanism but no clear idea of how or why it works.

It is remarkable that all of these complicated harmonic detection devices ultimately lead to very simple answers, as detailed in this paper. It is also remarkable that there are only three or four different types of symmetries; families with the same symmetry type often have different harmonic detectors, with different wrinkles at each new stage of off-diagonal, but somehow lead to answers which are structurally the same. It would be worthwhile to understand how this works.

Finally, we comment that the recipe we develop in this paper only uses the simplest diagonal harmonic detectors. Our formulas are expressed as combinatorial sums arising only from diagonal terms. We are well aware of the off-diagonal pieces, and we do not understand how they cancel and combine. What we do understand and what we are presenting here is a conjecture for the final simple answer that should emerge after all of the complicated cancellations between the increasingly subtle off-diagonal terms are taken into account. The reader needs to be aware of this to understand the goals and contents of this paper.

The paper is organized as follows. In the remainder of this section we give a detailed comparison between L-functions and characteristic polynomials of unitary matrices, summarize our previous work on the leading terms in the mean values of L-functions, and describe the more general moments considered in this paper. This allows us to state our main results and conjectures, which are given in § 1.5. We then give a detailed comparison with known results for the Riemann ζ-function.

In § 2 we give a detailed derivation of our conjectures in the case of moments on the critical line of a single L-function. We first write the conjecture in terms of a function defined by an infinite sum, and then write it as an Euler product and identify the leading-order poles. The local factors are also written in a concise form which is more suitable for computation. Both the L-function and random matrix calculations lead to expressions involving a sum over a set of partitions. These sums can be written in a concise form involving contour integrals, as described in § 2.5. We also show that the original results of Keating and Snaith [**30**, **31**] for the leading order term can be re-derived from the present work. In addition, we express the arithmetic factor in the moments of the Riemann zeta-function in an explicit form.

In § 3 we describe a particular notion of a family of L-functions which can be used to give a unified treatment of all of the mean values we have considered. These families are central to our method of conjecturing mean values and we give a detailed description of the method in § 4. As explicit examples we give the details of the calculations for L-function families with Unitary, Symplectic, and Orthogonal symmetry.

In § 5 we give numerical approximations for the coefficients in our conjectured mean values. We then report on numerical calculations of representative cases of the conjectures. Good agreement is found.

The calculations of the random matrix averages, which are based in part on [**3**] and [**4**], are complicated but elementary. Those results have been presented in [**9**]. In subsequent papers we will also present a fuller discussion of the terms which appear in our conjectures, give some more general conjectures, and describe the algorithms behind our numerical calculations.

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1.1. *Properties of L-functions*

We present the definition and key properties of L-functions. These properties are familiar, but a summary will be useful in our discussion of mean values and for the comparison with the characteristic polynomials of random matrices.

The definition of an L-function that we give below is a slight modification of what has come to be called the 'Selberg class' [**46**, **10**, **40**] of Dirichlet series. Let $s = \sigma + it$ with σ and t real. An *L-function* is a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.1.1}$$

with $a_n \ll_\varepsilon n^\varepsilon$ for every $\varepsilon > 0$, which has three additional properties.

Analytic continuation. The series $L(s)$ continues to a meromorphic function of finite order with at most finitely many poles, and all poles are located on the $\sigma = 1$ line.

Functional equation. There is a number ε with $|\varepsilon| = 1$, and a function $\gamma_L(s)$ of the form

$$\gamma_L(s) = P(s) Q^s \prod_{j=1}^{w} \Gamma(w_j s + \mu_j), \tag{1.1.2}$$

where $Q > 0$, $w_j > 0$, $\Re\mu_j \geqslant 0$, and P is a polynomial whose only zeros in $\sigma > 0$ are at the poles of $L(s)$, such that

$$\xi_L(s) := \gamma_L(s) L(s) \tag{1.1.3}$$

is entire, and

$$\xi_L(s) = \varepsilon\overline{\xi_L}(1-s), \tag{1.1.4}$$

where $\overline{\xi_L}(s) = \overline{\xi_L(\overline{s})}$ and $\overline{s}$ denotes the complex conjugate of s.

The number $2\sum_{j=1}^{w} w_j$ is called the *degree* of the L-function, and this is conjectured to be an integer. It is conjectured furthermore that each w_j can be taken to equal $\frac{1}{2}$, so w equals the degree of the L-function.

For the calculations we do in this paper, it is convenient to write the functional equation in asymmetric form:

$$L(s) = \varepsilon X_L(s)\overline{L}(1-s), \tag{1.1.5}$$

where $X_L(s) = \overline{\gamma_L}(1-s)/\gamma_L(s)$. Also we define the '$Z$-function' associated to an L-function:

$$Z_L(s) := \varepsilon^{-1/2}X_L^{-1/2}(s)L(s), \tag{1.1.6}$$

which satisfies the functional equation

$$Z_L(s) = \overline{Z_L}(1-s). \tag{1.1.7}$$

Note that here we define Z_L as a function of a complex variable, which is slightly different from the standard notation. Note also that $Z_L(\frac{1}{2}+it)$ is real when t is real, $X_L(\frac{1}{2}) = 1$, and $|X_L(\frac{1}{2}+it)| = 1$ if t is real.

Euler product. For $\sigma > 1$ we have

$$L(s) = \prod_p L_p(1/p^s), \tag{1.1.8}$$

where the product is over the primes p, and

$$L_p(1/p^s) = \sum_{k=0}^{\infty}\frac{a_{p^k}}{p^{ks}} = \exp\left(\sum_{k=1}^{\infty}\frac{b_{p^k}}{p^{ks}}\right), \tag{1.1.9}$$

where $b_n \ll n^\theta$ with $\theta < \frac{1}{2}$.

Note that $L(s) \equiv 1$ is the only constant L-function, the set of L-functions is closed under products, and if $L(s)$ is an L-function then so is $L(s+iy)$ for any real y. An L-function is called *primitive* if it cannot be written as a non-trivial product of L-functions, and it can be shown, assuming Selberg's orthonormality conjectures, that any L-function has a unique representation as a product of primitive L-functions. See [**10**]. It is believed that L-functions only arise from arithmetic objects, such as characters [**13**], automorphic forms [**24**, **25**], and automorphic representations [**2**, **5**]. Very little is known about L-functions beyond those cases which have been shown to be arithmetic.

There are several interesting consequences of the above properties, and many conjectures which have been established in a few (or no) cases. We highlight some additional properties of L-functions and then discuss their random matrix analogues.

Location of zeros. Since $\xi_L(s)$ is entire, $L(s)$ must vanish at the poles of the Γ-functions in the γ_L factor. These are known as the *trivial zeros* of the L-function. By the functional equation and the Euler product, the only other possible zeros of $L(s)$ lie in the *critical strip* $0 \leqslant \sigma \leqslant 1$. By the argument principle, the number of non-trivial zeros with $0 < t < T$ is asymptotically $(W/\pi)T\log T$, where $W = \sum w_j$. The *Riemann Hypothesis* for $L(s)$ asserts that the non-trivial zeros of $L(s)$ lie on the *critical line* $\sigma = \frac{1}{2}$. The much weaker (but still deep) assertion that $L(s) \neq 0$ on $\sigma = 1$ has been proven for arithmetic L-functions [**27**], which can be viewed as a generalization of the prime number theorem.

Average spacing of zeros. By the zero counting result described above, the average gap between consecutive zeros of $L(s)$ with imaginary part around T is $\pi/(W \log T)$.

Zeros of derivatives. If the Riemann Hypothesis is true then all zeros of the derivative $\xi'(s)$ lie on the critical line, while all zeros of $\zeta'(s)$ lie to the right of the critical line [**36**].

Critical values. The value $L(\frac{1}{2})$ is called the *critical value* of the L-function. The significance of $s = \frac{1}{2}$ is that it is the symmetry point of the functional equation. The mean values we study in this paper are averages of (powers of) critical values of L-functions, where the average is taken over a 'family' of L-functions. Examples of families and their corresponding mean values are given in § 1.3.

(*Note.* If the set $\{\mu_j\}$ is stable under complex conjugation and the a_n are real, then ε is commonly called the *sign of the functional equation*. If the sign is -1 then $L(s)$ has an odd order zero at $s = \frac{1}{2}$; more generally, if the sign is not 1 then $L(\frac{1}{2}) = 0$. When $L(\frac{1}{2})$ vanishes, it is common to use the term 'critical value' for the first non-zero derivative $L^{(j)}(\frac{1}{2})$, but in this paper we use 'critical value' to mean 'value at the critical point'.)

Log conductor. We measure the 'size' of an L-function by its *log conductor*, defined as $c(L) = \operatorname{cond}(L) = |X_L'(\frac{1}{2})|$. The conductor of an L-function has a conventional meaning in many contexts, and the log conductor is a simple function of the (logarithm of the) usual conductor. Other authors use similar names, such as 'analytic conductor', for similar quantities. By the argument principle, the density of zeros near the critical point is $2\pi c(L)^{-1}$.

Approximate functional equation. A standard tool for studying analytic properties of L-functions is an approximate functional equation for $L(s)$, which expresses the L-function as a sum of two Dirichlet series involving the Dirichlet coefficients of L multiplied by a smoothing function. See, for example, [**26**, 5.3]. For the purposes of the heuristics that we develop, we use a sharp cutoff and do not concern ourselves with the remainder,

$$L(s) = \sum_{m<x} \frac{a_m}{m^s} + \varepsilon X_L(s) \sum_{n<y} \frac{\overline{a_n}}{n^{1-s}} + \text{remainder}. \tag{1.1.10}$$

Here the product xy depends on parameters in the functional equation. The name comes from the fact that the right side looks like $L(s)$ if x is large, and like $\varepsilon X_L(s)\overline{L}(1-s)$ if x is small, which suggests the asymmetric form of the functional equation.

The approximate functional equation is the starting point of our approach to conjecturing the moments of L-functions. This is described in §§ 2.1 and 4.1.

1.2. *Properties of characteristic polynomials*

With the exception of the Euler product, all of the properties of L-functions have a natural analogue in the characteristic polynomials of unitary matrices. We note each property in turn.

Let

$$\Lambda(s) = \Lambda_A(s) = \det(I - A^* s) = \prod_{n=1}^{N}(1 - se^{-i\theta_n}) \tag{1.2.1}$$

denote the characteristic polynomial of an $N \times N$ matrix A. Throughout the paper we assume that A is unitary (that is, $A^*A = I$ where A^* is the Hermitian conjugate of A), so the eigenvalues of A lie on the unit circle and can be denoted by $e^{i\theta_n}$.

(*Note.* In our previous paper [**9**] we used a different definition of the characteristic polynomial.)

We can express $\Lambda(s)$ in expanded form:

$$\Lambda(s) = \sum_{n=0}^{N} a_n s^n, \tag{1.2.2}$$

which corresponds to the Dirichlet series representation for L-functions.

Analytic continuation. Since $\Lambda(s)$ is a polynomial, it is an entire function.

Functional equation. Since A is unitary, we have

$$\Lambda_A(s) = (-1)^N \det A^* \; s^N \det(I - As^{-1}), \tag{1.2.3}$$

and so, writing

$$\det A = e^{i\phi} \tag{1.2.4}$$

(where unitarity implies that $\phi \in \mathbb{R}$), we have

$$\begin{aligned} \Lambda_A(s) &= (-1)^N \det A^* s^N \Lambda_{A^*}(s^{-1}) \\ &= (-1)^N e^{-i\phi} s^N \,\overline{\Lambda_A}(s^{-1}). \end{aligned} \tag{1.2.5}$$

This plays the same role for $\Lambda(s)$ as the functional equation for L-functions: it represents a symmetry with respect to the unit circle ($s = re^{i\alpha} \to s^{-1} = r^{-1}e^{-i\alpha}$).

Also let

$$\mathcal{Z}_A(s) = ((-1)^N e^{i\phi})^{1/2} s^{-N/2} \Lambda_A(s), \tag{1.2.6}$$

in direct analogy to (1.1.6), the sign of the functional equation ε being identified with $(-1)^N e^{-i\phi} = (-1)^N \det A^*$. The functional equation becomes

$$\mathcal{Z}_A(s) = \overline{\mathcal{Z}}_A(s^{-1}). \tag{1.2.7}$$

Note that this implies that $\mathcal{Z}$ is real on the unit circle, and in analogy to the X_L factor from an L-function, the factor $s^{-N/2}$ equals 1 at the critical point $s = 1$, and has absolute value 1 on the unit circle.

Location of zeros. Since A is unitary, its eigenvalues all have modulus 1, so the zeros of $\Lambda(s)$ lie on the unit circle (that is, the Riemann Hypothesis is true). The unit circle is the 'critical line' for $\Lambda(s)$.

Average spacing of zeros. Since the $N \times N$ matrix A has N eigenvalues on the unit circle, the average spacing between zeros of $\Lambda_A(s)$ is $2\pi/N$.

Zeros of derivatives. Since the zeros of $\Lambda(s)$ lie on the unit circle, the zeros of the derivative Λ' lie inside the unit circle. This follows from the general fact that the zeros of the derivative of a polynomial lie in the convex hull of the zeros of the polynomial.

Critical values. The critical point for $\Lambda(s)$ is the symmetry point of the functional equation $s = 1 = e^{i \cdot 0}$, and $\Lambda(1)$ is the critical value.

Conductor. In analogy with the case of L-functions, we define the conductor of Λ to be the (absolute value of the) derivative of the factor in the asymmetric form of the functional equation, evaluated at the critical point $s = 1$. That is, the conductor of Λ is N. Also in analogy to the case of L-functions, the density of zeros on the unit circle is $2\pi/N$.

When modeling a family of L-functions, we choose N so that L and Λ have the same conductor. Equivalently, L and Λ have the same density of zeros near the critical point.

Approximate functional equation. Substituting the polynomial (1.2.2) into the functional equation (1.2.5), we have

$$\sum_{n=0}^{N} a_n s^n = (-1)^N e^{-i\phi} \sum_{n=0}^{N} \overline{a_n} s^{N-n}, \tag{1.2.8}$$

and so

$$a_n = (-1)^N e^{-i\phi}\, \overline{a_{N-n}}. \tag{1.2.9}$$

Hence, when N is odd, we have

$$\Lambda(s) = \sum_{m=0}^{(N-1)/2} a_m s^m + (-1)^N e^{-i\phi} s^N \sum_{n=0}^{(N-1)/2} \overline{a_n} s^{-n}, \tag{1.2.10}$$

which corresponds to the approximate functional equation for L-functions. When N is even, there is an additional term: $a_{N/2} s^{N/2}$.

Although we use the approximate functional equation in our calculations for L-functions, in our previous paper [**9**] we use other methods for the characteristic polynomials. In principle, it would be possible to use the approximate functional equation and compute averages of products of the coefficients a_n. Such a calculation would, presumably, mirror that for the L-functions. This would appear to be more cumbersome than the approach taken in [**9**], but might merit further investigation.

The above discussion applies to any unitary matrix. We also consider matrices which, in addition to being unitary, are either symplectic or orthogonal. We use these three ensembles of matrices to model families of L-functions. While the notion of 'family of L-functions' has not yet been made precise, we give several natural examples in the next section.

Associated to each family is a 'symmetry type' which identifies the matrix ensemble that will be used to model the family. This correspondence is most easily seen in terms of the sign of the functional equation, which is analogous to the determinant of the matrix. If A is unitary symplectic, then $\det A = 1$ (that is, $\phi = 0$), and if A is orthogonal, then $\det A = \pm 1$. Correspondingly, the functional equations for L-functions with unitary symmetry involve a (generally complex) phase factor, whereas for L-functions with symplectic symmetry this phase factor is unity, and in the case of orthogonal symmetry it is either $+1$ or -1.

While the sign of the functional equation can sometimes suggest the symmetry type of the family, in general it requires a calculation to determine the symmetry type. One possible calculation is to determine the moments of the family near the critical point, as described in this paper. Comparison with the corresponding random matrix average can then be used to determine the symmetry type. Another possibility is to determine the density of the low-lying zeros of the family.

1.3. *Example families and moments of L-functions*

We now give examples of families of primitive L-functions and describe the associated mean values. The families we consider here are of a special form, which is described in § 3. In preparation for the comparison with random matrices in the next section, we will classify the example families according to their symmetry type: Unitary, Orthogonal, and Symplectic. For the Orthogonal symmetry type we recognize three cases: SO, O^-, and O, corresponding respectively to Orthogonal families in which the functional equation has $\varepsilon = 1$, or $\varepsilon = -1$, or $\varepsilon = \pm 1$ equally often. Note that each family is a partially ordered set, and the order is determined by a quantity called the 'conductor' of the L-function. The mean values given below are conjectural for all but a few small values of k. For a general discussion of these mean values and some more examples, see [**8**].

Unitary examples.

(1) $\{L(s+iy) \mid y \geqslant 0\}$, ordered by y, where $L(s)$ is any primitive L-function. These are the only known continuous families of L-functions (Sarnak's rigidity conjecture).

(2) $\{L(s,\chi) \mid q \text{ a positive integer}, \chi \text{ a primitive character} \bmod q\}$ ordered by q.

An example conjectured mean value for integer k is

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = T\mathcal{P}_k(\log T) + O(T^{1/2+\varepsilon}), \tag{1.3.1}$$

for some polynomial $\mathcal{P}_k$ of degree k^2 with leading coefficient $g_k a_k / k^2!$, where

$$\begin{aligned} a_k &= \prod_p \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m} \\ &= \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \binom{k-1}{j}^2 p^{-j} \end{aligned} \tag{1.3.2}$$

and

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}. \tag{1.3.3}$$

(The placement of $k^2!$ is to ensure that g_k is an integer [**8**].) The above conjecture has been proven for $k = 1, 2$. See [**1**, **7**, **19**, **20**, **21**, **32**, **39**]. When $k = 2$ our conjectured error term of $O(T^{1/2+\varepsilon})$ has only been obtained in the case of a smooth weight function [**22**].

Symplectic examples.

(3) $\{L(s,\chi_d) \mid d \text{ a fundamental discriminant}, \chi_d(n) = (d/n)\}$ ordered by $|d|$.

(4) $\{L(s,\mathrm{sym}^2 f) \mid f \in S_k(\Gamma_0(1)), k \text{ a positive even integer}\}$, ordered by k.

An example conjectured mean value is

$$\sideset{}{^*}\sum_{|d| \leqslant D} L(\tfrac{1}{2},\chi_d)^k = \frac{6}{\pi^2} D\,\mathcal{Q}_k(\log D) + O(D^{1/2+\varepsilon}), \tag{1.3.4}$$

where $\sum^*$ is over fundamental discriminants, $\chi_d(n) = (d/n)$ is the Kronecker symbol, and the sum is over all real, primitive Dirichlet characters of conductor up to D. Here $\mathcal{Q}_k$ is a polynomial of degree $\frac{1}{2}k(k+1)$, with leading coefficient $g_k a_k/(\frac{1}{2}k(k+1))!$, where

$$a_k = \prod_p \frac{(1-1/p)^{k(k+1)/2}}{1+1/p}\left(\frac{(1-1/\sqrt{p})^{-k}+(1+1/\sqrt{p})^{-k}}{2}+\frac{1}{p}\right) \tag{1.3.5}$$

and

$$g_k = (\tfrac{1}{2}k(k+1))!\prod_{j=1}^{k}\frac{j!}{(2j)!}. \tag{1.3.6}$$

The main term of this conjecture has been proven for $k = 1, 2, 3$, and the case of $k = 4$ is almost within reach of current methods. See [**17**, **28**, **47**].

Orthogonal examples.

(5) $\{L(s,f) \mid f \in S_k(\Gamma_0(N)), N \text{ fixed}, k \text{ a positive even integer}\}$, ordered by k.

(6) $\{L(s,f) \mid f \in S_k(\Gamma_0(N)), k \text{ fixed}, N \text{ a positive integer}\}$, ordered by N.

An example conjectured mean value is

$$\sum_{f\in H_2(q)} L_f(\tfrac{1}{2},f)^k = \tfrac{1}{3}q\,\mathcal{R}_k(\log q) + O(q^{1/2+\varepsilon}), \tag{1.3.7}$$

where $H_2(q)$ is the collection of Hecke newforms of weight 2 and squarefree level q. Here $\mathcal{R}_k$ is a polynomial of degree $\frac{1}{2}k(k-1)$, with leading coefficient $g_k a_k/(\frac{1}{2}k(k-1))!$, where

$$\begin{aligned} a_k = \prod_{p\nmid q}\left(1-\frac{1}{p}\right)^{k(k-1)/2} \\ \times \frac{2}{\pi}\int_0^\pi \sin^2\theta\left(\frac{e^{i\theta}(1-e^{i\theta}/\sqrt{p})^{-1}-e^{-i\theta}(1-e^{-i\theta}/\sqrt{p})^{-1}}{e^{i\theta}-e^{-i\theta}}\right)^k d\theta \end{aligned} \tag{1.3.8}$$

and

$$g_k = 2^{k-1}(\tfrac{1}{2}k(k-1))!\prod_{j=1}^{k-1}\frac{j!}{(2j)!}. \tag{1.3.9}$$

The main term of this conjecture has been proven for $k = 1, 2, 3, 4$, in the case that q is prime. See [**15**, **16**, **34**]. Also see Ivić [**23**] for the analogous mean values for Maass forms.

The above examples are merely meant to give a flavor of the types of families which are of current interest.

The above cases, and their random matrix analogues, have been extensively discussed from the perspective of the *leading terms* in the asymptotic expansions. See [**8**, **30**, **31**]. In the present paper we extend that work to include all of the terms in the above mean values (that is, all coefficients in the conjectured polynomials), which we recover from a more general mean value involving a product of L-functions whose arguments are free parameters. In the next two sections we describe these more general mean values, discuss their random matrix analogues, and then state our results and conjectures.

1.4. *Shifted moments*

A key point in this paper is that the structure of mean values of L-functions is more clearly revealed if one considers the average of a product of L-functions, where each L-function is evaluated at a location slightly shifted from the critical point. The example mean values given in the previous section can be obtained by allowing the shifts to tend to zero.

Let $\alpha = (\alpha_1, \dots, \alpha_{2k})$, where throughout the paper we assume $|\Re\alpha_j| < \frac{1}{2}$, and suppose that $g(t)$ is a suitable weight function. The mean values we consider are

$$I_k(L, \alpha, g) = \int_{-\infty}^{\infty} Z_L(\tfrac{1}{2} + \alpha_1 + it) \dots Z_L(\tfrac{1}{2} + \alpha_{2k} + it) g(t)\, dt, \tag{1.4.1}$$

and, with $\alpha = (\alpha_1, \dots, \alpha_k)$,

$$S_k(\mathcal{F}, \alpha, g) = \sum_{L \in \mathcal{F}} Z_L(\tfrac{1}{2} + \alpha_1) \dots Z_L(\tfrac{1}{2} + \alpha_k) g(c(L)). \tag{1.4.2}$$

In the first case it is assumed that $L(s)$ is a primitive L-function, and in the second $\mathcal{F}$ is a family of primitive L-functions partially ordered by log conductor $c(L)$.

We refer to g as a 'suitable' weight function, but we leave that term undefined. An example of a suitable weight function is $g(x) = f(x/T)$, where f is real, non-negative, bounded, and integrable on the positive real line.

The random matrix analogs of the above expressions are

$$J_k(U(N), \alpha) = \int_{U(N)} \mathcal{Z}_A(e^{-\alpha_1}) \dots \mathcal{Z}_A(e^{-\alpha_{2k}})\, dA, \tag{1.4.3}$$

where $\alpha = (\alpha_1, \dots, \alpha_{2k})$ and the average is over Haar measure on $U(N)$, and

$$J_k(G(N), \alpha) = \int_{G(N)} \mathcal{Z}_A(e^{-\alpha_1}) \dots \mathcal{Z}_A(e^{-\alpha_k})\, dA, \tag{1.4.4}$$

where $G(N)$ is $\mathrm{USp}(2N)$, $O^-(2N)$, or $\mathrm{SO}(2N)$ and $\alpha = (\alpha_1, \dots, \alpha_k)$. Note that $O^-(2N)$ is defined as the collection of orthogonal $2N \times 2N$ matrices with determinant -1. Haar measure on $\mathrm{USp}(2N)$ and $O(2N)$ determines the weighting for the averages.

In the next section we compare our conjectures for the L-function mean values with exact formulae for the random matrix averages.

1.5. *Main results and example conjectures*

We state our main results and conjectures here. We give example conjectures for the full main term in shifted mean values of number-theoretic interest; these examples illustrate our methods and cover the three symmetry types of families of L-functions. We also give a corresponding theorem about the random matrix analogue of these mean values for each of the three compact matrix ensembles.

We present our results in pairs: a conjecture for an L-function mean value, followed by a theorem, quoted from [**9**], for the corresponding average of characteristic polynomials. For each pair the parts of each formula match according to the scheme described in § 1.2. In particular, the scaling of the large parameter is determined by equating log conductors. In the random matrix formula the integrand contains a term $(1 - e^{\pm z_m - z_\ell})^{-1}$, which has a simple pole at $z_\ell = \pm z_m$. In the L-function formula this corresponds to the term containing all of the arithmetic information, which is of the form $\zeta(1+z_i \mp z_j)$ times an Euler product, and so also has a simple pole at $z_i = \pm z_j$.

The formulae are written in terms of contour integrals and involve the Vandermonde:

$$\Delta(z_1,\dots,z_m) = \prod_{1\leqslant i<j\leqslant m}(z_j - z_i). \tag{1.5.1}$$

We also set $e(z) = e^{2\pi i z}$.

CONJECTURE 1.5.1. *Suppose $g(t)$ is a suitable weight function. Then*

$$\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it)|^{2k} g(t)\,dt = \int_{-\infty}^{\infty} P_k\Big(\log\frac{t}{2\pi}\Big)(1+O(t^{-1/2+\varepsilon}))g(t)\,dt, \tag{1.5.2}$$

where P_k is the polynomial of degree k^2 given by the $2k$-fold residue

$$P_k(x) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint\frac{G(z_1,\dots,z_{2k})\Delta^2(z_1,\dots,z_{2k})}{\prod_{j=1}^{2k} z_j^{2k}} \times e^{(x/2)\sum_{j=1}^{k} z_j - z_{k+j}}\,dz_1\dots dz_{2k}, \tag{1.5.3}$$

where one integrates over small circles about $z_i = 0$, with

$$G(z_1,\dots,z_{2k}) = A_k(z_1,\dots,z_{2k})\prod_{i=1}^{k}\prod_{j=1}^{k}\zeta(1+z_i-z_{k+j}), \tag{1.5.4}$$

and A_k is the Euler product

$$A_k(z) = \prod_p \prod_{i=1}^{k}\prod_{j=1}^{k}\Big(1-\frac{1}{p^{1+z_i-z_{k+j}}}\Big) \times \int_0^1 \prod_{j=1}^{k}\Big(1-\frac{e(\theta)}{p^{1/2+z_j}}\Big)^{-1}\Big(1-\frac{e(-\theta)}{p^{1/2-z_{k+j}}}\Big)^{-1}d\theta. \tag{1.5.5}$$

More generally,

$$I_k(\zeta,\alpha,g) = \int_{-\infty}^{\infty} P_k\Big(\log\frac{t}{2\pi},\alpha\Big)(1+O(t^{-1/2+\varepsilon}))g(t)\,dt, \tag{1.5.6}$$

where

$$P_k(x,\alpha) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint\frac{G(z_1,\dots,z_{2k})\Delta^2(z_1,\dots,z_{2k})}{\prod_{j=1}^{2k}\prod_{i=1}^{2k}(z_j-\alpha_i)} \times e^{(x/2)\sum_{j=1}^{k} z_j - z_{k+j}}\,dz_1\dots dz_{2k}, \tag{1.5.7}$$

with the path of integration being small circles surrounding the poles α_i.

A general version of the above conjecture is given in Conjecture 2.5.4.

THEOREM 1.5.2. *In the notation of § 1.4 we have*

$$J_k(U(N),0) = \prod_{j=0}^{k-1}\Big(\frac{j!}{(k+j)!}\prod_{i=1}^{k}(N+i+j)\Big). \tag{1.5.8}$$

More generally, with

$$G(z_1,\dots,z_{2k}) = \prod_{i=1}^{k}\prod_{j=1}^{k}(1-e^{-z_i+z_{j+k}})^{-1},$$

we have

$$J_k(U(N),\alpha) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint\frac{G(z_1,\dots,z_{2k})\Delta^2(z_1,\dots,z_{2k})}{\prod_{i=1}^{2k}\prod_{j=1}^{2k}(z_j-\alpha_i)} \times e^{(N/2)\sum_{j=1}^{k} z_j - z_{k+j}}\,dz_1\dots dz_{2k}. \tag{1.5.9}$$

Comments on the formulae. (1) Let $\alpha_i \to 0$ in the second part of Conjecture 1.5.1 to obtain the first part of Conjecture 1.5.1.

(2) The structures of $J_k(U(N),\alpha)$ and $P_k(x,\alpha)$ are identical in that the functions $G(z_1,\dots,z_{2k})$ have simple poles at $z_i = z_{k+j}$.

(3) The local factors of $A_k(\alpha)$ are polynomials in p^{-1} and $p^{-\alpha_i}$, for $i=1,\dots,k$, as seen from Theorem 2.6.2. Since $A_k(\alpha)$ comes from a symmetric expression, it is also a polynomial in $p^{\alpha_{i+k}}$, for $i=1,\dots,k$. This is discussed in § 2.6. Note also that a_k in (1.3.2) equals $A_k(0,\dots,0)$, as shown in § 2.7.

(4) That $P_k(x)$ is actually a polynomial of degree k^2 can be seen by considering the order of the pole at $z_j=0$. We wish to extract from the numerator of the integrand, the coefficient of $\prod z_i^{2k-1}$, a polynomial of degree $2k(2k-1)$. The Vandermonde determinant squared is a homogeneous polynomial of degree $2k(2k-1)$. However, the poles coming from the $\zeta(1+z_i-z_{k+j})$ cancel k^2 of the Vandermonde factors. This requires us, in computing the residue, to take, in the Taylor expansion of $\exp(\frac{1}{2}x\sum_1^k z_j - z_{k+j})$, terms up to degree k^2.

(5) The fact that $P_k(\log(t/2\pi))$ is a polynomial in $\log(t/2\pi)$ of degree k^2 corresponds nicely to the formula for $J_k(U(N),0)$, which is a polynomial of degree k^2 in N. Equating the density of the Riemann zeros at height t with the density of the random matrix eigenvalues suggests the familiar equivalence $N=\log(t/2\pi)$. In this paper we view this as equating conductors.

(6) The leading term of $P_k(x)$ coincides with the leading term conjectured by Keating and Snaith (see § 2.7). The full polynomial $P_k(x)$ agrees, when $k=1$ and $k=2$, with known theorems (see §§ 1.6 and 1.7).

(7) We can recover the polynomial $\mathcal{P}_k$ in (1.3.1) from P_k by taking $g(t)=\chi_{[0,T]}(t)$ in the conjecture.

(8) The multiple integrals in Theorem 1.5.2 and Conjecture 1.5.1 can be written as combinatorial sums. See § 2 where a detailed derivation of our conjecture is given.

(9) Our conjecture concerning the order of the error term is based on our numerical calculations (see § 5) and examination of examples in the literature.

Conjecture 1.5.3. *Suppose $g(u)$ is a suitable weight function with support in either $(0,\infty)$ or $(-\infty,0)$, and let $X_d(s)=|d|^{1/2-s}X(s,a)$ where $a=0$ if $d>0$ and $a=1$ if $d<0$, and*

$$X(s,a) = \pi^{s-1/2}\Gamma\Big(\frac{1+a-s}{2}\Big)\Big/\Gamma\Big(\frac{s+a}{2}\Big). \tag{1.5.10}$$

That is, $\chi_d(s)$ is the factor in the functional equation

$$L(s,\chi_d) = \varepsilon_d X_d(s) L(1-s,\chi_d).$$

Summing over fundamental discriminants d we have

$$\sum_d{}^{*} L(\tfrac12,\chi_d)^k g(|d|) = \sum_d{}^{*} Q_k(\log|d|)(1+O(|d|^{-1/2+\varepsilon}))g(|d|) \tag{1.5.11}$$

where Q_k is the polynomial of degree $\frac12 k(k+1)$ given by the k-fold residue

$$\begin{aligned} Q_k(x) = {} & \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint \frac{G(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} \\ & \times e^{(x/2)\sum_{j=1}^k z_j}\,dz_1\ldots dz_k, \end{aligned} \tag{1.5.12}$$

where

$$G(z_1,\ldots,z_k) = A_k(z_1,\ldots,z_k)\prod_{j=1}^k X(\tfrac12+z_j,a)^{-1/2}\prod_{1\leqslant i\leqslant j\leqslant k}\zeta(1+z_i+z_j), \tag{1.5.13}$$

and A_k is the Euler product, absolutely convergent for $|\Re z_j|<\frac12$, defined by

$$\begin{aligned} A_k(z_1,\ldots,z_k) = {} & \prod_p \prod_{1\leqslant i\leqslant j\leqslant k}\left(1-\frac{1}{p^{1+z_i+z_j}}\right) \\ & \times \left(\frac12\left(\prod_{j=1}^k\left(1-\frac{1}{p^{1/2+z_j}}\right)^{-1}+\prod_{j=1}^k\left(1+\frac{1}{p^{1/2+z_j}}\right)^{-1}\right)+\frac1p\right) \\ & \times \left(1+\frac1p\right)^{-1}. \end{aligned} \tag{1.5.14}$$

More generally, if $\mathcal{F}$ is the family of real primitive Dirichlet L-functions then

$$S_k(\mathcal{F},\alpha,g) = \sum_d{}^{*} Q_k(\log|d|,\alpha)(1+O(|d|^{-1/2+\varepsilon}))g(|d|), \tag{1.5.15}$$

in which

$$\begin{aligned} Q_k(x,\alpha) = {} & \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k} \\ & \times \oint\cdots\oint \frac{G(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{i=1}^k\prod_{j=1}^k(z_j-\alpha_i)(z_j+\alpha_i)} \\ & \times e^{(x/2)\sum_{j=1}^k z_j}\,dz_1\ldots dz_k, \end{aligned} \tag{1.5.16}$$

where the path of integration encloses the $\pm\alpha_i$.

THEOREM 1.5.4. *In the notation of § 1.4 we have*

$$J_k(\mathrm{USp}(2N),0) = \left(2^{k(k+1)/2}\prod_{j=1}^k\frac{j!}{(2j)!}\right)\prod_{1\leqslant i\leqslant j\leqslant k}(N+\tfrac12(i+j)). \tag{1.5.17}$$

More generally, with

$$G(z_1,\dots,z_k)=\prod_{1\leqslant i\leqslant j\leqslant k}(1-e^{-z_i-z_j})^{-1}$$

we have

$$\begin{aligned}J_k(\mathrm{USp}(2N),\alpha)=&\frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}\\&\times\oint\cdots\oint\frac{G(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{i=1}^k\prod_{j=1}^k(z_j-\alpha_i)(z_j+\alpha_i)}\\&\times e^{N\sum_{j=1}^k z_j}dz_1\dots dz_k,\end{aligned}\tag{1.5.18}$$

where the contours of integration enclose the $\pm\alpha_i$.

Comments. (1) When comparing Theorem 1.5.4 with Conjecture 1.5.3, equating log conductors (that is, the density of zeros) gives the equivalence

$$2N=\mathrm{cond}(d):=\log(|d|/\pi)+(\Gamma'/\Gamma)(\tfrac14+a).\tag{1.5.19}$$

The conductor we use here should be contrasted with the 'usual' conductor associated with Dirichlet L-functions: $\log(|d|/\pi)-\log(2)$. We believe this difference is significant, so we discuss it briefly.

The following manipulations show that our conductor arises naturally. In the derivation of the conjecture, one encounters the function $X_d(\frac12+z)^{-1/2}$, which can we rewritten in several ways:

$$\begin{aligned}X_d(\tfrac12+z)^{-1/2}&=e^{(\log d\cdot z)/2}X(\tfrac12+z,a)^{-1/2}\\&=e^{(\mathrm{cond}(d)\cdot z)/2}\mathcal{G}(z),\end{aligned}\tag{1.5.20}$$

where $\mathcal{G}(z)=1+O(z^3)$. In the statement of the conjecture we used the first line of (1.5.20), incorporating the product over $X(\frac12+z,a)^{-1/2}$ into the factor $G(z_1,\dots,z_k)$. If we chose instead to use the second line of (1.5.20), then the conjecture would be written as a sum over $Q_k(\mathrm{cond}(d))$. One would still find that Q_k is a polynomial of degree $\frac12k(k+1)$. Since $\mathcal{G}(z)=1+O(z^3)$, the first three leading terms in that polynomial would not explicitly depend on the factor X_d from the functional equation, although the lower degree terms would. This phenomenon does not occur for moments of L-functions in t-aspect.

(2) The function $\mathcal{Q}_k$ in (1.3.4) can be recovered from Q_k above by taking $g(|d|)=\chi_{[0,D]}(|d|)$, and using the estimate $\sum^*{}_{-D<d<0}1=3D/\pi^2+O(D^{1/2+\varepsilon})$; the same estimate holds for positive d.

(3) A heuristic derivation of Conjecture 1.5.3 is given in § 4.4.

(4) The leading term of Q_k coincides with the leading term conjectured by Keating and Snaith [**31**]. The calculation is analogous to the one given in § 2.7.

CONJECTURE 1.5.5. *Suppose q is squarefree, let $H_n(q)$ be the set of newforms in $S_n(\Gamma_0(q))$, and let*

$$X_{n,q}(s)=\Big(\frac{q}{4\pi^2}\Big)^{1/2-s}\frac{\Gamma(\frac12-s+\frac12n)}{\Gamma(s-\frac12+\frac12n)}\tag{1.5.21}$$

be the factor in the functional equation $L_f(s) = \varepsilon_{n,q} X_{n,q}(s) L_f(1-s)$ for the L-functions associated to $f \in H_n(q)$. Then

$$\sum_{f \in H_n(q)} L_f(\tfrac{1}{2})^k \langle f, f\rangle^{-1} = \sum_{f \in H_n(q)} R_k(n,q) \langle f, f\rangle^{-1} (1 + O(nq)^{-1/2+\varepsilon}) \tag{1.5.22}$$

as $nq \to \infty$, where $R_k(n,q)$ is given by the k-fold residue

$$\begin{aligned} R_k = \frac{(-1)^{k(k-1)/2} 2^{k-1}}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{G(z_1, \ldots, z_k) \Delta(z_1^2, \ldots, z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} \\ \times \prod_{j=1}^k X_{n,q}(\tfrac{1}{2} + z_j)^{-1/2} dz_1 \ldots dz_k, \end{aligned} \tag{1.5.23}$$

where

$$G(z_1, \ldots, z_k) = A_k(z_1, \ldots, z_k) \prod_{1 \leqslant i < j \leqslant k} \zeta(1 + z_i + z_j) \tag{1.5.24}$$

and A_k is the Euler product which is absolutely convergent for $|\Re z_j| < \frac{1}{2}$, with $j = 1, \ldots, k$, defined by

$$\begin{aligned} A_k&(z_1, \ldots, z_k) \\ &= \prod_{p \nmid q} \prod_{1 \leqslant i < j \leqslant k} \left(1 - \frac{1}{p^{1+z_i+z_j}}\right) \\ &\times \frac{2}{\pi} \int_0^\pi \sin^2\theta \prod_{j=1}^k \frac{e^{i\theta}\left(1 - e^{i\theta}/p^{1/2+z_j}\right)^{-1} - e^{-i\theta}\left(1 - e^{-i\theta}/p^{1/2+z_j}\right)^{-1}}{e^{i\theta} - e^{-i\theta}} \, d\theta. \end{aligned} \tag{1.5.25}$$

To state the more general version of Conjecture 1.5.5, involving a sum of products of $L_f(\frac{1}{2} + u_j)$, it is natural also to consider the sums over even f and odd f separately. See Conjectures 4.5.1 and 4.5.2.

THEOREM 1.5.6. *In the notation of* § 1.4 *we have*

$$J_k(\mathrm{SO}(2N), 0) = \left(2^{k(k+1)/2} \prod_{j=1}^{k-1} \frac{j!}{(2j)!}\right) \prod_{0 \leqslant i < j \leqslant k-1} (N + \tfrac{1}{2}(i+j)). \tag{1.5.26}$$

More generally, with

$$G(z_1, \ldots, z_k) = \prod_{1 \leqslant \ell < m \leqslant k} (1 - e^{-z_m - z_\ell})^{-1} \tag{1.5.27}$$

we have

$$\begin{aligned} J_k(\mathrm{SO}(2N), \alpha) = \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \\ \times \oint \cdots \oint \frac{G(z_1, \ldots, z_k) \Delta(z_1^2, \ldots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_j - \alpha_i)(z_j + \alpha_i)} \\ \times e^{N \sum_{j=1}^k z_j} dz_1 \ldots dz_k \end{aligned} \tag{1.5.28}$$

and

$$J_k(O^-(2N),\alpha) = i^{-k}\frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}$$
$$\times \oint\cdots\oint \frac{G(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^k \alpha_j}{\prod_{i=1}^k\prod_{j=1}^k (z_j-\alpha_i)(z_j+\alpha_i)}$$
$$\times e^{N\sum_{j=1}^k z_j}dz_1\ldots dz_k. \tag{1.5.29}$$

Comments. (1) The value of $R_k(n,q)$ does not actually depend on $f \in H_n(q)$. We write (1.5.22) in this manner to stress that $R_k(n,q)$ is the expected value of $L_f(\frac{1}{2})^k$.

(2) To compare Theorem 1.5.6 and Conjecture 1.5.5, equating conductors gives the equivalence

$$\begin{aligned}2N = \operatorname{cond}(n,q) &:= \log(q/4\pi^2) + (\Gamma'/\Gamma)(n/2)\\ &= \log(qn/8\pi^2) + O(n^{-1}).\end{aligned} \tag{1.5.30}$$

One can express the conjectured mean value in terms of the conductor in the following way. In (1.5.23) we can write

$$X_{n,q}(\tfrac{1}{2}+z_j)^{-1/2} = e^{(\operatorname{cond}(n,q)\cdot z_j)/2}\mathcal{G}(z_j), \tag{1.5.31}$$

where $\mathcal{G}(z_j) = 1 + O(z^3)$. As in Conjecture 1.5.3, we can express $R_k(n,q)$ as a polynomial in the conductor, the first three terms of which do not depend on the $X_{n,q}$ factor in the functional equation.

(3) All of our conjectures naturally contain a factor of the form $\prod X(\frac{1}{2}\pm z_j)^{-1/2}$; it just happens that in some cases $X(\frac{1}{2}\pm z_j)$ can be closely approximated by a simple function of the conductor. It is interesting that this same factor occurs in all of the random matrix moments. In that case $X(s) = s^{-M}$, where $M = N$ or $2N$, so in the formula for the moments there occurs $\prod X(e^{\pm z_j})^{-1/2} = e^{(M/2)\sum \pm z_j}$.

1.6. *Second moment of the Riemann zeta-function*

Now we consider the second moment of the Riemann zeta-function in detail, putting our results in the context of the literature.

Ingham's result [**21**] on the second moment can be stated as

$$\int_0^T \zeta(s+\alpha)\zeta(1-s-\beta)\,dt$$
$$= \int_0^T (\zeta(1+\alpha-\beta) + \tau^{\beta-\alpha}\zeta(1+\beta-\alpha))(1+O(t^{-1/2+\varepsilon}))\,dt \tag{1.6.1}$$

where $s = \frac{1}{2}+it$ and $\tau = \tau(t) = |t|/2\pi$; this is valid for $|\alpha|, |\beta| < \frac{1}{2}$. If we let α and β approach 0 here, we obtain Ingham's theorem

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = \int_0^T \left(\log\frac{t}{2\pi} + 2\gamma\right)dt + O(T^{1/2+\varepsilon}). \tag{1.6.2}$$

Our conjecture is compatible with these results, because, when $k=1$, the function $G(\alpha_1,\alpha_2)$ that appears in Conjecture 1.5.1 equals $\zeta(1+\alpha_1-\alpha_2)$. Computing the residue, we find that

$$P_1(x) = \frac{1}{4\pi^2}\oint\oint \frac{\zeta(1+z_1-z_2)(z_2-z_1)^2}{z_1^2 z_2^2} e^{(x/2)(z_1-z_2)}\, dz_1\, dz_2$$
$$= x + 2\gamma. \tag{1.6.3}$$

The second moment with a different weighting is now given; this theorem is a slight variation of the theorem of Kober presented by Titchmarsh in [**48**] and was inspired by the numerical calculations described in § 5.1.

Theorem 1.6.1. *Let*

$$I(\alpha,\beta,\delta) = \int_0^\infty \zeta(\tfrac{1}{2}+it+\alpha)\zeta(\tfrac{1}{2}-it-\beta)e^{-\delta t}\, dt. \tag{1.6.4}$$

Then, for any $\eta > 0$, $|\alpha|, |\beta| \leqslant \frac{1}{2} - \eta$ *and* $|\arg\delta| \leqslant \frac{1}{2}\pi - \eta$, *we have*

$$I(\alpha,\beta,\delta) = \int_0^\infty \left(\zeta(1+\alpha-\beta) + \left(\frac{t}{2\pi}\right)^{\alpha-\beta}\zeta(1-\alpha+\beta)\right)e^{-\delta t}\, dt$$
$$+ C_\delta(\alpha,\beta) + O(\delta\log 1/\delta) \tag{1.6.5}$$

uniformly in α, β *and* δ *where* $C_\delta(\alpha,\beta) \ll \log 1/\delta$ *uniformly in* α *and* β *and where* $C_\delta(\alpha,-\alpha) = -2\pi\zeta(2\alpha)$.

We restate the case $\beta = -\alpha$ as follows.

Corollary 1.6.2. *For any fixed* α *with* $|\alpha| < \frac{1}{2}$, *we have*

$$\lim_{\delta\to 0}\left(\int_0^\infty |\zeta(\tfrac{1}{2}+\alpha+it)|^2 e^{-\delta t}\, dt - \int_0^\infty \left(\zeta(1+2\alpha) + \left(\frac{t}{2\pi}\right)^{2\alpha}\zeta(1-2\alpha)\right)e^{-\delta t}\, dt\right)$$
$$= -\pi\zeta(2\alpha) - (2\pi)^{2\alpha}\zeta(1-2\alpha)\Gamma(1-2\alpha)\sin\pi\alpha$$
$$= -2\pi\zeta(2\alpha). \tag{1.6.6}$$

Note that

$$\lim_{x\to 0}(\zeta(1+x) + w^{-x}\Gamma(1-x)\zeta(1-x)) = (2\gamma + \log w). \tag{1.6.7}$$

Thus, letting $\alpha \to 0$, gives the following corollary.

Corollary 1.6.3.

$$\lim_{\delta\to 0}\left(\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-\delta t}\, dt - \int_0^\infty \left(2\gamma + \log\left(\frac{t}{2\pi}\right)\right)e^{-\delta t}\, dt\right) = \pi. \tag{1.6.8}$$

Remark. We discovered this corollary after seeing the numerical results of § 5.1. This result also follows from a result of Hafner and Ivić [**18**].

1.7. *Fourth moment of the Riemann zeta-function*

Now we consider the fourth moment of the Riemann zeta-function in detail. Our discussion here builds upon work of Atkinson [**1**], Heath-Brown [**20**], Conrey [**7**], and Motohashi [**39**].

Examining Motohashi's results in detail, consider

$$\int_{-\infty}^{\infty} \zeta(s+u_1)\dots\zeta(s+u_k)\zeta(1-s+v_1)\dots\zeta(1-s+v_k)g(t)\,dt \tag{1.7.1}$$

for a function $g(t)$ which is analytic in a horizontal strip $|\Im(t)| < c$ and decays sufficiently rapidly. Motohashi obtains an exact formula for these moments for $k = 1$ and $k = 2$. We reformulate Motohashi's theorem ($k = 2$) in our context. Let

$$C(v) = (2\pi)^v/(2\cos\tfrac{1}{2}\pi v) \tag{1.7.2}$$

and let

$$G_s(u,v) = \Gamma(s-u)/\Gamma(s-v). \tag{1.7.3}$$

Then, in notation analogous to Motohashi's, the $k = 2$ case of (1.7.1) equals

$$L_r + L_d + L_c + L_h, \tag{1.7.4}$$

where L_r is the (residual) main term which we are interested in here:

$$L_r(u,v) = \int_{-\infty}^{\infty} W(t,u,v)g(t)\,dt, \tag{1.7.5}$$

with

$$\begin{aligned} W(t,u,v) = C(0)&(G_s(0,0) + G_{1-s}(0,0))Z(u_1,u_2,v_1,v_2) \\ &+ C(u_1+v_1)(G_s(u_1,v_1) + G_{1-s}(u_1,v_1))Z(-v_1,u_2,-u_1,v_2) \\ &+ C(u_1+v_2)(G_s(u_1,v_2) + G_{1-s}(u_1,v_2))Z(-v_2,u_2,v_1,-u_1) \\ &+ C(u_2+v_1)(G_s(u_2,v_1) + G_{1-s}(u_2,v_1))Z(u_1,-v_1,-u_2,v_2) \\ &+ C(u_2+v_2)(G_s(u_2,v_2) + G_{1-s}(u_2,v_2))Z(u_1,-v_2,v_1,-u_2) \\ &+ C(u_1+u_2+v_1+v_2)(G_s(u_1,v_1)G_s(u_2,v_2) \\ &+ G_{1-s}(u_1,v_1)G_{1-s}(u_2,v_2))Z(-v_1,-v_2,-u_1,-u_2), \end{aligned} \tag{1.7.6}$$

where $s = \frac{1}{2} + it$, and

$$Z(u_1,u_2,v_1,v_2) = \frac{\zeta(1+u_1+v_1)\zeta(1+u_1+v_2)\zeta(1+u_2+v_1)\zeta(1+u_2+v_2)}{\zeta(2+u_1+u_2+v_2+v_2)}. \tag{1.7.7}$$

This formula may be obtained from Motohashi's work [**39**, pp. 174–178] by a careful analysis of his terms together with appropriate use of the functional equation in the form

$$\Gamma(s)\zeta(s) = \frac{(2\pi)^s}{2\cos\pi s/2}\zeta(1-s) \tag{1.7.8}$$

and some trigonometric identities.

If we use the approximation

$$\frac{\Gamma(s+\alpha)}{\Gamma(s+\beta)} = (i|s|)^{\alpha-\beta}(1 + O(1/|s|), \tag{1.7.9}$$

we have, using $\tau = |t|/(2\pi)$,

$$C(u)(G_s(u) + G_{1-s}(u)) = \tau^{-u}(1 + O(1/\tau)) \tag{1.7.10}$$

and

$$C(u+v)(G_s(u)G_s(v)+G_{1-s}(u)G_{1-s}(v)) = \tau^{-u-v}(1+O(1/\tau)). \tag{1.7.11}$$

We then have

$$\begin{aligned} W(t,u,v) = (&Z(u_1,u_2,v_1,v_2) \\ &+\tau^{-u_1-v_1}Z(-v_1,u_2,-u_1,v_2)+\tau^{-u_1-v_2}Z(-v_2,u_2,v_1,-u_1) \\ &+\tau^{-u_2-v_1}Z(u_1,-v_1,-u_2,v_2)+\tau^{-u_2-v_2}Z(u_1,-v_2,v_1,-u_2) \\ &+\tau^{-u_1-u_2-v_1-v_2}Z(-v_1,-v_2,-u_1,-u_2))(1+O(1/\tau)). \end{aligned} \tag{1.7.12}$$

By the formulae in §§ 2.2 and 2.5, the above agrees with the $k=2$ case of Conjecture 1.5.1.

A residue computation shows that our conjecture can be restated as

$$\begin{aligned} \int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt = \int_0^T \frac{1}{2}\frac{1}{(2\pi i)^2}\oint\oint \left(\frac{t}{2\pi}\right)^{x+y} \\ \times \frac{\zeta(1+x)^4\zeta(1+y)^4}{\zeta(1+x-y)\zeta(1+y-x)\zeta(2+2x+2y)}\,dx\,dy\,dt \\ + O(T^{1/2+\varepsilon}), \end{aligned} \tag{1.7.13}$$

where we integrate around small circles centered on the origin. This is in contrast to Conjecture 1.5.1, which when $k=2$ expresses the formula in terms of four contour integrals. It may be that our formulae can be similarly simplified for all k, but we have not succeeded in doing so.

2. *Moments in t-aspect*

The principle behind our method of conjecturing mean values is that the Dirichlet series coefficients of L-functions have an approximate orthogonality relation when averaged over a family. These orthogonality relations are used to identify the main terms in the mean values.

In this section we give a detailed account of the case of moments of a single primitive L-function. We describe the recipe for conjecturing the mean values, applying it first to the case of the Riemann ζ-function, and then to a general primitive L-function. In the remainder of this section we manipulate the formulas into a more usable form, and also obtain a generalization of Conjecture 1.5.1. Later in § 3 we recast our principles in a more general setting and consider the averages of various families of L-functions.

2.1. *The recipe*

The following is our recipe for conjecturing the $2k$th moment of an L-function:

(1) Start with a product of $2k$ shifted L-functions:

$$Z(s,\alpha_1,\dots,\alpha_{2k}) = Z(\tfrac{1}{2}+\alpha_1)\dots Z(\tfrac{1}{2}+\alpha_{2k}) \tag{2.1.1}$$

(here we have written the Z-function, but the examples below will show that the method applies to either the L- or the Z-function).

(2) Replace each L-function with the two terms from its approximate functional equation, ignoring the remainder term. Multiply out the resulting expression to obtain 2^{2k} terms.

(3) Keep the $\binom{2k}{k}$ terms for which the product of χ-factors from the functional equation is not rapidly oscillating. Use (2.2.9) to simplify the non-oscillating χ-factors.

(4) In each of those $\binom{2k}{k}$ terms, keep only the diagonal from the sum.

(5) Extend the sums to run over all positive integers, and call the total $M(s, \alpha_1, \ldots, \alpha_{2k})$.

(6) The conjecture is

$$\int_{-\infty}^{\infty} Z(\tfrac{1}{2}+it, \alpha_1, \ldots, \alpha_{2k}) g(t)\, dt = \int_{-\infty}^{\infty} M(\tfrac{1}{2}+it, \alpha_1, \ldots, \alpha_{2k})(1 + O(t^{-1/2+\varepsilon})) g(t)\, dt, \tag{2.1.2}$$

for all $\varepsilon > 0$, where g is a suitable weight function. In other words, $Z(s, \alpha)$ and $M(s, \alpha)$ have the same expected value if averaged over a sufficiently large range.

Notes. (1) In order to see the structure of these mean values, it is necessary to include the shifts α_j. One can obtain the moments of $L(\frac{1}{2}+it)$ by allowing the shifts to tend to 0. Because of the shifts α_j we avoid higher-order poles in our expressions.

(2) The recipe applies to either the L-function or the Z-function, and we give examples of both cases. The Z-function case can be directly obtained from the L-function, although the reverse is not true in general.

(3) For the approximate functional equations in the recipe, one can ignore the range of summation because it will just be extended to infinity in the final step.

(4) We do not define what is meant by a 'suitable weight function', but it is acceptable to take $g(t) = g_T(t) = f(t/T)$ for a fixed integrable function f. In particular, one can take f to be the characteristic function of the interval $[0, 1]$, obtaining the mean value $\int_0^T Z(\frac{1}{2}+it, \alpha)\, dt$. From this one can recover a fairly general weighted integral by partial integration.

(5) The error term $O(t^{-1/2+\varepsilon})$ fits with known examples and numerical evidence. See § 5.

(6) The above procedure is a recipe for conjecturing all of the main terms in the mean value of an L-function. It is not a heuristic, and the steps cannot be justified. In particular, some steps can throw away terms which are the same size as the main term, and other steps add main terms back in. Our conjecture is that all of those errors cancel.

2.2. *Moments of the Riemann ζ-function*

We illustrate our recipe in the case of the Riemann zeta-function. In this section we consider the most familiar case of moments of $\zeta(\frac{1}{2}+it)$. In § 2.3 we relate this to moments of $Z(\frac{1}{2}+it)$ and repeat the calculation for the Z-function of an arbitrary primitive L-function.

Consider

$$Z(s, \alpha) = \zeta(s+\alpha_1) \ldots \zeta(s+\alpha_k)\zeta(1-s-\alpha_{k+1}) \ldots \zeta(1-s-\alpha_{2k}), \tag{2.2.1}$$

where $\alpha = (\alpha_1, \dots, \alpha_{2k})$. Note that this is slightly different notation than given in (2.1.1). Our goal is a formula for

$$\int_{-\infty}^{\infty} Z(\tfrac{1}{2}+it, \alpha) g(t)\, dt. \tag{2.2.2}$$

For each ζ-function we use the approximate functional equation

$$\zeta(s) = \sum_m \frac{1}{m^s} + \chi(s) \sum_n \frac{1}{n^{1-s}} + \text{remainder}. \tag{2.2.3}$$

Recall that we ignore the remainder term and the limits on the sums. Multiplying out the resulting expression we obtain 2^{2k} terms, and the recipe tells us to keep those terms in which the product of χ-factors is not oscillating rapidly.

If $s = z + it$ with z bounded (but not necessarily real) then

$$\chi(s) = \Big(\frac{t}{2\pi}\Big)^{1/2-s} e^{it+\pi i/4}\Big(1 + O\Big(\frac{1}{t}\Big)\Big) \tag{2.2.4}$$

and

$$\chi(1-s) = \Big(\frac{t}{2\pi}\Big)^{s-1/2} e^{-it-\pi i/4}\Big(1 + O\Big(\frac{1}{t}\Big)\Big), \tag{2.2.5}$$

as $t \to +\infty$. We use the above formulas to determine which products of $\chi(s)$ and $\chi(1-s)$ are oscillating.

One term which does not have an oscillating factor is the one where we use the 'first part' of each approximate functional equation, for it does not have any χ-factors. With $s = \frac{1}{2} + it$, that term is

$$\begin{aligned}\sum_{\substack{m_1,\dots,m_k\\ n_1,\dots,n_k}} & m_1^{-s-\alpha_1} \dots m_k^{-s-\alpha_k} n_1^{s-1+\alpha_{k+1}} \dots n_k^{s-1+\alpha_{2k}} \\ &= \sum_{\substack{m_1,\dots,m_k\\ n_1,\dots,n_k}} m_1^{-1/2-\alpha_1} \dots m_k^{-1/2-\alpha_k} n_1^{-1/2+\alpha_{k+1}} \dots n_k^{-1/2+\alpha_{2k}} \Big(\frac{n_1 \dots n_k}{m_1 \dots m_k}\Big)^{it}. \end{aligned} \tag{2.2.6}$$

According to the recipe we keep the diagonal from the above sum, which is

$$\sum_{m_1 \dots m_k = n_1 \dots n_k} m_1^{-1/2-\alpha_1} \dots m_k^{-1/2-\alpha_k} n_1^{-1/2+\alpha_{k+1}} \dots n_k^{-1/2+\alpha_{2k}}. \tag{2.2.7}$$

If we define

$$R(s;\alpha) = \sum_{m_1 \dots m_k = n_1 \dots n_k} \frac{1}{m_1^{s+\alpha_1} \dots m_k^{s+\alpha_k} n_1^{s-\alpha_{k+1}} \dots n_k^{s-\alpha_{2k}}}, \tag{2.2.8}$$

where the sum is over all positive $m_1, \dots, m_k, n_1, \dots, n_k$ such that $m_1 \dots m_k = n_1 \dots n_k$, then $R(\frac{1}{2};\alpha)$ is the first piece which we have identified as contributing to the mean value. (The sum in equation (2.2.8) does not converge for $s = \frac{1}{2}$. See Theorem 2.4.1 for its analytic continuation.)

Note that the variable s in equation (2.2.8) should not be viewed the same as the variable $s = \frac{1}{2} + it$ from the previous equations. We are employing a trick of beginning with an expression involving s and $1-s$, noting that we will later be setting $s = \frac{1}{2}$, so instead we consider an expression only involving s, which later will be set equal to $\frac{1}{2}$. This same trick will appear in § 4.1 when we consider more general mean values.

Now consider one of the other terms, say the one where we use the second part of the approximate functional equation from $\zeta(s+\alpha_1)$ and the second part from $\zeta(1-s-\alpha_{k+1})$. By (2.2.4) and (2.2.5),

$$\chi(s+\alpha_1)\chi(1-s-\alpha_{k+1}) \sim \left(\frac{t}{2\pi}\right)^{-\alpha_1+\alpha_{k+1}}, \tag{2.2.9}$$

which is not rapidly oscillating. Using this and proceeding as above, we see that the contribution from this term will be

$$\left(\frac{t}{2\pi}\right)^{-\alpha_1+\alpha_{k+1}} R(\tfrac{1}{2};\alpha_{k+1},\alpha_2,\ldots,\alpha_k,\alpha_1,\alpha_{k+2},\ldots,\alpha_{2k}). \tag{2.2.10}$$

More generally, note that

$$\chi(s+\beta_1)\ldots\chi(s+\beta_J)\chi(1-s-\gamma_1)\ldots\chi(1-s-\gamma_K)$$
$$\sim \left(\frac{t}{2\pi e}\right)^{-i(J-K)t} e^{i(J-K)\pi/4}\left(\frac{t}{2\pi}\right)^{-\sum\beta_j+\sum\gamma_j}, \tag{2.2.11}$$

which is rapidly oscillating (because of the it in the exponent) unless $J=K$. Thus, the recipe tell us to keep those terms which involve an equal number of $\chi(s+\alpha_j)$ and $\chi(1-s-\alpha_{k+j})$ factors. This gives a total of

$$\binom{2k}{k} = \sum_{j=0}^{k}\binom{k}{j}^2$$

terms in the final answer.

We now describe a typical term of the conjectural formula. First note that the function $R(s;\alpha_1,\ldots,\alpha_k,\alpha_{k+1},\ldots,\alpha_{2k})$ is symmetric in $\alpha_1,\ldots,\alpha_k$ and in $\alpha_{k+1},\ldots,\alpha_{2k}$, so we can rearrange the entries so that the first k are in increasing order, as are the last k. Thus, the final result will be a sum of terms indexed by the $\binom{2k}{k}$ permutations $\sigma\in S_{2k}$ such that

$$\sigma(1)<\ldots<\sigma(k) \quad\text{and}\quad \sigma(k+1)<\ldots<\sigma(2k).$$

We denote the set of such permutations by Ξ. Second, note that the product of an equal number of $\chi(s+\alpha_j)$ and $\chi(1-s-\alpha_{k+j})$, as in (2.2.9), can be written as

$$\left(\frac{t}{2\pi}\right)^{(-\alpha_1-\ldots-\alpha_k+\alpha_{k+1}+\ldots+\alpha_{2k})/2}\left(\frac{t}{2\pi}\right)^{(\alpha_{\sigma(1)}+\ldots+\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}-\ldots-\alpha_{\sigma(2k)})/2}. \tag{2.2.12}$$

For example, (2.2.9) is the case $\sigma(1)=k+1$, $\sigma(k+1)=1$, and $\sigma(j)=j$ otherwise.

If we set

$$W(z,\alpha,\sigma) = \left(\frac{y}{2\pi}\right)^{(\alpha_{\sigma(1)}+\ldots+\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}-\ldots-\alpha_{\sigma(2k)})/2} R(x;\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(2k)}), \tag{2.2.13}$$

for $z=x+iy$ with x and y real, then combining all terms we have

$$M(z;\alpha) := \left(\frac{y}{2\pi}\right)^{(-\alpha_1-\ldots-\alpha_k+\alpha_{k+1}+\ldots+\alpha_{2k})/2}\sum_{\sigma\in\Xi} W(z,\alpha,\sigma). \tag{2.2.14}$$

The recipe has produced the conjecture

$$\int_{-\infty}^{\infty} Z(\tfrac{1}{2}+it,\alpha)g(t)\,dt = \int_{-\infty}^{\infty} M(\tfrac{1}{2}+it,\alpha)(1+O(t^{-1/2+\varepsilon}))g(t)\,dt, \tag{2.2.15}$$

with $Z(s,\alpha)$ given in (2.2.1) and $M(s;\alpha)$ given above.

Note that the exponent of $(t/2\pi)$ in (2.2.13) has half the α_j with a $+$ sign and the other half with a $-$ sign, and the same holds for $R(s,\alpha)$. This allows an alternate interpretation of Ξ as the set of ways of choosing k elements from $\{\alpha_1,\ldots,\alpha_{2k}\}$.

The general case of Conjecture 1.5.1 is stated in terms of the Z-function. We can recover the mean value of the Z-function directly from that of the L-function (in this case, the ζ-function). By the functional equation and (2.2.4) we see that

$$\begin{aligned} &Z(s+\alpha_1)\ldots Z(s+\alpha_{2k}) \\ &\quad = \left(\frac{t}{2\pi}\right)^{(\alpha_1+\ldots+\alpha_k-\alpha_{k+1}-\ldots-\alpha_{2k})/2}(1+O(1/t)) \\ &\qquad \times \zeta(s+\alpha_1)\ldots\zeta(s+\alpha_k)\zeta(1-s-\alpha_{k+1})\ldots\zeta(1-s-\alpha_{2k}). \end{aligned} \tag{2.2.16}$$

The factor $(t/2\pi)^{(\alpha_1+\ldots+\alpha_k-\alpha_{k+1}-\ldots-\alpha_{2k})/2}$ can be absorbed into the weight function $g(t)$, so we obtain the conjecture

$$\begin{aligned} &\int_{-\infty}^{\infty} Z(s+\alpha_1)\ldots Z(s+\alpha_{2k})g(t)\,dt \\ &\qquad = \int_{-\infty}^{\infty} \sum_{\sigma\in\Xi} W(s,\alpha,\sigma)(1+O(t^{-1/2+\varepsilon}))g(t)\,dt, \end{aligned} \tag{2.2.17}$$

where $s=\frac{1}{2}+it$.

In the next subsection we directly obtain the above conjecture for the Z-function of a general primitive L-function, and in the remainder of this section we perform various manipulations to put these in the form of Conjecture 1.5.1.

2.3. *Moments of a primitive L-function*

Consider the primitive L-function

$$\mathcal{L}(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s} = \prod_p \mathcal{L}_p\left(\frac{1}{p^s}\right), \quad (\sigma>1). \tag{2.3.1}$$

We assume a functional equation of the special form $\xi_L(s)=\gamma_L(s)L(s)=\varepsilon\overline{\xi}_L(1-s)$, where

$$\gamma_L(s) = Q^s\prod_{j=1}^{w}\Gamma(\tfrac{1}{2}s+\mu_j), \tag{2.3.2}$$

with $\{\mu_j\}$ stable under complex conjugation. Note that we have $w_j=\frac{1}{2}$, which is conjectured to hold for arithmetic L-functions. We also assume

$$\mathcal{L}_p(x) = \sum_{n=0}^{\infty} a_{p^n}x^n = \prod_{j=1}^{w}(1-\gamma_{p,j}x)^{-1}, \tag{2.3.3}$$

where w is the degree of $\mathcal{L}$ and where $|\gamma_{p,j}| = 0$ or 1. Again this is conjectured to hold for arithmetic L-functions.

We are going to evaluate the moments of the Z-function

$$Z_{\mathcal{L}}(s) = \varepsilon^{-1/2}\mathcal{X}(s)^{-1/2}\mathcal{L}(s,)$$

where

$$\mathcal{X}(s) = \frac{\overline{\gamma_{\mathcal{L}}(1-s)}}{\gamma_{\mathcal{L}}(s)} = Q^{1-2s}\prod_{j=1}^{w}\frac{\Gamma(\frac{1}{2}(1-s)+\overline{\mu_j})}{\Gamma(\frac{1}{2}s+\mu_j)}. \tag{2.3.4}$$

We will have to determine when products of $\mathcal{X}(s)$ and $\mathcal{X}(1-s)$ are not rapidly oscillating. By Stirling's formula

$$\Gamma(\sigma+it) = e^{-\pi t/2}t^{\sigma-1/2}\left(\frac{t}{e}\right)^{it}e^{(i\pi/2)(\sigma-1/2)} \times\left(1-\frac{i}{t}\left(\frac{1}{12}-\frac{\sigma}{2}+\frac{\sigma^2}{2}\right)+O\left(\frac{1}{t^2}\right)\right) \tag{2.3.5}$$

we obtain

$$\mathcal{X}(s) = Q^{1-2s}\left(\frac{t}{2}\right)^{w(1/2-s)}\left(\frac{t}{2\pi e}\right)^{-\sum\Im(\mu_j)}e^{w(it+i\pi/4)}\left(1+O\left(\frac{1}{t}\right)\right), \tag{2.3.6}$$

as $t\to+\infty$. Note that the above expression can be simplified because we have assumed $\sum\Im(\mu_j)=0$.

Now we are ready to produce a conjecture for

$$I_k(\mathcal{L},\alpha_1,\dots,\alpha_{2k},g) = \int_{-\infty}^{\infty} Z_{\mathcal{L}}(s+\alpha_1)\dots Z_{\mathcal{L}}(s+\alpha_{2k})g(t)\,dt. \tag{2.3.7}$$

with $s=\frac{1}{2}+it$.

By the definition of Z,

$$\prod_{j=1}^{2k} Z_{\mathcal{L}}(s+\alpha_j) = \prod_{j=1}^{2k}\varepsilon^{-1/2}\mathcal{X}(s+\alpha_j)^{-1/2}\prod_{j=1}^{2k}\mathcal{L}(s+\alpha_j). \tag{2.3.8}$$

According to the recipe, we replace each $\mathcal{L}(s+it)$ by its approximate functional equation and multiply out the product obtaining 2^{2k} terms. A typical term is a product of $2k$ sums arising from either the first piece or the second piece of the approximate functional equation. Consider a term where we have ℓ factors from the first piece of the approximate functional equation and $2k-\ell$ factors from the second piece. To take one specific example, suppose it is the first ℓ factors where we choose the first piece of the approximate functional equation, and the last

$2k-\ell$ factors where we take the second piece:

$$\varepsilon^{-k}\mathcal{X}(\tfrac12+\alpha_1+it)^{-1/2}\dots\mathcal{X}(\tfrac12+\alpha_\ell+it)^{-1/2}\sum_{n_1}\frac{a_{n_1}}{n_1^{1/2+\alpha_1+it}}\dots\sum_{n_\ell}\frac{a_{n_\ell}}{n_\ell^{1/2+\alpha_\ell+it}}$$
$$\times\,\varepsilon^{2k-\ell}\mathcal{X}(\tfrac12+\alpha_{\ell+1}+it)^{1/2}\dots\mathcal{X}(\tfrac12+\alpha_{2k}+it)^{1/2}$$
$$\times\sum_{n_{\ell+1}}\frac{\overline{a_{n_{\ell+1}}}}{n_{\ell+1}^{1/2-\alpha_{\ell+1}-it}}\dots\sum_{n_{2k}}\frac{\overline{a_{n_{2k}}}}{n_{2k}^{1/2-\alpha_{2k}-it}}$$
$$=\varepsilon^{k-\ell}\left(\frac{\mathcal{X}(\frac12+\alpha_1+it)\dots\mathcal{X}(\frac12+\alpha_\ell+it)}{\mathcal{X}(\frac12+\alpha_{\ell+1}+it)\dots\mathcal{X}(\frac12+\alpha_{2k}+it)}\right)^{-1/2}$$
$$\times\sum_{n_1}\dots\sum_{n_{2k}}\frac{a_{n_1}\dots a_{n_\ell}\overline{a_{n_{\ell+1}}\dots a_{n_k}}}{n_1^{1/2+\alpha_1}\dots n_\ell^{1/2+\alpha_\ell}n_{\ell+1}^{1/2-\alpha_{\ell+1}}\dots n_{2k}^{1/2-\alpha_{2k}}}\left(\frac{n_1\dots n_\ell}{n_{\ell+1}\dots n_{2k}}\right)^{-it}. \tag{2.3.9}$$

The recipe tells us to retain only the expressions of this sort where the factor involving $\mathcal{X}$ is not oscillating. By (2.3.6) the requirement is that $\ell=k$ (and in particular $2k$ has to be even), and we have

$$\left(\frac{\mathcal{X}(\frac12+\alpha_1+it)\dots\mathcal{X}(\frac12+\alpha_k+it)}{\mathcal{X}(\frac12+\alpha_{k+1}+it)\dots\mathcal{X}(\frac12+\alpha_{2k}+it)}\right)^{-1/2}$$
$$=\left(\frac{Q^{2/w}t}{2}\right)^{(w/2)(\alpha_1+\dots+\alpha_k-\alpha_{k+1}-\dots-\alpha_{2k})}\left(1+O\left(\frac1t\right)\right). \tag{2.3.10}$$

Now the recipe tells us to keep the diagonal from the remaining sums, which in (2.3.9) is the terms where $n_1\dots n_\ell=n_{\ell+1}\dots n_{2k}$. So in the same way as the ζ-function case in the previous section we let

$$R(s,\alpha)=\sum_{n_1\dots n_k=n_{k+1}\dots n_{2k}}\frac{a_{n_1}\dots a_{n_k}\overline{a_{n_{k+1}}\dots a_{n_k}}}{n_1^{s+\alpha_1}\dots n_k^{s+\alpha_k}n_{k+1}^{s-\alpha_{k+1}}\dots n_{2k}^{s-\alpha_{2k}}}, \tag{2.3.11}$$

and

$$W(z,\alpha,\sigma)=\left(\frac{Q^{2/w}y}{2}\right)^{(w/2)(\alpha_{\sigma(1)}+\dots+\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}-\dots-\alpha_{\sigma(2k)})}R(x;\alpha_{\sigma(1)},\dots,\alpha_{\sigma(2k)}), \tag{2.3.12}$$

for $\sigma\in\Xi$, the set of permutations of $\{1,\dots,2k\}$ with $\sigma(1)<\dots<\sigma(k)$ and $\sigma(k+1)<\dots<\sigma(2k)$. Then

$$M(z;\alpha)=\sum_{\sigma\in\Xi}W(z,\alpha,\sigma), \tag{2.3.13}$$

and we arrive at the conjecture

$$I_k(\mathcal{L},\alpha_1,\dots,\alpha_{2k},g)=\int_{-\infty}^{\infty}M(\tfrac12+it,\alpha)(1+O(t^{-1/2+\varepsilon}))g(t)\,dt, \tag{2.3.14}$$

which is the same form as we obtained in (2.2.17).

We will now examine the expressions in these conjectures in detail, rewriting them in a more explicit form.

2.4. *The arithmetic factor in the conjectures*

We retain the notation of the previous subsection. In particular, $\mathcal{L}(s)$ is a primitive L-function having the properties listed at the beginning of § 2.3 and $R(s;\alpha)$ is given in (2.3.11).

THEOREM 2.4.1. *Suppose* $|\alpha_j| < \delta$ *for* $j = 1, \ldots, 2k$. *Then* $R(s; \alpha_1, \ldots, \alpha_{2k})$ *converges absolutely for* $\sigma > \frac{1}{2} + \delta$ *and has a meromorphic continuation to* $\sigma > \frac{1}{4} + \delta$. *Furthermore,*

$$R(s; \alpha_1, \ldots, \alpha_{2k}) = \prod_{i,j=1}^{k} \zeta(2s + \alpha_i - \alpha_{k+j})\, A_k(s; \alpha_1, \ldots, \alpha_{2k}) \tag{2.4.1}$$

where

$$A_k(s; \alpha_1, \ldots, \alpha_{2k}) = \prod_p \left(\prod_{i,j=1}^{k} (1 - p^{-2s-\alpha_i+\alpha_{k+j}}) \right) B_p(s; \alpha_1, \ldots, \alpha_{2k}) \tag{2.4.2}$$

with

$$B_p(s; \alpha_1, \ldots, \alpha_{2k}) = \int_0^1 \prod_{j=1}^{k} \mathcal{L}_p\left(\frac{e(\theta)}{p^{s+\alpha_j}}\right) \overline{\mathcal{L}_p}\left(\frac{e(-\theta)}{p^{s-\alpha_{k+j}}}\right) d\theta. \tag{2.4.3}$$

Proof. We assumed $|\gamma_{p,j}| \leqslant 1$, so we have the Ramanujan bound $a_n \leqslant d_w(n) \ll n^{\varepsilon}$. That implies absolute convergence of $R(s;\alpha)$ for $\sigma > \frac{1}{2} + \delta + \varepsilon$.

Since the coefficients of $R(s;\alpha)$ are multiplicative, as is the condition $n_1 \ldots n_k = n_{k+1} \ldots n_{2k}$, we can write $R(s;\alpha)$ as an Euler product:

$$\begin{aligned}
R(s; \alpha_1, \ldots, \alpha_{2k}) &= \sum_{n_1 \ldots n_k = n_{k+1} \ldots n_{2k}} \frac{a_{n_1} \ldots a_{n_k} \overline{a_{n_{k+1}} \ldots a_{n_{2k}}}}{n_1^{s+\alpha_1} \ldots n_k^{s-\alpha_{2k}}} \\
&= \prod_p \sum_{\sum_{j=1}^k e_j = \sum_{j=1}^k e_{k+j}} \frac{a_{p^{e_1}} \ldots a_{p^{e_k}} \overline{a_{p^{e_{k+1}}} \ldots a_{p^{e_{2k}}}}}{p^{e_1(s+\alpha_1)} \ldots p^{e_{2k}(s-\alpha_{2k})}} \\
&= \prod_p \left(1 + |a_p|^2 \sum_{i,j=1}^{k} \frac{1}{p^{2s+\alpha_i-\alpha_{k+j}}} + \sum_{j=2}^{\infty} \frac{c_{p^j}(\alpha_1, \ldots, \alpha_{2k})}{p^{2js}} + \ldots \right) \\
&= \prod_{i,j=1}^{k} \zeta(2s + \alpha_i - \alpha_{k+j}) \\
&\quad \times \prod_p \Bigg(1 + (|a_p|^2 - 1) \sum_{i,j=1}^{k} \frac{1}{p^{2s+\alpha_i-\alpha_{k+j}}} \\
&\qquad\qquad + \sum_{j=2}^{\infty} \frac{c'_{p^j}(\alpha_1, \ldots, \alpha_{2k})}{p^{2js}} + \ldots \Bigg) \\
&= \prod_{i,j=1}^{k} \zeta(2s + \alpha_i - \alpha_{k+j}) A_k(s; \alpha_1, \ldots, \alpha_{2k}),
\end{aligned} \tag{2.4.4}$$

say. Above c_{p^j} and c'_{p^j} are just shorthand for the (complicated) coefficients in the Euler product. Estimating them trivially and using the fact that $|a_p|^2$ is 1 on average (which is conjectured to hold for primitive elements of the Selberg class), we find that $A_k(s;\alpha)$ is analytic in a neighborhood of $\sigma = \frac{1}{2}$.

Finally, we have

$$A_k(s;\alpha_1,\ldots,\alpha_{2k}) = \prod_p \prod_{i,j=1}^{k} (1 - p^{-2s-\alpha_i+\alpha_{k+j}}) B_p(s;\alpha_1,\ldots,\alpha_{2k}) \tag{2.4.5}$$

where

$$\begin{aligned} B_p(s;\alpha_1,\ldots,\alpha_{2k}) &= \sum_{\sum_{j=1}^k e_j = \sum_{j=1}^k e_{k+j}} \frac{a_{p^{e_1}} \ldots a_{p^{e_k}} \overline{a_{p^{e_{k+1}}} \ldots a_{p^{e_{2k}}}}}{p^{e_1(s+\alpha_1)} \ldots p^{e_{2k}(s-\alpha_{2k})}} \\ &= \int_0^1 \sum_{e_1,\ldots,e_{2k}} \frac{a_{p^{e_1}} \ldots a_{p^{e_k}} \overline{a_{p^{e_{k+1}}} \ldots a_{p^{e_{2k}}}}}{p^{e_1(s+\alpha_1)} \ldots p^{e_{2k}(s-\alpha_{2k})}} \\ &\quad \times e\bigg(\bigg(\sum_{j=1}^k e_j - \sum_{j=1}^k e_{k+j}\bigg)\theta\bigg)\, d\theta \\ &= \int_0^1 \prod_{j=1}^k \sum_{e_j=0}^{\infty} \frac{a_{p^{e_j}}}{p^{e_j(s+\alpha_j)}} e(e_j\theta) \prod_{j=1}^k \frac{\overline{a_{p^{e_{k+j}}}}}{p^{e_{k+j}(s-\alpha_{k+j})}} e(-e_{k+j}\theta)\, d\theta \\ &= \int_0^1 \prod_{j=1}^k \mathcal{L}_p\bigg(\frac{e(\theta)}{p^{s+\alpha_j}}\bigg) \overline{\mathcal{L}_p}\bigg(\frac{e(-\theta)}{p^{s-\alpha_{k+j}}}\bigg)\, d\theta, \end{aligned} \tag{2.4.6}$$

as claimed. □

To summarize, the conjecture for the general mean value $I_k(\mathcal{L},\alpha_1,\ldots,\alpha_{2k},g)$ involves the function $M(s;\alpha)$, which can be written as

$$M(s;\alpha) = \sum_{\sigma\in\Xi} W(s;\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(2k)}), \tag{2.4.7}$$

where we have written $W(s;\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(2k)})$ for $W(s;\alpha)$. And by Theorem 2.4.1,

$$\begin{aligned} W(s;\alpha_1,\ldots,\alpha_{2k}) = \bigg(\frac{Q^{2/w}t}{2}\bigg)^{(w/2)\sum_{j=1}^k \alpha_j - \alpha_{k+j}} & A_k(s;\alpha_1,\ldots,\alpha_{2k}) \\ \times \prod_{i,j=1}^{k} \zeta(2s+\alpha_i-\alpha_{k+j}). & \end{aligned} \tag{2.4.8}$$

One can see the above elements in Conjecture 1.5.1; in particular, the form of $B_p(s;\alpha)$, and $A_k(z)$ in that conjecture equals $A_k(\frac{1}{2},\alpha)$ given above. The overall structure is slightly different because Conjecture 1.5.1 is expressed as a multiple contour integral, as opposed to a sum over permutations. In the next subsection we show how to write the sum over permutations in a compact form. In the following subsection we return to the functions A_k and write them in a more explicit form.

2.5. *Concise form of permutation sums*

As we have seen, our methods naturally lead to an expression involving a sum over permutations. In this section we describe how to write those sums in a compact form involving contour integrals. Similar combinatorial sums arise from our matrix ensemble calculations, and we have previously stated our main results and conjectures in this compact form.

Note that in both of these lemmas, the terms in the sum on the left side have singularities. However, examining the right side of the formula makes it clear that those singularities all cancel.

LEMMA 2.5.1. *Suppose $F(a;b) = F(a_1,\dots,a_k;b_1,\dots,b_k)$ is a function of $2k$ variables, which is symmetric with respect to the first k variables and also symmetric with respect to the second set of k variables. Suppose also that F is regular near $(0,\dots,0)$. Suppose further that $f(s)$ has a simple pole of residue 1 at $s=0$ but is otherwise analytic in a neighborhood about $s=0$. Let*

$$K(a_1,\dots,a_k;b_1,\dots b_k) = F(a_1,\dots;\dots,b_k)\prod_{i=1}^{k}\prod_{j=1}^{k} f(a_i-b_j). \tag{2.5.1}$$

If for all $1\leqslant i,j\leqslant k$, $\alpha_i-\alpha_{k+j}$ is contained in the region of analyticity of $f(s)$ then

$$\begin{aligned}\sum_{\sigma\in\Xi} &K(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k)};\alpha_{\sigma(k+1)}\dots\alpha_{\sigma(2k)})\\ &=\frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\\ &\quad\times\oint\cdots\oint\frac{K(z_1,\dots,z_k;z_{k+1},\dots,z_{2k})\Delta(z_1,\dots,z_{2k})^2}{\prod_{i=1}^{2k}\prod_{j=1}^{2k}(z_i-\alpha_j)}\,dz_1\dots dz_{2k},\end{aligned} \tag{2.5.2}$$

where one integrates about small circles enclosing the α_j, and where Ξ is the set of $\binom{2k}{k}$ permutations $\sigma\in S_{2k}$ such that $\sigma(1)<\dots<\sigma(k)$ and $\sigma(k+1)<\dots<\sigma(2k)$.

The above lemma applies to the Unitary case, which has been the subject of this section. The next lemma is useful in the Symplectic and Orthogonal cases, which will be addressed beginning in § 4.4.

LEMMA 2.5.2. *Suppose F is a symmetric function of k variables, regular near $(0,\dots,0)$, and $f(s)$ has a simple pole of residue 1 at $s=0$ and is otherwise analytic in a neighborhood of $s=0$, and let*

$$K(a_1,\dots,a_k) = F(a_1,\dots,a_k)\prod_{1\leqslant i\leqslant j\leqslant k} f(a_i+a_j) \tag{2.5.3}$$

or

$$K(a_1,\dots,a_k) = F(a_1,\dots,a_k)\prod_{1\leqslant i<j\leqslant k} f(a_i+a_j). \tag{2.5.4}$$

If $\alpha_i+\alpha_j$ are contained in the region of analyticity of $f(s)$ then

$$\begin{aligned}\sum_{\epsilon_j=\pm1} K(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k) &= \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k}\frac{2^k}{k!}\oint\cdots\oint K(z_1,\dots,z_k)\\ &\quad\times\frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{i=1}^k\prod_{j=1}^k(z_i-\alpha_j)(z_i+\alpha_j)}\,dz_1\dots dz_k,\end{aligned} \tag{2.5.5}$$

and

$$\sum_{\epsilon_j=\pm1}\left(\prod_{j=1}^{k}\epsilon_j\right)K(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)$$
$$=\frac{(-1)^{k(k-1)/2}}{(2\pi i)^k}\frac{2^k}{k!}\oint\cdots\oint K(z_1,\ldots,z_k)$$
$$\times\frac{\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^k\alpha_j}{\prod_{i=1}^k\prod_{j=1}^k(z_i-\alpha_j)(z_i+\alpha_j)}\,dz_1\ldots dz_k, \qquad (2.5.6)$$

where the path of integration encloses the $\pm\alpha_j$*.*

The proofs of the lemmas come from the following result.

LEMMA 2.5.3. *Suppose that* $F(a;b)=F(a_1,\ldots,a_m;b_1,\ldots,b_n)$ *is symmetric in the* a *variables and in the* b *variables and is regular near* $(0,\ldots,0)$*. Suppose* $f(s)=s^{-1}+c+\ldots$ *and let*

$$G(a_1,\ldots,a_m;b_1,\ldots b_n)=F(a_1,\ldots;\ldots,b_n)\prod_{i=1}^{m}\prod_{j=1}^{n}f(a_i-b_j).$$

Let $\Xi_{m,n}$ *be as defined above. Then*

$$\sum_{\sigma\in\Xi_{m,n}}G(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(m)};\alpha_{\sigma(m+1)}\ldots\alpha_{\sigma(m+n)})$$
$$=\frac{(-1)^{(m+n)}}{m!n!}\sum_{\sigma\in\pi_{m+n}}\operatorname{Res}_{(z_1,\ldots,z_{m+n})=(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(m+n)})}\frac{G(z_1,\ldots,z_{m+n})\Delta(z_1,\ldots,z_{m+n})^2}{\prod_{i=1}^{m+n}\prod_{j=1}^{m+n}(z_i-\alpha_j)}.$$

Proof. It suffices to prove that

$$\operatorname{Res}_{(z_1,\ldots,z_{m+n})=(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(m+n)})}\frac{\Delta(z_1,\ldots,z_{m+n})^2}{\prod_{i=1}^{m+n}\prod_{j=1}^{m+n}(z_i-\alpha_j)}=(-1)^{m+n}$$

since each such term will appear $m!n!$ times. Consider the case where σ is the identity permutation. Then the residue is

$$\frac{\prod_{j<k}(\alpha_k-\alpha_j)^2}{\prod_{j\neq k}(\alpha_j-\alpha_k)}=(-1)^{m+n};$$

the answer will be the same for any permutation σ.

The residue above can be expressed as $(2\pi i)^{-m-n}$ times an $m+n$ fold integral, each path of which encircles all of the poles of the integrand; note that the value of such an integral may be calculated by summing the residues and note that there is no singularity when $z_j=z_k$ because of the factor $(z_k-z_j)^2$ in the numerator. □

To obtain the form of Conjecture 1.5.1 from the formulas at the end of §§ 2.3 and 2.4, apply Lemma 2.5.1 with $K(\alpha_1,\ldots,\alpha_{2k})=W(\frac{1}{2};\alpha_1,\ldots,\alpha_{2k})$. That is,

$f(z) = \zeta(1+z)$ and

$$\begin{aligned} F(\alpha_1, \dots, \alpha_{2k}) &= \left(\frac{Q^{2/w}t}{2}\right)^{(w/2)\sum_{j=1}^k \alpha_j - \alpha_{k+j}} A_k(\tfrac{1}{2}; \alpha_1, \dots, \alpha_{2k}) \\ &= \exp\left(w \log\left(\frac{Q^{2/w}t}{2}\right) \cdot \frac{1}{2} \sum_{j=1}^k \alpha_j - \alpha_{k+j}\right) \\ &\quad \times A_k(\tfrac{1}{2}; \alpha_1, \dots, \alpha_{2k}). \end{aligned} \tag{2.5.7}$$

We arrive at the general case of Conjecture 1.5.1.

Conjecture 2.5.4. *Suppose $\mathcal{L}(s)$ is a primitive L-function having the properties listed at the beginning of* § 2.3, *and the mean value $I_k(\mathcal{L}, \alpha_1, \dots, \alpha_{2k}, g)$ is given in* (2.3.7). *Then*

$$I_k(\mathcal{L}, \alpha_1, \dots, \alpha_{2k}, g) = \int_{-\infty}^{\infty} P_k\left(w \log\left(\frac{Q^{2/w}t}{2}\right), \alpha\right)(1 + O(t^{-1/2+\varepsilon}))g(t)\, dt, \tag{2.5.8}$$

where $P_k(x, \alpha)$ and $G(z_1, \dots, z_{2k})$ are as stated in Conjecture 1.5.1, *except that A_k is the Euler product*

$$A_k(z) = \prod_p \prod_{i=1}^k \prod_{j=1}^k \left(1 - \frac{1}{p^{1+z_i-z_{k+j}}}\right) \int_0^1 \prod_{j=1}^k \mathcal{L}_p\left(\frac{e(\theta)}{p^{1/2+z_j}}\right) \overline{\mathcal{L}_p}\left(\frac{e(-\theta)}{p^{1/2-z_{k+j}}}\right). \tag{2.5.9}$$

Note that for the Riemann ζ-function, $w = 1$ and $Q = 1/\sqrt{\pi}$ and $\mathcal{L}_p(x) = (1-x)^{-1}$, so Conjecture 1.5.1 is a special case of the above. Also note that $w \log(\frac{1}{2} Q^{2/w} t)$ is the mean density of zeros of $\mathcal{L}(\frac{1}{2} + it)$, or equivalently the log conductor, as expected.

It remains to express the arithmetic factor A_k in a more explicit form, which we do in the next section.

2.6. *Explicit versions of the arithmetic factor*

The factor $A_k(s, \alpha)$ in the $2k$th moment of a primitive L-function can be expressed in a simple form.

Recall, see Theorem 2.4.1, that A_k is the Euler product

$$A_k(s; \alpha) = \prod_p B_p(s; \alpha_1, \dots, \alpha_{2k}) \prod_{i=1}^k \prod_{j=1}^k \left(1 - \frac{1}{p^{2s+\alpha_i-\alpha_{k+j}}}\right), \tag{2.6.1}$$

where

$$B_p(s; \alpha_1, \dots, \alpha_{2k}) = \int_0^1 \prod_{j=1}^k \mathcal{L}_p\left(\frac{e(\theta)}{p^{s+\alpha_j}}\right) \overline{\mathcal{L}_p}\left(\frac{e(-\theta)}{p^{s-\alpha_{k+j}}}\right) d\theta. \tag{2.6.2}$$

LEMMA 2.6.1. *If* $\mathcal{L}_p(x) = (1-\gamma_p x)^{-1}$ *with* $|\gamma_p| = 1$ *then*

$$B_p(s;\alpha_1,\ldots,\alpha_{2k}) = \prod_{i=1}^{k}\prod_{j=1}^{k}\left(1-\frac{1}{p^{2s+\alpha_i-\alpha_{k+j}}}\right)^{-1} \times \sum_{m=1}^{k}\prod_{i\neq m}\left(\prod_{j=1}^{k}\left(1-\frac{1}{p^{2s+\alpha_j-\alpha_{k+i}}}\right)\right)\Big/(1-p^{\alpha_{k+i}-\alpha_{k+m}}). \tag{2.6.3}$$

COROLLARY 2.6.2. *If* $\mathcal{L}_p(x) = (1-\gamma_p x)^{-1}$ *with* $|\gamma_p| = 0$ *when* $p \mid N$ *and* $|\gamma_p| = 1$ *otherwise, then*

$$A_k(s;\alpha_1,\ldots,\alpha_{2k}) = \prod_{p\nmid N}\sum_{m=1}^{k}\prod_{i\neq m}\left(\prod_{j=1}^{k}\left(1-\frac{1}{p^{2s+\alpha_j-\alpha_{k+i}}}\right)\right)\Big/(1-p^{\alpha_{k+i}-\alpha_{k+m}}) \times \prod_{p\mid N}\prod_{i=1}^{k}\prod_{j=1}^{k}\left(1-\frac{1}{p^{2s+\alpha_i-\alpha_{k+j}}}\right). \tag{2.6.4}$$

In particular, if $\mathcal{L}(s) = L(s,\chi)$ *with* χ *a Dirichlet character of conductor* N, *where the Riemann* ζ*-function is the case* $N = 1$, *then*

$$A_1(s;\alpha_1,\alpha_2) = \prod_{p\mid N}\left(1-\frac{1}{p^{2s+\alpha_1-\alpha_2}}\right), \tag{2.6.5}$$

$$A_2(s;\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \zeta(4s+\alpha_1+\alpha_2-\alpha_3-\alpha_4)^{-1} \times \prod_{p\mid N}\prod_{i=1}^{2}\prod_{j=1}^{2}\left(1-\frac{1}{p^{2s+\alpha_i-\alpha_{2+j}}}\right), \tag{2.6.6}$$

and

$$\begin{aligned} &A_3(s;\alpha_1,\ldots,\alpha_6)\\ &\quad= \prod_{p\nmid N}(1-p^{-\sum_1^3\alpha_i-\alpha_{3+i}}(p^{\alpha_1}+p^{\alpha_2}+p^{\alpha_3})(p^{-\alpha_4}+p^{-\alpha_5}+p^{-\alpha_6})p^{-4s}\\ &\qquad + p^{-\sum_1^3\alpha_i-\alpha_{3+i}}((p^{\alpha_1}+p^{\alpha_2}+p^{\alpha_3})(p^{-\alpha_1}+p^{-\alpha_2}+p^{-\alpha_3})\\ &\qquad\qquad + (p^{\alpha_4}+p^{\alpha_5}+p^{\alpha_6})(p^{-\alpha_4}+p^{-\alpha_5}+p^{-\alpha_6})-2)p^{-6s}\\ &\qquad - p^{-\sum_1^3\alpha_i-\alpha_{3+i}}(p^{-\alpha_1}+p^{-\alpha_2}+p^{-\alpha_3})(p^{\alpha_4}+p^{\alpha_5}+p^{\alpha_6})p^{-8s}\\ &\qquad + p^{-2\sum_1^3\alpha_i-\alpha_{3+i}}p^{-12s})\\ &\quad\times \prod_{p\mid N}\prod_{i=1}^{3}\prod_{j=1}^{3}\left(1-\frac{1}{p^{2s+\alpha_i-\alpha_{3+j}}}\right). \end{aligned} \tag{2.6.7}$$

For $k \geqslant 3$ it is not possible to express A_k as a finite product of ζ-functions.

Proof of Lemma 2.6.1. Using $\mathcal{L}_p(x) = (1-\gamma_p x)^{-1}$ and setting

$$q_j = \frac{\gamma_p}{p^{s+\alpha_j}} \quad \text{and} \quad q_{k+j} = \frac{\overline{\gamma_p}}{p^{s-\alpha_{k+j}}} \quad \text{for } j = 1, \dots, k, \tag{2.6.8}$$

we have

$$\begin{aligned} B_p(s;\alpha_1,\dots,\alpha_{2k}) &= \int_0^1 \prod_{j=1}^k \big(1-e(\theta)q_j\big)^{-1}\big(1-e(-\theta)q_{k+j}\big)^{-1}\,d\theta \\ &= \frac{(-1)^k}{\prod_{j=1}^k q_j}\int_0^1 e(k\theta)\prod_{j=1}^k \big(e(\theta)-1/q_j\big)^{-1}\big(e(\theta)-q_{k+j}\big)^{-1}\,d\theta \\ &= \frac{(-1)^k}{\prod_{j=1}^k q_j}\frac{1}{2\pi i}\oint z^{k-1}\prod_{j=1}^k \big(z-1/q_j\big)^{-1}\big(z-q_{k+j}\big)^{-1}\,dz, \end{aligned} \tag{2.6.9}$$

where the path of integration is around the unit circle. Since $|q_j| < 1$, by the residue theorem we have a contribution from the poles at $q_{k+1},\dots,q_{2k}$, giving

$$\begin{aligned} B_p(s;\alpha_1,\dots,\alpha_{2k}) &= \frac{(-1)^k}{\prod_{j=1}^k q_j}\sum_{m=1}^k q_{k+m}^{k-1}\prod_{i=1}^k \big(q_{k+m}-q_i^{-1}\big)^{-1}\prod_{i\neq m}(q_{k+m}-q_{k+i})^{-1} \\ &= \sum_{m=1}^k\prod_{i=1}^k (1-q_iq_{k+m})^{-1}\prod_{i\neq m}\big(1-q_{k+i}q_{k+m}^{-1}\big)^{-1}. \end{aligned} \tag{2.6.10}$$

Since

$$\prod_{i,j=1}^k (1-q_iq_{k+j})\prod_{i=1}^k (1-q_iq_{k+m})^{-1} = \prod_{j\neq m}\prod_{i=1}^k (1-q_iq_{k+j}), \tag{2.6.11}$$

factoring out

$$\prod_{i,j=1}^k (1-q_iq_{k+j})^{-1}, \tag{2.6.12}$$

we have

$$B_p(s;\alpha) = \left(\prod_{i,j=1}^k (1-q_iq_{k+j})^{-1}\right)\sum_{m=1}^k\prod_{i\neq m}\frac{\prod_{j=1}^k(1-q_jq_{k+i})}{1-q_{k+i}q_{k+m}^{-1}}. \tag{2.6.13}$$

Since

$$q_jq_{k+i} = p^{-2s-\alpha_j+\alpha_{k+i}} \quad \text{and} \quad q_{k+i}q_{k+m}^{-1} = p^{\alpha_{k+i}-\alpha_{k+m}}, \tag{2.6.14}$$

we obtain the formula in the lemma. □

Notice that the special case $N = 1$, that is, the Riemann ζ function, reads in Corollary 2.6.2,

$$A_k(s;\alpha_1,\dots,\alpha_{2k}) = \prod_p\sum_{m=1}^k\prod_{i\neq m}\left(\prod_{j=1}^k\left(1-\frac{1}{p^{2s+\alpha_j-\alpha_{k+i}}}\right)\right)\Big/(1-p^{\alpha_{k+i}-\alpha_{k+m}}). \tag{2.6.15}$$

Each local factor

$$A_{p,k}(s;\alpha) = \sum_{m=1}^{k} \prod_{i\neq m} \left(\prod_{j=1}^{k} \left(1 - \frac{1}{p^{2s+\alpha_j-\alpha_{k+i}}} \right) \right) \Big/ (1 - p^{\alpha_{k+i}-\alpha_{k+m}}) \tag{2.6.16}$$

is actually a polynomial in p^{-2s}, $p^{-\alpha_j}$ and $p^{\alpha_{k+j}}$, for $j = 1, \ldots, k$. That this is so in p^{-2s} and $p^{-\alpha_j}$ is readily apparent from (2.6.16). The fact that it is also a polynomial in $p^{\alpha_{k+j}}$ follows from (2.4.2) and (2.4.3), from which

$$\begin{aligned} &A_{p,k}(s;\alpha_1,\ldots,\alpha_k,\alpha_{k+1},\ldots,\alpha_{2k}) \\ &\qquad = A_{p,k}(s;-\alpha_{k+1},\ldots,-\alpha_{2k},-\alpha_1,\ldots,-\alpha_k). \end{aligned} \tag{2.6.17}$$

Setting $\beta_1 = -\alpha_{k+1}, \ldots, \beta_k = -\alpha_{2k}$, one has, from the above discussion, that $A_{p,k}$ is a polynomial in $p^{-\beta_j}$, that is, in $p^{\alpha_{k+j}}$, for $j = 1, \ldots, k$. Finally, use the fact that if an analytic function of several variables is of polynomial growth in each variable separately, then it must be a polynomial.

2.7. *Recovering the leading order for moments of* ζ

Conjecture 1.5.1 contains, as a special case, a conjecture for the leading order term for the moments of the Riemann zeta-function. In this section we show that the leading order terms derived from Conjecture 1.5.1 agree with the leading order terms which have previously been conjectured by other methods.

As described in § 1.3, it is conjectured that the mean values of the Riemann zeta-function take the form

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt = T\mathcal{P}_k(\log T) + O(T^{1/2+\varepsilon}), \tag{2.7.1}$$

where $\mathcal{P}_k(\log T)$ is a polynomial in $\log T$ of degree k^2. Conrey and Ghosh conjectured that the coefficient of the $\log^{k^2} T$ term is of the form $g_k a_k/k^2!$, where a_k is given by (1.3.2). Keating and Snaith used random matrix theory to conjecture that g_k is given by (1.3.3). This leading order term $g_k a_k/k^2!$ will be re-derived here, starting with Conjecture 1.5.1.

Conjecture 1.5.1 implies that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt = \int_0^T P_k\left(\log\frac{t}{2\pi}\right) dt + O(T^{1/2+\varepsilon}), \tag{2.7.2}$$

where P_k is the polynomial of degree k^2 given by

$$\begin{aligned} P_k(x) = \frac{(-1)^k}{k!^2(2\pi i)^{2k}} \oint \cdots \oint A_k(z_1,\ldots,z_{2k}) \prod_{i=1}^{k}\prod_{j=1}^{k} \zeta(1+z_i-z_{k+j}) \\ \times \frac{\Delta^2(z_1,\ldots,z_{2k})}{\prod_{j=1}^{2k} z_j^{2k}} e^{(x/2)\sum_{j=1}^{k} z_j - z_{k+j}}\, dz_1 \ldots dz_{2k}. \end{aligned} \tag{2.7.3}$$

Our goal is to show that the leading order term of $P_k(x)$ is $(g_k a_k/k^2!)x^{k^2}$.

Using the fact that A_k is analytic in a neighborhood of $(0, \ldots, 0)$ and the ζ-function has a simple pole at 1 with residue 1, after a change of variables we have

$$\begin{aligned}
P_k(x) &= \frac{(-1)^k}{k!^2(2\pi i)^{2k}} \oint \cdots \oint A_k\Big(\frac{z_1}{x/2}, \ldots, \frac{z_{2k}}{x/2}\Big) \prod_{i=1}^{k}\prod_{j=1}^{k} \zeta\Big(1 + \frac{z_i - z_{k+j}}{x/2}\Big) \\
&\quad \times \frac{\Delta^2(z_1, \ldots, z_{2k})}{\prod_{j=1}^{2k} z_j^{2k}} e^{\sum_{j=1}^{k} z_j - z_{k+j}}\, dz_1 \ldots dz_{2k} \\
&= \frac{(-1)^k}{k!^2} \frac{A_k(0, \ldots, 0)}{(2\pi i)^{2k}} \Big(\frac{x}{2}\Big)^{k^2} (1 + O(x^{-1})) \\
&\quad \times \oint \cdots \oint \frac{\Delta^2(z_1, \ldots, z_{2k})}{\big(\prod_{i=1}^{k}\prod_{j=1}^{k}(z_i - z_{k+j})\big)\prod_{j=1}^{2k} z_j^{2k}} e^{\sum_{j=1}^{k} z_j - z_{k+j}}\, dz_1 \ldots dz_{2k} \\
&= \frac{A_k(0, \ldots, 0)}{k!^2 2^{k^2}(2\pi i)^{2k}} x^{k^2} (1 + O(x^{-1})) \\
&\quad \times \oint \cdots \oint \frac{\Delta(z_1, \ldots, z_{2k})\Delta(z_1, \ldots, z_k)\Delta(z_{k+1}, \ldots, z_{2k})}{\prod_{j=1}^{2k} z_j^{2k}} \\
&\quad \times e^{\sum_{j=1}^{k} z_j - z_{k+j}}\, dz_1 \ldots, dz_{2k}. \qquad (2.7.4)
\end{aligned}$$

Now we need only show that $A_k(0, \ldots 0) = a_k$ and the remaining factors give $g_k/k^2!$.

From Conjecture 1.5.1,

$$A_k(0, \ldots 0) = \prod_p \Big(1 - \frac{1}{p}\Big)^{k^2} \int_0^1 \Big(1 - \frac{e(\theta)}{p^{1/2}}\Big)^{-k} \Big(1 - \frac{e(-\theta)}{p^{1/2}}\Big)^{-k} d\theta. \qquad (2.7.5)$$

For a given p, we concentrate on the integral in the above expression, writing it as a contour integral around the unit circle:

$$(-p^{1/2})^k \frac{1}{2\pi i} \oint \frac{z^{k-1}(z - p^{1/2})^{-k}}{(z - p^{-1/2})^k} dz. \qquad (2.7.6)$$

After expanding the two factors in the numerator around $z = p^{-1/2}$ and calculating the residue we are left with the sum

$$\Big(1 - \frac{1}{p}\Big)^{-2k+1} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \binom{2k - \ell - 2}{k-1} p^{-k+\ell+1} \Big(1 - \frac{1}{p}\Big)^{\ell}. \qquad (2.7.7)$$

Next one can perform a binomial expansion of $(1 - p^{-1})^{\ell}$ and gather like powers of p^{-1} to obtain

$$\begin{aligned}
&\Big(1 - \frac{1}{p}\Big)^{-2k+1} \sum_{m=0}^{k-1} \Bigg(\sum_{q=0}^{m} (-1)^q \binom{k + q - m - 1}{q} \\
&\qquad \times \binom{k-1}{k + q - m - 1} \binom{k + m - q - 1}{k-1} \Bigg) p^{-m}. \qquad (2.7.8)
\end{aligned}$$

A simple manipulation of the binomial coefficients and replacing q by $m-q$ gives

$$\sum_{q=0}^{m}(-1)^q\binom{k+q-m-1}{q}\binom{k-1}{k+q-m-1}\binom{k+m-q-1}{k-1} = \binom{k-1}{m}\sum_{q=0}^{m}(-1)^{m-q}\binom{m}{q}\binom{k+q-1}{q}, \tag{2.7.9}$$

and this final sum over q is in fact just $\binom{k-1}{m}$ (see, for example, [**43**]). Thus,

$$A_k(0,\dots,0)=\prod_p\left(1-\frac{1}{p}\right)^{(k-1)^2}\sum_{m=0}^{k-1}\binom{k-1}{m}^2 p^{-m}, \tag{2.7.10}$$

and this is indeed equal to a_k defined in (1.3.2).

Now we must identify the remaining terms as $g_k/k^2!$ defined in (1.3.3), as $x\to\infty$. The method applied below as far as (2.7.14) follows closely that used for a similar purpose in [**4**]. Expanding the determinants $\Delta(z_1,\dots,z_k)=\det[z_j^{m-1}]_{j,m=1}^k$, we obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{P_k(x)}{a_k x^{k^2}} &= \frac{1}{(k!)^2 2^{k^2}(2\pi i)^{2k}}\oint\cdots\oint e^{\sum_{j=1}^k z_j - z_{k+j}}\\ &\quad\times\left(\sum_S \operatorname{sgn}(S) z_1^{S_0} z_2^{S_1}\dots z_k^{S_{k-1}} z_{k+1}^{S_k}\dots z_{2k}^{S_{2k-1}}\right)\\ &\quad\times\left(\sum_Q \operatorname{sgn}(Q) z_1^{Q_0}\dots z_k^{Q_{k-1}}\right)\left(\sum_R \operatorname{sgn}(R) z_{k+1}^{R_0}\dots z_{2k}^{R_{k-1}}\right)\\ &\quad\times z_1^{-2k}\dots z_{2k}^{-2k}\,dz_1\dots dz_{2k}.\end{aligned} \tag{2.7.11}$$

Here Q and R are permutations of $\{0,1,\dots,k-1\}$ and S is a permutation of $\{0,1,\dots,2k-1\}$.

Since the integrand is symmetric amongst $z_1,\dots,z_k$ and also amongst $z_{k+1},\dots,z_{2k}$, in each term of the sum over Q we permute the variables $z_1,\dots,z_k$ so that z_j appears with the exponent $j-1$, for $j=1,\dots,k$. In the sum over S the effect is to redefine the permutations, and the additional sign involved with this exactly cancels $\operatorname{sgn}(Q)$. We do the same with the sum over R, and as a result we are left with $k!^2$ copies of the sum over the permutation S:

$$\begin{aligned}\lim_{x\to\infty}\frac{P_k(x)}{a_k x^{k^2}} &= \frac{1}{2^{k^2}(2\pi i)^{2k}}\oint\cdots\oint e^{\sum_{j=1}^k z_j - z_{k+j}}\\ &\quad\times\sum_S \operatorname{sgn}(S) z_1^{-(2k-S_0)} z_2^{-(2k-S_1-1)}\dots z_k^{-(2k-S_{k-1}-(k-1))}\\ &\quad\times z_{k+1}^{-(2k-S_k)} z_{k+2}^{-(2k-S_{k+1}-1)}\dots z_{2k}^{-(2k-S_{2k-1}-(k-1))}\,dz_1\dots dz_{2k}.\end{aligned} \tag{2.7.12}$$

Since

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_C(-t)^{-z}e^{-t}(-dt), \tag{2.7.13}$$

where the path of integration C starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction and returns to the starting point, we can rewrite

(2.7.12) as

$$\begin{aligned}
&\lim_{x\to\infty}\frac{P_k(x)}{a_k x^{k^2}}\\
&= \frac{(-1)^k}{2^{k^2}}\sum_S \operatorname{sgn}(S)(\Gamma(2k-S_0)\Gamma(2k-S_1-1)\dots\Gamma(2k-S_{k-1}-(k-1))\\
&\qquad\qquad \times (-1)^{S_k}\Gamma(2k-S_k)(-1)^{S_{k+1}+1}\Gamma(2k-S_{k+1}-1)\dots\\
&\qquad\qquad \times (-1)^{S_{2k-1}+k-1}\Gamma(2k-S_{2k-1}-(k-1)))^{-1}\\
&= \frac{(-1)^k}{2^{k^2}}\begin{vmatrix}
\frac{1}{\Gamma(2k)} & \frac{1}{\Gamma(2k-1)} & \cdots & \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(2k)} & \frac{-1}{\Gamma(2k-1)} & \cdots & \frac{(-1)^{k-1}}{\Gamma(k+1)}\\
\frac{1}{\Gamma(2k-1)} & \frac{1}{\Gamma(2k-2)} & \cdots & \frac{1}{\Gamma(k)} & \frac{-1}{\Gamma(2k-1)} & \frac{1}{\Gamma(2k-2)} & \cdots & \frac{(-1)^{k}}{\Gamma(k)}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \cdots & \frac{1}{\Gamma(2-k)} & \frac{-1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \cdots & \frac{(-1)^{3k-2}}{\Gamma(2-k)}
\end{vmatrix}\\
&= \frac{(-1)^k}{2^{k^2}}\left(\prod_{\ell=0}^{k-1}\frac{\ell!}{(k+\ell)!}\right)\\
&\quad\times\begin{vmatrix}
\binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1} & \binom{0}{0} & -\binom{0}{1} & \cdots & (-1)^{k-1}\binom{0}{k-1}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\binom{2k-1}{0} & \binom{2k-1}{1} & \cdots & \binom{2k-1}{k-1} & -\binom{2k-1}{0} & \binom{2k-1}{1} & \cdots & (-1)^k\binom{2k-1}{k-1}
\end{vmatrix}.
\end{aligned}\tag{2.7.14}$$

The above is a $2k \times 2k$ determinant, the first k columns of which are identical to the first k columns of the matrix

$$\begin{pmatrix}
\binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{2k-1}\\
\vdots & \vdots & \ddots & \vdots\\
\binom{2k-1}{0} & \binom{2k-1}{1} & \cdots & \binom{2k-1}{2k-1}
\end{pmatrix}. \tag{2.7.15}$$

The matrix (2.7.15) is lower triangular and so can easily be seen to have determinant equal to 1. It is also the inverse of

$$\begin{pmatrix}
\binom{0}{0} & -\binom{0}{1} & \cdots & -\binom{0}{2k-1}\\
-\binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{2k-1}\\
\vdots & \vdots & \ddots & \vdots\\
-\binom{2k-1}{0} & \binom{2k-1}{1} & \cdots & \binom{2k-1}{2k-1}
\end{pmatrix}. \tag{2.7.16}$$

It so happens that matrix (2.7.16) has its k first columns identical to columns $k+1$ through $2k$ of the matrix in expression (2.7.14). Therefore we can multiply expression (2.7.14) by the determinant of (2.7.15) (which is equal to 1) and this simplifies the final k columns of the resulting determinant significantly:

$$\lim_{x\to\infty}\frac{P_k(x)}{a_k x^{k^2}}$$

$$= \frac{(-1)^k}{2^{k^2}}\left(\prod_{\ell=0}^{k-1}\frac{\ell!}{(k+\ell)!}\right)\begin{vmatrix}\binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{2k-1}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{2k-1}{0} & \binom{2k-1}{1} & \cdots & \binom{2k-1}{2k-1}\end{vmatrix}$$

$$\times\begin{vmatrix}\binom{0}{0} & \cdots & \binom{0}{k-1} & \binom{0}{0} & \cdots & (-1)^{k-1}\binom{0}{k-1}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \binom{2k-1}{0} & \cdots & \binom{2k-1}{k-1} & -\binom{2k-1}{0} & \cdots & (-1)^k\binom{2k-1}{k-1}\end{vmatrix}$$

$$= \frac{(-1)^k}{2^{k^2}}\left(\prod_{\ell=0}^{k-1}\frac{\ell!}{(k+\ell)!}\right)\begin{vmatrix}\binom{0}{0} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\ 2\binom{1}{0} & \binom{1}{1} & \cdots & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 2^{k-1}\binom{k-1}{0} & 2^{k-2}\binom{k-1}{1} & \cdots & \binom{k-1}{k-1} & 0 & 0 & \cdots & 1\\ 2^k\binom{k}{0} & 2^{k-1}\binom{k}{1} & \cdots & 2\binom{k}{k-1} & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 2^{2k-1}\binom{2k-1}{0} & 2^{2k-2}\binom{2k-1}{1} & \cdots & 2^k\binom{2k-1}{k-1} & 0 & 0 & \cdots & 0\end{vmatrix}$$

$$= \frac{(-1)^{k(k-1)/2}}{2^{k^2}}\left(\prod_{\ell=1}^{k-1}\frac{\ell!}{(k+\ell)!}\right)\begin{vmatrix}2^{2k-1}\binom{2k-1}{0} & 2^{2k-2}\binom{2k-1}{1} & \cdots & 2^k\binom{2k-1}{k-1}\\ \vdots & \vdots & \ddots & \vdots\\ 2^k\binom{k}{0} & 2^{k-1}\binom{k}{1} & \cdots & 2\binom{k}{k-1}\end{vmatrix}$$

$$= (-1)^{k(k-1)/2}\left(\prod_{\ell=1}^{k-1}\frac{\ell!}{(k+\ell)!}\right)\begin{vmatrix}\binom{2k-1}{0} & \binom{2k-1}{1} & \cdots & \binom{2k-1}{k-1}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k-1}\end{vmatrix}.$$

The matrix above can be decomposed as

$$\begin{pmatrix}\binom{2k-1}{0} & \binom{2k-1}{1} & \cdots & \binom{2k-1}{k-1}\\ \binom{2k-2}{0} & \binom{2k-2}{1} & \cdots & \binom{2k-2}{k-1}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k-1}\end{pmatrix}$$

$$= \begin{pmatrix}\binom{k-1}{0} & \binom{k-1}{1} & \cdots & \binom{k-1}{k-1}\\ \binom{k-2}{0} & \binom{k-2}{1} & \cdots & \binom{k-2}{k-1}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1}\end{pmatrix}\begin{pmatrix}\binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k-1}\\ \binom{k}{-1} & \binom{k}{0} & \cdots & \binom{k}{k-2}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{k}{-k+1} & \binom{k}{-k+2} & \cdots & \binom{k}{0}\end{pmatrix}.$$

The first matrix on the right side is zero in the lower right triangle, and the second matrix on the right side is upper triangular. Thus we read that the determinant of the matrix on the left-hand side is $(-1)^{k(k-1)/2}$. Therefore,

$$\lim_{x\to\infty} \frac{P_k(x)}{a_k x^{k^2}} = \prod_{\ell=1}^{k-1} \frac{\ell!}{(k+\ell)!}, \tag{2.7.17}$$

and this is $g_k/k^2!$ from (1.3.3), as required.

A similar method applies to the orthogonal and symplectic cases.

3. *Families of characters and families of L-functions*

We will describe a particular kind of 'family' of primitive L-functions based on the idea of twisting a single L-function by a family of 'characters'. In the next section we provide a general recipe for conjecturing the critical mean value of products of L-functions averaged over a family and we demonstrate the recipe in several examples.

Note that we use 'character' somewhat more generally than is usually covered by that term.

3.1. *Families of primitive characters*

We describe sets of arithmetic functions that we call 'families of characters'.

Let $\mathcal{F} = \{f\}$ be a collection of arithmetic functions $f(n)$, and assume that for each $f \in \mathcal{F}$ the associated L-function $L_f(s) = \sum f(n)\, n^{-s}$ is a primitive L-function with functional equation $L_f(s) = \varepsilon_f X_f(s) \overline{L}_f(1-s)$ and an Euler product of the form

$$L_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \prod_{j=1}^{v} (1 - \beta_{p,j}/p^s)^{-1}. \tag{3.1.1}$$

The quantity

$$c(f) = |(\varepsilon_f X_f)'(\tfrac{1}{2})| \tag{3.1.2}$$

is called the *log conductor* of f.

Note that if $f = \chi$, a primitive Dirichlet character of conductor q, then the log conductor is

$$c(\chi) = \begin{cases} \log q - \log \pi + (\Gamma'/\Gamma)(\frac{1}{4}) & \text{for } \chi \text{ even,} \\ \\ \log q - \log \pi + (\Gamma'/\Gamma)(\frac{3}{4}) & \text{for } \chi \text{ odd .} \end{cases} \tag{3.1.3}$$

If $f(n) = n^{-it}$ then the log conductor is $c(n^{-it}) = \log(t/2\pi) + O(t^{-1})$. Generally the log conductor $c(f)$ scales as the log of the 'usual' conductor of f.

In the case that $\mathcal{F}$ is finite, we require that the data Q, w_j, μ_j in the functional equation (1.1.2) is *the same* for all $f \in \mathcal{F}$. In particular, the conductor $c(f)$ is the same for all $f \in \mathcal{F}$.

In the case that $\mathcal{F}$ is infinite, we require that the data Q, w_j, μ_j in the functional equation (1.1.2) are *monotonic functions of the conductor* $c(f)$. Furthermore, we define the counting function $M(T) = \#\{f : c(f) \leqslant T\}$ and require that $M(\log(T)) = F(T^A, \log T) + O(T^{A/2+\epsilon})$ for all $\epsilon > 0$, where $A > 0$ and $F(\cdot,\cdot)$ is a polynomial.

If G is a function on $\mathcal{F}$, then we define the *expected value* of G by

$$\langle G(f)\rangle = \lim_{T\to\infty} M(T)^{-1} \sum_{\substack{f\in\mathcal{F}\\ c(f)<T}} G(f), \tag{3.1.4}$$

assuming the limit exists. In the case of a continuous family, the sum is an integral.

We require that if $m_1,\dots,m_k$ are integers then the expected value

$$\delta_\ell(m_1,\dots,m_k) = \langle f(m_1)\dots f(m_\ell)\overline{f(m_{\ell+1})\dots f(m_k)}\rangle \tag{3.1.5}$$

exists and is multiplicative. That is, if $(m_1m_2\dots m_k, n_1n_2\dots n_k)=1$, then

$$\delta_\ell(m_1n_1, m_2n_2,\dots,m_kn_k) = \delta_\ell(m_1,\dots,m_k)\delta_\ell(n_1,\dots,n_k). \tag{3.1.6}$$

We sometimes refer to δ as the 'orthogonality relation' of the family.

The practical use of being multiplicative is that a multiple Dirichlet series with δ_ℓ coefficients factors has an Euler product:

$$\sum_{m_1,\dots,m_\ell} \frac{\delta_\ell(m_1,\dots,m_\ell)}{m_1^{s_1}\dots m_\ell^{s_\ell}} = \prod_p \sum_{e_1,\dots,e_\ell} \frac{\delta_\ell(p^{e_1},\dots,p^{e_\ell})}{p^{e_1s_1+\dots+e_\ell s_\ell}}. \tag{3.1.7}$$

We will use the above relation in our calculations.

To summarize, a family of characters $\mathcal{F}=\{f\}$ is a collection of arithmetic functions, each of which are the coefficients of a particular kind of L-function. The characters are partially ordered by conductor $c(f)$, and the expected values $\delta_\ell(m_1,\dots,m_k)$ are multiplicative functions.

The following are examples of families of characters, two of which are finite and two are infinite. The term 'finite family' is somewhat misleading, because those families depend on a parameter, and the size of the family grows with the parameter.

1. *The family of t-twists.* This is

$$\mathcal{F}_{\mathrm{t}} = \{f_t(n) = n^{-it} : 0<t<T\}. \tag{3.1.8}$$

We have

$$\frac{1}{T}\int_0^T (m/n)^{it}\,dt = \begin{cases} 1 & \text{if } n=m,\\ \dfrac{(m/n)^{iT}-1}{T\log(m/n)} & \text{otherwise,}\end{cases} \tag{3.1.9}$$

leading to the expected values

$$\langle f_t(n)\overline{f_t(m)}\rangle = \langle n^{-it}m^{it}\rangle = \langle (m/n)^{it}\rangle = \begin{cases} 1 & \text{if } n=m,\\ 0 & \text{otherwise.}\end{cases} \tag{3.1.10}$$

Therefore the orthogonality relation is

$$\delta_\ell(n_1,\dots,n_k) = \delta(n_1\dots n_\ell = n_{\ell+1}\dots n_k). \tag{3.1.11}$$

2. *The family of primitive Dirichlet characters.* For each positive integer q we set

$$\mathcal{F}_{\mathrm{ch}}(q) = \{f_\chi(n)=\chi(n) : \chi \text{ is a primitive Dirichlet character mod } q\}. \tag{3.1.12}$$

We have

$$\frac{1}{q^*}\sum_{\chi \bmod q}{}^{*}\chi(n)\overline{\chi(m)} = \begin{cases} 1 & \text{if } n \equiv m \bmod q \text{ and } (mn,q)=1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1.13}$$

where the sum is over the primitive characters mod q, and q^* is the number of terms in the sum. This leads to the expected values

$$\langle f_\chi(n)\overline{f_\chi(m)}\rangle = \langle \chi(n)\overline{\chi(m)}\rangle = \begin{cases} 1 & \text{if } n = m \text{ and } (mn,q)=1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.14}$$

Since $\chi(m_1)\chi(m_2) = \chi(m_1 m_2)$ we obtain

$$\delta_\ell(m_1,\dots,m_k) = \delta(m_1\dots m_\ell = m_{\ell+1}\dots m_k \text{ and } (m_1\dots m_k,q)=1). \tag{3.1.15}$$

Note that the condition in the definition of δ_ℓ is *not* $m_1\dots m_\ell \equiv m_{\ell+1}\dots m_k \bmod q$. We are computing the expected value as a function of q, so one should think of the m_j as fixed and $q\to\infty$. The only way to have $m_1\dots m_\ell \equiv m_{\ell+1}\dots m_k \bmod q$ for sufficiently large q is to have actual equality.

Note that by our definition, $\mathcal{F}_{\rm ch}(q)$ is not a family, but it is the union of two families consisting of the even characters and the odd characters separately.

3. *The family of real primitive Dirichlet characters.* We define

$$\mathcal{F}_{\rm d} = \{f_d(n) = \chi_d(n) : \chi_d \text{ is a primitive real character mod } d,\ |d|<X\}, \tag{3.1.16}$$

where d runs over fundamental discriminants. We have expected values

$$\begin{aligned}\langle f_d(n) f_d(m)\rangle &= \langle \chi_d(n)\chi_d(m)\rangle = \langle \chi_d(nm)\rangle \\ &= \begin{cases} \prod_{p\mid nm}(1+p^{-1})^{-1} & \text{if } nm = \square, \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{3.1.17}$$

The calculation in the case $nm=\square$ is non-trivial and was first done by Jutila [**28**]. (If one were summing over all d then the expected value when $nm=\square$ would be $\varphi(nm)/nm$.)

In practice one encounters more restricted families, so we let

$$\mathcal{F}_{\rm d}(+) = \{f_d : d>0\} \quad \text{and } \mathcal{F}_{\rm d}(-) = \{f_d : d<0\},$$

and also

$$\mathcal{F}_{\rm d}(a,N,\pm) = \{f_d \in \mathcal{F}_{\rm d}(\pm) : d\equiv a \bmod N\}. \tag{3.1.18}$$

For the family $\mathcal{F}_{\rm d}(a,N,\pm)$, evaluating the expected value of $\chi_d(n)$ can be tricky, so we provide some useful asymptotics.

Below we restrict ourselves to $0<d<X$, but the same asymptotics hold if one restricts to $-X<d<0$.

THEOREM 3.1.1. *Let $Q=\gcd(a,N)$ not be divisible by the square of an odd prime. Then*

$$\sum_{\substack{0<d<X \\ d\equiv a \bmod N}}^{*} 1 \sim \frac{1}{\phi(4N/Q)}\frac{X}{Q}\frac{6}{\pi^2} h_2(a,N)\prod_{p\mid 2N}\frac{p}{p+1}. \tag{3.1.19}$$

Next, assume further that N is either odd or divisible by at least 8 (this condition is related to the fact that $\chi_d(2)$ is periodic mod 8*), and say $n=g\square$, with*

$(\square, N) = 1$, *and with all prime factors of* g *being prime factors of* N. *Then*

$$\sum_{\substack{0<d<X \\ d\equiv a \bmod N}}^{*} \chi_d(n) \sim \chi_a(g)\phi(\square)\frac{1}{\phi(4N\square/Q)}\frac{X}{Q}\frac{6}{\pi^2}h_2(a,N)\prod_{p|2N\square}\frac{p}{p+1}. \tag{3.1.20}$$

Here $h_2(a,N)$ *is determined according to Table* 3.1.1. *Consequently, for the family* $\mathcal{F}_{\mathrm{d}}(a,N,\pm)$,

$$\langle \chi_d(n)\rangle = \begin{cases} \chi_a(g)\prod_{p|\square}(1+p^{-1})^{-1} & n = g\square, \\ 0 & \text{otherwise.}\end{cases} \tag{3.1.21}$$

TABLE 3.1.1. *The function* $h_2(a,N)$ *that appears in Theorem* 3.1.1, *where* $N = 2^\beta N_0$, *with* N_0 *odd.*

β	a	$h_2(a,N)$
0	$a \in \mathbb{Z}$	3/2
1	$a \equiv 0 \bmod 2$	1
	$a \equiv 1 \bmod 2$	2
2	$a \equiv 0 \bmod 4$	2
	$a \equiv 1 \bmod 4$	4
	$a \equiv 2, 3 \bmod 4$	0
3	$a \equiv 0, 4 \bmod 8$	2
	$a \equiv 1, 5 \bmod 8$	4
	$a \equiv 2, 3, 6, 7 \bmod 8$	0
$\geqslant 4$	$a \equiv 1, 5, 8, 9, 12, 13 \bmod 16$	4
	otherwise	0

Proof. We first outline the proof for (3.1.19). One can count odd fundamental discriminants $|d| < X$ by using the Dirichlet series

$$\sum_{d\,\mathrm{odd}}^{*}\frac{1}{|d|^s} = \prod_{p\,\mathrm{odd}}\left(1+\frac{1}{p^s}\right) = \frac{\zeta(s)}{\zeta(2s)}\left(1+\frac{1}{2^s}\right)^{-1}. \tag{3.1.22}$$

As in the proof of the prime number theorem, the main contribution comes from the pole at $s = 1$, and one has

$$\sum_{\substack{|d|<X \\ d\,\mathrm{odd}}}^{*} 1 \sim \frac{4}{\pi^2}X. \tag{3.1.23}$$

Next, assume that N is odd and $(a,N) = 1$. To count odd fundamental discriminants in arithmetic progression, $d \equiv a \bmod N$, one imitates Dirichlet's theorem for primes in arithmetic progression, looking at linear combinations involving Dirichlet characters mod N of

$$\sum_{d\,\mathrm{odd}}^{*}\frac{\chi(d)}{|d|^s} = \prod_{p\,\mathrm{odd}}\left(1+\frac{\chi(p)}{p^s}\right). \tag{3.1.24}$$

If one wishes to further specify $d > 0$ or $d < 0$, one can restrict to $|d| \equiv 1 \bmod 4$ or $|d| \equiv 3 \bmod 4$ respectively, with χ ranging over Dirichlet characters mod $4N$. The main contribution comes from the trivial character whose corresponding Dirichlet series is

$$\sum_{d \text{ odd}}^{*} \frac{\chi_0(d)}{|d|^s} = \prod_{\substack{p \text{ odd} \\ p \nmid N}} \left(1 + \frac{1}{p^s}\right) = \frac{\zeta(s)}{\zeta(2s)} \prod_{p | 2N} \left(1 + \frac{1}{p^s}\right)^{-1}, \tag{3.1.25}$$

and whose main pole is at $s = 1$. Therefore, for N odd and $(a, N) = 1$, we have

$$\sum_{\substack{0<d<X \\ d \equiv a \bmod N \\ p \text{ odd}}}^{*} 1 \sim \frac{1}{\phi(4N)} X \frac{6}{\pi^2} \prod_{p|2N} \frac{p}{p+1}, \tag{3.1.26}$$

with the same result for $-X < d < 0$.

Next, for N odd and $(a, N) = Q > 1$, one can write, for $d \equiv a \bmod N$, $d = d_1 Q$. Apply the above method to d_1 with $0 < d_1 < X/Q$, $d_1 \equiv (a/Q) \bmod (N/Q)$, d_1 odd, and, because d is squarefree, the extra condition that $(d_1, Q) = 1$. Because of this last condition, the Euler product that we need to take in (3.1.25) is not just over odd $p \nmid (N/Q)$ but also over $p \nmid Q$, that is, it is still

$$\prod_{\substack{p \text{ odd} \\ p \nmid N}} \left(1 + \frac{1}{p^s}\right). \tag{3.1.27}$$

Hence, if $(a, N) = Q$,

$$\sum_{\substack{0<d<X \\ d \equiv a \bmod N \\ d \text{ odd}}}^{*} 1 \sim \frac{1}{\phi(4N/Q)} \frac{X}{Q} \frac{6}{\pi^2} \prod_{p|2N} \frac{p}{p+1}, \tag{3.1.28}$$

and we have the same result for $-X < d < 0$.

Finally, we wish to take into account even d. The set of even fundamental discriminants consists of -4 and ± 8 times the odd fundamental discriminants.

Again, assume N is odd. One can count discriminants, $d \equiv a \bmod N$, lying in the interval $(0, X)$ by counting odd discriminants lying in $(0, X)$, together with odd discriminants in $(-X/4, 0)$, $(0, X/8)$ and $(-X/8, 0)$. Overall, this gives the same asymptotics as before, but with an extra factor of $(1 + 1/4 + 2/8) = 3/2$. This accounts for line 1 in Table 3.1.1. The other lines in the table can be obtained by similar considerations.

We now apply (3.1.19) to obtain (3.1.20) and (3.1.21). Consider

$$\sum_{\substack{0<d<X \\ d \equiv a \bmod N}}^{*} \chi_d(n) \Big/ \sum_{\substack{0<d<X \\ d \equiv a \bmod N}}^{*} 1 \tag{3.1.29}$$

(the following analysis also holds for $-X < d < 0$).

Write $N = N_1^{r_1} \cdot \ldots \cdot N_m^{r_m}$, the prime factorization of N, and let $g = N_1^{u_1} \cdot \ldots \cdot N_m^{u_m}$. Then

$$\chi_d(n) = \chi_d(N_1)^{u_1} \ldots \chi_d(N_m)^{u_m} \chi_d(\square). \tag{3.1.30}$$

Now, if N_i is an odd prime, $\chi_d(N_i) = \chi_a(N_i)$, since $d \equiv a \bmod N$, and so $d \equiv a \bmod N_i$. If $N_i = 2$ we need to be careful because $\chi_d(2)$ is periodic mod 8. Now we are assuming that if N is even it is at least divisible by 8, that is, that $d \equiv a \bmod 8$, and thus that $\chi_d(2) = \chi_a(2)$.

Therefore

$$\chi_d(n) = \chi_a(g)\chi_d(\square) = \begin{cases} \chi_a(g) & \text{if } (d,\square)=1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1.31}$$

and one finds that (3.1.29) equals

$$\chi_a(g) \sideset{}{^*}\sum_{\substack{0<d<X \\ d\equiv a \bmod N \\ (d,\square)=1}} 1 \Big/ \sideset{}{^*}\sum_{\substack{0<d<X \\ d\equiv a \bmod N}} 1. \tag{3.1.32}$$

Since $(\square, N)=1$, the sum in the numerator can be split into sums $d \bmod N\square$. Naively, one expects to have $\phi(\square)$ sums, one for each residue class $(d,\square)=1$. However, if $\square$ is even, then only half of these residue classes, namely those that have $d \equiv 1 \bmod 4$, contain fundamental discriminants, so one only gets $\phi(\square)/2$ sums. We thus consider the case that $\square$ is odd separately from the case that it is even. Both cases end up giving the same answer.

Assume that $\square$ is odd. To apply our formula (3.1.19) to each of the $\phi(\square)$ residue classes mod $N\square$, one needs to compute the various components that go into the formula.

Given $d \equiv a \bmod N$ and $d \equiv b \bmod \square$, one has via the chinese remainder theorem $d \equiv \widetilde{a} \bmod N\square$. Now, $Q=(a,N)=(d,N)$, and $(d,\square)=1$, so $(\widetilde{a}, N\square) = (d, N\square) = Q$.

One also needs to evaluate $h_2(\widetilde{a}, N\square)$. Let $N = 2^\beta N_0$, with N_0 odd. Now $\square$ is odd, and so $h_2(\widetilde{a}, N\square)$ only depends on $\widetilde{a}$ mod 2^β, but this is determined by a mod N. So $h_2(\widetilde{a}, N\square) = h_2(a,N)$. Therefore, the numerator of (3.1.32) is asymptotically

$$\chi_a(g)\phi(\square)\frac{1}{\phi(4N\square/Q)}\frac{X}{Q}\frac{6}{\pi^2}h_2(a,N)\prod_{p|2N\square}\frac{p}{p+1}. \tag{3.1.33}$$

Canceling factors appearing in the asymptotics (3.1.19) of the denominator of (3.1.32) we get

$$\chi_a(g)\prod_{p|\square}\frac{p}{p+1}. \tag{3.1.34}$$

If $\square$ is even, write $\square = 2^\lambda \square_0$, with $\lambda \geqslant 2$, and $\square_0$ odd. Now, $(d,\square)=1$, so d is odd. In all cases, according to Table 3.1.1, $h_2(\widetilde{a}, N\square)$ is therefore 4. Furthermore, as in the odd case, $(\widetilde{a}, N\square) = Q$.

Hence, one gets asymptotically for the numerator of (3.1.32)

$$\chi_a(g)\frac{\phi(\square)}{2}\frac{1}{\phi(4N\square/Q)}\frac{X}{Q}\frac{6}{\pi^2}4\prod_{p|2N\square}\frac{p}{p+1}. \tag{3.1.35}$$

Since $\square$ is even, N is odd. Hence $h_2(a,N)=3/2$, and the denominator of (3.1.32) is asymptotically

$$\frac{1}{\phi(4N/Q)}\frac{X}{Q}\frac{6}{\pi^2}\frac{3}{2}\prod_{p|2N}\frac{p}{p+1}. \tag{3.1.36}$$

Canceling numerator and denominator, taking special care for powers of 2 appearing in $\square$, we get

$$\chi_a(g)\frac{2}{3}\prod_{p|\square_0}\frac{p}{p+1} = \chi_a(g)\prod_{p|\square}\frac{p}{p+1}. \tag{3.1.37}$$

$\square$

4. *The family of coefficients of holomorphic newforms.* Define

$$\mathcal{F}_{\mathrm{mod}}(k,q) = \left\{f(n) = \lambda_f(n) : \sum n^{(k-1)/2}\lambda_f(n) \in H_k(q)\right\}, \tag{3.1.38}$$

where $H_k(q)$ the set of newforms in $S_k(\Gamma_0(q))$. A good reference for these functions is Iwaniec [**24**]. In this family the parameter tending to infinity can be either k, or q, or some combination.

The orthogonality relation here is somewhat subtle, and in fact there are two natural ways to average over these characters. In both cases the starting point is the Hecke relation

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|m,d|n\\(d,q)=1}} \lambda_f(mn/d^2), \tag{3.1.39}$$

which imply that any product

$$\lambda_f(m_1)\dots\lambda_f(m_k) \tag{3.1.40}$$

can be expressed as a linear combination

$$\sum_{j\geqslant 1} b_j\lambda_f(j) \tag{3.1.41}$$

for some integers b_j, and in fact only for j a prime power. Thus, we need only determine the expected value of $\lambda_f(p^j)$.

If one averages over $H_k(q)$ in the most straightforward way, then for $p\nmid q$,

$$\langle\lambda_f(p^j)\rangle = \begin{cases} p^{-j/2} & \text{for } j \text{ even,}\\ 0 & \text{for } j \text{ odd,}\end{cases} \tag{3.1.42}$$

and more generally, if $(n,q)=1$,

$$\langle\lambda_f(n)\rangle = \begin{cases} n^{-1/2} & \text{if } n=\square,\\ 0 & \text{otherwise.}\end{cases} \tag{3.1.43}$$

This follows from the Selberg trace formula. However, if one averages with respect to a weighting by the Petersson norm,

$$\sideset{}{^{\mathrm{h}}}\sum_{f\in H_k(q)} * \quad = \sum_{f\in H_k(q)} */\langle f,f\rangle, \tag{3.1.44}$$

then

$$\langle\lambda_f(p^j)\rangle = \begin{cases} 1 & \text{if } j=0,\\ 0 & \text{otherwise,}\end{cases} \tag{3.1.45}$$

and more generally, if $(n,q)=1$,

$$\langle\lambda_f(n)\rangle = \begin{cases} 1 & \text{if } n=1\ ,\\ 0 & \text{otherwise.}\end{cases} \tag{3.1.46}$$

This follows from the Petersson formula (see [**24**]), if $(mn,q)=1$,

$$\sum_{f\in H_k(q)}^{\mathrm{h}} \lambda_f(m)\lambda_f(n) = \delta(m,n) + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(m,n;cq)J_{k-1}(4\pi\sqrt{mn}/cq)}{cq}. \tag{3.1.47}$$

Here J_k is the Bessel function and

$$S(m,n;c) = \sum_{ad=1 \bmod c} e\Big(\frac{ma+nc}{c}\Big) \tag{3.1.48}$$

is the Kloosterman sum. Since the Petersson weighting leads to a somewhat simpler expression, we will consider that weighting in our example. When passing from the Petersson formula to the expected value, using the Weil bound for the Kloosterman sum and the fact that J_k has a kth order zero at 0, we see that for fixed m and n the sum on the right side of (3.1.47) vanishes as $k\to\infty$ or $q\to\infty$.

Let

$$\delta(m_1,\ldots,m_k) = \langle \lambda_f(m_1)\ldots\lambda_f(m_k)\rangle. \tag{3.1.49}$$

So in the Petersson weighting, $\delta(m_1,\ldots,m_k)$ is the coefficient b_1 of $\lambda_f(1)=1$ in (3.1.41). One can use the Hecke relations to show by induction that δ is multiplicative in the sense of 3.1.6. Thus, we only need to know δ on prime powers.

LEMMA 3.1.2. *With respect to the Petersson weighting, if $p\nmid q$ then*

$$\begin{aligned}\delta(p^{m_1},\ldots,p^{m_k}) &= \frac{2}{\pi}\int_0^{\pi} \sin^2\theta \prod_{j=1}^{k} \frac{\sin(m_j+1)\theta}{\sin\theta}\,d\theta \\ &= \frac{2}{\pi}\int_0^{\pi} \sin^2\theta \prod_{j=1}^{k} \frac{e^{i(m_j+1)\theta}-e^{-i(m_j+1)\theta}}{e^{i\theta}-e^{-i\theta}}\,d\theta.\end{aligned} \tag{3.1.50}$$

For the unweighted sum we have

$$\delta(p^{m_1},\ldots,p^{m_k}) = \frac{4}{\pi}\int_0^{\pi} \frac{\sin^2\theta}{1-(2\cos\theta)/\sqrt{p}+p^{-1}} \prod_{j=1}^{k} \frac{\sin(m_j+1)\theta}{\sin\theta}\,d\theta. \tag{3.1.51}$$

If $p\mid q$ then $\delta(p^{m_1},\ldots,p^{m_k})=0$ unless $m_1=\ldots=m_k=0$.

Proof. We only give the details for (3.1.50). Beginning from

$$\mathcal{L}_p(x) = \sum_{j=0}^{\infty} \lambda_f(p^j)x^j = \Big(1-e^{i\theta_{p,f}}x\Big)^{-1}\Big(1-e^{-i\theta_{p,f}}x\Big)^{-1} \tag{3.1.52}$$

we have

$$\lambda_f(p^j) = \frac{\sin(j+1)\theta_{f,p}}{\sin\theta_{f,p}} = U_j(\cos\theta_{f,p})$$

where U_j is the usual Tchebychev polynomial. Then $\delta(p^{m_1},\ldots,p^{m_k})=c_0$ where

$$U_{m_1}U_{m_2}\ldots U_{m_k} = \sum_{e\geqslant 0} c_e U_e. \tag{3.1.53}$$

If we evaluate both sides of this equation at $\cos\theta$ and integrate from 0 to π with respect to the measure $(2/\pi)\sin^2\theta\,d\theta$, then the result follows from the orthogonality of the Tchebychev polynomials with respect to this measure. □

5. *The family of coefficients of Maass newforms.* This is

$$\mathcal{F}_{\mathrm{M}}(q) = \{f(n) = \lambda_f(n) : \sqrt{y}\sum \lambda_f(n)K_{iR}(2\pi|n|y)e^{2\pi inx} \in H(q)\}, \qquad (3.1.54)$$

where $H(q)$ is the set of Maass newforms on $\Gamma_0(q)$. A good reference for these functions is Iwaniec [**25**]. The orthogonality relation is derived from the Kuznetsov trace formula. See Chapter 9 of [**25**].

3.2. *Families of L-functions*

We use a family of characters to create a family of L-functions in the following manner.

Begin with a fixed primitive L-function

$$\mathcal{L}(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s} = \prod_p \mathcal{L}_p\left(\frac{1}{p^s}\right) \quad (\sigma > 1). \qquad (3.2.1)$$

We assume that

$$\mathcal{L}_p(x) = \sum_{n=0}^{\infty} a_{p^n}x^n = \prod_{j=1}^{w}(1-\gamma_{p,j}x)^{-1}, \qquad (3.2.2)$$

where w is the degree of $\mathcal{L}$ and where $|\gamma_{p,j}| = 0$ or 1. Assume $\mathcal{L}(s)$ satisfies the functional equation

$$\mathcal{L}(s) = \varepsilon\mathcal{X}(s)\overline{\mathcal{L}}(1-s), \qquad (3.2.3)$$

as described in § 1.1.

We create a family of L-functions by twisting $\mathcal{L}$ by a family of characters. Let $\mathcal{F} = \{f\}$ be a family of characters, with the properties described in § 3.1. The twist of $\mathcal{L}$ by f is denoted by $\mathcal{L}(s,f)$ and is given by a Rankin–Selberg convolution:

$$\mathcal{L}(s,f) = \prod_p\prod_{i=1}^{v}\prod_{j=1}^{w}(1-\beta_{p,i}\gamma_{p,j}/p^s)^{-1} = \sum_{n=1}^{\infty}\frac{a_n(f)}{n^s}. \qquad (3.2.4)$$

Note that if $w=1$ or $v=1$, as will be the case in our detailed examples,

$$\mathcal{L}(s,f) = \sum_{n=1}^{\infty}\frac{a_n f(n)}{n^s}. \qquad (3.2.5)$$

We require that $\mathcal{L}(s,f)$ is an L-function. That is, our family of L-functions must consist of L-functions! In particular, $\mathcal{L}(s,f)$ satisfies a functional equation

$$\mathcal{L}(s,f) = \varepsilon_f\mathcal{X}_f(s)\overline{\mathcal{L}}(1-s,f), \qquad (3.2.6)$$

as described in § 1.1. As part of our definition of 'family', we make a restrictive, but natural, assumption on $\mathcal{X}_f$. We have $\mathcal{X}_f(s) = \overline{\gamma_f}(1-s)/\gamma_f(s)$ where

$$\gamma_f(s) = Q_f^s\prod_{j=1}^{w}\Gamma(\tfrac{1}{2}s+\mu_{j,f}). \qquad (3.2.7)$$

We assume that w is constant, and each of Q_j, $\Re(\mu_{j,f})$ and $\Im(\mu_{j,f})$ is a monotonic function of the conductor $c(f)$. In practice this will mean that each of those quantities will either be constant or be tending to infinity with the conductor.

For example, the collection of all real primitive Dirichlet L-functions $L(s,\chi_d)$ is not a family, because for $d>0$ we have $\mu_{1,d}=0$ and for $d<0$ we have $\mu_{1,d}=1$, so $\mu_{1,d}$ is not a monotonic function of $c(\chi_d)$. So we consider these two families separately.

Finally, we will make use of an approximate functional equation of shape

$$\mathcal{L}(s,f)=\sum\frac{a_n(f)}{n^s}+\varepsilon_f\mathcal{X}_f(s)\sum\frac{\overline{a_n(f)}}{n^{1-s}}+\text{remainder}. \tag{3.2.8}$$

(*Note*. We are not claiming that the 'remainder' in the above equation is small; nevertheless we will ignore the remainder in our calculations.)

4. *A recipe for conjecturing moments, with examples*

We give a general recipe for conjecturing the moments of a primitive family of L-functions, and then apply the recipe to several interesting examples.

4.1. *The general recipe*

Suppose $\mathcal{L}$ is an L-function and f is a character with conductor $c(f)$, as described in § 3. So

$$Z_{\mathcal{L}}(s,f)=\varepsilon_f^{-1/2}\mathcal{X}_f(s)^{-1/2}\mathcal{L}(s,f), \tag{4.1.1}$$

which satisfies the functional equation

$$Z_{\mathcal{L}}(s,f)=\overline{Z_{\mathcal{L}}}(1-s,f), \tag{4.1.2}$$

so $Z_{\mathcal{L}}(s,f)$ is real on the $\frac{1}{2}$-line. Note that $\varepsilon_f^{-1/2}$ involves a choice of sign which needs to be chosen consistently in the discussion below. We consider the moment

$$\sum_{f\in\mathcal{F}}Z_{\mathcal{L}}(\tfrac{1}{2}+\alpha_1,f)\dots Z_{\mathcal{L}}(\tfrac{1}{2}+\alpha_k,f)g(c(f)) \tag{4.1.3}$$

where g is a suitable test function. The recipe below also applies to averages of products of $\mathcal{L}(\frac{1}{2}+\alpha,f)$. The sum is an integral when $\mathcal{F}=\mathcal{F}_t$.

We now give a recipe for conjecturing a formula for the above moment.

(1) Start with a product of k shifted L-functions:

$$Z_f(s,\alpha_1,\dots,\alpha_k)=Z_{\mathcal{L}}(s+\alpha_1,f)\dots Z_{\mathcal{L}}(s+\alpha_k,f). \tag{4.1.4}$$

As we will demonstrate in our examples, the recipe applies to the Z-function as well as the L-function.

(2) Replace each L-function with the two terms from its approximate functional equation (3.2.8), ignoring the remainder term. Multiply out the resulting expression to obtain 2^k terms. Write those terms as

$$(\text{product of }\varepsilon_f\text{ factors})(\text{product of }\mathcal{X}_f\text{ factors})\sum_{n_1,\dots,n_k}(\text{summand}). \tag{4.1.5}$$

(3) Replace each product of ε_f-factors by its expected value when averaged over the family.

(4) Replace each summand by its expected value when averaged over the family.

(5) Complete the resulting sums, and call the total $M(s,\alpha_1,\dots,\alpha_{2k})$.

(6) The conjecture is

$$\sum_{f\in\mathcal{F}} Z_f(\tfrac{1}{2},\alpha_1,\dots,\alpha_{2k})g(c(f)) = \sum_{f\in\mathcal{F}} M_f(\tfrac{1}{2},\alpha_1,\dots,\alpha_{2k})(1+O(e^{(-1/2+\varepsilon)c(f)}))g(c(f)), \tag{4.1.6}$$

for all $\varepsilon > 0$, where g is a suitable weight function.

In other words, $Z_f(s,\alpha)$ and $M_f(s,\alpha)$ have the same value distribution if averaged over a sufficiently large portion of the family. Note that the dependence of M_f on f only occurs in the product of $\mathcal{X}_f$ factors.

As we mentioned earlier, some of the individual steps in this recipe cannot be rigorously justified. Only by using the entire recipe does one arrive at a reasonable conjecture. In particular, we ignore off-diagonal terms which actually make a contribution. However, comparison with examples in the literature, random matrix moments, and numerical data, suggests that the various errors in our recipe all cancel. The underlying cause for this remains a mystery.

We will apply the recipe to several examples, but first we do the initial steps of the recipe in some generality.

For each $Z_{\mathcal{L}}$ substitute the expression in (4.1.1). After replacing each $\mathcal{L}(s,f)$ by its approximate functional equation (3.2.8), multiply out the product. A typical term is a product of k sums arising from either the first piece or the second piece of the approximate functional equation. Consider a term where we have ℓ factors from the first piece of an approximate functional equation and $k-\ell$ factors from the second piece. To take one specific example, suppose it is the first ℓ factors from which we choose the first piece of the approximate functional equation, and the last $k-\ell$ factors from which we take the second piece of the approximate functional equation:

$$\varepsilon_f^{-\ell/2}\mathcal{X}_f(s+\alpha_1)^{-1/2}\dots\mathcal{X}_f(s+\alpha_\ell)^{-1/2}\sum_{n_1}\frac{a_{n_1}(f)}{n_1^{s+\alpha_1}}\dots\sum_{n_\ell}\frac{a_{n_\ell}(f)}{n_\ell^{s+\alpha_\ell}} \times \varepsilon_f^{(k-\ell)/2}\mathcal{X}_f(s+\alpha_{\ell+1})^{1/2}\dots\mathcal{X}_f(s+\alpha_k)^{1/2}\sum_{n_{\ell+1}}\frac{\overline{a_{n_{\ell+1}}(f)}}{n_{\ell+1}^{1-s-\alpha_{\ell+1}}}\dots\sum_{n_k}\frac{\overline{a_{n_k}(f)}}{n_k^{1-s-\alpha_k}}. \tag{4.1.7}$$

Rearranging this expression and using the fact that $\mathcal{X}_f(s) = \mathcal{X}_f(1-s)^{-1}$, we have

$$\varepsilon_f^{k/2-\ell}\prod_{j=1}^{\ell}\mathcal{X}_f(s+\alpha_j)^{-1/2}\prod_{j=\ell+1}^{k}\mathcal{X}_f(1-s-\alpha_j)^{-1/2} \times \sum_{n_1,\dots,n_k}\frac{a_{n_1}(f)\dots a_{n_\ell}(f)\overline{a_{n_{\ell+1}}(f)\dots a_{n_k}(f)}}{n_1^{s+\alpha_1}\dots n_\ell^{s+\alpha_\ell}n_{\ell+1}^{1-s-\alpha_{\ell+1}}\dots n_k^{1-s-\alpha_k}}. \tag{4.1.8}$$

A little trick: since we will eventually set $s=\frac{1}{2}$, we replace the above expression by

$$\varepsilon_f^{k/2-\ell}\prod_{j=1}^{\ell}\mathcal{X}_f(s+\alpha_j)^{-1/2}\prod_{j=\ell+1}^{k}\mathcal{X}_f(s-\alpha_j)^{-1/2} \times \sum_{n_1,\dots,n_k}\frac{a_{n_1}(f)\dots a_{n_\ell}(f)\overline{a_{n_{\ell+1}}(f)\dots a_{n_k}(f)}}{n_1^{s+\alpha_1}\dots n_\ell^{s+\alpha_\ell}n_{\ell+1}^{s-\alpha_{\ell+1}}\dots n_k^{s-\alpha_k}}. \tag{4.1.9}$$

It is expression (4.1.9), and the corresponding pieces from the other terms when multiplying out the approximate functional equation, which will appear in the final conjecture, evaluated at $s = \frac{1}{2}$.

Now consider the product of ε_f factors $\varepsilon_f^{k/2-\ell}$, which according to the recipe should be replaced by its expected value. An important issue is the choice of the square root. We believe that there is a natural choice of $\varepsilon_f^{1/2}$ so that the following hold.

(a) *Unitary case*: the ε_f are uniformly distributed on the unit circle, and $\langle \varepsilon_f^{k/2-\ell} \rangle = 0$ unless $\frac{1}{2}k - \ell = 0$. In particular, k must be even. There will be $\binom{k}{k/2}$ terms in the final answer.

(b) *Orthogonal case*: either $\varepsilon_f = 1$ is constant (1 or -1) over the family, or $\varepsilon_f = 1$ for approximately half the f and $\varepsilon_f = -1$ for the other half. Here $\langle \varepsilon_f^{k/2-\ell} \rangle = 0$ unless $\frac{1}{2}k - \ell$ is even. In particular, k must be even and there will be 2^{k-1} terms in the final answer.

(c) *Symplectic case*: $\varepsilon_f = 1$ for all f, and $\langle \varepsilon_f^{k/2-\ell} \rangle = 1$ for all k and ℓ. There is no restriction and there will be 2^k terms in the final answer.

Note that if we are considering the L-function, instead of the Z-function, then the issue of $\varepsilon_f^{1/2}$ does not arise and the calculation is somewhat easier. See (4.5.4) and the discussion following it. Also note that in the Unitary and Orthogonal cases, odd powers of the Z-function will average to zero, while odd powers of the L-function will not.

The recipe now tells us to replace the summand by its expected value when averaged over the family. That is, we replace

$$a_{n_1}(f) \ldots a_{n_\ell}(f) \overline{a_{n_{\ell+1}}(f) \ldots a_{n_k}(f)} \tag{4.1.10}$$

by its expected value when averaged over the family. In practice, this will be of the form

$$c(\mathcal{F}) \delta_\ell(n_1, \ldots, n_k) \tag{4.1.11}$$

where $c(\mathcal{F})$ depends only on the family, and where the δ_ℓ are multiplicative functions, that is,

$$\delta_\ell(m_1 n_1, \ldots, m_k n_k) = \delta_\ell(m_1, \ldots, m_k) \delta_\ell(n_1, \ldots, n_k) \tag{4.1.12}$$

whenever $(m_1 \ldots m_k, n_1 \ldots n_k) = 1$.

The final step is to extend the range of summation. This produces one term in the conjecture. By considering the other terms when multiplying out the approximate functional equations, one arrives at a conjecture for the original mean value.

Although the above steps have produced an answer, it is not written in a particularly usable form. There are three more steps to put the conjecture in the form of Conjecture 1.5.1: writing the main terms as an Euler product, identifying the polar part, and expressing the combinatorial sum as a multiple integral.

Since the δ_ℓ are multiplicative, we can write the main term as an Euler product. Specifically,

$$\sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{\delta_\ell(n_1, \ldots, n_k)}{n_1^{s+\alpha_1} \ldots n_k^{s+\alpha_k}} = \prod_p \sum_{e_1, \ldots, e_k = 0}^{\infty} \frac{\delta_\ell(p^{e_1}, \ldots, p^{e_k})}{p^{e_1(s+\alpha_1) + \ldots + e_k(s+\alpha_k)}} \tag{4.1.13}$$

assuming that $\Re s$ is sufficiently large.

Next we determine the leading order poles. It usually turns out that $\delta_\ell(p^{e_1}, \ldots, p^{e_k}) = 0$ when $\sum e_j = 1$. Thus, the first poles come from those terms where $\sum e_j = 2$. This happens in two ways, so the rightmost poles are the same as the poles of

$$\prod_{1\leqslant i<j\leqslant k} \prod_p \left(1 + \frac{\delta_{\ell,i,j}(p,p)}{p^{2s+\alpha_i+\alpha_j}}\right) \times \prod_{j=1}^{k} \prod_p \left(1 + \frac{\delta_{\ell,j}(p^2)}{p^{2s+2\alpha_j}}\right). \tag{4.1.14}$$

In practice the first factor has simple poles at $\frac{1}{2} - \frac{1}{2}(\alpha_i + \alpha_j)$, and the second factor is either regular in a neighborhood of $\sigma = \frac{1}{2}$, or else it has a simple pole at $s = \frac{1}{2} - \alpha_j$. Accordingly, we factor out either

$$\prod_{1\leqslant i<j\leqslant k} \zeta(2s+\alpha_i+\alpha_j) \quad \text{or} \quad \prod_{1\leqslant i\leqslant j\leqslant k} \zeta(2s+\alpha_i+\alpha_j). \tag{4.1.15}$$

The remainder is the A_k in our conjectures, and it is regular in a neighborhood of $\sigma = \frac{1}{2}$.

Having identified the polar part of our main terms, we can apply the lemmas in § 2.5 to express the sum of terms as a contour integral. The result is an expression similar to Conjecture 1.5.1.

We have already seen this procedure in § 2.2 for the case of mean values of the zeta-function. In the following sections we carry out example calculations for families of each of the three symmetry types.

4.2. *Unitary: moments of a primitive L-function*

The recipe for mean values in § 2.1 is a special case of the general recipe. To see this, note that if $f_t \in \mathcal{F}_t$ then $f_t(n) = n^{-it}$, so $\mathcal{L}(s, f_t) = \mathcal{L}(s+it)$. From the functional equation $\mathcal{L}(s) = \varepsilon\mathcal{X}(s)\overline{\mathcal{L}}(1-s)$ we obtain the functional equation

$$\mathcal{L}(s, f_t) = \varepsilon_t \mathcal{X}_t(s)\overline{\mathcal{L}}(1-s, f_t), \tag{4.2.1}$$

where

$$\varepsilon_t = \varepsilon\mathcal{X}(\tfrac{1}{2}+it) \quad \text{and} \quad \mathcal{X}_t(s) = \frac{\mathcal{X}(s+it)}{\mathcal{X}(\frac{1}{2}+it)}. \tag{4.2.2}$$

Note that these satisfy the requirements $|\mathcal{X}_t(\frac{1}{2}+iy)| = 1$ for y real, with $\mathcal{X}_t(\frac{1}{2}) = 1$ and $|\varepsilon_t| = 1$. Also note that the log conductor of $\mathcal{L}(s, f_t)$, defined as $|(\varepsilon_t\mathcal{X}_t)'(\frac{1}{2})|$, equals $|\mathcal{X}'(\frac{1}{2}+it)|$, in agreement with the usual notion of conductor in t-aspect.

Replacing the product of ε_t-factors by their expected value is the same as 'keep the terms where the product of the χ-factors is not oscillating'. Thus, after multiplying out the approximate functional equations there will be $\binom{2k}{k}$ terms which contribute. In each of those terms replacing the summand by its expected value is the same as 'keeping the diagonal'. Thus, we arrive at the same conjecture as before.

4.3. *Unitary: all Dirichlet L-functions*

We apply our recipe to conjecture the average

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi \text{ even or odd}}} Z_\chi(\tfrac{1}{2}; \alpha_1, \ldots, \alpha_{2k}), \tag{4.3.1}$$

where the sum is over either the even or the odd primitive Dirichlet characters mod q and

$$Z_\chi(s; \alpha_1, \ldots, \alpha_{2k}) = Z(s+\alpha_1, \chi) \ldots Z(s+\alpha_{2k}, \chi). \tag{4.3.2}$$

Here $Z(s+\alpha_k,\chi)=(\varepsilon_\chi X_\chi(s))^{-1/2}L(s,\chi)$ where $L(s,\chi)=\varepsilon_\chi X_\chi(s)L(1-s,\chi)$. Note that $\varepsilon_\chi=\tau(\chi)q^{-1/2}$, which is uniformly distributed on the unit circle.

Following the general discussion in § 4.1, equation (4.1.9) specializes in this case to

$$\varepsilon_\chi^{k-\ell}\prod_{j=1}^{\ell}X_\chi(\tfrac{1}{2}+\alpha_j)^{-1/2}\prod_{j=\ell+1}^{2k}X_\chi(\tfrac{1}{2}-\alpha_j)^{-1/2} \times\sum_{n_1,\dots,n_{2k}}\frac{1}{n_1^{1/2+\alpha_1}\dots n_{2k}^{1/2-\alpha_{2k}}}\,\chi(n_1)\dots\overline{\chi}(n_{2k}). \tag{4.3.3}$$

According to the recipe, we replace $\varepsilon_\chi^{k-\ell}$ by its expected value. Since the ε_χ are uniformly distributed on the unit circle, the expected value is 1 if $\ell=k$ and 0 otherwise, so we keep $\binom{2k}{k}$ terms.

Next we replace the summand by its expected value, which is

$$\delta(n_1,\dots,n_{2k})=\langle\chi(n_1)\dots\chi(n_k)\overline{\chi}(n_{k+1})\dots\overline{\chi}(n_{2k})\rangle = \begin{cases}1 & \text{if } n_1\dots n_k=n_{k+1}\dots n_{2k} \text{ and } (n_1\dots n_{2k},q)=1,\\ 0 & \text{otherwise.}\end{cases} \tag{4.3.4}$$

The above is almost identical to the conjectures obtained for the mean values, in t-aspect, for a primitive L-function. So one obtains the same formulas as appear in Conjectures 1.5.1 and 2.5.4, the only changes being that one omits the factors $p\,|\,q$ in the Euler product A_k, and one must use the factors $X_\chi(\frac{1}{2}\pm\alpha_j)^{-1/2}$. Specifically, in Conjectures 1.5.1 and 2.5.4 a simplification occurred by use of equations (2.2.4) and (2.3.6). If those conjectures were written in terms of $\prod X(\frac{1}{2}\pm z_j)^{-1/2}$, then the Dirichlet L-function moment conjecture would be obtained by substituting with $\prod X_\chi(\frac{1}{2}\pm z_j)^{-1/2}$. Note that we are considering the averages over the even and odd primitive characters separately, so in the sum X_χ only depends on the conductor of χ. See the comments following the theorems in § 1.5 for more discussion on these X-factors and conductors.

4.4. *Symplectic and Orthogonal: quadratic twists of a real L-function*

Next we consider what happens when we average the shifts of central values of $\mathcal{L}(s)$ twisted by the family of quadratic characters

$$\chi_d(n)=\left(\frac{d}{n}\right),$$

with $d<0$ a fundamental discriminant. Here $\chi_d(n)=\left(\frac{d}{n}\right)$ is the Kronecker symbol which is a primitive Dirichlet character of conductor $|d|$. We will see that the family can be either Symplectic or Orthogonal, depending on the particular L-function that we start with.

Again, let

$$\mathcal{L}(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}=\prod_p\mathcal{L}_p(1/p^s) \tag{4.4.1}$$

be a primitive L-function and note that

$$\mathcal{L}(s,\chi_d)=\sum_{n=1}^{\infty}a_n\left(\frac{d}{n}\right)/n^s. \tag{4.4.2}$$

We assume that $\mathcal{L}$ is real, that is, $\mathcal{L}=\overline{\mathcal{L}}$, as this case is relatively easy to deal with from a fairly general perspective. Thus,

$$\mathcal{L}(s)=\varepsilon\mathcal{X}(s)\mathcal{L}(1-s), \tag{4.4.3}$$

where $\varepsilon=\pm1$. The twisted L-function is expected to satisfy a functional equation of the form

$$\mathcal{L}(s,\chi_d)=\varepsilon_d\mathcal{X}_d(s)\mathcal{L}(1-s,\chi_d). \tag{4.4.4}$$

It is further expected that

$$\mathcal{X}_d(s)=|d|^{w(1/2-s)}X(s,d), \tag{4.4.5}$$

where there are only finitely many possibilities for $X(s,d)$. By our definition of 'family' we require that the parameters in the functional equation be monotonic functions of the conductor. Since there are only finitely many choices for $X(s,d)$, we must restrict to averages over sets of d for which $X(s,d)$ is constant (as a function of d). In the situation described here, it is believed that there exists an integer N, depending on $\mathcal{L}$, such that ε_d and $X(s,d)$ only depend on the sign of d and on $(d \bmod N)$. Thus, we will consider the averages

$$\sideset{}{^*}\sum_{\substack{d<0\\ d\equiv a \bmod N}} L_d(\tfrac{1}{2};\alpha_1,\dots,\alpha_k)g(|d|), \tag{4.4.6}$$

(the following analysis holds also for $d>0$) where $\sum^*$ denotes a sum over fundamental discriminants d, and

$$L_d(s;\alpha_1,\dots,\alpha_k)=Z_{\mathcal{L}}(s+\alpha_1,\chi_d)\dots Z_{\mathcal{L}}(s+\alpha_k,\chi_d). \tag{4.4.7}$$

Note that $\varepsilon_d=\varepsilon_a$, which may depend on the sign of d. If N is even we are insisting further that it be divisible by at least 8.

Following the general discussion in § 4.1, equation (4.1.9) specializes in this case to

$$\begin{aligned}\varepsilon_f^{k/2-\ell}\prod_{j=1}^{\ell}\mathcal{X}_d(s+\alpha_j)^{-1/2}&\prod_{j=\ell+1}^{k}\mathcal{X}_d(s-\alpha_j)^{-1/2}\\ &\times\sum_{n_1,\dots,n_k}\frac{a_{n_1}\dots a_{n_k}}{n_1^{s+\alpha_1}\dots n_k^{s-\alpha_k}}\chi_d(n_1)\dots\chi_d(n_k).\end{aligned} \tag{4.4.8}$$

According to the recipe, we replace $\varepsilon_f^{k/2-\ell}$ by its expected value. We have assumed (by our choice of $a \bmod N$) that $\varepsilon_d=\varepsilon_a$ for all d, so the expected value is $\varepsilon_a^{k/2-\ell}$ and we will have a contribution from all 2^k terms. (That expression is more transparent if one considers the cases $\varepsilon_a=1$ and $\varepsilon_a=-1$ separately.)

The next step in the recipe is to replace the summand by its expected value. Since $\chi_d(n_1)\dots\chi_d(n_k)=\chi_d(n_1\dots n_k)$, from equation (3.1.21) we have the expected value

$$\langle \chi_d(n_1)\dots\chi_d(n_k)\rangle = \begin{cases} \chi_a(g)\prod_{p|\square}\left(1+\frac{1}{p}\right)^{-1} & \text{if } n_1\dots n_k = g\square, \\ 0 & \text{otherwise,} \end{cases} \tag{4.4.9}$$

where $(N,\square)=1$, and with all the prime factors of g also being prime factors of N. So the contribution from the term where we use the first part of the approximate functional equation for the first ℓ factors, and the second part for the rest, is

$$\varepsilon_d^{k/2-\ell}\prod_{j=1}^{\ell}\mathcal{X}_d(s+\alpha_j)^{-1/2}\prod_{j=\ell+1}^{k}\mathcal{X}_d(s-\alpha_j)^{-1/2} \\ \times R_{k,N}(s;\alpha_1,\dots,\alpha_\ell,-\alpha_{\ell+1},\dots,-\alpha_k) \tag{4.4.10}$$

where

$$R_{k,N}(s;\alpha_1,\dots,\alpha_k) = \sum_{g\square}\chi_a(g)\sum_{n_1\dots n_k=g\square}\frac{a_{n_1}\dots a_{n_k}}{n_1^{s+\alpha_1}\dots n_k^{s+\alpha_k}}\prod_{p|\square}\left(1+\frac{1}{p}\right)^{-1}. \tag{4.4.11}$$

Adding up all 2^k terms we obtain

$$M(s;\alpha_1,\dots,\alpha_k) = \sum_{\epsilon_i=\pm1}\operatorname{sign}(\{\epsilon_j\})\prod_{j=1}^{k}\mathcal{X}_d(\tfrac12+\epsilon_j\alpha_j)^{-1/2} \\ \times R_{k,N}(s;\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k), \tag{4.4.12}$$

where

$$\operatorname{sign}(\{\epsilon_j\}) = \begin{cases} 1 & \text{if } \varepsilon_a = 1, \\ (-1)^{(\Sigma\epsilon_i)/2} & \text{if } \varepsilon_a = -1\ . \end{cases} \tag{4.4.13}$$

So the recipe has produced the conjecture

$$\sum_{\substack{d<0 \\ d\equiv a \bmod N}}^{*} L_d(\tfrac12,\alpha_1,\dots,\alpha_k)g(d) \\ = \sum_{\substack{d<0 \\ d\equiv a \bmod N}}^{*} M(\tfrac12;\alpha_1,\dots,\alpha_k)(1+O(|d|^{-1/2+\varepsilon}))g(d). \tag{4.4.14}$$

To put the conjecture in a more useful form, we now write $R_{k,N}$ as an Euler product, and then express the main term as a contour integral.

We have $R_{k,N}=\prod_p R_{k,N,p}$, which naturally separates into a product over the primes which divide N and a product over the primes which do not divide N. The

p-factor when $p\nmid N$ is

$$\begin{aligned}R_{k,N,p}(s) &= \left(1+\left(1+\frac{1}{p}\right)^{-1}\sum_{j=1}^{\infty}\sum_{e_1+\ldots+e_k=2j}\prod_{i=1}^{k}\frac{a_{p^{e_i}}}{p^{e_i(s+\alpha_i)}}\right)\\ &= \left(1+\frac{1}{p}\right)^{-1}\left(\frac{1}{p}+\sum_{j=0}^{\infty}\sum_{e_1+\ldots+e_k=2j}\prod_{i=1}^{k}\frac{a_{p^{e_i}}}{p^{e_i(s+\alpha_i)}}\right)\\ &= \left(1+\frac{1}{p}\right)^{-1}\left(\frac{1}{p}+\frac{1}{2}\left(\prod_{j=1}^{k}\mathcal{L}_p\left(\frac{1}{p^{s+\alpha_j}}\right)+\prod_{j=1}^{k}\mathcal{L}_p\left(\frac{-1}{p^{s+\alpha_j}}\right)\right)\right). \qquad (4.4.15)\end{aligned}$$

Similarly, the p-factor when $p\mid N$ is

$$R_{k,N,p}=\prod_{j=1}^{k}\mathcal{L}_p\left(\frac{\chi_a(p)}{p^{s+\alpha_j}}\right). \qquad (4.4.16)$$

The above expression will enable us to locate the leading poles of $R_{k,N}$. Consider the expansion of $R_{k,N,p}$ (for $p\nmid N$) in powers of $1/p$. The expansion is of the form

$$\begin{aligned}1+\sum_{j=1}^{k}\frac{a_{p^2}}{p^{2s+2\alpha_j}}+\sum_{1\leqslant i<j\leqslant k}\frac{(a_p)^2}{p^{2s+\alpha_i+\alpha_j}}+O(p^{-1-2s+\varepsilon})+O(p^{-3s+\varepsilon})\\ =\prod_{j=1}^{k}\left(1+\frac{a_{p^2}}{p^{2s+\alpha_j}}\right)\times\prod_{1\leqslant i<j\leqslant k}\left(1+\frac{(a_p)^2}{p^{2s+\alpha_i+\alpha_j}}\right)\\ \times\left(1+O(p^{-1-2s+\varepsilon})+O(p^{-3s+\varepsilon})\right). \qquad (4.4.17)\end{aligned}$$

We assume that

$$\prod_p\left(1+\frac{(a_p)^2}{p^s}\right) \qquad (4.4.18)$$

has a simple pole at $s=1$. This is conjectured to be equivalent to $\mathcal{L}(s)$ being a primitive L-function, and this is the key place where the assumption of primitivity enters the calculation. We also assume that

$$\prod_p\left(1+\frac{a_{p^2}}{p^s}\right) \qquad (4.4.19)$$

has a pole of order $\delta=0$ or 1 at $s=1$.

In general, δ is expected to be 0 or 1 according to whether the symmetric square L-function of $\mathcal{L}(s)$ is analytic at $s=1$ or has a simple pole at $s=1$. If $\mathcal{L}(s)$ is a degree 1 L-function (that is, the Riemann ζ-function or a Dirichlet L-function), then $\delta=1$. If $\mathcal{L}(s)$ is associated to a GL(2) automorphic form, then $\delta=0$ in general (except possibly when $\mathcal{L}$ is a dihedral Artin L-function associated to a weight 1 modular form).

Note that $\prod_p(1+O(p^{-1-2s})+O(p^{-3s}))$ is regular for $\sigma>\frac{1}{3}$. Thus the total order of the pole of the above product at $s=\frac{1}{2}$ when $\alpha_1=\ldots=\alpha_k=0$ is $\frac{1}{2}k(k-1)+\delta k$. Accordingly, we factor out appropriate zeta-factors and write the above product as

$$R_{k,N}(s) = \prod_{1\leqslant i<j\leqslant k} \zeta(2s+\alpha_i+\alpha_j) \prod_p R_{k,N,p}(s) \prod_{1\leqslant i<j\leqslant k} \left(1 - \frac{1}{p^{2s+\alpha_i+\alpha_j}}\right) \tag{4.4.20}$$

if $\delta = 0$, and as

$$R_{k,N}(s) = \prod_{1\leqslant i\leqslant j\leqslant k} \zeta(2s+\alpha_i+\alpha_j) \prod_p R_{k,N,p}(s) \prod_{1\leqslant i\leqslant j\leqslant k} \left(1 - \frac{1}{p^{2s+\alpha_i+\alpha_j}}\right) \tag{4.4.21}$$

if $\delta = 1$. In the first case above the family is Orthogonal, and in the second case it is Symplectic.

In summary, we are led to conjecture that

$$\begin{aligned} &\sum_{\substack{d<0\\ d\equiv a \bmod N}}^{*} Z_{\mathcal{L}}(\tfrac{1}{2}+\alpha_1,\chi_d)\dots Z_{\mathcal{L}}(\tfrac{1}{2}+\alpha_k,\chi_d)g(|d|) \\ &\quad = \sum_{\epsilon_i=\pm1} \operatorname{sign}(\{\epsilon_i\}) \prod_{j=1}^{k} X(\tfrac{1}{2}+\epsilon_j\alpha_j,a)^{-1/2} \\ &\qquad \times \sum_{\substack{d<0\\ d\equiv a \bmod N}}^{*} R_{k,N}(\tfrac{1}{2},\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k)|d|^{(w/2)\sum_{j=1}^{k}\epsilon_j\alpha_j}(1+O(|d|^{-1/2+\varepsilon}))\,g(|d|). \end{aligned} \tag{4.4.22}$$

The analogous sum over $d>0$ leads to a similar conjecture. Here $\operatorname{sign}(\{\epsilon_i\})$ is given in (4.4.13) and in either case we can use Lemma 2.5.2 to write the sum as a contour integral.

In the case that $\mathcal{L}(s)$ is the Riemann zeta-function, the above reduces to Conjecture 1.5.3.

4.5. *Orthogonal: L-functions associated with cusp forms*

Recall that the set of primitive newforms $f \in S_n(\Gamma_0(q))$ is denoted by $H_n(q)$. In this section we consider the shifted moments of $L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n)n^{-s}$ near the critical point $s = \frac{1}{2}$ averaged over $f \in H_n(q)$. Note that in the language of § 3.2 these L-functions are the twists of $\zeta(s)$ by the family of characters $H_k(q)$ and would be denoted as $\zeta(s,f)$. However, we use the more common notation here.

The functional equation is

$$L_f(s) = \varepsilon_f X(s) L_f(1-s), \tag{4.5.1}$$

where $\varepsilon_f = -\sqrt{q}\,\lambda_f(q) = \pm1$.

We consider the 'harmonic average'

$$\sum_{f\in H_n(q)}^{\mathrm{h}} L_f(\tfrac{1}{2}+\alpha_1)\dots L_f(\tfrac{1}{2}+\alpha_k) \tag{4.5.2}$$

which attaches a weight $\langle f,f\rangle^{-1}$ to each summand. That is,

$$\sum_{f\in H_n(q)}^{\mathrm{h}} * \quad = \sum_{f\in H_n(q)} */\langle f,f\rangle. \tag{4.5.3}$$

Following the general discussion in § 4.1, equation (4.1.9) specializes in this case to

$$\varepsilon_f^{k-\ell} \prod_{j=\ell+1}^{k} X(s-\alpha_j)^{-1} \sum_{n_1,\dots,n_k} \frac{\lambda_f(n_1)\dots\lambda_f(n_k)}{n_1^{s+\alpha_1}\dots n_k^{s-\alpha_k}}. \tag{4.5.4}$$

According to the recipe, we replace $\varepsilon_f^{k-\ell}$ by its expected value. Since ε_f is randomly ±1, the expected value is 0 unless $k-\ell$ is even. Thus, we will have 2^{k-1} terms in the final answer.

Next we replace $\lambda_f(n_1)\dots\lambda_f(n_k)$ by its expected value. This is given in Lemma 3.1.2. After factoring into an Euler product and summing the relevant geometric series we see that (4.5.4) contributes

$$\prod_{j=1}^{k} X(s-\alpha_j)^{-1/2} R(\alpha_1,\dots,\alpha_\ell,-\alpha_{\ell+1},\dots,-\alpha_k),$$

where

$$\begin{aligned} R(s,\alpha_1,\dots,\alpha_k) = \prod_{j=1}^{k} X(s+\alpha_j)^{-1/2} \prod_p \frac{2}{\pi}\int_0^{\pi} \sin^2\theta \\ \times \prod_{j=1}^{k} \frac{e^{i\theta}(1-e^{i\theta}/p^{s+\alpha_j})^{-1} - e^{-i\theta}(1-e^{-i\theta}/p^{s+\alpha_j})^{-1}}{e^{i\theta}-e^{-i\theta}}\, d\theta. \end{aligned} \tag{4.5.5}$$

Here remember that s will eventually be set to $\frac{1}{2}$ and $X(s) = X(1-s)^{-1}$. Adding up all 2^{k-1} terms we obtain

$$M(s;\alpha_1,\dots,\alpha_k) = \prod_{j=1}^{k} X(s-\alpha_j)^{-1/2} \sum_{\substack{\epsilon_i=\pm1 \\ \prod_{j=1}^k \epsilon_j=1}} R(s;\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k), \tag{4.5.6}$$

so the recipe has produced the conjecture

$$\begin{aligned} \sum_{f\in H_n(q)}^{h} L_f(\tfrac12+\alpha_1)\dots L_f(\tfrac12+\alpha_k) &= \sum_{f\in H_n(q)}^{h} M(\tfrac12;\alpha_1,\dots,\alpha_k)(1+O(nq)^{-1/2+\varepsilon}) \\ &= (1+O(nq)^{-1/2+\varepsilon}) M(\tfrac12;\alpha_1,\dots,\alpha_k). \end{aligned} \tag{4.5.7}$$

Summarizing, we have the following conjecture.

Conjecture 4.5.1. *With $A_k(\alpha_1,\dots,\alpha_k)$ as in Conjecture 1.5.5, we have*

$$\begin{aligned} &\sum_{f\in H_n(q)}^{h} L_f(\tfrac12+\alpha_1)\dots L_f(\tfrac12+\alpha_k) \\ &\quad= \prod_{j=1}^{k} X(\tfrac12-\alpha_j)^{-1/2} \sum_{\substack{\epsilon_j=\pm1 \\ \prod_{j=1}^k \epsilon_j=1}} \prod_{j=1}^{k} X(\tfrac12+\epsilon_j\alpha_j)^{-1/2} \\ &\qquad\times \prod_{1\leqslant i<j\leqslant k} \zeta(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j) A_k(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k)(1+O(nq)^{-1/2+\varepsilon}). \end{aligned} \tag{4.5.8}$$

For the purpose of considering averages of even forms or odd forms separately, we note that

$$\begin{aligned}
&\sum_{f\in H_n(q)}^{\mathrm{h}} \varepsilon_f L_f(\tfrac12+\alpha_1)\dots L_f(\tfrac12+\alpha_k)\\
&\qquad = X(\tfrac12+\alpha_1)\sum_{f\in H_n(q)}^{\mathrm{h}} L_f(\tfrac12-\alpha_1)L_f(\tfrac12+\alpha_2)\dots L_f(\tfrac12+\alpha_k)\\
&\qquad = X(\tfrac12+\alpha_1)M(\tfrac12;-\alpha_1,\alpha_2,\dots,\alpha_k)(1+O(nq)^{-1/2+\varepsilon}). \qquad (4.5.9)
\end{aligned}$$

By looking at the combinations

$$\sum_{f\in H_n(q)}^{\mathrm{h}} L_f(\tfrac12+\alpha_1)\dots L_f(\tfrac12+\alpha_k) \pm \sum_{f\in H_n(q)}^{\mathrm{h}} \varepsilon_f L_f(\tfrac12+\alpha_1)\dots L_f(\tfrac12+\alpha_k) \qquad (4.5.10)$$

we see that this leads to the following.

CONJECTURE 4.5.2. *With $A_k(\alpha_1,\dots,\alpha_k)$ as in Conjecture 1.5.5, we have*

$$\begin{aligned}
&\sum_{\substack{f\in H_n(q)\\ f \text{ even}}}^{\mathrm{h}} L_f(\tfrac12+\alpha_1)\dots L_f(\tfrac12+\alpha_k)\\
&\quad = \frac12\prod_{j=1}^{k} X(\tfrac12-\alpha_j)^{-1/2}\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k} X(\tfrac12+\epsilon_j\alpha_j)^{-1/2}\\
&\qquad \times \prod_{1\leqslant i<j\leqslant k}\zeta(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)A_k(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k)(1+O(nq)^{-1/2+\varepsilon}), \qquad (4.5.11)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{f\in H_n(q)\\ f \text{ odd}}}^{\mathrm{h}} L_f(\tfrac12+\alpha_1)\dots L_f(\tfrac12+\alpha_k)\\
&\quad = \frac12\prod_{j=1}^{k} X(\tfrac12-\alpha_j)^{-1/2}\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k} \epsilon_j X(\tfrac12+\epsilon_j\alpha_j)^{-1/2}\\
&\qquad \times \prod_{1\leqslant i<j\leqslant k}\zeta(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)A_k(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k)(1+O(nq)^{-1/2+\varepsilon}). \qquad (4.5.12)
\end{aligned}$$

The above formulae can be written as contour integrals using Lemma 2.5.2, giving expressions analogous to those in Conjecture 1.5.5. In particular, expressing (4.5.11) as a contour integral and then letting $\alpha_j \to 0$ gives Conjecture 1.5.5.

5. *Numerical calculations*

We compare our conjectures with some numerical calculations. The agreement is very good. These calculations involve numerically approximating the coefficients in the conjectured formulae, and numerically evaluating the mean value. Both of those calculations are interesting and we will give more details and examples in a subsequent paper.

5.1. *Unitary: Riemann zeta-function*

The coefficients of $P_2(x)$ in Conjecture 1.5.1 can be written explicitly in terms of known constants. When $k=2$ the function $G(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ that appears in Conjecture 1.5.1 equals

$$\zeta(2+\alpha_1+\alpha_2-\alpha_3-\alpha_4)^{-1}\prod_{i,j=1}^{2}\zeta(1+\alpha_i-\alpha_{k+j}), \tag{5.1.1}$$

which is given in § 2.2. But

$$\zeta(1+s)=s^{-1}+\gamma-\gamma_1 s+\frac{\gamma_2}{2!}s^2-\frac{\gamma_3}{3!}s^3+\dots \tag{5.1.2}$$

and

$$\zeta(2+s)^{-1}=\frac{6}{\pi^2}-\frac{36\zeta'(2)}{\pi^4}s+\frac{-3\pi^2\zeta''(2)+36\zeta'(2)^2}{\pi^6}s^2+\dots. \tag{5.1.3}$$

In the latter, the terms up to s^4 were evaluated using MAPLE. Computing the residue in Conjecture 1.5.1 gives

$$\begin{aligned}
P_2(x)=&\ \frac{1}{2\pi^2}x^4+\frac{8}{\pi^4}(\gamma\pi^2-3\zeta'(2))x^3\\
&+\frac{6}{\pi^6}\big(-48\gamma\zeta'(2)\pi^2-12\zeta''(2)\pi^2+7\gamma^2\pi^4+144\zeta'(2)^2-2\,\gamma_1\pi^4\big)x^2\\
&+\frac{12}{\pi^8}(6\gamma^3\pi^6-84\gamma^2\zeta'(2)\pi^4+24\gamma_1\zeta'(2)\pi^4-1728\zeta'(2)^3+576\gamma\zeta'(2)^2\pi^2\\
&\qquad+288\zeta'(2)\zeta''(2)\pi^2-8\zeta'''(2)\pi^4-10\gamma_1\gamma\pi^6-\gamma_2\pi^6-48\gamma\zeta''(2)\pi^4)x\\
&+\frac{4}{\pi^{10}}(-12\zeta''''(2)\pi^6+36\gamma_2\zeta'(2)\pi^6+9\gamma^4\pi^8+21\gamma_1^2\pi^8+432\zeta''(2)^2\pi^4\\
&\qquad+3456\gamma\zeta'(2)\zeta''(2)\pi^4+3024\gamma^2\zeta'(2)^2\pi^4-36\gamma^2\gamma_1\pi^8-252\gamma^2\zeta''(2)\pi^6\\
&\qquad+3\gamma\gamma_2\pi^8+72\gamma_1\zeta''(2)\pi^6+360\gamma_1\gamma\zeta'(2)\pi^6-216\gamma^3\zeta'(2)\pi^6\\
&\qquad-864\gamma_1\zeta'(2)^2\pi^4+5\gamma_3\pi^8+576\zeta'(2)\zeta'''(2)\pi^4-20736\gamma\zeta'(2)^3\pi^2\\
&\qquad-15552\zeta''(2)\zeta'(2)^2\pi^2-96\gamma\zeta'''(2)\pi^6+62208\zeta'(2)^4),
\end{aligned} \tag{5.1.4}$$

in agreement with a result implied in the work of Heath-Brown [**20**] (see [**7**] where, using [**20**], the same polynomial is worked out, although there are some slight errors). Numerically,

$$\begin{aligned}
P_2(x)=&\ 0.0506605918211688857219397316048638\,x^4\\
&+0.6988698848789799698470962842765850 2\,x^3\\
&+2.4259621988466820047565753101606663\,x^2\\
&+3.2279079649012547643806898512746668\,x\\
&+1.3124243859616692261684400662299978.
\end{aligned} \tag{5.1.5}$$

There are several ways that one can numerically compute the coefficients of $P_3(x)$, and these will be described in a future paper. We found

$$\begin{aligned} P_3(x) = {} & 0.000005708527034652788398376841445252313\, x^9 \\ & + 0.00040502133088411440331215332025984\, x^8 \\ & + 0.011072455215246998350410400826667\, x^7 \\ & + 0.14840073080150272680851401518774\, x^6 \\ & + 1.0459251779054883439385323798059\, x^5 \\ & + 3.9843850948235347247479640734 29\, x^4 \\ & + 8.6073191457812067561483476 3629\, x^3 \\ & + 10.274330830703446134183009522\, x^2 \\ & + 6.5939130206497581046571339 2\, x \\ & + 0.91651550763789305901785 43. \end{aligned} \tag{5.1.6}$$

One notices that the leading coefficient is much smaller than the lower order coefficients, which means that, in numerical calculations, the lower order terms will contribute significantly. One might suppose that the coefficients of $P_k(x)$ are always positive. Unfortunately, while this is true for $P_1, \ldots, P_4$, by $k = 5$, negative coefficients begin to appear (see Table 5.1.1).

Table 5.1.2 depicts

$$\int_C^D |\zeta(\tfrac{1}{2} + it)|^6 \, dt \tag{5.1.7}$$

as compared to

$$\int_C^D P_3(\log(t/2\pi)) \, dt, \tag{5.1.8}$$

along with their ratio, for various blocks $[C, D]$ of length 50000, as well as a larger block of length 2,350,000. The data agree with our conjecture and are consistent with a remainder of size $|D - C|^{1/2} D^{\varepsilon}$.

One can also look at smoothed moments, for example,

$$\int_0^\infty |\zeta(\tfrac{1}{2} + it)|^{2k} \exp(-t/T) \, dt \tag{5.1.9}$$

as compared to

$$\int_0^\infty P_k(\log(t/2\pi)) \exp(-t/T) \, dt. \tag{5.1.10}$$

Table 5.1.3 compares these with $T = 10000$, for $k = 4, 3, 2, 1$.

For $k = 3, 4$ the data agrees to roughly half the decimal places. This supports our conjecture that the error term in the conjectured mean values is $O(T^{1/2+\varepsilon})$. For $k = 1$ the numerics suggest Corollary 1.6.3.

5.2. *Symplectic: quadratic Dirichlet L-functions*

We have computed the polynomials Q_k of Conjecture 1.5.3 for $k = 1, 2, \ldots, 8$, separately for $d < 0$ and $d > 0$. Table 5.2.1 lists these polynomials for $d < 0$, while

TABLE 5.1.1. *Coefficients of* $P_k(x) = c_0(k)x^{k^2} + c_1(k)x^{k^2-1} + \ldots + c_{k^2}(k)$, *for* $k = 4, 5, 6, 7$. *Notice the relatively small size of* $c_0(k)$. *We believe the coefficients to be correct to the number of places listed, except in the cases indicated by question marks, where the numerics have not quite stabilized. Two different methods were used to compute the coefficients. The former, for* $0 \leqslant r \leqslant 7$, *gave us higher precision but was less efficient, while the latter for* $r \leqslant 49$, *was more efficient but required using less precision.*

r	$c_r(4)$	$c_r(5)$	$c_r(6)$	$c_r(7)$
0	.24650183919342276e-12	.141600102062273e-23	.512947340914913e-39	.658228478760010e-59
1	.54501405731171861e-10	.738041275649445e-21	.530673280992642e-36	.120414305554514e-55
2	.52877296347912035e-8	.177977962351965e-18	.260792077114835e-33	.106213557174925e-52
3	.29641143179993979e-6	.263588660966072e-16	.810161321577902e-31	.601726537601586e-50
4	.1064595006812847e-4	.268405453499975e-14	.178612973800931e-28	.246062876732400e-47
5	.25702983342426343e-3	.199364130924990e-12	.297431671086361e-26	.773901216652114e-45
6	.42639216163116947e-2	.111848551249336e-10	.388770829115587e-24	.194786494949524e-42
7	.48941424514215989e-1	.484279755304480e-9	.409224261406863e-22	.403076849263637e-40
8	.38785267	.16398013e-7	.35314664e-20	.69917763e-38
9	2.1091338	.43749351e-6	.25306377e-18	.1031402e-35
10	7.8325356	.92263335e-5	.15198191e-16	.13082869e-33
11	19.828068	.00015376778	.77001514e-15	.14392681e-31
12	33.888932	.0020190278	.3306121e-13	.13825312e-29
13	38.203306	.020772707	.12064042e-11	.11657759e-27
14	25.604415	.16625059	.37467193e-10	.86652477e-26
15	10.618974	1.0264668	.99056943e-9	.56962227e-24
16	.708941	4.8485893	.22273886e-7	.33197649e-22
17		17.390876	.42513729e-6	.1718397e-20
18		47.040877	.68674336e-5	.79096789e-19
19		95.116618	.9351583e-4	.32396929e-17
20		141.44446	.0010683164	.11809579e-15
21		149.35697	.010180702	.3830227e-14
22		105.88716	.080418679	.11044706e-12
23		44.1356	.52296142	.28282258e-11
24		20.108	2.7802018	.64210662e-10
25		-1.27	12.001114	.12898756e-8
26			41.796708	.22869667e-7
27			116.72309	.35683995e-6
28			259.39898	.48834071e-5
29			452.491	.58391045e-4
30			601.17	.00060742037
31			573.54	.0054716438
32			374.8	.042465904
33			246.5	.28245494
34			248.	1.6013331
35			1.6e+02 ?	7.6966995
36			-4.e+01 ?	31.20352
37				106.19714
38				301.91363
39				711.742
40				1370.10
41				2083.
42				2356.
43				1.9e+03
44				1.8e+03
45				3.e+03
46				3.e+03
47				8.e+01 ?
48				-1.e+03 ?
49				-2.e+02 ?

in Table 5.2.2 we consider $d > 0$. Again notice the small size of the leading coefficients.

Table 5.2.3 compares, for $d < 0$, conjectured moments for $k = 1, \ldots, 8$ against numerically computed moments,

$$\sum_{d<0}^{*} L(\tfrac{1}{2}, \chi_d)^k g(|d|) \tag{5.2.1}$$

TABLE 5.1.2. *Sixth moment of* ζ *versus Conjecture* 1.5.1. *The 'reality' column, that is, integrals involving* ζ, *was computed using* MATHEMATICA.

$[C, D]$	Conjecture (5.1.8)	Reality (5.1.7)	Ratio
[0,50000]	7236872972.7	7231005642.3	.999189
[50000,100000]	15696470555.3	15723919113.6	1.001749
[100000,150000]	21568672884.1	21536840937.9	.998524
[150000,200000]	26381397608.2	26246250354.1	.994877
[200000,250000]	30556177136.5	30692229217.8	1.004453
[250000,300000]	34290291841.0	34414329738.9	1.003617
[300000,350000]	37695829854.3	37683495193.0	.999673
[350000,400000]	40843941365.7	40566252008.5	.993201
[400000,450000]	43783216365.2	43907511751.1	1.002839
[450000,500000]	46548617846.7	46531247056.9	.999627
[500000,550000]	49166313161.9	49136264678.2	.999389
[550000,600000]	51656498739.2	51744796875.0	1.001709
[600000,650000]	54035153255.1	53962410634.2	.998654
[650000,700000]	56315178564.8	56541799179.3	1.004024
[700000,750000]	58507171421.6	58365383245.2	.997577
[750000,800000]	60619962488.2	60870809317.1	1.004138
[800000,850000]	62661003164.6	62765220708.6	1.001663
[850000,900000]	64636649728.0	64227164326.1	.993665
[900000,950000]	66552376294.2	65994874052.2	.991623
[950000,1000000]	68412937271.4	68961125079.8	1.008013
[1000000,1050000]	70222493232.7	70233393177.0	1.000155
[1050000,1100000]	71984709805.4	72919426905.7	1.012985
[1100000,1150000]	73702836332.4	72567024812.4	.984589
[1150000,1200000]	75379769148.4	76267763314.7	1.011780
[1200000,1250000]	77018102997.5	76750297112.6	.996523
[1250000,1300000]	78620173202.6	78315210623.9	.996121
[1300000,1350000]	80188090542.5	80320710380.9	1.001654
[1350000,1400000]	81723770322.2	80767881132.6	.988303
[1400000,1450000]	83228956776.3	83782957374.3	1.006656
[0,2350000]	3317437762612.4	3317496016044.9	1.000017

TABLE 5.1.3. *Smoothed moment of* ζ *versus Conjecture* 1.5.1.

k	(5.1.9)	(5.1.10)	Difference	Relative difference
1	79499.9312635	79496.7897047	3.14156	3.952×10^{-5}
2	5088332.55512	5088336.43654	-3.8814	-7.628×10^{-7}
3	708967359.4	708965694.5	1664.9	2.348×10^{-6}
4	143638308513.0	143628911646.6	9396866.4	6.542×10^{-5}

versus

$$\sum_{d<0}^{*} Q_k(\log|d|)g(|d|) \tag{5.2.2}$$

where g is the smooth test function

$$g(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < 850000, \\ \exp\left(1 - \left(1 - \dfrac{(t-850000)^2}{(150000)^2}\right)^{-1}\right) & \text{if } 850000 \leqslant t \leqslant 1000000, \\ 0 & \text{if } 1000000 < t\ . \end{cases} \tag{5.2.3}$$

Table 5.2.4 compares the same quantities, but for $d > 0$.

Table 5.2.1. *Coefficients of* $Q_k(x) = d_0(k)x^{k(k+1)/2} + d_1(k)x^{k(k+1)/2} + \ldots$, *for* $k = 1, \ldots, 8$, *odd twists,* $d < 0$.

r	$d_r(1)$	$d_r(2)$	$d_r(3)$	$d_r(4)$
0	.3522211004995828	.1238375103096e-1	.1528376099282e-4	.31582683324433e-9
1	.61755003361406	.18074683511868	.89682763979959e-3	.50622013406082e-7
2		.3658991414081	.17014201759477e-1	.32520704779144e-5
3		−.13989539029	.10932818306819	.10650782552992e-3
4			.13585569409025	.18657913487212e-2
5			−.23295091113684	.16586741288851e-1
6			.47353038377966	.59859999105052e-1
7				.52311798496e-2
8				−.1097356195
9				.55812532
10				.19185945

r	$d_r(5)$	$d_r(6)$	$d_r(7)$	$d_r(8)$
0	.671251761107e-16	.1036004645427e-24	.886492719e-36	.337201e-49
1	.23412332535824e-13	.67968140667178e-22	.98944375081241e-33	.59511917e-46
2	.35711692341033e-11	.20378083365099e-19	.51762930260135e-30	.500204322e-43
3	.31271184907852e-9	.36980514080794e-17	.16867245856115e-27	.2664702284e-40
4	.17346173129392e-7	.45348387982697e-15	.38372675160809e-25	.1010164552e-37
5	.63429411057027e-6	.39728668850800e-13	.64746354773372e-23	.29004988867e-35
6	.15410644373832e-4	.2563279107877e-11	.84021141030379e-21	.65555882460e-33
7	.2441498848698e-3	.12372292296e-9	.85817644593981e-19	.11966099802e-30
8	.2390928284571e-2	.44915158297e-8	.70024645896E-17	.17958286298e-28
9	.127561073626e-1	.1222154548e-6	.4607034349989e-15	.22443685425e-26
10	.24303820161e-1	.2461203700e-5	.2455973970377e-13	.2357312577e-24
11	−.333141763e-1	.3579140509e-4	.106223013225e-11	.20942850060e-22
12	.25775611e-1	.3597968761e-3	.3719625461492e-10	.15805997923e-20
13	.531596583	.230207769e-2	.1048661496741e-8	.10159435845e-18
14	−.325832	.7699469185e-2	.2357398870407e-7	.55665248752e-17
15	−1.34187	.4281359929e-2	.416315210727e-6	.25985097519e-15
16		−.2312387714e-1	.564739434674e-5	.103134457e-13
17		.109503	.56831273239e-4	.346778002e-12
18		.2900464	.40016131254e-3	.982481680e-11
19		−.9016	.1755324808e-2	.232784142e-9
20		−.89361	.340409901e-2	.456549799e-8
21		−.181	−.2741804e-2	.7309216472e-7
22			.353555e-3	.9368893764e-6
23			.117734	.9348804928e-5
24			.20714e-1	.69517414e-4
25			−.9671	.356576507e-3
26			−.284	.1059852e-2
27			1.3	.8242527e-3
28			−1.	−.206921e-2
29				.181031e-1
30				.862815e-1
31				−.14025
32				−.91619
33				−.942
34				−.153e-1
35				−.3?
36				?

Figure 1 depicts, for $k = 1, \ldots, 8$ and $X = 10000, 20000, \ldots, 10^7$,

$$\sum_{-X<d<0}^{*} L(\tfrac{1}{2}, \chi_d)^k \tag{5.2.4}$$

divided by

$$\sum_{-X<d<0}^{*} Q_k(\log |d|). \tag{5.2.5}$$

TABLE 5.2.2. *Coefficients of* $Q_k(x) = e_0(k)x^{k(k+1)/2} + e_1(k)x^{k(k+1)/2} + \ldots$, *for* $k = 1, \ldots, 8$, *even twists,* $d > 0$.

r	$e_r(1)$	$e_r(2)$	$e_r(3)$	$e_r(4)$
0	.3522211004995828	.1238375103096e-1	.1528376099282e-4	.31582683324433e-9
1	−.4889851881547	.6403273133043e-1	.60873553227400e-3	.40700020814812e-7
2		−.403098546303	.51895362572218e-2	.19610356347280e-5
3		.878472325297	−.20704166961612e-1	.4187933734219e-4
4			−.4836560144296e-1	.32338329823195e-3
5			.6305676273171	−.7264209058150e-3
6			−1.23114954368	−.97413031149e-2
7				.6254058547e-1
8				.533803934e-1
9				−1.125788
10				2.125417

r	$e_r(5)$	$e_r(6)$	$e_r(7)$	$e_r(8)$
0	.671251761107e-16	.1036004645427e-24	.886492719e-36	.337201e-49
1	.2024913313373e-13	.6113326104277e-22	.91146378e-33	.556982629e-46
2	.261100345555e-11	.16322243213252e-19	.437008961e-30	.43686422e-43
3	.187088892376e-9	.2605311255687e-17	.1297363095e-27	.216465856e-40
4	.8086250862418e-8	.2766415183453e-15	.2670392090e-25	.7604817313e-38
5	.2126496335545e-6	.2056437432502e-13	.404346681e-23	.201532781e-35
6	.319415704903e-5	.10957094998959e-11	.46631481394e-21	.418459324e-33
7	.21201987479e-4	.42061728711797e-10	.41831543311e-19	.698046515e-31
8	−.33900555230e-4	.11491097182922e-8	.29548572643e-17	.951665168e-29
9	−.775061385e-3	.21545094604323e-7	.1652770327e-15	.1073015400e-26
10	.333997849e-2	.25433712247032e-6	.73192383650e-14	.1008662234e-24
11	.22204682e-1	.1448397731463e-5	.25506469557e-12	.7945270901e-23
12	−.1538433	−.2179868777201e-5	.6901276286e-11	.5257922143e-21
13	−.19794e-1	−.54298634893e-4	.141485467e-9	.2924082555e-19
14	2.01541	.1698771341e-3	.210241720e-8	.1363867915e-17
15	−4.451	.22887524e-2	.20651382e-7	.5311448709e-16
16		−.1042e-1	.101650951e-6	.1714154659e-14
17		−.4339429e-1	−.16979129e-6	.453180963e-13
18		.343054	−.37367e-5	.9644403068e-12
19		−.1947171	.97069e-5	.160742335e-10
20		−3.16910	.18351e-3	.200188929e-9
21		7.31266	−.54878e-3	.16931900e-8
22			−.5621e-2	.7257434e-8
23			.284e-1	−.14329111e-7
24			.639e-1	−.25913136e-6
25			−.7	.6473933e-6
26			.86	.138673e-4
27			5.	−.2339e-4
28			−.1e2	−.48124e-3
29				.162e-2
30				.976e-2
31				−.83e-1
32				−.62e-1
33				2.
34				−2.
35				−9.
36				30.?

One sees the graphs fluctuating above and below 1. Interestingly, the graphs have a similar shape as k varies. This is explained by the fact that large values of $L(\frac{1}{2}, \chi_d)$ tend to skew the moments, and this gets amplified as k increases.

Figure 2 depicts the same but for $0 < d \leqslant X$.

The values $L(\frac{1}{2}, \chi_d)$ were computed using a smoothed form of the approximate functional equation which expresses the L-function in terms of the incomplete Gamma-function (see, for example, [**35**]).

TABLE 5.2.3. *Smoothed moment of* $L(\frac{1}{2}, \chi_d)$ *versus Conjecture 1.5.3, for fundamental discriminants* $-1000000 < d < 0$, *and* $k = 1, \ldots, 8$.

k	Reality (5.2.1)	Conjecture (5.2.2)	Ratio
1	1460861.8	1460891.	0.99998
2	17225813.8	17226897.5	0.999937
3	316065502.1	316107868.6	0.999866
4	7378585496.	7380357447.1	0.99976
5	198754711593.6	198809762196.4	0.999723
6	5876732216291.7	5877354317291.3	0.999894
7	185524225881950.	185451557119001.	1.000392
8	6149876164696600	6141908614344770	1.0013

TABLE 5.2.4. *Smoothed moment of* $L(\frac{1}{2}, \chi_d)$ *versus Conjecture 1.5.3, for fundamental discriminants* $0 < d < 1000000$, *and* $k = 1, \ldots, 8$.

k	Reality (5.2.1)	Conjecture (5.2.2)	Ratio
1	1144563.5	1144535.5	1.000024
2	9252479.6	9252229.9	1.000027
3	109917867.0	109917367.9	1.0000045
4	1622521963.4	1622508843.4	1.0000081
5	27321430060.	27320230686.	1.000043
6	501621762060.6	501542204848.7	1.000159
7	9787833470714.1	9783848274459.6	1.000407
8	199831160877919	199664775232854	1.000833

5.3. *Orthogonal: twists of a* GL(2) *L-function*

Let

$$L_{11}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^{1/2+s}} \tag{5.3.1}$$

be the L-function of conductor 11 of the elliptic curve

$$y^2 + y = x^3 - x^2. \tag{5.3.2}$$

The coefficients a_n are obtained from the cusp form of weight 2 and level 11 given by

$$\sum_{n=1}^{\infty} a_n q^n = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2. \tag{5.3.3}$$

Expanding the right side using Euler's pentagonal theorem provides an efficient means to compute the a_n.

The function $L_{11}(s)$ satisfies an even functional equation (that is, $\varepsilon = +1$),

$$\left(\frac{11^{1/2}}{2\pi}\right)^{s} \Gamma(s + \tfrac{1}{2}) L_{11}(s) = \left(\frac{11^{1/2}}{2\pi}\right)^{1-s} \Gamma(\tfrac{3}{2} - s) L_{11}(1 - s), \tag{5.3.4}$$

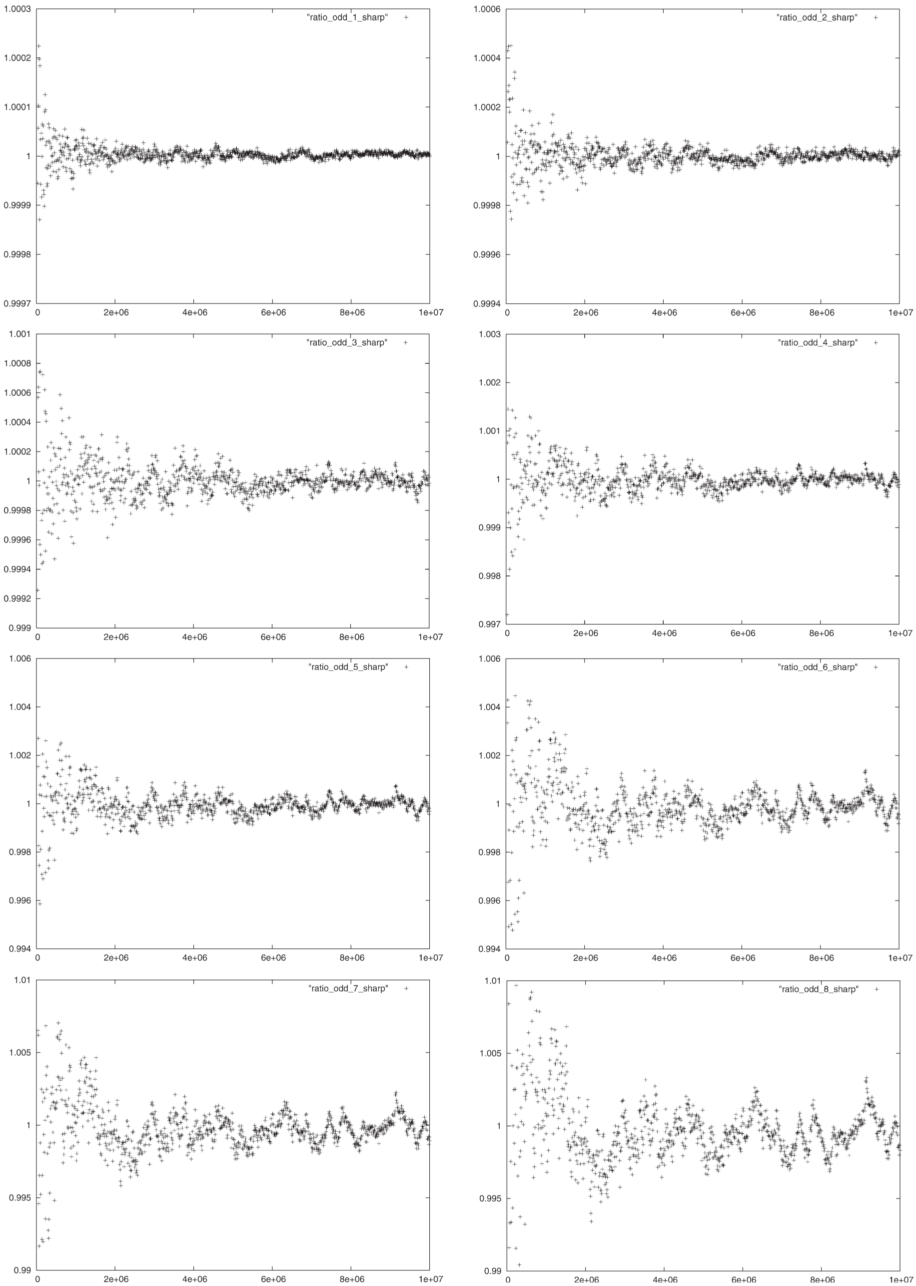

FIGURE 1. *Horizontal axis in each graph is X. These graphs depict the first eight moments, sharp cutoff, of $L(\frac{1}{2}, \chi_d)$, for $-X \leqslant d < 0$, divided by the conjectured value, sampled at $X = 10000, 20000, \ldots, 10^7$. One sees the graphs fluctuating above and below 1. Notice that the vertical scale varies from graph to graph.*

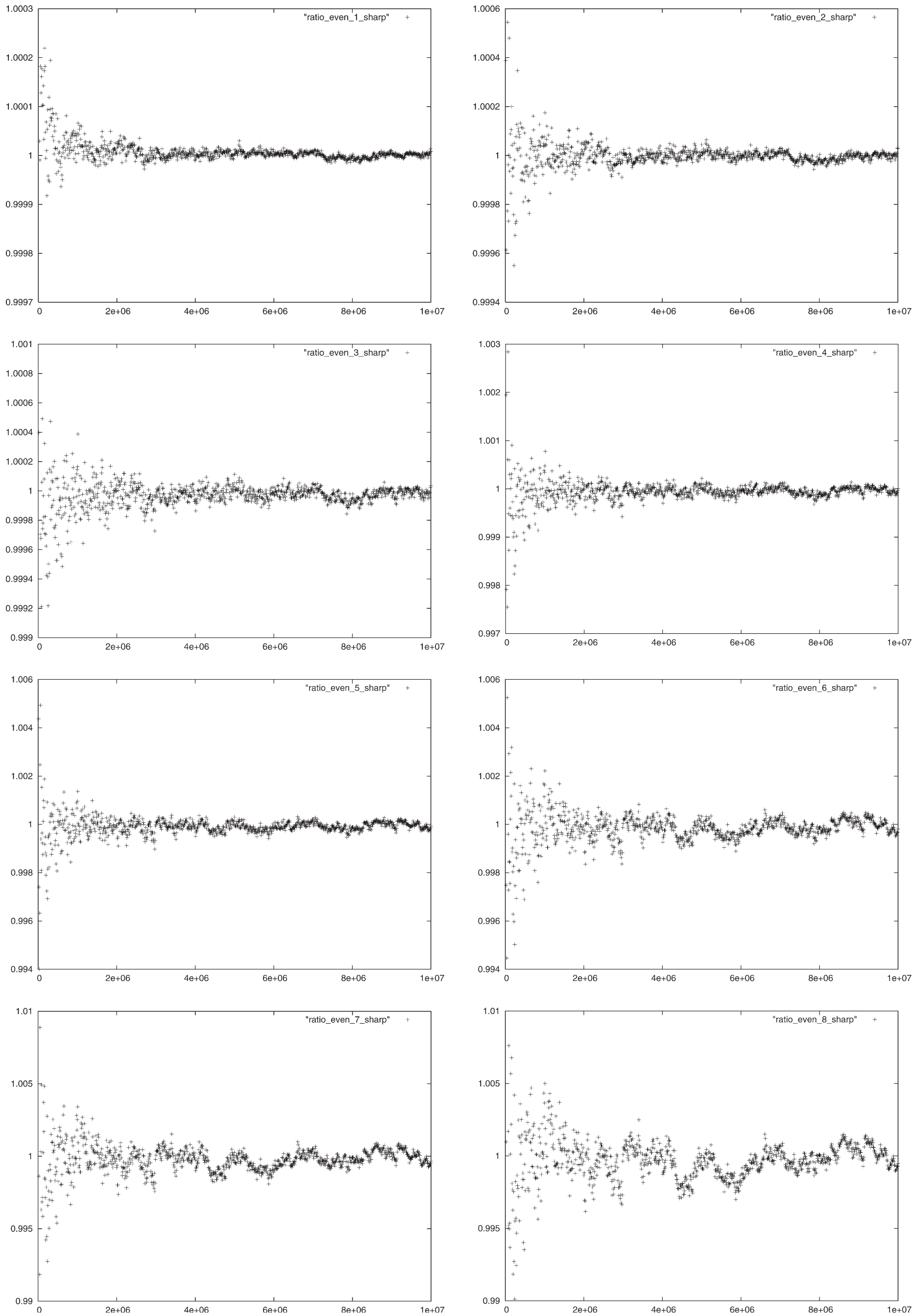

FIGURE 2. *Same as the previous figure, but for* $0 < d \leqslant X$.

and may be written as a product over primes

$$L_{11}(s) = \frac{1}{1 - 11^{-s-1/2}} \prod_{p \neq 11} \frac{1}{1 - a_p p^{-s-1/2} + p^{-2s}}. \tag{5.3.5}$$

Consider now quadratic twists of $L_{11}(s)$,

$$L_{11}(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a_n}{n^{1/2+s}} \chi_d(n). \tag{5.3.6}$$

with $(d, 11) = 1$. Here $L_{11}(s, \chi_d)$ satisfies the functional equation

$$L_{11}(s, \chi_d) = \chi_d(-11) \frac{\Gamma(\frac{3}{2} - s)}{\Gamma(s + \frac{1}{2})} \left(\frac{2\pi}{11^{1/2}} \right)^{2s-1} |d|^{2(1/2-s)} L_{11}(1 - s, \chi_d). \tag{5.3.7}$$

We wish to look at moments of $L_{11}(\frac{1}{2}, \chi_d)$ but only for those $L(s, \chi_d)$ that have an even functional equation, that is, $\chi_d(-11) = 1$. We further only look at $d < 0$ since in that case a theorem of Kohnen and Zagier [**33**] enables us to easily gather numerical data for $L_{11}(\frac{1}{2}, \chi_d)$ with which to check our conjecture.

When $d < 0$, $\chi_d(-1) = -1$; hence, in order to have an even functional equation, we require $\chi_d(11) = -1$, that is, $d = 2, 6, 7, 8, 10 \bmod 11$. Conjectured formula (4.4.22) combined with Lemma 2.5.2 gives an estimate for the sum over fundamental discriminants

$$\sideset{}{^*}\sum_{\substack{-D<d<0 \\ d=2,6,7,8,10 \bmod 11}} L_{11}(\tfrac{1}{2}, \chi_d)^k = \sideset{}{^*}\sum_{\substack{-D<d<0 \\ d=2,6,7,8,10 \bmod 11}} \Upsilon_k(\log |d|) + O(D^{1/2+\varepsilon}) \tag{5.3.8}$$

where, as in § 4.4, Υ_k is the polynomial of degree $\frac{1}{2}k(k-1)$ given by the k-fold residue

$$\begin{aligned} \Upsilon_k(x) = {} & \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \\ & \times \oint \cdots \oint \frac{R_{11}(z_1, \ldots, z_k) \Delta(z_1^2, \ldots, z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} e^{x \sum_{j=1}^k z_j} \, dz_1 \ldots dz_k, \end{aligned} \tag{5.3.9}$$

where

$$\begin{aligned} R_{11}(z_1, \ldots, z_k) = {} & A_k(z_1, \ldots, z_k) \prod_{j=1}^k \left(\frac{\Gamma(1 + z_j)}{\Gamma(1 - z_j)} \left(\frac{11}{4\pi^2} \right)^{z_j} \right)^{1/2} \\ & \times \prod_{1 \leqslant i < j \leqslant k} \zeta(1 + z_i + z_j), \end{aligned} \tag{5.3.10}$$

and A_k is the Euler product which is absolutely convergent for $\sum_{j=1}^k |z_j| < \frac{1}{2}$,

$$A_k(z_1, \ldots, z_k) = \prod_p R_{11,p}(z_1, \ldots, z_k) \prod_{1 \leqslant i < j \leqslant k} \left(1 - \frac{1}{p^{1+z_i+z_j}} \right) \tag{5.3.11}$$

with, for $p \neq 11$,

$$R_{11,p} = \left(1+\frac{1}{p}\right)^{-1}\left(\frac{1}{p}+\frac{1}{2}\left(\prod_{j=1}^{k}\frac{1}{1-a_p p^{-1-z_j}+p^{-1-2z_j}} + \prod_{j=1}^{k}\frac{1}{1+a_p p^{-1-z_j}+p^{-1-2z_j}}\right)\right) \tag{5.3.12}$$

and

$$R_{11,11} = \prod_{j=1}^{k}\frac{1}{1+11^{-1-z_j}}. \tag{5.3.13}$$

Numerically, it is more challenging to compute the polynomials Υ_k. First, using

$$\prod_{1\leqslant i<j\leqslant k}\zeta(1+z_i+z_j) \tag{5.3.14}$$

to estimate the sum over primes of (4.4.17) makes a poor approximation and one would do better to use the Rankin–Selberg convolution L-function of $L_{11}(s)$ with itself. However, it is simpler to work with ζ, and we thus computed the first four moment polynomials of $L_{11}(\frac{1}{2},\chi_d)$ but to low accuracy. The coefficients of these polynomials are given to 2 to 5 decimal place accuracy in Table 5.3.1.

TABLE 5.3.1. *Coefficients of* $\Upsilon_k(x) = f_0(k)x^{k(k-1)/2} + f_1(k)x^{k(k-1)/2-1} + \ldots$, *for* $k = 1,2,3,4$.

r	$f_r(1)$	$f_r(2)$	$f_r(3)$	$f_r(4)$
0	1.2353	.3834	.00804	.0000058
1		1.850	.209	.000444
2			1.57	.0132
3			2.85	.1919
4				1.381
5				4.41
6				4.3

In Table 5.3.2 we compare moments computed numerically with moments estimated by our conjecture. The two agree to within the accuracy we have for the moment polynomial coefficients. We believe that if one were to compute the coefficients to higher accuracy, one would see an even better agreement with the data.

While one can compute $L_{11}(\frac{1}{2},\chi_d)$ using standard techniques (see [**6**]), in our case we can exploit a theorem of Kohnen and Zagier [**33**] which relates $L_{11}(\frac{1}{2},\chi_d)$, for fundamental discriminants $d<0$, $d \equiv 2,6,7,8,10 \bmod 11$, to the coefficients $c_{11}(|d|)$ of a weight $\frac{3}{2}$ modular form

$$L_{11}(\tfrac{1}{2},\chi_d) = \kappa_{11}c_{11}(|d|)^2/\sqrt{d} \tag{5.3.15}$$

where κ_{11} is a constant. The weight $\frac{3}{2}$ form in question was determined by

TABLE 5.3.2. *Moments of* $L_{11}(\frac{1}{2}, \chi_d)$ *versus their conjectured values, for fundamental discriminants* $-85{,}000{,}000 < d < 0$, $d = 2, 6, 7, 8, 10 \bmod 11$, *and* $k = 1, \ldots, 4$. *The data agree with our conjectures to the accuracy to which we have computed the moment polynomials* Υ_k.

k	Left side (5.3.8)	Right side (5.3.8)	Ratio
1	14628043.5	14628305.	0.99998
2	100242348.8	100263216.	0.9998
3	1584067116.8	1587623419.	0.998
4	41674900434.9	41989559937.	0.993

Rodriguez-Villegas (private communication):

$$\sum_{n=1}^{\infty} c_{11}(n) q^n = \tfrac{1}{2}(\theta_1(q) - \theta_2(q))$$
$$= -q^3 + q^4 + q^{11} + q^{12} - q^{15} - 2q^{16} - q^{20} \ldots \tag{5.3.16}$$

where

$$\theta_1(q) = \sum_{\substack{(x,y,z)\in\mathbb{Z}^3 \\ x \equiv y \bmod 2}} q^{x^2+11y^2+11z^2}$$
$$= 1 + 2q^4 + 2q^{11} + 4q^{12} + 4q^{15} + 2q^{16} + 4q^{20} \ldots \tag{5.3.17}$$

and

$$\theta_2(q) = \sum_{\substack{(x,y,z)\in\mathbb{Z}^3 \\ x \equiv y \bmod 3 \\ y \equiv z \bmod 2}} q^{(x^2+11y^2+33z^2)/3} = 1 + 2q^3 + 2q^{12} + 6q^{15} + 6q^{16} + 6q^{20} \ldots . \tag{5.3.18}$$

This was used to compute the $c_{11}(|d|)$ for $d < 85{,}000{,}000$.

Evaluating the left side of (5.3.8) in a more traditional manner for $d = -3$, and comparing with the right side, we determined

$$\kappa_{11} = 2.917633233876991. \tag{5.3.19}$$

References

1. F. V. ATKINSON, 'The mean value of the zeta-function on the critical line', *Proc. London Math. Soc.* (2) 47 (1941) 174–200.

2. A. BOREL and W. CASSELMAN (eds), *Automorphic forms, representations, and L-functions*, Proceedings of Symposia in Pure Mathematics 33 (American Mathematical Society, Providence, RI, 1979).

3. E. L. BASOR and P. J. FORRESTER, 'Formulas for the evaluation of Toeplitz determinants with rational generating functions', *Math. Nachr.* 170 (1994) 5–18.

4. E. BRÉZIN and S. HIKAMI, 'Characteristic polynomials of random matrices', *Comm. Math. Phys.* 214 (2000) 111–135.

5. D. BUMP, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics 55 (Cambridge University Press, 1997).

6. H. COHEN, *A course in computational algebraic number theory* (Springer, Berlin, 1995).

7. J. B. Conrey, 'A note on the fourth power moment of the Riemann zeta-function', *Analytic number theory*, Allerton Park, IL, 1995 (ed. B. C. Berndt, H. G. Diamond and A. J. Hildebrand), Progress in Mathematics 138 (Birkhäuser, Boston, 1996).

8. J. B. Conrey and D. W. Farmer, 'Mean values of L-functions and symmetry', *Internat. Math. Res. Notices* 17 (2000) 883–908.

9. J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, 'Autocorrelation of random matrix polynomials', *Commun. Math. Phys* 237 (2003) 365–395.

10. J. B. Conrey and A. Ghosh, 'On the Selberg class of Dirichlet series: small degrees', *Duke Math. J.* 72 (1993) 673–693.

11. J. B. Conrey and A. Ghosh, 'A conjecture for the sixth power moment of the Riemann zeta-function', *Internat. Math. Res. Notices* 15 (1998) 775–780.

12. J. B. Conrey and S. M. Gonek, 'High moments of the Riemann zeta-function', *Duke Math. J.* 107 (2001) 577–604.

13. H. Davenport, *Multiplicative number theory*, Graduate Texts in Mathematics 74 (Springer, New York, 2000).

14. A. Diaconu, D. Goldfeld and J. Hoffstein, 'Multiple Dirichlet series and moments of zeta- and L-functions', *Compositio Math.* 139 (2003) 297–360.

15. W. Duke, 'The critical order of vanishing of automorphic L-functions with large level', *Invent. Math.* 119 (1995) 165–174.

16. W. Duke, J. B. Friedlander and H. Iwaniec, 'Bounds for automorphic L-functions. II', *Invent. Math.* 115 (1994) 219–239.

17. D. Goldfeld and C. Viola, 'Mean values of L-functions associated to elliptic, Fermat and other curves at the centre of the critical strip', *J. Number Theory* 11 (1979) S. Chowla Anniversary Issue 305–320.

18. J. L. Hafner and A. Ivić, 'On the mean-square of the Riemann zeta-function on the critical line', *J. Number Theory* 32 (1989) 151–191.

19. G. H. Hardy and J. E. Littlewood, 'Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes', *Acta Math* 41 (1918) 119–196.

20. D. R. Heath-Brown, 'The fourth power moment of the Riemann zeta function', *Proc. London Math. Soc.* (3) 38 (1979) 385–422.

21. A. E. Ingham, 'Mean-value theorems in the theory of the Riemann zeta-function', *Proc. London Math. Soc.* (2) 27 (1926) 273–300.

22. A. Ivić, 'On the fourth moment of the Riemann zeta function', *Publ. Inst. Math.* (*Beograd*) *N.S.* 57 (71) (1995) 101–110.

23. A. Ivić, 'Moments of Hecke series at central points', *Funct. Approx. Comment. Math.* 30 (2002) 49–82.

24. H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics 17 (American Mathematical Society, Providence, RI, 1997).

25. H. Iwaniec, *Spectral methods of automorphic forms*, 2nd edn, Graduate Studies in Mathematics 53 (American Mathematical Society, Providence, RI, 1997).

26. H. Iwaniec and E. Kowalksi, *Analytic number theory* (American Mathematical Society, Providence, RI, 2004).

27. H. Jacquet and J. A. Shalika, 'A nonvanishing theorem for zeta functions on GL_n', *Invent. Math.* 38 (1976) 1–16.

28. M. Jutila, 'On the mean value of $L(\frac{1}{2}, \chi)$ for real characters', *Analysis* 1 (1981) 149–161.

29. N. M. Katz and P. Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, American Mathematical Society Colloquium Publications 45 (AMS, Providence, RI, 1999).

30. J. P. Keating and N. C. Snaith, 'Random matrix theory and $\zeta(\frac{1}{2}+it)$', *Comm. Math. Phys.* 214 (2000) 57–89.

31. J. P. Keating and N. C. Snaith, 'Random matrix theory and L-functions at $s=\frac{1}{2}$', *Comm. Math. Phys.* 214 (2000) 91–110.

32. H. Kober, 'Eine Mittelwortformel der Riemannschen Zetafunktion', *Compositio Math.* 3 (1936) 174–189.

33. W. Kohnen and D. Zagier, 'Values of L-series of modular forms at the center of the critical strip', *Invent. Math.* 64 (1981) 175–198.

34. E. Kowalski, P. Michel and J. M. VanderKam, 'Mollification of the fourth moment of automorphic L-functions and arithmetic applications', *Invent. Math.* 142 (2000) 95–151.

35. A. F. Lavrik, 'Approximate functional equation for Dirichlet Functions', *Izv. Akad. Nauk SSSR* 32 (1968) 134–185.

36. N. Levinson and H. Montgomery, 'Zeros of the derivatives of the Riemann zeta-function', *Acta Math.* 133 (1974) 49–65.

37. M. Mehta, *Random matrix theory* (Academic Press, Boston, 1991).

38. H. L. Montgomery, 'The pair correlation of zeros of the Riemann zeta-function', *Analytic number theory* (ed. H. G. Diamond), Proceedings of Symposia in Pure Mathematics 24 (American Mathematical Society, Providence, RI, 1973) 181–193.

39. Y. Motohashi, *Spectral theory of the Riemann zeta-function*, Cambridge Tracts in Mathematics 127 (Cambridge University Press, 1997).

40. M. R. Murty, 'Selberg's conjectures and Artin L-functions', *Bull. Amer. Math. Soc.* (*N.S.*) 31 (1994) 1–14.

41. A. Odlyzko, 'The 10^{20}th zero of the Riemann zeta-function and 70 million of its neighbors', Preprint, www.dtc.umn.edu/∼odlyzko/unpublished/index.html, 1989.

42. A. Özlük and C. Snyder, 'Small zeros of quadratic L-functions', *Bull. Austral. Math. Soc.* 47 (1993) 307–319.

43. J. Riordan, *Combinatorial identities* (John Wiley, New York, 1968).

44. M. O. Rubinstein, 'Evidence for a spectral interpretation of the zeros of L-functions', Thesis, Princeton University, 1998.

45. Z. Rudnick and P. Sarnak, 'Zeros of principal L-functions and random matrix theory', *Duke Math. J.* 81 (1996) 269–322.

46. A. Selberg, 'Old and new conjectures and results about a class of Dirichlet series', *Collected papers*, Vol. 2 (Springer, Berlin, 1991) 47–63.

47. K. Soundararajan, 'Non-vanishing of quadratic Dirichlet L-functions at $s = \frac{1}{2}$', *Ann. of Math.* (2) 152 (2000) 447–488.

48. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd edn, edited and with a preface by D. R. Heath-Brown (Oxford University Press, New York, 1986).

49. Q. Zhang, 'On the cubic moment of quadratic Dirichlet L-functions', Preprint.

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