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Pseudomoments of the Riemann zeta-function and pseudomagic squares

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Abstract

We compute integral moments of partial sums of the Riemann zeta function on the critical line and obtain an expression for the leading coefficient as a product of the standard arithmetic factor and a geometric factor. The geometric factor is equal to the volume of the convex polytope of substochastic matrices and is equal to the leading coefficient in the expression for moments of truncated characteristic polynomial of a random unitary matrix.

1. Introduction

1.1. Moments of the Riemann zeta function

The Riemann zeta-function is defined for $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1}$$

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As is well-known [33], ζ has meromorphic continuation to the whole complex plane with a single simple pole at $s = 1$ with residue 1. Further, it satisfies a functional equation, relating the value of $\zeta(s)$ and the value of $\zeta(1-s)$,

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{2}$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s). \tag{3}$$

Following the standard notation we write $s = \sigma + it$.

The problem of computing the moments of ζ on the critical line $\sigma = \frac{1}{2}$ is fundamental, difficult and longstanding.

The second moment was obtained by Hardy and Littlewood [19] in 1918:

$$\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim \log T, \tag{4}$$

the fourth moment was obtained by Ingham [22] in 1926:

$$\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \sim \frac{1}{2\pi^2} \log^4 T. \tag{5}$$

The asymptotics of higher moments is not known. It has long been conjectured that

$$\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \sim c_k \log^{k^2} T. \tag{6}$$

In 1984 Conrey and Ghosh [7] gave the moment conjecture a more precise form; namely, they conjectured that there should be a factorization

$$c_k = \frac{g_k a_k}{\Gamma(1+k^2)}, \tag{7}$$

where a_k is an arithmetic factor given by

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{j=0}^{\infty} \frac{d_k(p^j)^2}{p^j}, \tag{8}$$

and g_k is a geometric factor, which should be an integer. Using Dirichlet polynomial techniques Conrey and Ghosh [8] conjectured that $g_3 = 42$ and Conrey and Gonek [9] conjectured that $g_4 = 24,024$.

1.2. The Riemann zeta-function and characteristic polynomials of random matrices

In the past few years, following the work of Keating and Snaith [23], Conrey and Farmer [4], Hughes et al. [20,21], and Conrey, Farmer, Keating, Rubinstein, and Snaith [6] it has become clear that the leading order asymptotic of the moments of the Riemann zeta function can be conjecturally understood in terms of corresponding quantities of the characteristic polynomial of the random unitary matrices. Let M be a matrix in $U(N)$ chosen uniformly with respect to Haar measure, denote by $e^{i\theta_1}, \dots, e^{i\theta_N}$ its eigenvalues, and consider the characteristic polynomial of M:

$$P_M(z) = \det(M - zI) = \prod_{j=1}^{N} (e^{i\theta_j} - z). \tag{9}$$

Keating and Snaith (see also [1]) computed the moments of P_M with respect to Haar measure on $U(N)$ and found that

$$M_N(s) = E_{U(N)} |P_M(z)|^{2s} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2s)}{\Gamma(j+s)^2}. \tag{10}$$

They also showed that

$$\lim_{N\to\infty} \frac{M_N(s)}{N^{s^2}} = \frac{G(1+s)^2}{G(1+2s)}, \tag{11}$$

where $G(s)$ is Barnes double Gamma function satisfying $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$. For $s = k$ an integer

$$\frac{G(1+k)^2}{G(1+2k)} = \prod_{j=0}^{k-1} \frac{j!}{(j+n)!}. \tag{12}$$

For $k = 1, 2, 3$ the quantity above is in agreement with the value of g_k in the theorems of Hardy and Littlewood, and Ingham and the conjecture of Conrey and Ghosh.

The conjecture of Keating and Snaith [23] (considerably refined and extended in [6]) is as follows:

Conjecture 1 (*Keating and Snaith [23]*).

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \sim a_k g_k \log^{k^2} T, \tag{13}$$

where a_k is an arithmetic factor given by (8) and g_k is a "geometric" factor (here the notation is different from (7)) given by

$$g_k = \lim_{N\to\infty} \frac{E_{U(N)}|P_M(z)|^{2k}}{N^{k^2}} = \prod_{j=0}^{k-1} \frac{j!}{(j+n)!}. \tag{14}$$

1.3. Characteristic polynomials of unitary matrices and magic squares

The moments of the secular coefficients of the random unitary matrices have also been recently investigated. If M is a random unitary matrix, following the notation preceding Eq. (9) we write

$$P_M(z) = \det(M - zI) = \prod_{j=1}^{N} (e^{i\theta_j} - z) = (-1)^N \sum_{j=0}^{N} \mathrm{Sc}_j(M) z^{N-j}(-1)^j, \tag{15}$$

where $\mathrm{Sc}_j(M)$ is the jth *secular coefficient* of the characteristic polynomial. Note that

$$\mathrm{Sc}_1(M) = \mathrm{Tr}(M) \tag{16}$$

and

$$\mathrm{Sc}_N(M) = \det(M). \tag{17}$$

Moments of the higher secular coefficients were studied by Haake and collaborators [17,18] who obtained:

$$\mathbb{E}_{U(N)}\mathrm{Sc}_j(M) = 0, \quad \mathbb{E}_{U(N)}|\mathrm{Sc}_j(M)|^2 = 1; \tag{18}$$

and posed the question of computing the higher moments. The answer is given by Theorem 1, which we state below after pausing to give the following definition.

Definition 1. If A is an $m \times n$ matrix with nonnegative integer entries and with row and column sums

$$r_i = \sum_{j=1}^{n} a_{ij},$$

$$c_j = \sum_{i=1}^{m} a_{ij};$$

then the row-sum vector $\mathrm{row}(A)$ and column-sum vector $\mathrm{col}(A)$ are defined by

$$\mathrm{row}(A) = (r_1, \dots, r_m),$$

$$\mathrm{col}(A) = (c_1, \dots, c_n).$$

Given two partitions $\mu = (\mu_1, \dots, \mu_m)$ and $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_n)$ (see [26] for the partition notation) we denote by $N_{\mu\tilde{\mu}}$ the number of nonnegative integer matrices A with $\mathrm{row}(A) = \mu$ and $\mathrm{col}(A) = \tilde{\mu}$.

For example, for $\mu = (2, 1, 1)$ and $\tilde{\mu} = (3, 1)$ we have $N_{\mu\tilde{\mu}} = 3$; and the matrices in question are

$$\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

For $\mu = (2, 2, 1)$ and $\tilde{\mu} = (3, 1, 1)$ we have $N_{\mu\tilde{\mu}} = 8$; and the matrices in question are

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We are ready to state the following theorem, proved in [10].

Theorem 1 (*Diaconis and Gamburd [10]*). (a) *Consider* $\mathbf{a} = (a_1, \dots, a_l)$ *and* $\mathbf{b} = (b_1, \dots, b_l)$ *with* a_j, b_j *nonnegative natural numbers. Then for* $N \geqslant \max\left(\sum_1^l j a_j, \sum_1^l j b_j\right)$ *we have*

$$\mathbb{E}_{U_N} \prod_{j=1}^{l} (\mathrm{Sc}_j(M))^{a_j} \overline{(\mathrm{Sc}_j(M))}^{b_j} = N_{\mu\tilde{\mu}}. \tag{19}$$

Here μ *and* $\tilde{\mu}$ *are partitions* $\mu = \langle 1^{a_1} \dots l^{a_l} \rangle$, $\tilde{\mu} = \langle 1^{b_1} \dots l^{b_l} \rangle$ *and* $N_{\mu\tilde{\mu}}$ *is the number of nonnegative integer matrices* A *with* $\mathrm{row}(A) = \mu$ *and* $\mathrm{col}(A) = \tilde{\mu}$.

(b) *In particular, for* $N \geqslant jk$ *we have*[4]

$$E_{U(N)} |\mathrm{Sc}_j(M)|^{2k} = H_k(j), \tag{20}$$

[4] We remark that in [15] the answer is also obtained in the case $N < jk$: it is related to enumeration of magic squares with certain additional constraints.

where $H_k(j)$ is the number of $k \times k$ nonnegative integer matrices with each row and column summing up to j – "magic squares".

1.4. Magic squares

The reader is likely to have encountered objects, which following Ehrhart [14] are referred to as "historical magic squares". These are square matrices of order k, whose entries are nonnegative integers $(1, \ldots, k^2)$ and whose rows and columns sum up to the same number. The oldest such object,

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix} \tag{21}$$

first appeared in ancient Chinese literature under the name *Lo Shu* in the third millennium BC and repeatedly reappeared in the cabbalistic and occult literature in the middle ages. Not knowing ancient Chinese, Latin, or Hebrew it is difficult to understand what is "magic" about *Lo Shu*; it is quite easy to understand however why it keeps reappearing: there is (modulo reflections) only one historic magic square of order 3.

Following MacMahon [27] and Stanley [29], what is referred to as magic squares in modern combinatorics are square matrices of order k, whose entries are nonnegative integers and whose rows and columns sum up to the same number j. The number of magic squares of order k with row and column sum j, denoted by $H_k(j)$, is of great interest; see [11] and references therein. The first few values are easily obtained:

$$H_k(1) = k!, \tag{22}$$

corresponding to all k by k permutation matrices (this is the kth moment of the traces of powers leading in the work of Diaconis and Shahshahani [12] to the result on the asymptotic normality);

$$H_1(j) = 1, \tag{23}$$

corresponding to 1×1 matrix $[j]$ (this is the result of Haake and collaborators given in Eq. (18)). We also easily obtain $H_2(j) = j + 1$, corresponding to $\left[\begin{smallmatrix} i & j-i \\ j-i & i \end{smallmatrix}\right]$, but the value of $H_3(j)$ is considerably more involved:

$$H_3(j) = \binom{j+2}{4} + \binom{j+3}{4} + \binom{j+4}{4}. \tag{24}$$

This expression was first obtained by Mac Mahon in 1915 and a simple proof was found only a few years ago by M. Bona. The main results on $H_k(j)$ are given by the following theorems, proved by Stanley and Ehrhart (see [13,14,29–31]):

Theorem (*Stanley*). *$H_k(j)$ is a polynomial in j of degree $(k-1)^2$, having "trivial zeroes" at the negative integers,*

$$H_k(-1) = H_k(-2) = \cdots = H_k(-k+1) = 0, \tag{25}$$

and satisfying the following "functional equation":

$$H_k(-k-j) = (-1)^{k-1} H_k(j). \tag{26}$$

It can be shown that the statements above are equivalent to

$$\sum_{j \geqslant 0} H_k(j) x^j = \frac{h_0 + h_1 x + \cdots + h_d x^d}{(1-x)^{(k-1)^2+1}}, \quad d = k^2 - 3k + 2, \tag{27}$$

with $h_0 + h_1 + \cdots + h_d \neq 0$ and $h_i = h_{d-i}$.

For example,

$$H_3(j) = \tfrac{1}{8} j^4 + \tfrac{3}{4} j^3 + \tfrac{15}{8} j^2 + \tfrac{9}{4} j + 1$$

and

$$\sum_{j \geqslant 0} H_3(j) x^j = \frac{1 + x + x^2}{(1-x)^5},$$

$$\sum_{j \geqslant 0} H_4(j) x^j = \frac{1 + 14x + 87x^2 + 148x^3 + 87x^4 + 14x^5 + x^6}{(1-x)^{10}}.$$

Theorem (*Ehrhart*). *The leading coefficient of $H_k(j)$ is the relative volume of $\mathcal{B}_k$—the kth Birkhoff polytope, i.e. leading coefficient is equal to $\frac{\mathrm{vol}(\mathcal{B}_k)}{k^{k-1}}$.*

By definition, the kth Birkhoff polytope is the convex hull of permutation matrices:

$$\mathcal{B}_k = \left\{ (x_{ij}) \in \mathbb{R}^{k^2} \,\middle|\, x_{ij} \geqslant 0; \quad \sum_{i=1}^{k} x_{ij} = 1; \quad \sum_{j=1}^{k} x_{ij} = 1 \right\}. \tag{28}$$

In the example above, $\mathrm{vol}(\mathcal{B}_3) = \frac{1}{8} \times 9$.

1.5. Pseudomoments of the Riemann zeta-function and pseudomagic squares

The purpose of this paper is to prove the following result:

Theorem 2. *Let a_k be the arithmetic factor given by Eq.* (8). *Then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left|\sum_{n=1}^{X}\frac{1}{n^{\frac{1}{2}+it}}\right|^{2k}dt = a_k\gamma_k(\log X)^{k^2}+O\left((\log X)^{k^2-1}\right). \tag{29}$$

Here γ_k is the geometric factor, $\gamma_k=\mathrm{vol}(\mathcal{P}_k)$, where $\mathcal{P}_k$ is the convex polytope in $\mathbb{R}^{k^2}$ defined by the following inequalities:

$$\mathcal{P}_k=\left\{(x_{ij})\in\mathbb{R}^{k^2}\ \middle|\ x_{ij}\geqslant 0;\quad \sum_{i=1}^{k}x_{ij}\leqslant 1;\quad \sum_{j=1}^{k}x_{ij}\leqslant 1\right\}. \tag{30}$$

The connection with the characteristic polynomials of unitary matrices is as follows. From Theorem 1 it follows that if we consider *truncated* characteristic polynomial

$$P_{M,l}(z)=\sum_{j=0}^{l}\mathrm{Sc}_j(M)z^{N-j}(-1)^j, \tag{31}$$

we have for $N\geqslant lk$

$$\mathbb{E}_{U(N)}|P_{M,l}(z)|^{2k}=G_k(l), \tag{32}$$

where $G_k(l)$ denotes the number of $k\times k$ nonnegative integer matrices with row and column sums less than or equal to l (referred to as "pseudomagic squares" by Ehrhart [14]):

$$G_k(l)=\mathrm{card}\left\{(x_{ij})\in\mathbb{Z}^{k^2}\ \middle|\ x_{ij}\geqslant 0;\quad \sum_{i=1}^{k}x_{ij}\leqslant l;\quad \sum_{j=1}^{k}x_{ij}\leqslant l\right\}. \tag{33}$$

Ehrhart [14] proved that $G_k(l)$ is a polynomial in l of degree k^2 with leading coefficient given by $\gamma_k=\mathrm{vol}(\mathcal{P}_k)$; in fact $G_k(l)=\mathrm{card}\left(l\mathcal{P}_k\cap\mathbb{Z}^{k^2}\right)$. For example,

$$G_2(l)=\tfrac{1}{6}(l+1)(l+2)(l^2+3l+3)$$

and

$$\mathrm{vol}(\mathcal{P}_2) = \tfrac{1}{6}.$$

Hence we can rewrite the geometric factor γ_k in a manner similar to the expression for g_k in (14) as follows:

$$\gamma_k = \lim_{l \to \infty} \frac{E_{U(lk)} |P_{M,l}(z)|^{2k}}{l^{k^2}}. \tag{34}$$

The proof proceeds as follows. In Section 2 we obtain an expression for γ_k in terms of a multiple complex integral. In Section 3 we express the left-hand side of (29) as a multiple complex integral and then show that the leading terms in the two resulting expressions are equal.

2. Pseudomagic squares

Let $G_k(l)$ denote the number of $k \times k$ nonnegative integer matrices with row and column sums less than or equal to l given by (33) (we remark that $H_{k+1}(l)$, the number of magic squares, is obtained by imposing an additional diophantine inequality $\sum_{i,j} x_{ij} \geqslant (k-1)l$).

We have the following expression for $G_k(l)$ as a multiple complex integral:

Proposition 1. *Notation being as above we have*

$$G_k(l) = \frac{1}{(2\pi i)^{2k}} \int \cdots \int_{\substack{|w_i| = \varepsilon_i \\ |z_j| = \varepsilon_j}} \frac{(w_1 \ldots w_k z_1 \ldots z_k)^{-l-1} \prod_{i=1}^k dw_i \prod_{j=1}^k dz_j}{\prod_{i,j} (1 - w_i z_j) \prod_{i=1}^k (1 - w_i) \prod_{j=1}^k (1 - z_j)}. \tag{35}$$

The proof follows the approach in [2], which we now review.

Let $\mathbb{Z}^n$ denote an n-dimensional integer lattice in $\mathbb{R}^n$ and let $\mathcal{P}$ be a convex polytope in $\mathbb{R}^n$ whose vertices are on the lattice $\mathbb{Z}^n$ ($\mathcal{P}_k$ is a convex lattice polytope in $\mathbb{Z}^{k^2}$).

Any convex lattice polytope situated in the nonnegative orthant can be described as an intersection of finitely many half-spaces:

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^n_{\geqslant 0} \,|\, \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \right\}, \tag{36}$$

where $\mathbf{A}$ is an $m \times n$ integer matrix and $\mathbf{b} \in \mathbb{Z}^m$. Consider now the function of an integer-valued variable l describing the number of lattice points that lie inside the dilated polytope $l\mathcal{P}$:

$$L(\mathcal{P}, l) = \mathrm{card}\{l\mathcal{P} \cap \mathbb{Z}^n\}; \tag{37}$$

with this notation $G_k(l) = L(\mathcal{P}_k, l)$. Denote the columns of $\mathbf{A}$ by $\mathbf{c}_1, \ldots, \mathbf{c}_n$. Using multivariate generating functions it is proved in [2] that for the lattice polytope $\mathcal{P}$ given by (36) we have the following expression for $L(\mathcal{P}, l)$:

$$L(\mathcal{P}, l) = \frac{1}{(2\pi i)^m} \int \cdots \int_{|z_j|=\varepsilon_j} \frac{\prod_{j=1}^m z_j^{-lb_j-1}}{\prod_{l=1}^n (1-\mathbf{z}^{\mathbf{c}_l}) \prod_{j=1}^m (1-z_j)} \, d\mathbf{z}. \tag{38}$$

In the expression above we use the standard multivariate notation $\mathbf{x}^{\mathbf{y}} = x_1^{y_1} \ldots x_n^{y_n}$.

Now for $\mathcal{P}_k$ the defining system of diophantine inequalities is given in (30); the corresponding $\mathbf{A}$ is a $(2k \times k^2)$ matrix given by

$$\mathbf{A} = \begin{pmatrix} 1 \ \ldots \ 1 & & & \\ & 1 \ \ldots \ 1 & & \\ & & \ddots & \\ & & & 1 \ \ldots \ 1 \\ 1 \quad\quad & 1 \quad\quad & & 1 \quad\quad \\ \ddots \quad & \ddots \quad & \ldots & \ddots \quad \\ \quad\quad 1 & \quad\quad 1 & & \quad\quad 1 \end{pmatrix}, \tag{39}$$

and $\mathbf{b} = (1, \ldots, 1) \in \mathbb{Z}^{2k}$. Proposition 1 now follows from (38); for notational convenience we have split the variables into two groups $w_1, \ldots, w_k$ and $z_1, \ldots, z_k$.

3. Proof of the Theorem

By the mean-value theorem for Dirichlet polynomials due to Montgomery and Vaughan [28] we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \sum_{n=1}^X \frac{1}{n^{\frac{1}{2}+it}} \right|^{2k} dt = \sum_{n=1}^X \frac{d_{k,X}^2(n)}{n}, \tag{40}$$

where $d_{k,X}(n)$ is defined by

$$d_{k,X}(n) = \sum_{\substack{l_1 \ldots l_k = n \\ l_1, \ldots, l_k \leqslant X}} 1.$$

Consequently, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \sum_{n=1}^X \frac{1}{n^{\frac{1}{2}+it}} \right|^{2k} dt = \sum_{\substack{1 \leqslant l_i \leqslant X \\ 1 \leqslant m_j \leqslant X \\ l_1 \ldots l_k = m_1 \ldots m_k}} \frac{1}{\sqrt{l_1 \ldots l_k m_1 \ldots m_k}}. \tag{41}$$

Now we use the discontinuous integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{X^s}{s}\,ds=\begin{cases}0 & \text{if } 0<X<1,\\ 1 & \text{if } X>1,\end{cases} \tag{42}$$

where $c>0$ to pick the terms of the Dirichlet series which are less than X. Denoting the integral in Eq. (42) by $\int_{(c)}$ we can now express the right-hand side of (41) as follows:

$$\begin{aligned}&\frac{1}{(2\pi i)^{2k}}\int_{(2)}\cdots\int_{(2)}\prod_{i=1}^{k}\frac{X^{u_i}}{u_i}\prod_{j=1}^{k}\frac{X^{v_j}}{v_j}F(u_1,\dots,u_k,v_1,\dots,v_k)\\ &\quad\times du_1\dots du_k\,dv_1\dots dv_k,\end{aligned} \tag{43}$$

where

$$F(u_1,\dots,u_k,v_1,\dots,v_k)=\sum_{\substack{l_i\geqslant 1\\ m_j\geqslant 1\\ l_1\dots l_k=m_1\dots m_k}}\frac{1}{l_1^{\frac12+u_1}\dots l_k^{\frac12+u_k}m_1^{\frac12+v_1}\dots m_k^{\frac12+v_k}}. \tag{44}$$

To simplify notation let $\mathbf{u}=(u_1,\dots,u_k)$, $\mathbf{v}=(v_1,\dots,v_k)$, $d\mathbf{u}=du_1\dots du_k$ and $d\mathbf{v}=dv_1\dots dv_k$.

Now since for a multiplicative function $g(n)$ we have the Euler product identity:

$$\sum_{n=1}^{\infty}g(n)=\prod_p\left(1+g(p)+g(p^2)+g(p^3)+\cdots\right), \tag{45}$$

it follows that

$$\begin{aligned}F(u_1,\dots,u_k,v_1,\dots,v_k)&=\prod_p\left(\sum_{n=1}^{\infty}\sum_{\substack{\alpha_1+\cdots+\alpha_k=n\\ \beta_1+\cdots+\beta_k=n\\ \alpha_i\geqslant 0,\,\beta_j\geqslant 0}}\frac{1}{p^{\alpha_1(\frac12+u_1)+\cdots+\beta_k(\frac12+v_k)}}\right)\\ &=\prod_p\left(1+\sum_{i,j}\frac{1}{p^{1+u_i+v_j}}+\cdots\right)\\ &=G(\mathbf{u},\mathbf{v})\prod_{i,j}\zeta(1+u_i+v_j),\end{aligned} \tag{46}$$

where G is an Euler product which is absolutely convergent for $|u_i|<\frac14$, $|v_j|<\frac14$.

Since

$$\sum_{\substack{\alpha_1+\cdots+\alpha_k=n\\ \beta_1+\cdots+\beta_k=n\\ \alpha_i\geqslant 0,\,\beta_j\geqslant 0}} 1 = d_k^2(p^n),$$

if we let all u_i and v_j be equal to δ we obtain

$$G(\delta,\dots\delta) = \prod_p \left(1-\frac{1}{p^{2\delta+1}}\right)^{k^2} \sum_{n=0}^{\infty} d_k^2(p^n)p^{-2n\delta-n}, \tag{47}$$

and consequently

$$\lim_{\mathbf{u},\mathbf{v}\to 0} G(\mathbf{u},\mathbf{v}) = \prod_p \left(1-\frac{1}{p}\right)^{k^2} \sum_{n=0}^{\infty} \frac{d_k(p^n)^2}{p^n} = a_k. \tag{48}$$

Summarizing, we have obtained the following expression for the left-hand side of (29):

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|\sum_{n=1}^{X} \frac{1}{n^{\frac{1}{2}+it}}\right|^{2k} dt \\
&\quad = \frac{1}{(2\pi i)^{2k}} \int_c \cdots \int_c G(\mathbf{u},\mathbf{v}) \prod_{i,j} \zeta(1+u_i+v_j) \frac{X^{\sum(u_i+v_j)}}{\prod_{i,j} u_i v_j}\, d\mathbf{u}\, d\mathbf{v}.
\end{aligned} \tag{49}$$

Now using the fact that $(s-1)\zeta(s)$ is analytic in the entire complex plane together with the standard techniques and bounds pertaining to ζ, we obtain that the leading term in (49) is given by

$$\frac{a_k}{(2\pi i)^{2k}} \int_c \cdots \int_c \frac{X^{\sum(u_i+v_j)}}{\prod_{i,j}(1-e^{-u_i-v_j})\prod_{i,j} u_i v_j}\, d\mathbf{u}\, d\mathbf{v}, \tag{50}$$

where we have used (48).

Write

$$\frac{1}{\prod_{i,j}(1-e^{-u_i-v_j})} = \prod_{i,j}\left[\sum_{a_{ij}\geqslant 0} (e^{-u_i-v_j})^{a_{ij}}\right]. \tag{51}$$

A term $e^{-\mathbf{u}\boldsymbol{\alpha}}e^{-\mathbf{v}\boldsymbol{\beta}}$ in this expansion is obtained by choosing an $\mathbb{N}$-matrix $A^t = (a_{ij})^t$ of finite support with $\mathrm{row}(A) = \boldsymbol{\alpha}$ and $\mathrm{col}(A) = \boldsymbol{\beta}$. Hence the coefficient of $e^{-\mathbf{u}\boldsymbol{\alpha}}e^{-\mathbf{v}\boldsymbol{\beta}}$ in (51) is the number $N_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ of $\mathbb{N}$-matrices A with $\mathrm{row}(A) = \boldsymbol{\alpha}$ and $\mathrm{col}(A) = \boldsymbol{\beta}$:

$$\frac{1}{\prod_{i,j}(1 - e^{-u_i - v_j})} = \sum_{\boldsymbol{\alpha}\boldsymbol{\beta}} N_{\boldsymbol{\alpha}\boldsymbol{\beta}} e^{-\mathbf{u}\boldsymbol{\alpha}} e^{-\mathbf{v}\boldsymbol{\beta}}. \tag{52}$$

Further, let $l = \log X$ and rewrite the integral (42) as follows:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ls}}{s}\, ds = \begin{cases} 0 & \text{if } l < 0, \\ 1 & \text{if } l > 0. \end{cases} \tag{53}$$

We now express the integral appearing in (50) using (52) and apply (53) to obtain

$$\begin{aligned} &\frac{1}{(2\pi i)^{2k}} \int_c \cdots \int_c \frac{X^{\sum(u_i+v_j)}}{\prod_{i,j}(1 - e^{-u_i - v_j}) \prod_{i,j} u_i v_j}\, d\mathbf{u}\, d\mathbf{v} \\ &= \frac{1}{(2\pi i)^{2k}} \int_c \cdots \int_c \prod_i \frac{e^{lu_i}\, du_i}{u_i} \prod_j \frac{e^{lv_j}\, dv_j}{v_j} \sum_{\boldsymbol{\alpha}\boldsymbol{\beta}} N_{\boldsymbol{\alpha}\boldsymbol{\beta}} e^{-\mathbf{u}\boldsymbol{\alpha}} e^{-\mathbf{v}\boldsymbol{\beta}} \\ &= \sum_{\substack{\boldsymbol{\alpha} \leqslant \mathbf{l} \\ \boldsymbol{\beta} \leqslant \mathbf{l}}} N_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \mathrm{card}\left\{ (x_{ij}) \in \mathbb{N}^{k^2} \;\middle|\; \sum_{i=1}^{k} x_{ij} \leqslant l;\quad \sum_{j=1}^{k} x_{ij} \leqslant l \right\} = G_k(l). \end{aligned} \tag{54}$$

We remark that this proves that the integrals given by (54) and (35) are equal; a direct proof using, for example, a change of variables has thus far eluded us. We also remark that the integral expression for $G_k(l)$ given by (35) has served only as a motivation for the proof presented above. We also note that sums related to the expression given by the right-hand side of (40) were considered in [16].

4. Generalizations

Note that in fact we have proved

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left| \sum_{n=1}^{X} \frac{1}{n^{\frac{1}{2}+it}} \right|^{2k} dt = a_k G_k(\log X) + O\left((\log X)^{k^2-1} \right). \tag{55}$$

The proof given in the previous section easily generalizes to yield the following result:

Theorem 3. *Let a_k be the arithmetic factor given by Eq.* (8). *Then up to the lower order terms we have*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left|\sum_{n=1}^{X_1}\frac{1}{n^{\frac{1}{2}+it}}\right|^2\cdots\left|\sum_{n=1}^{X_k}\frac{1}{n^{\frac{1}{2}+it}}\right|^2 dt\sim a_k G_k(\log X_1,\dots,\log X_k). \tag{56}$$

Here we assume that $X_i=Y^{m_i}$ with $m_i=O(1)$ and $Y\to\infty$;

$$G_k(l_1,\dots,l_k)=\mathrm{card}\left\{(x_{ij})\in\mathbb{N}^{k^2}\;\middle|\;\sum_{i=1}^k x_{ij}\leqslant l_j;\quad\sum_{j=1}^k x_{ij}\leqslant l_i\right\}. \tag{57}$$

Finally, we note that in [10] results analogous to Theorem 1 are proved for orthogonal and symplectic group; for example the result for symplectic group is as follows:

Theorem 4. (a) *Consider $\mathbf{a}=(a_1,\dots,a_l)$ with a_j nonnegative natural numbers. Let μ be a partition $\mu=\langle 1^{a_1}\dots l^{a_l}\rangle$. Then for $N\geqslant\sum_1^l ja_j$ and $|\mu|$ even we have*

$$\mathbb{E}_{\mathrm{Sp}(2N)}\prod_{j=1}^l(\mathrm{Sc}_j(M))^{a_j}=\mathrm{NSP}_\mu. \tag{58}$$

Here NSP_μ *is the number of nonnegative symmetric integer matrices A with* $\mathrm{row}(A)=\mathrm{col}(A)=\mu$ *and with all diagonal entries of A even.*

(b) *In particular, for $N\geqslant jk$ and jk even we have*

$$E_{\mathrm{Sp}(2N)}\mathrm{Sc}_j(M)^k=S_k^{\mathrm{sp}}(j), \tag{59}$$

where $S_k^{\mathrm{sp}}(j)$ is the number of $k\times k$ symmetric nonnegative integer matrices with each row and column summing up to j and all diagonal entries even (*equivalently*, *the number of j-regular graphs on k vertices with loops and multiple edges*).

We will present analogues of Theorem 2 for L-functions with orthogonal and symplectic symmetries in a forthcoming paper. Here we state a representative result for $L(s,\chi_d)$ with $\chi_d(n)=(\frac{d}{n})$ where d is a fundamental discriminant, which has symplectic symmetry.

Theorem 5. *Let b_k be the arithmetic factor given by*

$$b_k=\prod_p\frac{\left(1-\frac{1}{p}\right)^{\frac{k(k+1)}{2}}}{1+\frac{1}{p}}\left(\frac{\left(1-\frac{1}{\sqrt{p}}\right)^{-k}+\left(1+\frac{1}{\sqrt{p}}\right)^{-k}}{2}+\frac{1}{p}\right).$$

Then

$$\lim_{T\to\infty}\frac{1}{T^*}\sum_{d<T}\left(\sum_{n<X}\frac{\chi_d(n)}{\sqrt{n}}\right)^k=\frac{6}{\pi^2}\,b_kF_k(\log X)+O(\log X^{k^2+k-2/2}). \tag{60}$$

Here $F_k(l)$ is the polynomial in l of degree $k(k+1)/2$ equal to the number of $k\times k$ symmetric nonnegative integer matrices with row and column sums less than or equal to l and all diagonal entries even.

The connection with the characteristic polynomials of symplectic matrices is as follows. From Theorem 4 it follows that if we consider *truncated* characteristic polynomial

$$P_{M,l}(z)=\sum_{j=0}^{l}\mathrm{Sc}_j(M)z^{N-j}(-1)^j,$$

we have for $N\geqslant lk$

$$\mathbb{E}_{\mathrm{Sp}(2N)}P_{M,l}(z)^k=F_k(l);$$

from results of Ehrhart [14] it follows that $F_k(l)$ is a polynomial in l of degree $k(k+1)/2$.

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