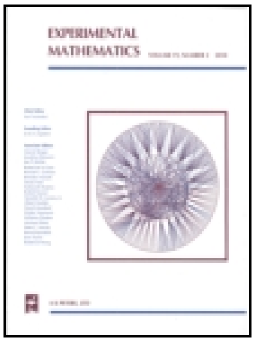

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CONTENTS

Conjectured links between the distribution of values taken by the characteristic polynomials of random orthogonal matrices and that for certain families of L-functions at the center of the critical strip are used to motivate a series of conjectures concerning the value distribution of the Fourier coefficients of half-integral-weight modular forms related to these L-functions. Our conjectures may be viewed as being analogous to the Sato–Tate conjecture for integral-weight modular forms. Numerical evidence is presented in support of them.

1. INTRODUCTION

The limiting value distribution of the Fourier coefficients of integral-weight modular forms is given by the celebrated Sato–Tate conjecture. Our purpose here is to identify the implications of some recent conjectures concerning the value distribution of certain families of L-functions at the center of the critical strip for the distribution of the Fourier coefficients of related half-integral-weight modular forms.

These conjectures concern the relationship between properties of L-functions and random matrix theory. It was conjectured by Montgomery [Montgomery 73] that correlations between the zeros of the Riemann zeta function on the scale of the mean zero separation coincide with those between the phases of the eigenvalues of unitary matrices, chosen at random, uniformly with respect to Haar measure on the unitary group, in the limit of large matrix size. This is supported both by theoretical [Montgomery 73, Rudnick and Sarnak 96, Bogomolny and Keating 95, Bogomolny and Keating 96] and extensive numerical [Odlyzko 89, Rubinstein 98] evidence. Katz and Sarnak [Katz and Sarnak 99] then generalized the connection by suggesting that statistical properties

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of the zeros within various families of L-functions, computed by averaging over a given family, coincide with those of the eigenvalues of random matrices from the various classical compact groups, the particular group being determined by the family in question. Based on these ideas, a link was proposed in [Keating and Snaith 00a] between the leading-order asymptotics of the value distribution of the Riemann zeta function on its critical line and that of the characteristic polynomials of random unitary matrices, giving, for example, an explicit conjecture for the leading-order asymptotics of the moments of the zeta function. This approach was then extended to relate the value distribution of L-functions, in families, at the center of the critical strip, to that of the characteristic polynomials of matrices from the various classical compact groups [Conrey and Farmer 00, Keating and Snaith 00b]. It has also recently been extended to include all lower-order terms in the asymptotics [Conrey et al. 05]. (For other related results, see [Hughes et al. 00, Hughes et al. 01].) These developments have recently been reviewed in [Conrey 01, Keating and Snaith 03].

Our strategy here is to combine these random-matrix-theory-inspired conjectures for the value distribution of L-functions with formulas due to Waldspurger [Waldspurger 1981], Shimura [Shimura 73], Kohnen and Zagier [Kohnen and Zagier 81], and Baruch and Mao [Baruch and Mao 03] that relate the values taken by L-functions associated with elliptic curves at the center of the critical strip to the Fourier coefficients of certain half-integral-weight modular forms. This approach was first outlined in [Conrey et al. 02], where its implications for the vanishing of L-functions were examined. Here we use the same ideas in order to develop conjectures for the value distribution of the Fourier coefficients.

The outline of this paper is as follows. In Section 2 we give a brief overview of the relationship between modular forms and L-functions associated with elliptic curves. In Section 3 we review the results from random matrix theory that we shall need, and some conjectures they suggest for the value distribution of L-functions associated with elliptic curves. In Section 4, we combine results from Section 2 and Section 3 to motivate conjectures for the moments of the Fourier coefficients, the value distribution of the logarithm of the Fourier coefficients, and the distribution of the coefficients themselves. In Section 5 we review the implications, first outlined in [Conrey et al. 02], for the vanishing of L-functions at the center of the critical strip. Finally, in Section 6 we describe numerical experiments whose results support our conjectures.

2. MODULAR FORMS AND L-FUNCTIONS

We review in this section the connection between the quadratic twists of modular L-functions and the Fourier coefficients of half-integral-weight modular forms. This is central to the motivation underlying the conjectures we make in subsequent sections. In order to be concrete, we concentrate on the case of quadratic twists of L-functions associated with elliptic curves.

Let $L_E(s)$ be the L-function associated with an elliptic curve E over $\mathbb{Q}$ with Dirichlet series and Euler product given by

$$\begin{aligned} L_E(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad (2\text{–}1) \\ &= \prod_{p|\Delta} \left(1 - a_p p^{-s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1} \\ &= \prod_p \mathcal{L}_p(1/p^s), \qquad \mathrm{Re}(s) > \frac{3}{2}, \end{aligned}$$

with Δ the discriminant of E, and $a_p = p + 1 - \#E(\mathbb{F}_p)$, where $\#E(\mathbb{F}_p)$ denotes the number of points on E regarded over $\mathbb{F}_p$. It is a consequence of the recently solved Taniyama–Shimura Conjecture [Wiles 95, Taylor and Wiles 95, Breuil et al. 01] that $L_E(s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation of the form

$$\left(\frac{2\pi}{\sqrt{Q}}\right)^{-s} \Gamma(s) L_E(s) = w_E \left(\frac{2\pi}{\sqrt{Q}}\right)^{s-2} \Gamma(2-s) L_E(2-s), \qquad (2\text{–}2)$$

where Q is the conductor of the elliptic curve E and $w_E = \pm 1$.

We let $\chi_d(n) = \left(\frac{d}{n}\right)$ for a fundamental discriminant d, where $\left(\frac{d}{n}\right)$ is the Kronecker symbol. The twisted L-function

$$L_E(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n^s} \qquad (2\text{–}3)$$

is the L-function of the elliptic curve E_d, the quadratic twist of E by d. If $(d, Q) = 1$, then $L_E(s, \chi_d)$ satisfies the functional equation

$$\begin{aligned} &\left(\frac{2\pi}{\sqrt{Q}|d|}\right)^{-s} \Gamma(s) L_E(s, \chi_d) \qquad (2\text{–}4) \\ &\quad = \chi_d(-Q) w_E \left(\frac{2\pi}{\sqrt{Q}|d|}\right)^{s-2} \Gamma(2-s) L_E(2-s, \chi_d). \end{aligned}$$

We shall be interested in the case in which $w_E \chi_d(-Q) = 1$, since otherwise $L_E(1, \chi_d)$ is trivially equal to zero.

We now come to a key point. The L-functions above are related to half-integer-weight modular forms via formulas due to Waldspurger [Waldspurger 1981], Shimura

[Shimura 73], Kohnen and Zagier [Kohnen and Zagier 81], and Baruch and Mao [Baruch and Mao 03]. One must distinguish between positive and negative d, and one must also sort them according to various residue classes. This has been worked out explicitly for thousands of examples by Rodriguez-Villegas and Tornaria in the case that Q is square-free and they kindly supplied a database of such forms to the authors [Rodriquez-Villegas and Tornaria 04].

Specifically, for Q square-free, assume that $d < 0$ and that $\chi_d(p) = -a_p$ for every $p \mid Q$ ($a_p = \pm 1$ for $p \mid Q$ when Q is square-free). Notice that such d are restricted to $\prod_{\substack{p \mid Q \\ p \text{ odd}}} ((p-1)/2)$ residue classes mod Q or $4Q$, depending on whether Q is odd or even. Then, for such d,

$$L_E(1, \chi_d) = \kappa_E \frac{c_E(|d|)^2}{\sqrt{|d|}}, \tag{2–5}$$

where $c_E(|d|) \in \mathbb{Z}$ are the Fourier coefficients of a weight-$\frac{3}{2}$ modular form, and where κ_E depends on E.

For $d > 0$, the work of Baruch, Mao, Rodriguez-Villegas, and Tornaria [Baruch and Mao 03] [Mao et al. 06] gives the relevant weight-$\frac{3}{2}$ form for Q prime and d satisfying $\chi_d(Q) = a_Q$. One has the same relation as above, but with a different proportionality constant. Some examples are listed in Section 6. In either case, given a coefficient of the weight-$\frac{3}{2}$ form, the constant κ_E can be evaluated either by comparison with the Birch and Swinnerton-Dyer conjecture, or by numerically computing $L_E(1, \chi_d)$ as a series involving the exponential function.

Our strategy will be to write down conjectures for the value distribution of the Fourier coefficients $c(|d|)$ by coupling the connection (2–5) with conjectures motivated by random matrix theory for the value distribution of $L_E(1, \chi_d)$.

3. *L*-FUNCTIONS AND RANDOM MATRICES

It was conjectured by Montgomery [Montgomery 73] that the zeros of the Riemann zeta function are distributed statistically like the eigenvalues of random Hermitian (self-adjoint) matrices, or, equivalently, like the phases of the eigenvalues of random unitary matrices. This extends to the zeros of any given principal L-function [Rudnick and Sarnak 96]. It was conjectured by Katz and Sarnak [Katz and Sarnak 99] that the distribution of zeros defined by averaging over families of L-functions with the height up the critical line fixed coincides with the distribution of the phases of the eigenvalues of matrices from one of the classical compact groups, depending on the family in question.

These ideas motivated the Conjecture [Keating and Snaith 00a] that, asymptotically, the moments of the Riemann zeta function (or any other principal L-function) averaged high on its critical line coincide, up to a simple arithmetical factor, with the moments of the characteristic polynomials of random unitary matrices. This suggestion was then extended to relate the moments of families of L-functions at the center of the critical strip to those of the characteristic polynomials of matrices from the various classical compact groups [Conrey and Farmer 00, Keating and Snaith 00b].

For any elliptic curve E it is conjectured that the family of even-functional-equation quadratic twists

$$\Phi_E = \{L_E(s, \chi_d) : w_E \chi_d(-Q) = +1\} \tag{3–1}$$

is orthogonal. Specifically, this family conjecturally has symmetry type O^+. Thus the value distribution of $L_E(1, \chi_d)$ should be related to that of the characteristic polynomials of matrices in $\mathrm{SO}(2N)$, at the spectral symmetry point, with $N \sim \log(|d|)$.

For an orthogonal matrix A, the characteristic polynomial may be defined by

$$Z(A, \theta) = \det\left(I - A\mathrm{e}^{-i\theta}\right). \tag{3–2}$$

The eigenvalues of A form complex conjugate pairs $\mathrm{e}^{\pm i\theta_n}$, and so the symmetry point is at $\theta = 0$. The moments of $Z(A, 0)$ are defined by averaging over A with respect to normalized Haar measure for $\mathrm{SO}(2N)$, dA:

$$M_O(N, s) = \int_{\mathrm{SO}(2N)} |\det(I - A)|^s dA. \tag{3–3}$$

It was shown in [Keating and Snaith 00b] that for $\mathrm{Re}\, s > -\frac{1}{2}$,

$$M_O(N, s) = 2^{2Ns} \prod_{j=1}^{N} \frac{\Gamma(N + j - 1)\Gamma(s + j - 1/2)}{\Gamma(j - 1/2)\Gamma(s + j + N - 1)}, \tag{3–4}$$

and that as $N \to \infty$,

$$M_O(N, s) \sim g_s(O^+) N^{s(s-1)/2}, \tag{3–5}$$

with

$$g_s(O^+) = \frac{2^{s^2/2} G(1 + s)\sqrt{\Gamma(1 + 2s)}}{\sqrt{G(1 + 2s)\Gamma(1 + s)}}, \tag{3–6}$$

where G is Barnes's G-function:

$$G(z + 1) = (2\pi)^{z/2} \exp(-((\gamma + 1)z^2 + z)/2) \times \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n \exp\left(-z + z^2/2n\right). \tag{3–7}$$

It follows from the fact that $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$ that, for integer k,

$$g_k(O^+) = 2^{k(k+1)/2} \prod_{j=1}^{k-1} \frac{j!}{2j!}. \tag{3–8}$$

Note that the right-hand side of (3–4) has a meromorphic continuation to the whole complex s-plane.

For integer k, $M_O(N,k)$ is a polynomial of order $k(k-1)/2$:

$$M_O(N,k) = \left(2^{k(k+1)/2} \prod_{j=1}^{k-1} \frac{j!}{(2j)!}\right) \times \prod_{0\le i<j\le k-1} \left(N + \frac{i+j}{2}\right). \tag{3–9}$$

It can also be written in terms of a multiple contour integral [Conrey et al. 03], a form that will be useful later for comparison with L-functions:

$$M_O(N,k) = \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint e^{N\sum_{j=1}^k z_j} \times \prod_{1\le \ell<m\le k} (1-e^{-z_m-z_\ell})^{-1} \frac{\Delta(z_1^2,\dots,z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} dz_1\cdots dz_k. \tag{3–10}$$

Here $\Delta(x_1,\dots,x_n) = \prod_{1\le i<j\le n}(x_j - x_i)$ and the contours enclose the poles at zero.

The value distributions of the characteristic polynomial and its logarithm can be written down directly using (3–4). Let $P_N(t)$ denote the probability density function associated with the value distribution of $|Z(A,0)|$, i.e.,

$$\text{meas}\{A \in SO(2N) : \alpha < |\det(I-A)| \le \beta\} = \int_\alpha^\beta P_N(t)dt. \tag{3–11}$$

Then

$$P_N(t) = \frac{1}{2\pi i t} \int_{c-i\infty}^{c+i\infty} M_O(N,s) t^{-s} ds, \tag{3–12}$$

for any $c > 0$.

The asymptotics of $P_N(t)$ as $t \to 0$ come from the pole of $M_O(N,s)$ at $s = -\frac{1}{2}$. The residue there is

$$h(N) = \frac{2^{-N}}{\Gamma(N)} \prod_{j=1}^N \frac{\Gamma(N+j-1)\Gamma(j)}{\Gamma(j-1/2)\Gamma(j+N-3/2)}, \tag{3–13}$$

which, as $N\to\infty$, is given asymptotically by

$$h(N) \sim 2^{-7/8} G(1/2) \pi^{-1/4} N^{3/8}. \tag{3–14}$$

Thus as $t \to 0$,

$$P_N(t) \sim h(N) t^{-1/2}. \tag{3–15}$$

Importantly for us, one may deduce a central limit theorem for $\log|Z(A,0)|$ from equation (3–12) (see [Keating and Snaith 00b]):

$$\lim_{N\to\infty} \text{meas}\left\{A \in SO(2N) : \frac{\log|\det(I-A)| + \frac{1}{2}\log N}{\sqrt{\log N}} \in (\alpha,\beta)\right\} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta \exp\left(-\frac{t^2}{2}\right) dt. \tag{3–16}$$

The above results, proved for the characteristic polynomials of random matrices, suggest the following conjectures for $L_E(1,\chi_d)$ (these are special cases of those made in [Keating and Snaith 00b], but with the number-theoretic details worked out explicitly).

Recall that we are assuming that Q is square-free. Let $(d,Q) = 1$. For $L_E(s,\chi_d)$ to have an even functional equation, one needs $w_E\chi_d(-Q) = 1$. This imposes a condition, in the case that Q is odd, on d mod Q, and, in the case that Q is even on d mod $4Q$ ($4Q$ because $\chi_d(2)$ is periodic with period 8). Let

$$\tilde{Q} = \begin{cases} Q & \text{if } Q \text{ is odd and square-free,} \\ 4Q & \text{if } Q \text{ is even and square-free.} \end{cases} \tag{3–17}$$

Next, we focus our attention on a subset of the d's according to certain residue classes mod $\tilde{Q}$. We let

$$\begin{aligned} S^-(X) &= S_E^-(X) \\ &= \{-X \le d < 0; \chi_d(p) = -a_p \text{ for all } p \mid Q\}, \end{aligned} \tag{3–18}$$

i.e., the set of negative fundamental discriminants d up to $-X$, but restricted according to a condition on d mod $\tilde{Q}$. Let

$$\int_0^\alpha W_E^-(X,t)dt = \frac{|\{d \in S^-(X); L_E(1,\chi_d) < \alpha\}|}{|S^-(X)|} \tag{3–19}$$

and

$$M_E^-(X,s) = \frac{1}{|S^-(X)|} \sum_{d\in S^-(X)} L_E(1,\chi_d)^s, \tag{3–20}$$

i.e., the sth moment of $L_E(1,\chi_d)$.

For elliptic curves E whose conductor Q is prime, we also look at the set of positive fundamental discriminants

$$S^+(X) = S_E^+(X) = \{0 < d \le X; \chi_d(Q) = a_Q\} \tag{3–21}$$

and define $W_E^\pm(X,t), M_E^\pm(X,s)$ as in the negative case.

Note that

$$W_E^\pm(X,t) = \frac{1}{2\pi i t} \int_{c-i\infty}^{c+i\infty} M_E^\pm(X,s) t^{-s} ds. \tag{3–22}$$

The central conjecture is that as $X \to \infty$,

$$M_E^{\pm}(X,s) \sim A^{\pm}(s) M_O(\log X, s), \tag{3–23}$$

where

$$A^{\pm}(s) = \prod_{p \nmid Q} \left(1-p^{-1}\right)^{s(s-1)/2} \left(\frac{p}{p+1}\right) \times \left(\frac{1}{p} + \frac{1}{2}\left(\mathcal{L}_p\left(\frac{1}{p}\right)^s + \mathcal{L}_p\left(\frac{-1}{p}\right)^s\right)\right) \times \prod_{p \mid Q} \left(1-p^{-1}\right)^{s(s-1)/2} \mathcal{L}_p(\pm a_p/p)^s \tag{3–24}$$

is an arithmetical factor that depends on E. A heuristic for $A^{\pm}(s)$ is given in [Conrey et al. 05]. The following conjectures are then motivated by this.

First we consider the moments of elliptic curve L-functions.

Conjecture 3.1. *For $M_E^{\pm}(X,s)$ defined as in (3–20),*

$$\lim_{X\to\infty} \frac{M_E^{\pm}(X,s)}{(\log X)^{s(s-1)/2}} = A^{\pm}(s) g_s(O^+). \tag{3–25}$$

Second, for large X, as $t \to 0$ we have

$$W_E^{\pm}(X,t) \sim A^{\pm}(-\tfrac{1}{2}) h(\log X) t^{-1/2}, \tag{3–26}$$

and thus by scaling t so that $t = O((\log X)^{-\gamma})$, for $\gamma > 1$, we are led to the following conjecture.

Conjecture 3.2. *If* $\log X f(\log X) \to 0$ *as* $X \to \infty$*, then*

$$\lim_{X\to\infty} \frac{\sqrt{f(\log X)}}{(\log X)^{\frac{3}{8}}} W_E^{\pm}(X, f(\log X) y) = B y^{-1/2}, \tag{3–27}$$

where

$$B = 2^{-7/8} G(1/2) \pi^{-1/4} A^{\pm}(-1/2). \tag{3–28}$$

In addition, we have the following.

Conjecture 3.3.

$$\lim_{X\to\infty} \frac{1}{|S^{\pm}(X)|} \times \left| \left\{ d \in S^{\pm}(X); \frac{\log L_E(1,\chi_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha,\beta) \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta \exp\left(-\frac{t^2}{2}\right) dt. \tag{3–29}$$

We take $\log L_E(1,\chi_d) + \frac{1}{2}\log\log|d|$ *to lie outside the interval if* $L_E(1,\chi_d) = 0$.

The above conjecture is similar to central limit theorems for the Riemann zeta function and other L-functions usually attributed to Selberg [Selberg 46a, Selberg 46b]. An analogous conjecture is made in [Keating and Snaith 00b] for quadratic Dirichlet L-functions.

Further, we have a conjecture, closely related to the preceding one, for the distribution of the full range of values of $L_E(1,\chi_d)$.

Conjecture 3.4.

$$\lim_{X\to\infty} \frac{1}{|S^{\pm}(X)|} \times \left| \left\{ d \in S^{\pm}(X); \alpha \le \left(\sqrt{\log|d|} L_E(1,\chi_d)\right)^{\frac{1}{\sqrt{\log\log|d|}}} \le \beta \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta \frac{1}{t} e^{-\frac{1}{2}(\log t)^2} dt \tag{3–30}$$

for fixed $0 \le \alpha \le \beta$.

Finally, it is discussed in [Conrey et al. 05] how (3–10) leads to conjectures for mean values of L-functions, not just at leading order as in (3–25), but including all terms in the expansion down to the constant term.

Conjecture 3.5. *With k an integer,*

$$M_E^{\pm}(X,k) = \frac{1}{X}\int_0^X \Upsilon_k^{\pm}(\log(t))\, dt + O\left(X^{-\frac{1}{2}+\epsilon}\right) \tag{3–31}$$

as $X \to \infty$, where Υ_k is a polynomial of degree $k(k-1)/2$ given by the k-fold residue

$$\Upsilon_k^{\pm}(x) = \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{F_k^{\pm}(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} \times e^{x\sum_{j=1}^k z_j} dz_1 \cdots dz_k, \tag{3–32}$$

where the contours above enclose the poles at zero and

$$F_k^{\pm}(z_1,\dots,z_k) = A_k^{\pm}(z_1,\dots,z_k) \prod_{j=1}^k \left(\frac{\Gamma(1+z_j)}{\Gamma(1-z_j)}\left(\frac{Q}{4\pi^2}\right)^{z_j}\right)^{1/2} \times \prod_{1\le i<j\le k} \zeta(1+z_i+z_j) \tag{3–33}$$

and $A_k^{\pm}$, which depends on E, is the Euler product, which is absolutely convergent for $\sum_{j=1}^{k}|z_j| < \frac{1}{2}$,

$$A_k^{\pm}(z_1,\ldots,z_k) = \prod_p F_{k,p}(z_1,\ldots,z_k) \times \prod_{1\le i<j\le k}\left(1-\frac{1}{p^{1+z_i+z_j}}\right) \quad (3\text{–}34)$$

with, for $p \nmid Q$,

$$F_{k,p} = \left(1+\frac{1}{p}\right)^{-1}\left(\frac{1}{p}+\frac{1}{2}\left(\prod_{j=1}^{k}\mathcal{L}_p\left(\frac{1}{p^{1+z_j}}\right) + \prod_{j=1}^{k}\mathcal{L}_p\left(\frac{-1}{p^{1+z_j}}\right)\right)\right), \quad (3\text{–}35)$$

and, for $p \mid Q$,

$$F_{k,p} = \prod_{j=1}^{k}\mathcal{L}_p\left(\frac{\pm a_p}{p^{1+z_j}}\right). \quad (3\text{–}36)$$

4. CONJECTURES RELATING TO THE VALUE DISTRIBUTION OF THE FOURIER COEFFICIENTS OF HALF-INTEGRAL-WEIGHT FORMS

Our goal now is to use the conjectures listed at the end of the previous section for the value distribution of the L-functions associated with elliptic curves to motivate conjectures for the value distribution of the Fourier coefficients of half-integral-weight forms. These follow straightforwardly from the connection (2–5).

In each case, we let $c(|d|)$ refer to the coefficients of the corresponding weight-$\frac{3}{2}$ modular form, as in (2–5), and $\kappa_E^{\pm}$ refer to the corresponding proportionality constant.

Our first conjecture then follows from (3–29):

Conjecture 4.1. (Central limit conjecture.)

$$\lim_{X\to\infty}\frac{1}{|S^{\pm}(X)|} \times \left|\left\{d\in S^{\pm}(X); \frac{2\log c(|d|) - \frac{1}{2}\log|d| + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha,\beta)\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta}\exp\left(-\frac{t^2}{2}\right)dt. \quad (4\text{–}1)$$

We take $2\log c(|d|) - \frac{1}{2}\log(|d|) + \frac{1}{2}\log\log|d|$ to lie outside the interval if $c(|d|) = 0$.

This leads directly to a conjecture for the appropriately normalized distribution of the coefficients themselves, which is analogous to (3–30).

Conjecture 4.2.

$$\lim_{X\to\infty}\frac{1}{|S^{\pm}(X)|}\left|\left\{d\in S^{\pm}(X); \alpha \le \left(\frac{\kappa_E^{\pm}\sqrt{\log|d|}}{\sqrt{|d|}}c(|d|)^2\right)^{\frac{1}{\sqrt{\log\log|d|}}} \le \beta\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta}\frac{1}{t}\, e^{-\frac{1}{2}(\log t)^2}dt \quad (4\text{–}2)$$

for fixed $0 \le \alpha \le \beta$.

Our third conjecture follows from (2–5) and (3–25):

Conjecture 4.3. (Moment conjecture.)

$$\lim_{X\to\infty}\frac{1}{(\log X)^{s(s-1)/2}}\frac{1}{|S^{\pm}(X)|}\sum_{d\in S^{\pm}(X)}\frac{c(|d|)^{2s}}{|d|^{s/2}} = (\kappa_E^{\pm})^{-s}A^{\pm}(s)g_s(O^+). \quad (4\text{–}3)$$

Further, we have the conjecture following from (2–5) and (3–27).

Conjecture 4.4. *If $\log X f(\log X) \to 0$ as $X\to\infty$, then*

$$\lim_{X\to\infty}\frac{\sqrt{\kappa_E^{\pm}f(\log X)}}{(\log X)^{\frac{3}{8}}}\frac{1}{|S^{\pm}(X)|} \times \left|\left\{d\in S^{\pm}(X); \frac{c(|d|)^2}{\sqrt{|d|}} < f(\log X)y\right\}\right| = By^{-1/2}, \quad (4\text{–}4)$$

where

$$B = 2^{-7/8}G(1/2)\pi^{-1/4}A^{\pm}(-1/2). \quad (4\text{–}5)$$

Lastly, we have the analogue of (3–31).

Conjecture 4.5. *With k an integer, and summing over fundamental discriminants,*

$$\frac{(\kappa_E^{\pm})^k}{|S^{\pm}(X)|}\sum_{d\in S^{\pm}(X)}\frac{c(|d|)^{2k}}{|d|^{k/2}} = \frac{1}{X}\int_0^X \Upsilon_k^{\pm}(\log(t))\,dt + O\left(X^{-(1/2)+\epsilon}\right) \quad (4\text{–}6)$$

as $X \to \infty$, where $\Upsilon_k^{\pm}$ is the polynomial of degree $k(k-1)/2$ given in Conjecture 3.5.

The numerical evidence that supports these conjectures is amassed in Section 6.

5. FREQUENCY OF VANISHING OF L-FUNCTIONS

Examining the frequency of $L_E(1,\chi_d)=0$ as d varies, as well as the order of the zeros, has particular significance in the context of the conjecture of Birch and Swinnerton-Dyer, which says that $L_E(s)$ has a zero at $s=0$ with order exactly equal to the rank of the elliptic curve E. Random matrix theory appears to have a role in predicting the frequency of such zeros. We have argued in [Conrey et al. 02] that since (2–5) implies a discretization of the values of $L_E(1,\chi_d)$ and

$$\int_0^{\alpha} W_E^{\pm}(X,t)dt \tag{5–1}$$

is the probability that $L_E(1,\chi_d)$ has a value of α or smaller, then the combination of (3–22) and (3–23) suggests the following.

Conjecture 5.1. *There is a constant $c_E^{\pm}\geq 0$ such that*

$$\frac{\sum_{\substack{d\in S^{\pm}(X)\\ |d|\ prime\\ L_E(1,\chi_d)=0}} 1}{\sum_{\substack{d\in S^{\pm}(X)\\ |d|\ prime}} 1} \sim c_E^{\pm} X^{-1/4}(\log X)^{3/8}. \tag{5–2}$$

This conjecture first appeared in [Conrey et al. 02], but was stated with $c_E^{\pm}>0$. However, it became clear in preparing numerics for this paper that $c_E^{\pm}$ can equal zero, and an arithmetic explanation has been given for one of the examples in our data by Delaunay [Delaunay 06].

Here the fundamental discriminants have been restricted to prime values to avoid extra two-divisibility issues being placed on the coefficients $c(|d|)$. The constant $c_E^{\pm}$ remains somewhat mysterious. The random matrix model suggests that it should be proportional to $A^{\pm}\left(\frac{-1}{2}\right)\sqrt{\kappa_E^{\pm}}$. However, when one attempts to apply the random matrix model to the problem of the discrete values taken on by the $c(|d|)$, one ignores subtle arithmetic. It seems, from numerical experiments, that one needs to take into account further correction factors that depend on the size of the torsion subgroup of E, but this is still not understood. See Section 6.

Let $q\nmid Q$ be a fixed prime. Another conjecture that follows from this approach concerns sorting the d's for which $L_E(1,\chi_d)=0$ according to residue classes mod q, according to whether $\chi_d(q)=1$ or -1. Let

$$R_q^{\pm}(X) = \frac{\sum_{\substack{d\in S^{\pm}(X)\\ L_E(1,\chi_d)=0\\ \chi_d(q)=1}} 1}{\sum_{\substack{d\in S^{\pm}(X)\\ L_E(1,\chi_d)=0\\ \chi_d(q)=-1}} 1}. \tag{5–3}$$

Conjecture 5.2. [Conrey et al. 02] *Let*

$$R_q = \left(\frac{q+1-a_q}{q+1+a_q}\right)^{1/2}. \tag{5–4}$$

Then for $q\nmid Q$,

$$\lim_{X\to\infty} R_q^{\pm}(X) = R_q. \tag{5–5}$$

We believe this conjecture to hold even if we allow d to range over different sets of discriminants, such as $|d|$ restricted to primes (though in the latter case we must be sure to rule out there being no vanishings at all due to arithmetic reasons).

A more precise conjecture given in [Conrey et al. 06] incorporates the next term. The lower terms do depend on whether we are looking at $S^+(X)$ as opposed to $S^-(X)$. We require some notation. Let p be prime. For $p\mid Q$ set

$$\beta(p)=\beta^{\pm}(p)=\begin{cases}\log(p)/(1+p) & \text{in the } + \text{ case,}\\ \log(p)/(1-p) & \text{in the } - \text{ case,}\end{cases} \tag{5–6}$$

and for $p\nmid Q$ set

$$\beta(p)=\log(p)\left(\frac{(2-a_p)f_1(p)^{-1/2}+(2+a_p)f_2(p)^{-1/2}}{2+p\left(f_1(p)^{1/2}+f_2(p)^{1/2}\right)}\right), \tag{5–7}$$

where

$$f_1(p)=1-a_p/p+1/p,$$
$$f_2(p)=1+a_p/p+1/p.$$

Next, let

$$\lambda_\nu(q)=\frac{\log(q)(\nu a_q-2)}{\nu a_q-q-1}-\frac{3\log(q)}{2(q-1)}-\frac{5\gamma}{2} + \sum_{p\neq q}\left(\beta(p)-\frac{3\log(p)}{2(p-1)}\right), \tag{5–8}$$

where γ is Euler's constant.

Conjecture 5.3. [Conrey et al. 06] *For* $q \nmid Q$,

$$R_q^{\pm}(X) = \left(\frac{q+1-a_q}{q+1+a_q}\right)^{1/2} \frac{g+\lambda_1(q)}{g+\lambda_{-1}(q)} + O(1/\log(X)^2) \tag{5–9}$$

where

$$g = \frac{8}{3}\log\left(\frac{XQ^{1/2}}{2\pi}\right) - 1. \tag{5–10}$$

These conjectures are supported by numerical evidence that will be described in the next section.

6. NUMERICAL EXPERIMENTS

We present numerics for 2398 elliptic curves and millions of quadratic twists for each curve ($|d| < 10^8$) confirming the aforementioned conjectures. To test these conjectures we used the relation (2–5). To make our examples explicit, we list in Tables 3-3 relevant data for 26 of the 2398 elliptic curves examined. The remaining data may be obtained from the L-function database of one of the authors [Rubinstein 04]. We used as the starting point for our computations a database of Tornaria and Rodriguez-Villegas [Rodriquez-Villegas and Tornaria 04] that lists, for thousands of elliptic curves, the relevant ternary quadratic forms.

Each entry in this table contains the following data:

name	$[a_1, a_2, a_3, a_4, a_6]$	κ	no. of d	no. ternary forms
relevant residue classes $\lvert d\rvert$ mod $\tilde{Q}$				
linear combination				
ternary forms				

The name we use for an elliptic curve E corresponds to Cremona's table [Cremona], but with an extra subscript, either i or r, standing for imaginary or real respectively, to specify whether we are looking at quadratic twists $L_E(s, \chi_d)$ with $d < 0$ or $d > 0$. The naming convention used by Cremona includes the conductor Q in the name for the elliptic curve. So, for example, the first entry has name $11A_i$, which is the elliptic curve of conductor 11. The i indicates that we are looking at quadratic twists of $11A$ with $d < 0$.

The notation $[a_1, a_2, a_3, a_4, a_6]$ refers to the coefficients defining the equation of E,

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{6–1}$$

For each curve, our data consists of values of $L_E(1, \chi_d)$ with $|d| < 10^8$, and d restricted to $\prod_{\substack{p|Q \\ p \text{ odd}}}((p-1)/2)$ residue classes mod Q or $4Q$, depending on whether Q is odd or even. For imaginary twists, all the Tornaria–Rodriguez-Villegas examples have Q square-free, and in the case of real twists, all their examples have Q prime. The relevant residue classes and relevant modulus are listed in the second line of each entry, and the number of such d up to 10^8 is given by "no. of d."

Our $L(1, \chi_d)$ values are expressed in terms of the coefficients of a weight-$\frac{3}{2}$ modular form of level $4Q$ in the case of imaginary quadratic twists, $d < 0$, and of level $4Q^2$ in the case of real quadratic twists, $d > 0$. This weight-$\frac{3}{2}$ form is expressed as a linear combination of theta series attached to positive definite ternary quadratic forms. The number of forms, say r, is given in the last part of the first line, while the linear combination, $[\alpha_1, \ldots, \alpha_r]$, and ternary forms occupy the last two lines of each entry. Each ternary form is specified as a sextuple of integers $\beta = [\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6]$. The ternary form is

$$f_\beta(x, y, z) = \beta_1x^2 + \beta_2y^2 + \beta_3z^2 + \beta_4yz + \beta_5xz + \beta_6xy \tag{6–2}$$

and the associated theta series is given by

$$\theta_\beta(w) = \sum_{(x,y,z)\in\mathbb{Z}^3} w^{f_\beta(x,y,z)}. \tag{6–3}$$

Given this data, one defines

$$\sum_{n=1}^{\infty} c(n)w^n = \sum_{j=1}^{r} \alpha_j\theta_{\beta_j}(w). \tag{6–4}$$

Then, for fundamental discriminants lying in the relevant residue classes in the table (and $d < 0$ or $d > 1$ according to the name of the entry), one has

$$L_E(1, \chi_d) = \frac{\kappa c(|d|)^2}{|d|^{1/2}}. \tag{6–5}$$

Note that our values of κ and $c(|d|)$ differ slightly from the values given in [Mao et al. 06] in that in our tables, we ignored the value of $c(1)$, for example in the case that $d > 0$, and normalized the α_j's so that the gcd of all the $c(n)$'s, for $n \neq 1$, is equal to one, absorbing if necesary an extra square factor into κ. For example, while [Mao et al. 06] gives for the curve $11A_r$ a value of $\kappa = .25384186\ldots$, we list a value of κ that is $5^2 = 25$ times as big, but give $c(n)$'s that are $\frac{1}{5}$ as large.

The most comprehensive test carried out [Conrey et al. 05] for moments involved looking at Conjecture 3.5 for millions of values of $L_{E_{11A}}(1, \chi_d)$, with $d < 0$ and $|d| = 1, 3, 4, 5, 9 \mod 11$. The coefficients from [Conrey et al. 05] of the polynomials Υ_k^- in Conjecture 3.5 are given by

r	$f_r(1)$	$f_r(2)$	$f_r(3)$	$f_r(4)$
0	1.2353	.3834	.00804	.0000058
1		1.850	.209	.000444
2			1.57	.0132
3			2.85	.1919
4				1.381
5				4.41
6				4.3

TABLE 1. Coefficients of $\Upsilon_k^-(x) = f_0(k)x^{k(k-1)/2} + f_1(k)x^{k(k-1)/2-1} + \cdots$, for $k = 1, 2, 3, 4$.

Table 1. In checking Conjecture 3.5, we compared

$$\sum_{\substack{-850000<d<0 \\ |d|=1,3,4,5,9 \mod 11}} L_{11A_i}(\frac{1}{2}, \chi_d)^k \tag{6–6}$$

to

$$\sum_{\substack{-850000<d<0 \\ |d|=1,3,4,5,9 \mod 11}} \Upsilon_k^-\left(\log(|d|)\right). \tag{6–7}$$

This comparison is depicted in Table 2.

k	(6–6)	(6–7)	ratio
1	14628043.5	14628305.	0.99998
2	100242348.8	100263216.	0.9998
3	1584067116.8	1587623419.	0.998
4	41674900434.9	41989559937.	0.993

TABLE 2. Moments of $L_{11A_i}(1, \chi_d)$ versus their conjectured values, for fundamental discriminants $-85\,000\,000 < d < 0$, $|d| = 1, 3, 4, 5, 9 \mod 11$, and $k = 1, \ldots, 4$. The data agree with our conjecture to the accuracy to which we have computed the moment polynomials Υ_k^-.

The first part of Conjecture 3.1 is, for integral moments, a weaker form of Conjecture 3.5, while Conjectures 4.3 and 4.5 follow from Conjectures 3.1 and 3.5.

In Figure 1 we depict the numerical value distribution of $L_{11A_i}(1, \chi_d)$, for fundamental discriminants

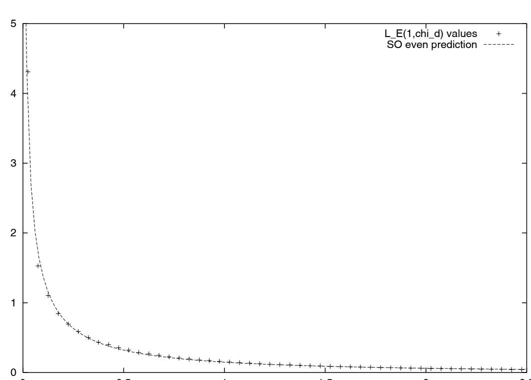

FIGURE 1. Value distribution of $L_{E_{11}}(1, \chi_d)$, with $-85\,000\,000 < d < 0$, $d = 2, 6, 7, 8, 10 \mod 11$.

$-85\,000\,000 < d < 0$, $|d| = 1, 3, 4, 5, 9 \mod 11$, compared to $P_N(t)$ obtained by taking the inverse Mellin transform of (3–4) with $N = 20$. Because we are neglecting the arithmetic factor in computing the density, a slight cheat was used to get a better fit. The histograms were rescaled by a constant along both axes until the histogram displayed matched up nicely with the solid curve. Considering that we are compensating for leaving out the arithmetic factor in such a naive way, it is a bit surprising how nicely the two fit. The main point we wish to make is that the histogram does exhibit $t^{-1/2}$ behavior near the origin in support of part 2 of Conjecture 3.1.

In Figure 2 we verify the central limit theorem described in Conjecture 3.3. The first picture shows the

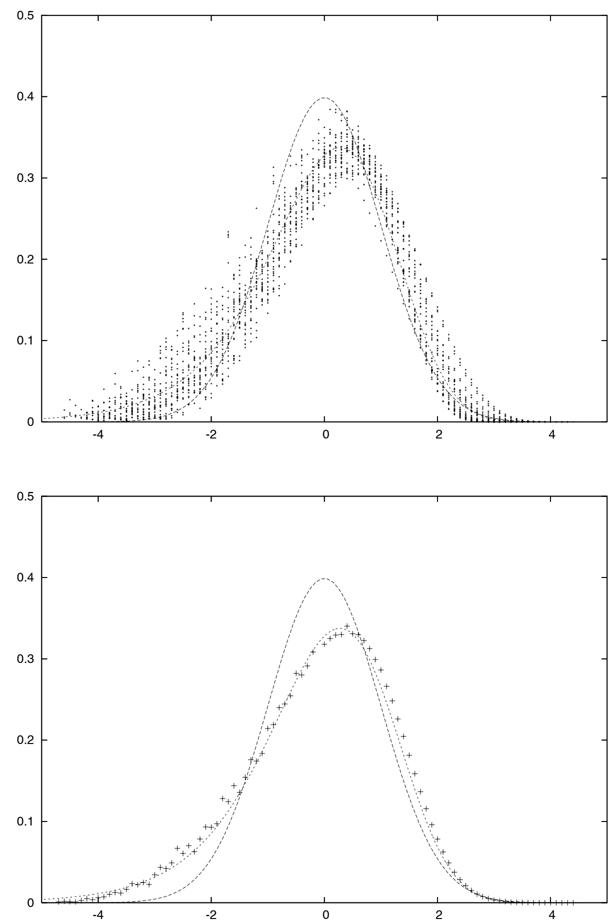

FIGURE 2. Value distribution of $\left(\log L_E(1, \chi_d) + \frac{1}{2}\log\log|d|\right)/\sqrt{\log\log|d|}$ compared to the random matrix theory counterpart $P_N(g(t))g'(t)$ with $N = 20$ (rescaled as explained in the text), and its limit the standard Gaussian. In the first picture we superimpose the value distributions for the twenty-six curves described in Table 3. The second picture shows the average value distribution of the twenty-six.

value distributions, superimposed, of

$$\frac{\log L_E(1,\chi_d)+\frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}}, \quad |d|<10^8, \tag{6–8}$$

for the twenty-six curves described in Table 3. The second plot depicts the average value distribution of the twenty-six. These are compared against the standard Gaussian predicted in the conjecture and also against the density function associated with the value distribution of

$$\frac{\log|Z(A,0)|+\frac{1}{2}\log N}{\sqrt{\log N}} \tag{6–9}$$

with $N=20$. This density is given by

$$P_N(g(t))g'(t), \tag{6–10}$$

where

$$g(t)=\frac{\exp(t\sqrt{\log N})}{N^{1/2}}, \tag{6–11}$$

and is shown in [Keating and Snaith 00b] to tend to the standard Gaussian as $N\to\infty$. To get a better fit to the numeric value distribution, one would also need to incorporate the arithmetic factor. In the limit, this factor has no effect, but the convergence to the limit is extremely slow. The variability of the arithmetic factor explains why the twenty-six densities superimposed in Figure 2 don't fall exactly on top of one another. To get a better fit to the average, we set $p(t)=P_N(g(t))g'(t)$ and plot instead $\alpha p(\alpha t)$, with $\alpha=1.21$ chosen so that the density function visually lines up with the data.

Conjecture 5.2 is verified numerically in the top plot in Figure 3, which compares, for the first one hundred elliptic curves E in our database and the sets $S_E^{\pm}(X)$, the predicted value of R_q to the actual value $R_q^{\pm}(X)$, with $X=10^8$.

The horizontal axis is q. For each q and each of the one hundred elliptic curves E we plot $R_q^{\pm}(X)-R_q$, with $X=10^8$ and $q\leq 3571$. For each q on the horizontal there are 100 points corresponding to the 100 values, one for each elliptic curve, of $R_q^{\pm}(X)-R_q$. We see the values fluctuating about zero, most of the time agreeing to within about 0.02. The convergence in X is predicted from secondary terms to be logarithmically slow, and one gets a better fit by including more terms as predicted in Conjecture 5.3.

This is depicted in the second plot of Figure 3, which shows the difference

$$R_q^{\pm}(X)-\left(\frac{q+1-a_q}{q+1+a_q}\right)^{1/2}\frac{g+\lambda_1(q)}{g+\lambda_{-1}(q)}, \tag{6–12}$$

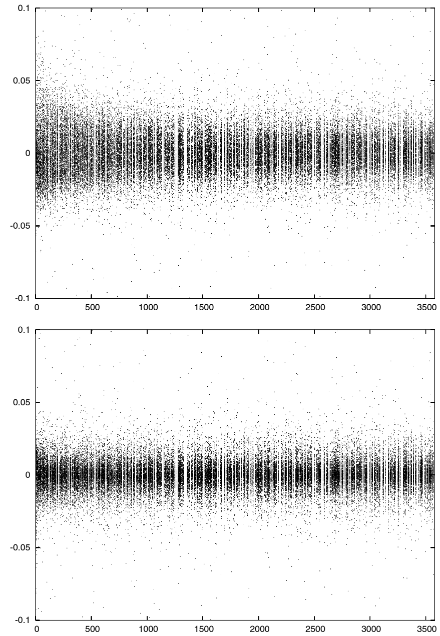

FIGURE 3. A plot [Conrey et al. 04] for one hundred elliptic curves of $R_q^{\pm}(X)-R_q$, left plot, and of (6–12), right plot, for $2\leq q\leq 3571$, $X=10^8$.

again with $X=10^8$, $q\leq 3571$, and the same one hundred elliptic curves E. We see an improvement to the first plot, which uses just the main term.

This improvement is emphasized in Figure 4, which compares the distribution of $R_q^{\pm}(X)-R_q$ for all our 2398 elliptic curves, $X=10^8$, $q\leq 3571$, versus the distribution of (6–12). The latter has smaller variance. These distributions are not Gaussian. They depict the remainder of $R_q^{\pm}(X)$ compared to the first and second conjectured approximations. There are yet further lower terms and these are given by complicated sums involving the Dirichlet coefficients of $L_E(s)$.

To verify Conjecture 5.1 we computed the left-hand side of (5–2) divided by

$$A^{\pm}(-1/2)\sqrt{\kappa_E^{\pm}}X^{-1/4}(\log X)^{3/8} \tag{6–13}$$

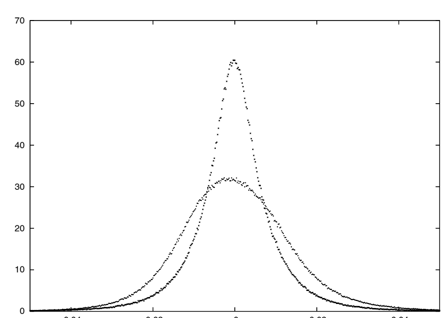

FIGURE 4. Distribution first approximation versus second approximation for ratio of vanishings [Conrey et al. 04]

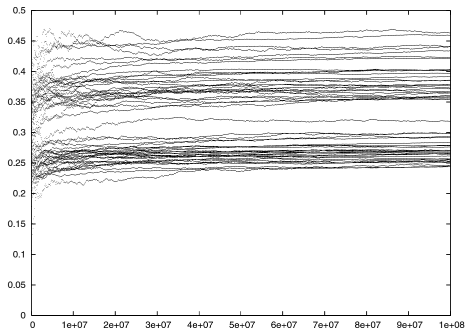

FIGURE 5. A test of Conjecture 5.1 for 55 elliptic curves in our database: we are plotting for each curve the left-hand side of (5–2) divided by $A^{\pm}\left(\frac{-1}{2}\right)\sqrt{\kappa_E^{\pm}}X^{-1/4}(\log X)^{3/8}$, $X = 100\,000, 200\,000, \ldots, 10^8$. The graphs appear relatively flat.

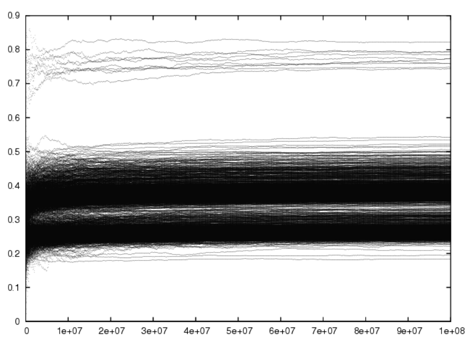

FIGURE 6. The same as the previous figure, but for all elliptic curves in our database.

for our 2398 curves (545 of these have $c_E^{\pm} = 0$ and these are omitted) and $X = 100\,000, 200\,000, \ldots, 10^8$. The resulting values are depicted in Figure 6. We also display in Figure 5 the same data for a subset of 55 of these curves (those in our database with $c_E^{\pm} \neq 0$ and whose conductor have leading digits 11). The graphs displayed in these two figures appear relatively flat.

To measure how flat these graphs are, we show in Figure 7 the distribution of the slopes of the graphs in Figure 6, measured by sampling each one at $X = 5 \times 10^7$ and $X = 10^8$. Most have slopes that are of size less than 10^{-10}, and this suggests that the power of X, namely $\frac{3}{4}$, in our conjecture is not off by more that 0.01, and that the power of $\log(X)$, namely $\frac{3}{8}$ is not off by more than 0.1. The mean does appear slightly to the right of 0, occurring at 0.5×10^{-10}, but this might be the result of using a limited number of curves and may also perhaps be due to lower-order terms.

Next we sort the graphs in Figure 6 by their rightmost values at $X = 10^8$, and, in Figure 8, plot these values against the order of the torsion subgroup of the corresponding elliptic curves. The plot shows that curves with trivial torsion tend to have smaller constants, followed by curves of torsion size equal to 7 or 5 or 3, then 2 and 1 again, etc. We do not yet have an explanation for this phenomenon and do not know how to incorporate it into our model. It seems [Rubinstein 04] that to nail down the constant $c_E^{\pm}$ one would need to incorporate into the model Delaunay's heuristics for Tate–Shavarevich groups [Delaunay 01]. However, it appears that for primes p dividing the order of the torsion subgroup, the probability that $c(|d|)$ is divisible by p deviates in a way we do not yet understand from a prediction made by Delaunay for the probability that the order of the Tate–Shavarevich group is divisible by a given prime. Delaunay's predictions are for all elliptic curves sorted by conductor, and here we are examining a skinny set of elliptic curves, namely quadratic twists of a fixed elliptic curve.

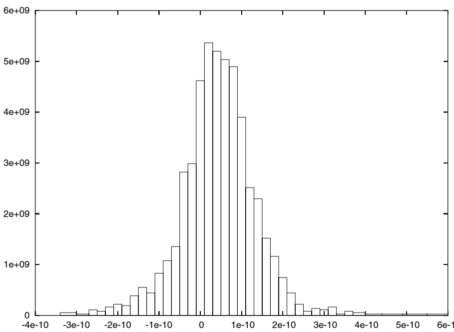

FIGURE 7. Distribution of the slopes of the graphs in Figure 6 from $X = 5\times10^7$ to $X = 10^8$. This tells us that the graphs are relatively flat.

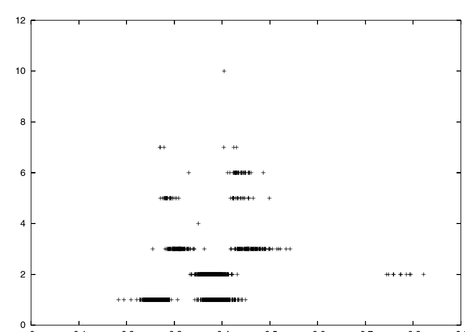

FIGURE 8. The rightmost values of the graphs in Figure 6 plotted against the order of the torsion subgroups of the corresponding elliptic curves. We see, for example, that the smallest constants tend to go along with the curves whose torsion is trivial. The few curves at the far right with torsion size equal to 2 have $c(|d|)$ divisible by 2 when $|d| \neq 4$, and hence, because the corresponding discretization is twice as large, the $c(|d|)$'s are four times as likely to vanish.

$11A_i$	[0, -1, 1, -10, -20]	2.91763323388	13931691	2
1, 3, 4, 5, 9 mod 11				
[1/2, -1/2]				
[3, 15, 15, -14, -2, -2], [4, 11, 12, 0, -4, 0]				
$11A_r$	$[0, -1, 1, -10, -20]$	6.34604652140	13931649	6
1, 3, 4, 5, 9 mod 11				
[1/10, -1/10, 3/10, -3/10, -2/10, 2/10]				
[1, 44, 132, -44, 0, 0], [4, 12, 121, 0, 0, -4], [4, 33, 45, -22, -4, 0], [5, 9, 124, -8, -4, -2], [5, 36, 36, 28, 4, 4], [16, 16, 25, -4, -4, -12]				
$14A_i$	[1, 0, 1, 4, -6]	5.30196495873	4432803	2
15, 23, 39 mod 56				
[1/4, -1/4]				
[4, 15, 15, 2, 4, 4], [7, 8, 16, -8, 0, 0]				
$15A_i$	[1, 1, 1, -10, -10]	3.19248444426	4749434	2
2, 8 mod 15				
[1/4, -1/4]				
[3, 20, 20, -20, 0, 0], [8, 8, 15, 0, 0, -4]				
$17A_i$	[1, -1, 1, -1, -14]	2.74573911809	14353828	2
3, 5, 6, 7, 10, 11, 12, 14 mod 17				
[1/2, -1/2]				
[3, 23, 23, -22, -2, -2], [7, 11, 20, -8, -4, -6]				
$19A_i$	[0, 1, 1, -9, -15]	4.12709239172	14438275	2
1, 4, 5, 6, 7, 9, 11, 16, 17 mod 19				
[1/2, -1/2]				
[4, 19, 20, 0, -4, 0], [7, 11, 23, -10, -6, -2]				
$19A_r$	[0, 1, 1, -9, -15]	4.07927920046	14438248	12
1, 4, 5, 6, 7, 9, 11, 16, 17 mod 19				
[-1/6, 1/6, -1/6, 1/6, -1/3, -1/3, -1/3, 1/3, 1/3, 1/3, 1/3, -1/3]				
[1, 76, 380, -76, 0, 0], [4, 20, 361, 0, 0, -4], [4, 77, 96, 40, 4, 4], [5, 16, 365, 16, 2, 4], [5, 61, 92, 16, 4, 2], [5, 76, 92, -76, -4, 0], [9, 44, 77, 28, 6, 8], [16, 24, 77, 20, 8, 4], [17, 44, 44, 12, 16, 16], [20, 24, 73, 4, 8, 20], [20, 36, 45, 20, 16, 12], [25, 36, 36, -4, -16, -16]				
$21A_i$	[1, 0, 0, -4, -1]	3.82197956150	4986931	2
10, 13, 19 mod 21				
[1/4, -1/4]				
[3, 28, 28, -28, 0, 0], [7, 12, 24, -12, 0, 0]				
$26A_i$	[1, 0, 1, -5, -8]	3.47934348343	4704178	2
7, 15, 31, 47, 63, 71 mod 104				
[-1/2, 1/2]				
[7, 15, 31, -14, -6, -2], [8, 15, 28, 4, 8, 8]				
$26B_i$	[1, -1, 1, -3, 3]	1.80405719338	4704185	2
3, 27, 35, 43, 51, 75 mod 104				
[-1/2, 1/2]				
[3, 35, 35, -34, -2, -2], [4, 27, 27, 2, 4, 4]				
$30A_i$	[1, 0, 1, 1, 2]	18.5342737810	1583137	2
31, 79 mod 120				
[1/8, -1/8]				
[4, 31, 31, 2, 4, 4], [15, 16, 16, -8, 0, 0]				
$33A_i$	[1, 1, 0, -11, 0]	2.74463335747	5224398	3
5, 14, 20, 23, 26 mod 33				
[1/4, 1/4, -1/2]				
[3, 44, 44, -44, 0, 0], [11, 12, 36, -12, 0, 0], [15, 20, 20, -4, -12, -12]				
$34A_i$	[1, 0, 0, -3, 1]	7.45670022989	4784604	2
19, 35, 43, 59, 67, 83, 115, 123 mod 136				
[1/4, -1/4]				
[4, 35, 35, 2, 4, 4], [8, 19, 36, 4, 8, 8]				
$35A_i$	[0, 1, 1, 9, 1]	2.20504427610	5540991	2
1, 4, 9, 11, 16, 29 mod 35				
[-1/2, 1/2]				
[4, 35, 36, 0, -4, 0], [11, 15, 39, -10, -6, -10]				

TABLE 3. Some of the ternary forms from [Rodriquez-Villegas and Tornaria 04].

$37B_i$	[0, 1, 1, -23, -50]	7.07044268094	14798224	2
2, 5, 6, 8, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 29, 31, 32, 35 mod 37				
[1/2, -1/2]				
[8, 19, 39, 2, 8, 4], [15, 20, 23, -8, -14, -4]				
$43A_r$	[0, 1, 1, 0, 0]	10.9373790599	14852746	22
2, 3, 5, 7, 8, 12, 18, 19, 20, 22, 26, 27, 28, 29, 30, 32, 33, 34, 37, 39, 42 mod 43				
[1/4, 1/4, -1/4, -1/4, 1/4, 1/4, -1/4, 1/4, -1/4, -1/4, -1/4, -1/4, -1/4, 1/4, 1/4, 1/4, -1/4, 1/4, 1/4, 1/4, -1/4, -1/4]				
[5, 69, 929, -34, -2, -2], [5, 241, 276, 104, 4, 2], [8, 65, 624, 44, 4, 4], [8, 108, 389, 88, 8, 4], [12, 29, 932, -28, -4, -4], [12, 73, 373, 30, 4, 8], [20, 29, 621, 22, 12, 16], [20, 61, 277, 54, 12, 8], [28, 32, 377, 28, 16, 12], [28, 33, 376, 28, 12, 16], [29, 77, 157, 14, 18, 26], [29, 77, 161, 46, 10, 26], [32, 69, 157, 26, 12, 24], [37, 45, 237, -2, -26, -34], [37, 89, 104, 44, 16, 10], [37, 93, 104, -60, -16, -2], [45, 77, 113, 62, 42, 10], [48, 77, 104, -56, -4, -32], [48, 77, 108, -48, -28, -32], [61, 80, 89, 76, 42, 16], [69, 76, 77, -4, -58, -32], [69, 77, 80, -44, -8, -58]				
$67A_i$	[0, 1, 1, -12, -21]	6.05993680291	14974655	3
1, 4, 6, 9, 10, 14, 15, 16, 17, 19, 21, 22, 23, 24, 25, 26, 29, 33, 35, 36, 37, 39, 40, 47, 49, 54, 55, 56, 59, 60, 62, 64, 65 mod 67				
[1/2, -1, 1/2]				
[4, 67, 68, 0, -4, 0], [15, 36, 39, -16, -14, -4], [16, 19, 71, 6, 16, 12]				
$67A_r$	[0, 1, 1, -12, -21]	1.27377003655	14974660	70
1, 4, 6, 9, 10, 14, 15, 16, 17, 19, 21, 22, 23, 24, 25, 26, 29, 33, 35, 36, 37, 39, 40, 47, 49, 54, 55, 56, 59, 60, 62, 64, 65 mod 67				
[-1/2, 1/2, 1/2, -1, -1/2, -1, -1, 1, -1, -1, -1, -1, 1, 1, -1, 1, -1, -1, 1, -1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, 1, 1, 1, -1, 1, 1, 1, 1, -1, 1, -1, 1, -1, 1, 1, -1, 1, 1, -1, 1, -1, -1, -1, -1, 1, 1, -1, 1]				
[1, 268, 4556, -268, 0, 0], [4, 68, 4489, 0, 0, -4], [4, 269, 1140, 136, 4, 4], [9, 269, 508, 92, 8, 6], [16, 17, 4493, 2, 16, 4], [17, 237, 332, -172, -8, -6], [17, 268, 332, -268, -8, 0], [21, 77, 753, -38, -2, -6], [21, 217, 308, 192, 12, 2], [21, 220, 308, 196, 12, 16], [24, 157, 328, 48, 20, 8], [24, 236, 261, -196, -20, -12], [25, 248, 248, 228, 12, 12], [29, 56, 760, -44, -16, -8], [29, 89, 504, 20, 24, 26], [29, 205, 216, 88, 20, 14], [29, 205, 224, 124, 16, 14], [29, 224, 224, 180, 16, 16], [33, 144, 292, -92, -32, -28], [33, 173, 236, 124, 8, 18], [33, 173, 237, -118, -14, -18], [36, 68, 509, 48, 20, 12], [36, 173, 216, 120, 4, 16], [37, 77, 464, 68, 8, 26], [37, 116, 293, -64, -22, -4], [37, 188, 189, 84, 10, 32], [37, 189, 189, 110, 10, 10], [40, 77, 449, 2, 4, 40], [40, 77, 457, 18, 36, 40], [40, 149, 237, -130, -20, -16], [49, 104, 260, -92, -24, -4], [60, 77, 277, -62, -4, -16], [60, 84, 277, 64, 4, 44], [60, 89, 285, -42, -44, -56], [64, 129, 160, 4, 36, 44], [65, 68, 301, -20, -2, -40], [65, 68, 304, -12, -28, -40], [65, 140, 140, 12, 32, 32], [65, 140, 160, -84, -44, -32], [65, 149, 149, 30, 58, 58], [65, 149, 157, 110, 34, 58], [68, 77, 237, 10, 12, 24], [68, 84, 265, 8, 16, 68], [68, 96, 217, -28, -8, -60], [68, 96, 237, 92, 12, 60], [68, 140, 149, 72, 44, 52], [73, 77, 240, 32, 20, 46], [73, 92, 181, 12, 18, 8], [73, 132, 132, -4, -32, -32], [76, 77, 272, 48, 12, 72], [76, 93, 173, -6, -8, -20], [76, 93, 188, 16, 68, 20], [77, 84, 193, 40, 22, 12], [77, 89, 220, -8, -36, -70], [77, 129, 157, 70, 74, 66], [77, 132, 148, 92, 52, 56], [77, 132, 157, 100, 74, 56], [81, 93, 196, 4, 16, 74], [84, 116, 153, 88, 32, 60], [84, 116, 169, -48, -80, -60], [88, 92, 173, 76, 44, 36], [88, 96, 173, 80, 44, 52], [89, 132, 132, -4, -72, -72], [89, 132, 148, -92, -48, -72], [92, 92, 181, -12, -12, -84], [93, 93, 173, -6, -6, -82], [93, 93, 193, -46, -46, -82], [96, 96, 157, -16, -16, -76], [100, 104, 149, 64, 96, 44], [104, 104, 149, -64, -64, -60]				
$73A_i$	[1, -1, 0, 4, -3]	2.79278430294	14992818	4
5, 7, 10, 11, 13, 14, 15, 17, 20, 21, 22, 26, 28, 29, 30, 31, 33, 34, 39, 40, 42, 43, 44, 45, 47, 51, 52, 53, 56, 58, 59, 60, 62, 63, 66, 68 mod 73				
[-1/2, 1/2, -1, 1]				
[7, 43, 84, -40, -4, -6], [11, 28, 80, 28, 4, 8], [15, 39, 40, 20, 8, 2], [20, 31, 44, -28, -4, -12]				
$79A_r$	[1, 1, 1, -2, 0]	11.9016007052	15008174	40
3, 6, 7, 12, 14, 15, 17, 24, 27, 28, 29, 30, 33, 34, 35, 37, 39, 41, 43, 47, 48, 53, 54, 56, 57, 58, 59, 60, 61, 63, 66, 68, 69, 70, 71, 74, 75, 77, 78 mod 79				
[-1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2, -1/2, 1/2]				

TABLE 3. (continued) Some of the ternary forms from [Rodriquez-Villegas and Tornaria 04].

$89B_i$	[1, 1, 0, 4, 5]	2.18489393577	15029294	7
[12, 369, 449, 54, 8, 4], [17, 133, 896, -12, -16, -14], [17, 224, 521, 20, 6, 8], [17, 264, 449, -12, -14, -16], [24, 185, 461, 106, 4, 8], [28, 192, 385, -112, -12, -4], [29, 33, 2088, -8, -28, -6], [33, 116, 544, 92, 32, 12], [37, 153, 376, 8, 36, 34], [41, 56, 1044, 4, 24, 40], [41, 116, 425, -4, -38, -8], [41, 185, 265, -38, -22, -2], [41, 185, 276, -84, -32, -2], [41, 216, 236, 20, 28, 36], [41, 217, 229, -18, -30, -14], [48, 112, 377, 28, 40, 4], [48, 193, 224, -32, -36, -20], [53, 201, 233, -182, -10, -30], [56, 96, 377, -44, -32, -4], [60, 157, 216, -20, -4, -32], [61, 96, 372, -4, -32, -44], [68, 113, 284, 104, 12, 20], [68, 172, 185, -92, -32, -4], [69, 137, 217, 10, 22, 34], [69, 161, 193, -74, -38, -14], [77, 113, 249, -62, -46, -26], [77, 137, 201, -38, -50, -22], [85, 113, 233, 82, 46, 42], [93, 145, 161, 70, 22, 46], [96, 136, 197, -132, -60, -20], [96, 140, 185, -128, -16, -44], [108, 120, 161, -52, -36, -4], [108, 145, 161, 70, 36, 104], [112, 116, 185, 100, 76, 12], [113, 116, 185, -100, -14, -56], [113, 161, 165, 138, 2, 110], [116, 137, 185, 134, 100, 96], [132, 137, 149, 98, 64, 108], [132, 153, 156, -44, -124, -100], [140, 145, 153, 134, 132, 64]				
3, 6, 7, 12, 13, 14, 15, 19, 23, 24, 26, 27, 28, 29, 30, 31, 33, 35, 37, 38, 41, 43, 46, 48, 51, 52, 54, 56, 58, 59, 60, 61, 62, 63, 65, 66, 70, 74, 75, 76, 77, 82, 83, 86 mod 89				
[1/2, -1/2, 1/2, 1/2, 1/2, -1/2, -1]				
[3, 119, 119, -118, -2, -2], [7, 51, 103, -50, -6, -2], [12, 31, 92, 4, 12, 8], [15, 24, 95, 24, 2, 4], [15, 27, 96, -20, -8, -14], [19, 23, 95, -14, -10, -18], [23, 31, 48, 16, 12, 2]				
$109A_i$	[1, -1, 0, -8, -7]	5.94280424076	15060017	3
2, 6, 8, 10, 11, 13, 14, 17, 18, 19, 23, 24, 30, 32, 33, 37, 39, 40, 41, 42, 44, 47, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 62, 65, 67, 68, 69, 70, 72, 76, 77, 79, 85, 86, 90, 91, 92, 95, 96, 98, 99, 101, 103, 107 mod 109				
[1/2, 1/2, -1]				
[11, 40, 119, 40, 2, 4], [19, 23, 119, -22, -18, -2], [24, 39, 56, 4, 12, 16]				
$113A_i$	[1, 1, 1, 3, -4]	2.85781203904 15064917	7	
3, 5, 6, 10, 12, 17, 19, 20, 21, 23, 24, 27, 29, 33, 34, 35, 37, 38, 39, 40, 42, 43, 45, 46, 47, 48, 54, 55, 58, 59, 65, 66, 67, 68, 70, 71, 73, 74, 75, 76, 78, 79, 80, 84, 86, 89, 90, 92, 93, 94, 96, 101, 103, 107, 108, 110 mod 113				
[-1/2, 1/2, -1/2, -1, -1/2, 1, 1]				
[3, 151, 151, -150, -2, -2], [12, 39, 116, 4, 12, 8], [19, 24, 119, 24, 2, 4], [20, 47, 68, -44, -4, -12], [23, 24, 119, 24, 10, 20], [23, 40, 59, 20, 2, 8], [35, 39, 47, -10, -34, -6]				
$139A_i$	[1, 1, 0, -3, -4]	5.80133204474	15089693	5
1, 4, 5, 6, 7, 9, 11, 13, 16, 20, 24, 25, 28, 29, 30, 31, 34, 35, 36, 37, 38, 41, 42, 44, 45, 46, 47, 49, 51, 52, 54, 55, 57, 63, 64, 65, 66, 67, 69, 71, 77, 78, 79, 80, 81, 83, 86, 89, 91, 96, 99, 100, 106, 107, 112, 113, 116, 117, 118, 120, 121, 122, 124, 125, 127, 129, 131, 136, 137 mod 139				
[-1/2, 1/2, 1/2, 1/2, -1]				
[7, 80, 159, 80, 2, 4], [11, 52, 152, 52, 4, 8], [20, 28, 139, 0, 0, -4], [20, 31, 144, 8, 20, 16], [24, 47, 71, 2, 12, 8]				
$179A_i$	[0, 0, 1, -1, -1]	5.10909732904	15113724	7
1, 3, 4, 5, 9, 12, 13, 14, 15, 16, 17, 19, 20, 22, 25, 27, 29, 31, 36, 39, 42, 43, 45, 46, 47, 48, 49, 51, 52, 56, 57, 59, 60, 61, 64, 65, 66, 67, 68, 70, 74, 75, 76, 77, 80, 81, 82, 83, 85, 87, 88, 89, 93, 95, 100, 101, 106, 107, 108, 110, 116, 117, 121, 124, 125, 126, 129, 135, 138, 139, 141, 142, 144, 145, 146, 147, 149, 151, 153, 155, 156, 158, 161, 168, 169, 171, 172, 173, 177 mod 179				
[1/2, 1/2, 1/2, 1/2, -1/2, -1/2, -1]				
[4, 179, 180, 0, -4, 0], [15, 48, 191, 48, 2, 4], [15, 51, 192, -44, -8, -14], [16, 47, 183, 6, 16, 12], [19, 39, 191, -34, -14, -10], [20, 39, 184, 8, 20, 16], [39, 56, 76, -52, -20, -12]				
$233A_i$	[1, 0, 1, 0, 11]	1.63933561519	15133226	13
3, 5, 6, 10, 11, 12, 17, 20, 21, 22, 24, 27, 34, 35, 39, 40, 41, 42, 43, 44, 45, 47, 48, 53, 54, 57, 59, 61, 65, 67, 68, 69, 70, 73, 75, 77, 78, 79, 80, 82, 83, 84, 86, 87, 88, 90, 93, 94, 95, 96, 97, 99, 103, 106, 108, 111, 114, 115, 118, 119, 122, 125, 127, 130, 134, 136, 137, 138, 139, 140, 143, 145, 146, 147, 149, 150, 151, 153, 154, 155, 156, 158, 160, 163, 164, 165, 166, 168, 172, 174, 176, 179, 180, 185, 186, 188, 189, 190, 191, 192, 193, 194, 198, 199, 206, 209, 211, 212, 213, 216, 221, 222, 223, 227, 228, 230 mod 233				
[1/2, -1/2, 1/2, 1, 1/2, 3/2, -3/2, 1, -1, -3, 1, 1, -1]				
[3, 311, 311, -310, -2, -2], [11, 87, 255, -82, -6, -10], [12, 79, 236, 4, 12, 8], [20, 95, 140, -92, -4, -12], [24, 39, 239, 2, 24, 4], [24, 43, 239, 10, 24, 20], [27, 39, 244, -28, -16, -22], [35, 80, 84, 28, 24, 4], [39, 80, 96, -68, -8, -36], [40, 47, 119, 2, 20, 8], [44, 68, 87, -36, -20, -28], [47, 68, 87, 36, 38, 40], [48, 59, 79, 2, 16, 12]				

TABLE 3. (continued) Some of the ternary forms from [Rodriquez-Villegas and Tornaria 04].

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REFERENCES

[Baruch and Mao 03] E. Baruch and Z. Mao. "Central Values of Automorphic L-Functions." Preprint, 2003.

[Bogomolny and Keating 95] E.B. Bogomolny and J.P. Keating. "Random Matrix Theory and the Riemann Zeros I: Three- and Four-Point Correlations." *Nonlinearity* 8 (1995), 1115–1131.

[Bogomolny and Keating 96] E.B. Bogomolny and J.P. Keating. "Random Matrix Theory and the Riemann Zeros II: n-Point Correlations." *Nonlinearity*, 9 (1996), 911–935.

[Breuil et al. 01] C. Breuil, B. Conrad, F. Diamond, and R. Taylor. "On the Modularity of Elliptic Curves over $\mathbb{Q}$." *J. Amer. Math. Soc.* 14:4 (2001), 843–939.

[Conrey 01] J.B. Conrey. "L-Functions and Random Matrices." In *Mathematics Unlimited—2001 and Beyond*, pp. 331–352. Berlin: Springer, 2001.

[Conrey and Farmer 00] J.B. Conrey and D.W. Farmer. "Mean Values of L-Functions and Symmetry." *Int. Math. Res. Notices* 17 (2000), 883–908.

[Conrey et al. 02] J.B. Conrey, J.P. Keating, M.O. Rubinstein, and N.C. Snaith. "On the Frequency of Vanishing of Quadratic Twists of Modular L-Functions." In *Number Theory for the Millennium I: Proceedings of the Millennial Conference on Number Theory*, edited by, M. A. Bennett et al., pp. 301–315. Natick: A K Peters, Ltd., 2002.

[Conrey et al. 03] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, and N.C. Snaith. "Autocorrelation of Random Matrix Polynomials." *Comm. Math. Phys.* 237:3 (2003), 365–395.

[Conrey et al. 05] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, and N.C. Snaith. "Integral Moments of L-Functions." *Proceedings of the London Mathematical Society* 91 (2005), 33–104.

[Conrey et al. 06] B. Conrey, A. Pokharel, M. Rubinstein, and M. Watkins. "Secondary Terms in the Number of Vanishings of Quadratic Twists of Elliptic Curve L-Functions." In *Ranks of Elliptic Curves and Random Matrix Theory*, edited by J. B. Conrey, D. W. Farmer, F. Mezzadri, and N. C. Snaith. Cambridge: Cambridge University Press, 2006.

[Cremona] J. Cremona. *Algorithms for Modular Elliptic Curves.* Available at http://www.maths.nott.ac.uk/personal/jec.

[Delaunay 01] C. Delaunay. "Heuristics on Tate–Shafarevitch Groups of Elliptic Curves Defined over $\mathbb{Q}$." *Experimental Math.* 10:2 (2001), 191–196.

[Delaunay 06] C. Delaunay. "Note on the Frequency of Vanishing of L-Functions of Elliptic Curves in a Family of Quadratic Twists." In *Ranks of Elliptic Curves and Random Matrix Theory*, edited by J.B. Conrey, D.W. Farmer, F. Mezzadri, and N.C. Snaith. Cambridge: Cambridge University Press, 2006.

[Hughes et al. 00] C.P. Hughes, J.P. Keating, and N. O'Connell. "Random Matrix Theory and the Derivative of the Riemann Zeta Function." *Proc. R. Soc. Lond. A* 456 (2000), 2611–2627.

[Hughes et al. 01] C.P. Hughes, J.P. Keating, and N. O'Connell. "On the Characteristic Polynomial of a Random Unitary Matrix." *Commun. Math. Phys.* 220:2 (2001), 429–451.

[Katz and Sarnak 99] N.M. Katz and P. Sarnak. *Random Matrices, Frobenius Eigenvalues and Monodromy*, American Mathematical Society Colloquium Publications, 45. Providence: American Mathematical Society, 1999.

[Keating and Snaith 00a] J.P. Keating and N.C. Snaith. "Random Matrix Theory and $\zeta\left(\frac{1}{2}+it\right)$." *Commun. Math. Phys.* 214 (2000), 57–89.

[Keating and Snaith 00b] J.P. Keating and N.C. Snaith. "Random Matrix Theory and L-Functions at $s=\frac{1}{2}$." *Commun. Math. Phys.* 214 (2000), 91–110.

[Keating and Snaith 03] J.P. Keating and N.C. Snaith. "Random matrices and L-functions." *J. Phys. A* 36:12 (2003), 2859–2881.

[Kohnen and Zagier 81] W. Kohnen and D. Zagier. "Values of L-Series of Modular Forms at the Center of the Critical Strip." *Invent. Math.* 64 (1981), 175–198.

[Mao et al. 06] Z. Mao, F. Rodriguez-Villegas, and G. Tornaria. "Computation of Central Value of Quadratic Twists of Modular L-Functions." In *Ranks of Elliptic Curves and Random Matrix Theory*, edited by J. B. Conrey, D. W. Farmer, F. Mezzadri, and N. C. Snaith, pp. 115–129. Cambridge: Cambridge University Press, 2006.

[Montgomery 73] H.L. Montgomery. "The Pair Correlation of the Zeta Function." *Proc. Symp. Pure Math.* 24 (1973), 181–93.

[Odlyzko 89] A.M. Odlyzko, "The 10^{20}th Zero of the Riemann Zeta Function and 70 Million of Its Neighbors." Preprint, 1989.

[Rodriquez-Villegas and Tornaria 04] F. Rodriguez-Villegas and G. Tornaria. Private communication, 2004.

[Rubinstein 98] M. Rubinstein. "Evidence for a Spectral Interpretation of Zeros of L-functions." PhD diss., Princeton University, 1998.

[Rubinstein 04] M. Rubinstein. "The L-Function Database." Available at www.math.uwaterloo.ca/˜mrubinst.

[Rudnick and Sarnak 96] Z. Rudnick and P. Sarnak, "Zeros of Principal L-Functions and Random Matrix Theory." *Duke Mathematical Journal* 81:2 (1996), 269–322.

[Selberg 46a] A. Selberg. "Contributions to the Theory of Dirichlet's L-Functions." *Skr. Norske Vid. Akad. Oslo* 1 (1946), 1–62.

[Selberg 46b] A. Selberg. "Contributions to the Theory of the Riemann Zeta-Function." *Arch. for Math. og Naturv. B* 48:5), 1946.

[Shimura 73] G. Shimura. "On Modular Forms of Half Integral Weight." *Ann. Math.* 97:2 (1973), 440–481.

[Taylor and Wiles 95] R. Taylor and A. Wiles. "Ring-Theoretic Properties of Certain Hecke Algebras." *Ann. of Math. (2)* 141:3 (1995), 553–572.

[Waldspurger 1981] J.-L. Waldspurger. "Sur les coefficients de Fourier des formes modulaires de poids demi-entier." *J. Math. Pures Appl.* 60:9 (1981), 375–484.

[Wiles 95] A. Wiles. "Modular Elliptic Curves and Fermat's Last Theorem." *Ann. of Math. (2)* 141:3 (1995), 443–551.

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