Commun. Math. Phys. 267, 611–629 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-006-0090-5

Communications in
**Mathematical
Physics**

Moments of the Derivative of Characteristic Polynomials with an Application to the Riemann Zeta Function

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Received: 25 August 2005 / Accepted: 21 April 2006

Abstract: We investigate the moments of the derivative, on the unit circle, of characteristic polynomials of random unitary matrices and use this to formulate a conjecture for the moments of the derivative of the Riemann ζ function on the critical line. We do the same for the analogue of Hardy's Z-function, the characteristic polynomial multiplied by a suitable factor to make it real on the unit circle. Our formulae are expressed in terms of a determinant of a matrix whose entries involve the I-Bessel function and, alternately, by a combinatorial sum.

1. Introduction

Characteristic polynomials of unitary matrices serve as extremely useful models for the Riemann zeta-function $\zeta(s)$. The distribution of their eigenvalues give insight into the distribution of zeros of the Riemann zeta-function and the values of these characteristic polynomials give a model for the value distribution of $\zeta(s)$. See the works [KS] and [CFKRS] for detailed descriptions of how these models work. The important fact is that formulas for the moments of the Riemann zeta-function are suggested by the moments of the characteristic polynomials of unitary matrices.

We consider two problems here: the moments of the derivative of the characteristic polynomial $\Lambda_A(s)$ of an $N \times N$ unitary matrix A, and also the moments of the analogue of the derivative of Hardy's Z-function, the characteristic polynomial multiplied by a suitable factor to make it real on the unit circle.

In its simplest form our problem is to give an exact formula, valid for complex r with $\Re r > 0$, of the moments of the absolute value of the derivative of characteristic polynomials

$$\int_{U(N)} |\Lambda_A'(1)|^r \, dA_N \tag{1.1}$$

or of

$$\int_{U(N)} |\mathcal{Z}_A'(1)|^r dA_N. \tag{1.2}$$

Here we are integrating against Haar measure on the unitary group, and $\mathcal{Z}_A(s)$ is equal to $\Lambda_A(s)$ times a rotation factor that makes it real on the unit circle. See the next section for the precise definition.

Unfortunately, we cannot yet solve either of these problems. However, we can give asymptotic formulas when $r = 2k$ for positive integer values of k. The first two of these formulas involve the Maclaurin series coefficients of a certain $k \times k$ determinant, while the third involves a combinatorial sum.

Theorem 1. *For fixed k and $N \to \infty$ we have*

$$\int_{U(N)} |\Lambda_A'(1)|^{2k} dA_N \sim b_k N^{k^2+2k}, \tag{1.3}$$

where

$$b_k = (-1)^{k(k+1)/2} \sum_{h=0}^{k} \binom{k}{h} \left(\frac{d}{dx}\right)^{k+h} \left(e^{-x} x^{-k^2/2} \det_{k\times k} \left(I_{i+j-1}(2\sqrt{x})\right)\right)\Bigg|_{x=0}, \tag{1.4}$$

and $I_\nu(z)$ denotes the modified Bessel function of the first kind.

Theorem 2. *For fixed k and $N \to \infty$ we have*

$$\int_{U(N)} |\mathcal{Z}_A'(1)|^{2k} dA_N \sim b_k' N^{k^2+2k}, \tag{1.5}$$

where

$$b_k' = (-1)^{k(k+1)/2} \left(\frac{d}{dx}\right)^{2k} \left(e^{-\frac{x}{2}} x^{-k^2/2} \det_{k\times k} \left(I_{i+j-1}(2\sqrt{x})\right)\right)\Bigg|_{x=0}. \tag{1.6}$$

We also have combinatorial description of b_k'.

Theorem 3.

$$b_k' = (-1)^{k(k+1)/2} \sum_{m \in P_O^{k+1}(2k)} \binom{2k}{m} \left(\frac{-1}{2}\right)^{m_0} \times \left(\prod_{i=1}^{k} \frac{1}{(2k-i+m_i)!}\right) \left(\prod_{1\le i<j\le k} (m_j - m_i + i - j)\right), \tag{1.7}$$

where $P_O^{k+1}(2k)$ denotes the set of partitions $m = (m_0, \ldots, m_k)$ of $2k$ into $k+1$ non-negative parts.

We have computed some values of b_k and b_k'; these are tabulated at the end of the paper.

Applying the random matrix theory philosophy suggests the conjecture:

Conjecture 1.

$$\frac{1}{T}\int_0^T |\zeta'(1/2+it)|^{2k}dt \sim a_k b_k \log(T)^{k^2+2k}, \tag{1.8}$$

and, similarly for Hardy's Z function,

$$\frac{1}{T}\int_0^T |Z'(1/2+it)|^{2k}dt \sim a_k b'_k \log(T)^{k^2+2k}, \tag{1.9}$$

where a_k is the arithmetic factor

$$a_k = \prod_p \left(1-\tfrac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}. \tag{1.10}$$

Remarks.

(1) The factor a_k is the same arithmetic contribution that arises in the moments of the Riemann zeta function itself, see [KS] or [CFKRS]. For an explanation of why these moments factor, asymptotically, into the product of a contribution from the primes, a_k, and a coefficient calculated via random matrix theory, see [GHK].

(2) In this paper we are only concerned with the leading asymptotics of the moments of $\zeta'(1/2+it)$ and $Z'(1/2+it)$. Consequently, we use the k-fold integrals for moments given below in Lemma 3. If one wishes to study lower order terms one would need to use the full moment conjecture for ζ and Z given in [CFKRS] as a $2k$ fold integral.

(3) Forrester and Witte have taken our Theorems 1-2 and managed to find an alternate expression for b_k and b'_k involving a Painlevé III′ equation, and also an expression involving a certain generalised hypergeometric function [FW, Sect. 5].

(4) In his PhD thesis, Chris Hughes gives a similar conjecture for a more general problem involving mixed moments [Hug, Conj. 6.1]. What is new in our paper are the formulas for b_k and b'_k. For comparison, we state Hughes' formulation of the conjecture for Hardy's Z function. Let

$$I(h,k) = \int_0^T |Z(t)|^{2k-2h}\left|Z'(t)\right|^{2h}\, dt.$$

Hughes conjectures that

$$I(h,k) \sim B(h,k) a_k f_k T (\log T)^{k^2+2h},$$

where a_k is given above,

$$f_k = \prod_{j=0}^{k-1} \frac{j!}{(k+j)!},$$

and $B(h,k)$ is the constant that Hughes obtains from the analogous moments for $\mathcal{Z}_A$ via random matrix theory:

$$B(h,k) = \lim_{\beta\to 0} \frac{1}{\beta^{2h}} \sum_{n=0}^{2h} (-1)^{n-h} \binom{2h}{n} e^{-n\beta/2} \det\{b_{i,j}\},$$

where

$$b_{i,j} = \sum_{m=0}^{2h} \frac{(2k-n+i-1)!}{(2k-n+i-1+m)!} \binom{i+k-n-1+m}{m} \binom{i+m-1}{j-1} \beta^m.$$

To get Hughes' conjecture in this form, see (6.60), (6.51), and (6.52) of [Hug] and replace β by $\beta/(iN)$.

(5) Brezin and Hikami [BH] attempt to use a similar approach to obtain a theorem for moments of derivatives of characteristic polynomials, but there is an error in their paper.

(6) Numerically, we have observed that $b_k \sim 4^{-k} f_k$, as $k \to \infty$. Other than the power of 4, the r.h.s. is the constant that appears in the moments of characteristic polynomials [KS]:

$$\int_{U(N)} |\Lambda_A(1)|^{2k} dA_N \sim f_k N^{k^2}, \quad \text{as } N \to \infty.$$

A heuristic explanation is as follows. The large values of $|\Lambda_A'(1)|$ occur near the large values of $|\Lambda_A(1)|$, namely when all the eigenvalues are close to -1. The derivative of Λ_A involves a sum of N terms, each of which is missing, when the eigenvalues are close to -1, one factor of size roughly 2. A comparison with the $2k^{\text{th}}$ moment of $|\Lambda_A|$ thus gives an extra N^{2k} and a factor of 2^{-2k}. We have not attempted to make this argument rigorous. We have also not attempted to show that b_k and b_k' are non-zero.

(7) The problem of moments of the derivative, is related, through Jensen's formula, to the problem of zeros of the derivative. This approach requires knowledge of the complex moments of the derivative and we are only able to obtain integer moments. For characteristic polynomials one is interested in studying the radial distribution of the zeros of the derivative. Francesco Mezzadri has the best results in this direction [Mez]. On the number theory side, one is interested in the horizontal distribution of the zeros of ζ'. Partial results have been obtained by Levinson and Montgomery [LM], Conrey and Ghosh [CG], Soundararajan [Sou], and Zhang [Z].

2. Notation

If A is an $N \times N$ matrix with complex entries $A = (a_{jk})$, we let A^* be its conjugate transpose, i.e. $A^* = (b_{jk})$, where $b_{jk} = \overline{a_{kj}}$. A is said to be unitary if $AA^* = I$. We let $U(N)$ denote the group of all $N \times N$ unitary matrices. This is a compact Lie group and has a Haar measure.

All of the eigenvalues of $A \in U(N)$ have absolute value 1; we write them as

$$e^{i\theta_1}, e^{i\theta_2}, \ldots e^{i\theta_N}. \tag{2.1}$$

The eigenvalues of A^* are $e^{-i\theta_1}, \ldots, e^{-i\theta_N}$. Clearly, the determinant, $\det A = \prod_{n=1}^{N} e^{i\theta_n}$ of a unitary matrix is a complex number with absolute value equal to 1.

We are interested in computing various statistics about these eigenvalues. Consequently, we identify all matrices in $U(N)$ that have the same set of eigenvalues. The collection of matrices with the same set of eigenvalues constitutes a conjugacy class in

$U(N)$. Weyl's integration formula [Weyl, p. 197] gives a simple way to perform averages over $U(N)$ for functions f that are constant on conjugacy classes. Weyl's formula asserts that for such an f,

$$\int_{U(N)} f(A)\, d\text{Haar} = \int_{[0,2\pi]^N} f(\theta_1, \ldots, \theta_N) dA_N, \tag{2.2}$$

where

$$dA_N = \prod_{1 \le j < k \le N} \left|e^{i\theta_k} - e^{i\theta_j}\right|^2 \frac{d\theta_1 \ldots d\theta_N}{N!(2\pi)^N}. \tag{2.3}$$

The characteristic polynomial of a matrix A is denoted $\Lambda_A(s)$ and is defined by

$$\Lambda_A(s) = \det(I - sA^*) = \prod_{n=1}^{N} (1 - se^{-i\theta_n}). \tag{2.4}$$

The roots of $\Lambda_A(s)$ are the eigenvalues of A and are on the unit circle. Notice that this definition of the characteristic polynomial differs slightly from the usual definition in that it has an extra factor of $\det(A^*)$. We regard $\Lambda_A(s)$ as an analogue of the Riemann zeta-function and this definition is chosen so as to resemble the Hadamard product of ζ.

The characteristic polynomial satisfies the functional equation

$$\begin{aligned} \Lambda_A(s) &= (-s)^N \prod_{n=1}^{N} e^{-i\theta_n} \prod_{n=1}^{N} (1 - e^{i\theta_n}/s) && (2.5) \\ &= (-1)^N \det A^*\, s^N\, \Lambda_{A^*}(1/s). && (2.6) \end{aligned}$$

We define the $\mathcal{Z}$-function by

$$\mathcal{Z}_A(s) = e^{-\pi i N/2} e^{i \sum_{n=1}^{N} \theta_n/2} s^{-N/2} \Lambda_A(s); \tag{2.7}$$

here if N is odd, we use the branch of the square-root function that is positive for positive real s. The functional equation for $\mathcal{Z}$ is

$$\mathcal{Z}_A(s) = (-1)^N \mathcal{Z}_{A^*}(1/s). \tag{2.8}$$

Note that

$$\overline{\mathcal{Z}_A(e^{i\theta})} = \mathcal{Z}_A(e^{i\theta}) \tag{2.9}$$

so that $\mathcal{Z}_A(e^{i\theta})$ is real when θ is real. We regard $\mathcal{Z}_A(e^{i\theta})$ as an analogue of Hardy's function $Z(t)$.

We let I_n be the usual modified Bessel function with power series expansion

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j}(n+j)!j!}. \tag{2.10}$$

The way that the I-Bessel function enters our calculation is through the following formula:

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{e^{Lz+t/z}}{z^{2k}}\, dz = \frac{L^{2k-1} I_{2k-1}(2\sqrt{Lt})}{(Lt)^{k-1/2}}. \tag{2.11}$$

This formula can be proven by comparing the coefficient of z^{2k-1} in $e^{Lz+t/z}$ with the power series formula for I_{2k-1}.

We let $\Delta(z_1,\dots,z_k)$ denote the Vandermonde determinant

$$\Delta(z_1,\dots,z_k)=\det_{k\times k}\left(z_i^{j-1}\right). \tag{2.12}$$

We often omit the subscripts and write $\Delta(z)$ in place of $\Delta(z_1,\dots,z_k)$. Also, we allow differential operators as the arguments, such as

$$\Delta\left(\frac{d}{dL}\right)=\Delta\left(\frac{d}{dL_1},\dots,\frac{d}{dL_k}\right)=\det_{k\times k}\left(\left(\frac{d}{dL_i}\right)^{j-1}\right). \tag{2.13}$$

The key fact about the Vandermonde is that

$$\Delta(z_1,\dots,z_k)=\prod_{1\le i<j\le k}(z_j-z_i). \tag{2.14}$$

We let

$$z(x)=\frac{1}{1-e^{-x}}=\frac{1}{x}+O(1). \tag{2.15}$$

The function $z(x)$ plays the role for random matrix theory that $\zeta(1+x)$ plays in the theory of moments of the Riemann zeta-function. See for example pp. 371–372 of [CFKRS2].

We let Ξ denote the subset of permutations $\sigma\in S_{2k}$ of $\{1,2,\dots,2k\}$ for which

$$\sigma(1)<\sigma(2)<\cdots<\sigma(k) \tag{2.16}$$

and

$$\sigma(k+1)<\sigma(k+2)<\cdots<\sigma(2k). \tag{2.17}$$

We let $P_O^{k+1}(2k)$ be the set of partitions $m=(m_0,\dots,m_k)$ of $2k$ into $k+1$ non-negative parts. This quantity arises from the multinomial expansion

$$(x_0+x_1+\cdots+x_k)^{2k}=\sum_{m\in P_O^{k+1}(2k)}\binom{2k}{m}x_0^{m_0}\cdots x_k^{m_k}, \tag{2.18}$$

where

$$\binom{2k}{m}=\frac{(2k)!}{m_0!\dots m_k!}. \tag{2.19}$$

3. Lemmas

The main tool in proving Theorems 1–2 is to take formulas (Lemma 3) for moments of characteristic polynomials with shifts, differentiate these with respect to the shifts, and then set the shifts equal to zero. This gives k-fold contour integrals. To separate the integrals involved, we introduce extra parameters and differential operators to pull out a portion of these integrands.

Lemma 1. *Assume that* $\alpha_1, \ldots, \alpha_{2k}$ *are distinct complex numbers. We have*

$$\int_{U(N)} \prod_{j=1}^{k} \Lambda_A(e^{-\alpha_j}) \Lambda_{A^*}(e^{\alpha_{j+k}})\, dA_N = \sum_{\sigma \in \Xi} e^{N \sum_{j=1}^{k} (\alpha_{\sigma(j)} - \alpha_j)} \prod_{1 \le i, j \le k} z(\alpha_{\sigma(j)} - \alpha_{\sigma(k+i)}). \tag{3.1}$$

This is proven in Sect. 2 of [CFKRS2]. See formulas (2.5), (2.16), and (2.21) of that paper. The definition given there of the characteristic polynomial differs slightly from the one we use here, and that introduces some extra exponential factors in (2.21) of the aforementioned paper, and also necessitates replacing the α's by $-\alpha$'s.

Since

$$\mathcal{Z}_A(e^{-\alpha_j}) \mathcal{Z}_{A^*}(e^{\alpha_{j+k}}) = (-1)^N e^{N(\alpha_j - \alpha_{j+k})/2} \Lambda_A(e^{-\alpha_j}) \Lambda_{A^*}(e^{\alpha_{j+k}}) \tag{3.2}$$

we can write a corresponding lemma for $\mathcal{Z}$.

Lemma 2. *Assume that* $\alpha_1, \ldots, \alpha_{2k}$ *are distinct complex numbers. Then*

$$\int_{U(N)} \prod_{j=1}^{k} \mathcal{Z}_A(e^{-\alpha_j}) \mathcal{Z}_{A^*}(e^{\alpha_{j+k}})\, dA_N = (-1)^{Nk} e^{-\frac{N}{2} \sum_{j=1}^{2k} \alpha_j} \sum_{\sigma \in \Xi} e^{N \sum_{j=1}^{k} \alpha_{\sigma(j)}} \prod_{1 \le i, j \le k} z(\alpha_{\sigma(j)} - \alpha_{\sigma(k+i)}). \tag{3.3}$$

We can express the sums in the last two lemmas as integrals. Thus we have

Lemma 3. *Assume that all of the* α_j *are smaller than* 1 *in absolute value. Then*

$$\int_{U(N)} \prod_{j=1}^{k} \Lambda_A(e^{-\alpha_j}) \Lambda_{A^*}(e^{\alpha_{j+k}})\, dA_N = \frac{1}{k!(2\pi i)^k} \int_{|w_i|=1} e^{N \sum_{j=1}^{k} (w_j - \alpha_j)} \prod_{\substack{1 \le i \le k \\ 1 \le j \le 2k}} z(w_i - \alpha_j) \prod_{i \ne j} z(w_i - w_j)^{-1} \prod_{j=1}^{k} dw_j \tag{3.4}$$

and

$$\begin{aligned}&\int_{U(N)}\prod_{j=1}^{k}\mathcal{Z}_A(e^{-\alpha_j})\mathcal{Z}_{A^*}(e^{\alpha_{j+k}})\,dA_N\\&=(-1)^{Nk}\frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{j=1}^{k}w_j}\prod_{\substack{1\le i\le k\\1\le j\le 2k}}z(w_i-\alpha_j)\prod_{i\neq j}z(w_i-w_j)^{-1}\prod_{j=1}^{k}dw_j.\end{aligned}\tag{3.5}$$

In this lemma, and its corollary below, we do not require the $\alpha_j's$ to be distinct. The proof of Lemma 3 is a straightforward evaluation of the residues in the integral in (3.4), arising from the factor $z(w_i-\alpha_j)$, to obtain the $\binom{2k}{k}$ terms in (3.1). Each of the k integrals in (3.4) results in a sum over $2k$ residues, but due to the factor $\prod_{i\neq j}z(w_i-w_j)^{-1}$, any one of these $2k^2$ terms is zero if the residue of two of the integrals, say w_i and w_j, are evaluated at the same point α_ℓ.

Using the fact that $z(w)=1/w+O(1)$ we easily deduce

Corollary 1. *Suppose that* $\alpha_j=\alpha_j(N)$ *and* $|\alpha_j|\ll 1/N$ *for each* j. *Then*

$$\begin{aligned}&\int_{U(N)}\prod_{j=1}^{k}\Lambda_A(e^{-\alpha_j})\Lambda_{A^*}(e^{\alpha_{j+k}})\,dA_N\\&=\frac{1}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{j=1}^{k}(w_j-\alpha_j)}\frac{\prod_{i\neq j}(w_i-w_j)}{\prod_{\substack{1\le i\le k\\1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^{k}dw_j+O(N^{k^2-1})\end{aligned}\tag{3.6}$$

with an implicit constant independent of N; *similarly*,

$$\begin{aligned}&\int_{U(N)}\prod_{j=1}^{k}\mathcal{Z}_A(e^{-\alpha_j})\mathcal{Z}_{A^*}(e^{\alpha_{j+k}})\,dA_N\\&=(-1)^{Nk}\frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{j=1}^{k}w_j}\frac{\prod_{i\neq j}(w_i-w_j)}{\prod_{\substack{1\le i\le k\\1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^{k}dw_j+O(N^{k^2-1}).\end{aligned}\tag{3.7}$$

Lemma 4. *Let* f *be* $k-1$ *times differentiable*, $k\ge 1$. *Then*

$$\Delta(\frac{d}{dL})\prod_{i=1}^{k}f(L_i)=\det_{k\times k}\left(f^{(j-1)}(L_i)\right),$$

where by $\Delta(d/dL)$ *we mean the differential operator*

$$\prod_{1\le i<j\le k}\left(\frac{d}{dL_j}-\frac{d}{dL_i}\right)=\det_{k\times k}\left(\frac{d^{j-1}}{dL_i^{j-1}}\right).\tag{3.8}$$

Proof. This follows using the definition of the Vandermonde determinant. Noticing that row i of the matrix only involves L_i, we factor the product into the determinant.

Lemma 5. *Let f be $2k-2$ times differentiable. Then*

$$\Delta^2\left(\frac{d}{dL}\right)\left(\prod_{i=1}^{k} f(L_i)\right)\Bigg|_{L_i=L} = k!\det_{k\times k}\left(f^{(i+j-2)}(L)\right). \tag{3.9}$$

More generally, suppose that $g(L_1,\dots,L_k)=\sum_{r=1}^{R} a_r \prod_{i=1}^{k} f_{r,i}(L_i)$ is a symmetric function of its k variables. Then

$$\Delta^2\left(\frac{d}{dL}\right)g(L_1,\dots,L_K)\Bigg|_{L_j=L} = k!\sum_{r=1}^{R} a_r \det_{k\times k}\left(f_{r,i}^{(i+j-2)}(L)\right). \tag{3.10}$$

Proof. Applying the Vandermonde a second time to Lemma 4 we get

$$\Delta\left(\frac{d}{dL}\right)\det_{k\times k}\left(f^{(j-1)}(L_i)\right). \tag{3.11}$$

Expand the determinant as a sum over all permutations μ of the numbers $1,2,\dots,k$:

$$\det_{k\times k}\left(f^{(j-1)}(L_i)\right) = \sum_{\mu}\operatorname{sgn}(\mu)\prod_{i=1}^{k} f^{\mu_i-1}(L_i). \tag{3.12}$$

Apply Lemma 4 to find that a typical term above equals

$$\operatorname{sgn}(\mu)\det_{k\times k}\left(f^{(\mu_i+j-2)}(L_i)\right). \tag{3.13}$$

Setting $L_i=L$ for $1\le i\le k$, we may rearrange the rows so as to undo the permutation μ. This introduces another $\operatorname{sgn}(\mu)$ in front of the determinant and gives

$$\det_{k\times k}\left(f^{(i+j-2)}(L)\right). \tag{3.14}$$

Since there are $k!$ permutations μ, we get

$$k!\det_{k\times k}\left(f^{(i+j-2)}(L)\right). \tag{3.15}$$

The proof of the second part of the lemma is left to the reader.

Lemma 6. *Suppose that P and Q are polynomials with $Q(w)=\prod_{j=1}^{2k}(w-\alpha_j)$ and $\max|\alpha_j|<c$. Then*

$$\frac{1}{2\pi i}\int_{|w|=c}\frac{e^{wL}}{w}\frac{P(w)}{Q(w)}dw = P\left(\frac{d}{dL}\right)\int_{\sum_{j=1}^{2k}x_j\le L} e^{\sum_{j=1}^{2k}x_j\alpha_j}\prod_{j=1}^{2k}dx_j. \tag{3.16}$$

Proof. Since

$$P\left(\frac{d}{dL}\right)e^{wL} = e^{wL}P(w), \tag{3.17}$$

the derivatives can be pulled outside the integral immediately. With the Laplace transform pair $e^{x\alpha}$ and $\frac{1}{w-\alpha}$, related by

$$e^{x\alpha} = \frac{1}{2\pi i}\int_{|w|=c}\frac{e^{wx}}{w-\alpha}dw, \tag{3.18}$$

we merely apply repeatedly the Laplace convolution formula, which for Laplace transform pairs f_i and ϕ_i states that

$$\frac{1}{2\pi i}\int_{|w|=c}\phi_1(s)\phi_2(s)e^{sx}ds=\int_0^x f_1(y)f_2(x-y)dy, \tag{3.19}$$

to evaluate the Laplace transform of the product $\frac{1}{w\,\prod_{j=1}^{2k}(w-\alpha_j)}$.

Lemma 7. *We have*

$$\int_{\sum_{j=1}^{2k}x_j\le L} x_1\dots x_n\prod_{j=1}^{2k}dx_j=\frac{L^{2k+n}}{(2k+n)!}. \tag{3.20}$$

This lemma can be proved in a straightforward manner by induction.

4. Proofs

We now give the proofs of our identities for the leading terms of the moments of the derivatives of Λ and $\mathcal{Z}$. We begin with the proof of Theorem 2 for $\mathcal{Z}$ as it is slightly easier.

Proof of Theorem 2. A differentiated form of the second formula of Corollary 1 gives us

$$\begin{aligned}&\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\int_{U(N)}\prod_{h=1}^{k}\mathcal{Z}_A(e^{-\alpha_h})\mathcal{Z}_{A^*}(e^{\alpha_{k+h}})dA_N\\&\quad=(-1)^{\frac{k(k-1)}{2}+kN}\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{j=1}^{k}w_j}\frac{\Delta^2(w)}{\prod_{\substack{1\le i\le k\\1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^{k}dw_j\\&\qquad+O(N^{k^2+2k-1}),\end{aligned} \tag{4.1}$$

provided that $\alpha_j=\alpha_j(N)\ll 1/N$. Notice that

$$\frac{d}{d\alpha}\mathcal{Z}_A(e^{-\alpha})\big|_{\alpha=0}=-\frac{d}{ds}\mathcal{Z}_A(s)\big|_{s=1}=-\mathcal{Z}_A'(1) \tag{4.2}$$

and

$$\frac{d}{d\alpha}\mathcal{Z}_{A^*}(e^{\alpha})\big|_{\alpha=0}=\mathcal{Z}_{A^*}'(1)=(-1)^N\overline{\mathcal{Z}_A'(1)}. \tag{4.3}$$

So,

$$\begin{aligned}&\int_{U(N)}|\mathcal{Z}_A'(1)|^{2k}dA_N\\&\quad=(-1)^{\frac{k(k+1)}{2}}\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{j=1}^{k}w_j}\frac{\Delta^2(w)}{\prod_{\substack{1\le i\le k\\1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^{k}dw_j\Bigg|_{\alpha=0}\\&\qquad+O(N^{k^2+2k-1}).\end{aligned} \tag{4.4}$$

The sign here arises as the $(-1)^{kN}$ from (4.3) cancels the same factor in (4.1), we have a $(-1)^k$ from (4.2) and we pick up the $(-1)^{\frac{k(k-1)}{2}}$ in (4.1) through writing the factor $\prod_{i\neq j}(w_i - w_j)$ in (3.7) as $\Delta^2(w)$ above.

To separate the integrals, we introduce extra parameters L_i and move the Vandermonde polynomial outside the integral as a differential operator, getting

$$\begin{aligned}&(-1)^{\frac{k(k+1)}{2}}\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\frac{\Delta^2(d/dL)e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k}\\ &\times\int_{|w_j|=1}\frac{e^{\sum_{i=1}^{k}L_i w_i}}{\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^{k}dw_j\Bigg|_{\alpha=0,L_i=N}\\ &+O(N^{k^2+2k-1}).\end{aligned}\tag{4.5}$$

Next, we observe that

$$\frac{d}{d\alpha}\frac{e^{-\frac{N}{2}\alpha}}{\prod_{1\le i\le k}(w_i-\alpha)}\Bigg|_{\alpha=0}=\frac{1}{\prod_{i=1}^{k}w_i}\left(\sum_{j=1}^{k}\frac{1}{w_j}-\frac{N}{2}\right)\tag{4.6}$$

so that (4.5) equals, without the O term,

$$(-1)^{\frac{k(k+1)}{2}}\frac{\Delta^2(d/dL)}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^{k}L_i w_i}\left(\sum_{j=1}^{k}\frac{1}{w_j}-\frac{N}{2}\right)^{2k}}{\prod_{i=1}^{k}w_i^{2k}}\prod_{j=1}^{k}dw_j\Bigg|_{L_i=N}.\tag{4.7}$$

Introducing another auxiliary variable t, this can be expressed as

$$(-1)^{\frac{k(k+1)}{2}}\frac{\Delta^2(d/dL)\left(\frac{d}{dt}\right)^{2k}e^{-Nt/2}}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^{k}L_i w_i+t/w_i}}{\prod_{i=1}^{k}w_i^{2k}}\prod_{j=1}^{k}dw_j\Bigg|_{L_i=N,t=0}.\tag{4.8}$$

This allows us to separate the integrals and we get

$$(-1)^{\frac{k(k+1)}{2}}\frac{\Delta^2(d/dL)\left(d/dt\right)^{2k}e^{-Nt/2}}{k!}\prod_{i=1}^{k}\left(\frac{1}{2\pi i}\int_{|w|=1}\frac{e^{L_i w+t/w}}{w^{2k}}dw\right)\Bigg|_{L_i=N,t=0}.\tag{4.9}$$

The integral evaluates to

$$\frac{L_i^{2k-1}I_{2k-1}(2\sqrt{L_i t})}{(L_i t)^{k-1/2}}\tag{4.10}$$

as noted earlier. Thus,

$$\int_{U(N)} |\mathcal{Z}'_A(1)|^{2k} dA_N$$
$$= (-1)^{\frac{k(k+1)}{2}} \frac{\Delta(d/dL)\big(d/dt\big)^{2k} e^{-Nt/2}}{k!} \left(\prod_{i=1}^{k} \frac{L_i^{2k-1} I_{2k-1}(2\sqrt{L_i t})}{(L_i t)^{k-1/2}} \right)\Bigg|_{L_i=N,t=0}$$
$$+ O(N^{k^2+2k-1}). \tag{4.11}$$

So, letting

$$f_t(L_i) = \frac{L_i^{2k-1} I_{2k-1}(2\sqrt{L_i t})}{(L_i t)^{k-1/2}}, \tag{4.12}$$

we have, by Lemma 5, that (4.11) equals

$$(-1)^{\frac{k(k+1)}{2}} \left(\frac{d}{dt}\right)^{2k} e^{-Nt/2} \left(\det_{k\times k} \big(f_t^{(i+j-2)}(N)\big) \right)\Bigg|_{t=0} + O(N^{k^2+2k-1}). \tag{4.13}$$

Now we see, from (2.10), that

$$f_t(L) = \sum_{r=0}^{\infty} \frac{t^r L^{2k-1+r}}{r!(2k-1+r)!}, \tag{4.14}$$

so that if $\mu \le 2k-1$, then

$$f^{(\mu)}(L) = \sum_{r=0}^{\infty} \frac{t^r L^{2k-1-\mu+r}}{r!(2k-1-\mu+r)!} = \left(\frac{L}{t}\right)^{(2k-1-\mu)/2} I_{2k-1-\mu}(2\sqrt{Lt}). \tag{4.15}$$

Therefore, (4.13) equals

$$(-1)^{\frac{k(k+1)}{2}} \left(\frac{d}{dt}\right)^{2k} e^{-Nt/2} \det_{k\times k} \left(\left(\frac{N}{t}\right)^{(2k+1-i-j)/2} I_{2k+1-i-j}(2\sqrt{Nt}) \right)\Bigg|_{t=0}$$
$$+O(N^{k^2+2k-1}). \tag{4.16}$$

Clearly $\det_k(a_{i,j}) = \det_k(a_{k+1-i,k+1-j})$, therefore (4.16) can be written, without the remainder term, as

$$(-1)^{\frac{k(k+1)}{2}} \left(\frac{d}{dt}\right)^{2k} e^{-Nt/2} \det_{k\times k} \left(\left(\frac{N}{t}\right)^{(i+j-1)/2} I_{i+j-1}(2\sqrt{Nt}) \right)\Bigg|_{t=0}. \tag{4.17}$$

If we substitute $x = Nt$, then $d/dt = Nd/dx$ and we get

$$(-1)^{\frac{k(k+1)}{2}} N^{2k} \left(\frac{d}{dx}\right)^{2k} e^{-x/2} \det_{k\times k} \left(\left(\frac{N^2}{x}\right)^{(i+j-1)/2} I_{i+j-1}(2\sqrt{x}) \right)\Bigg|_{x=0}$$
$$= (-1)^{\frac{k(k+1)}{2}} N^{k^2+2k} \left(\frac{d}{dx}\right)^{2k} \left(e^{-x/2} x^{-k^2/2} \det_{k\times k} \big(I_{i+j-1}(2\sqrt{x})\big) \right)\Bigg|_{x=0}, \tag{4.18}$$

since $\det_k(M^{i+j-1}a_{i,j}) = M^{k^2}\det_k(a_{i,j})$ as is seen by factoring M^j out of the j^{th} column and M^{i-1} out of the i^{th} row. This completes the proof of Theorem 2.

Proof of Theorem 1. Turning to Theorem 1's proof, we begin as before, but with a differentiated form of the first formula of Corollary 1:

$$\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\int_{U(N)}\prod_{h=1}^{k}\Lambda_A(e^{-\alpha_h})\Lambda_{A^*}(e^{\alpha_{k+h}})dA_N$$
$$=(-1)^{\frac{k(k-1)}{2}}\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\frac{1}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{j=1}^{k}(w_j-\alpha_j)}\frac{\Delta^2(w)}{\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^{k}dw_j$$
$$+O(N^{k^2+2k-1}), \tag{4.19}$$

provided that $\alpha_j \ll 1/N$. Now

$$\frac{d}{d\alpha}\Lambda_A(e^{-\alpha})\big|_{\alpha=0}=-\frac{d}{ds}\Lambda_A(s)\big|_{s=1}=-\Lambda'_A(1) \tag{4.20}$$

and

$$\frac{d}{d\alpha}\Lambda_{A^*}(e^{\alpha})\big|_{\alpha=0}=\Lambda'_{A^*}(1)=\overline{\Lambda'_A(1)}, \tag{4.21}$$

hence setting $\alpha=0$, (4.19) becomes

$$\int_{U(N)}|\Lambda'_A(1)|^{2k}dA_N$$
$$=(-1)^{\frac{k(k+1)}{2}}\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\frac{1}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{j=1}^{k}(w_j-\alpha_j)}\frac{\Delta^2(w)}{\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^{k}dw_j\Bigg|_{\alpha_j=0}$$
$$+O(N^{k^2+2k-1}). \tag{4.22}$$

Introducing variables L_i as before, the above equals, without the O term

$$(-1)^{\frac{k(k+1)}{2}}\frac{\prod_{j=1}^{2k}(d/d\alpha_j)\Delta^2(d/dL)}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^{k}(L_iw_i-N\alpha_i)}}{\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^{k}dw_j\Bigg|_{\alpha_j=0,L_i=N}. \tag{4.23}$$

Performing the differentiations with respect to the α_j leads us to

$$(-1)^{\frac{k(k+1)}{2}}\frac{\Delta^2(d/dL)}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^{k}L_iw_i}\left(\sum_{j=1}^{k}\frac{1}{w_j}-N\right)^k\left(\sum_{j=1}^{k}\frac{1}{w_j}\right)^k}{\prod_{i=1}^{k}w_i^{2k}}\prod_{j=1}^{k}dw_j\Bigg|_{L_i=N}. \tag{4.24}$$

Now we write

$$\left(\sum_{j=1}^{k}\frac{1}{w_j}-N\right)^k\left(\sum_{j=1}^{k}\frac{1}{w_j}\right)^k=\left(\sum_{j=1}^{k}\frac{1}{w_j}-N\right)^k\left(\sum_{j=1}^{k}\frac{1}{w_j}-N+N\right)^k$$
$$=\sum_{h=0}^{k}\binom{k}{h}N^{k-h}\left(\sum_{j=1}^{k}\frac{1}{w_j}-N\right)^{k+h}. \tag{4.25}$$

Introducing the auxiliary variable t, (4.24) can be expressed as

$$\begin{aligned}
&(-1)^{\frac{k(k+1)}{2}} \sum_{h=0}^{k} \binom{k}{h} N^{k-h} \frac{\Delta^2(d/dL)\left(\frac{d}{dt}\right)^{k+h} e^{-Nt}}{k!(2\pi i)^k} \\
&\quad \times \int_{|w_j|=1} \frac{e^{\sum_{i=1}^{k} L_i w_i + t/w_i}}{\prod_{i=1}^{k} w_i^{2k}} \prod_{j=1}^{k} dw_j \bigg|_{L_i=N,t=0} \\
&= (-1)^{\frac{k(k+1)}{2}} \sum_{h=0}^{k} \binom{k}{h} N^{k-h} \frac{\Delta^2(d/dL)\big(d/dt\big)^{k+h} e^{-Nt}}{k!} \\
&\quad \times \prod_{i=1}^{k} \left(\frac{1}{2\pi i} \int_{|w|=1} \frac{e^{L_i w + t/w}}{w^{2k}} dw \right) \bigg|_{L_i=N,t=0}. \qquad (4.26)
\end{aligned}$$

Proceeding as before we arrive at

$$\begin{aligned}
&\int_{U(N)} |\Lambda_A'(1)|^{2k} dA_N \\
&= (-1)^{\frac{k(k+1)}{2}} N^{k^2+2k} \sum_{h=0}^{k} \binom{k}{h} \left(\frac{d}{dx}\right)^{k+h} \left(e^{-x} x^{-k^2/2} \det_{k\times k} \big(I_{i+j-1}(2\sqrt{x}) \big) \right) \bigg|_{x=0} \\
&\quad + O(N^{k^2+2k-1}). \qquad (4.27)
\end{aligned}$$

Proof of Theorem 3. We now give the proof of Theorem 3. We rewrite Eq. (4.1) as

$$\begin{aligned}
&\prod_{j=1}^{2k} \frac{d}{d\alpha_j} \int_{U(N)} \prod_{h=1}^{k} \mathcal{Z}_A(e^{-\alpha_h}) \mathcal{Z}_{A^*} dA_N \\
&= (-1)^{\frac{k(k-1)}{2}+kN} \prod_{j=1}^{2k} \frac{d}{d\alpha_j} \frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k} \int_{|w_i|=1} e^{N\sum_{i=1}^{k} w_i} \frac{\Delta^2(w)\prod_{i=1}^{k} w_i}{\prod_{\substack{1\le i\le k \\ 1\le j\le 2k}} (w_i-\alpha_j)} \prod_{i=1}^{k} \frac{dw_i}{w_i} \\
&\quad + O(N^{k^2+2k-1}). \qquad (4.28)
\end{aligned}$$

Introducing variables L_i as before, we can rewrite the main term above as

$$\begin{aligned}
&\frac{(-1)^{\frac{k(k-1)}{2}+kN}}{k!} \prod_{j=1}^{2k} \frac{d}{d\alpha_j} e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j} \\
&\quad \times \Delta^2\left(\frac{d}{dL}\right) \prod_{i=1}^{k} \left(\frac{d}{dL_i}\right) \prod_{i=1}^{k} \left(\frac{1}{2\pi i} \int_{|w|=1} \frac{e^{L_i w}}{\prod_{j=1}^{2k}(w-\alpha_j)} \frac{dw}{w} \right). \qquad (4.29)
\end{aligned}$$

Now, by Lemma 6, the integral is

$$\int_{\sum_{j=1}^{2k} x_j \le L_i} e^{\sum_{j=1}^{2k} x_j \alpha_j} \prod_{1\le j\le 2k} dx_j. \qquad (4.30)$$

Letting the variables in the i^{th} integral be $x_{i,j}$ we may express the product of the k integrals as

$$\int_{\sum_{j=1}^{2k} x_{1,j} \le L_1} \cdots \int_{\sum_{j=1}^{2k} x_{k,j} \le L_k} e^{\sum_{i=1}^{k} \sum_{j=1}^{2k} x_{i,j} \alpha_j} \prod_{\substack{1 \le i \le k \\ 1 \le j \le 2k}} dx_{i,j}. \tag{4.31}$$

We incorporate the factor $e^{-\frac{N}{2} \sum_{j=1}^{2k} \alpha_j}$ into this product and have

$$\int_{\sum_{j=1}^{2k} x_{1,j} \le L_1} \cdots \int_{\sum_{j=1}^{2k} x_{k,j} \le L_k} e^{\sum_{j=1}^{2k} \alpha_j \left(\sum_{i=1}^{k} x_{i,j} - N/2 \right)} \prod_{\substack{1 \le i \le k \\ 1 \le j \le 2k}} dx_{i,j}. \tag{4.32}$$

We differentiate this product of integrals with respect to each α_j and set each α_j equal to 0 yielding

$$\int_{\sum_{j=1}^{2k} x_{1,j} \le L_1} \cdots \int_{\sum_{j=1}^{2k} x_{k,j} \le L_k} \prod_{j=1}^{2k} \left(\sum_{i=1}^{k} x_{i,j} - \frac{N}{2} \right) \prod_{\substack{1 \le i \le k \\ 1 \le j \le 2k}} dx_{i,j}. \tag{4.33}$$

We want to compute this integral by multiplying out the product and using Lemma 7. A good way to think about this is as follows. By Eq. (2.18)

$$(A_1 + \cdots + A_k - A)^{2k} = \sum_{m \in P_O^{k+1}(2k)} \binom{2k}{m} (-A)^{m_0} A_1^{m_1} \cdots A_k^{m_k}. \tag{4.34}$$

When we multiply out the product we will have a sum of $(k+1)^{2k}$ terms, each term being a product of some number of factors $(-N/2)$ and $x_{i,j}$. Let $m \in P_O^{k+1}(2k)$ represent a generic term in which $(-N/2)$ appears m_0 times, and factors $x_{1,j}$ appear for m_1 values of j, and $x_{2,j}$ for m_2 values of j and so on. When we apply Lemma 7 to this term, when we perform the integration over the variables $x_{1,j}$ the answer is solely determined by m_1, the number of different $x_{1,j}$ that appear in this term. Therefore, we find that the product of integrals evaluates as

$$\sum_{m \in P_O^{k+1}(2k)} \binom{2k}{m} \left(-\frac{N}{2} \right)^{m_0} \frac{L_1^{2k+m_1}}{(2k+m_1)!} \cdots \frac{L_k^{2k+m_k}}{(2k+m_k)!}. \tag{4.35}$$

We now have that the quantity in Eq. (4.29) is equal to

$$\frac{(-1)^{\frac{k(k-1)}{2}+kN}}{k!} \times \Delta^2\left(\frac{d}{dL}\right) \prod_{i=1}^{k} \left(\frac{d}{dL_i}\right) \sum_{m \in P_O^{k+1}(2k)} \binom{2k}{m} \left(-\frac{N}{2} \right)^{m_0} \frac{L_1^{2k+m_1}}{(2k+m_1)!} \cdots \frac{L_k^{2k+m_k}}{(2k+m_k)!}. \tag{4.36}$$

Now we need to carry out the differentiations with respect to the L_i and set the L_i equal to N. We perform the differentiations $\prod_{i=1}^{k} d/dL_i$ and obtain

$$\frac{(-1)^{\frac{k(k-1)}{2}+kN}}{k!}\Delta^2\left(\frac{d}{dL}\right)\sum_{m\in P_O^{k+1}(2k)}\binom{2k}{m}\left(-\frac{N}{2}\right)^{m_0}\frac{L_1^{2k-1+m_1}}{(2k-1+m_1)!}\cdots\frac{L_k^{2k+m_k}}{(2k+m_k)!}. \tag{4.37}$$

Now the sum over $m_1,\ldots,m_k$ is a symmetric function of the variables L_i. Therefore, we can apply the second part of Lemma 5 to obtain that the above, evaluated at $L_i = N$ is

$$\begin{aligned}&\int_{U(N)}|\mathcal{Z}_A'(1)|^{2k}dA_N\\ &\quad=(-1)^{\frac{k(k+1)}{2}}\sum_{m\in P_O^{k+1}(2k)}\binom{2k}{m}\left(-\frac{N}{2}\right)^{m_0}\det_{k\times k}\left(\frac{N^{2k+1+m_i-i-j}}{(2k+1+m_i-i-j)!}\right)+O(N^{k^2+2k-1}),\end{aligned} \tag{4.38}$$

which we rewrite as

$$(-1)^{\frac{k(k+1)}{2}}N^{k^2+2k}\sum_{m\in P_O^{k+1}(2k)}\binom{2k}{m}(-2)^{-m_0}\det_{k\times k}\left(\frac{1}{(2k+1+m_i-i-j)!}\right)+O(N^{k^2+2k-1}). \tag{4.39}$$

Here the signs work out as in (4.4). We factor $1/(2k-i+m_i)!$ out of the i^{th} row. The remaining determinant has i^{th} row

$$1,\,(2k-i+m_i),\,(2k-i+m_i)(2k-i-1+m_i),\,\ldots,\,\prod_{j=1}^{k-1}(2k-i-j+1+m_i). \tag{4.40}$$

This determinant is a polynomial in the m_i of degree $0+1+\cdots+(k-1)=k(k-1)/2$ which vanishes whenever $m_j-m_i=j-i$; moreover the part of it with degree $k(k-1)/2$ is precisely $\Delta(m_1,\ldots,m_k)=\prod_{1\le i<j\le k}(m_j-m_i)$. Consequently the determinant evaluates to

$$\prod_{1\le i<j\le k}(m_j-m_i-j+i). \tag{4.41}$$

This concludes the evaluation of b_k'.

5. Numerical Evaluation of b_k and b_k'

We have the following values for b_k:

$$b_1 = \frac{1}{3}$$

$$b_2 = \frac{61}{2^5 \cdot 3^2 \cdot 5 \cdot 7}$$

$$b_3 = \frac{277}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11}$$

$$b_4 = \frac{2275447}{2^{18} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13}$$

$$b_5 = \frac{3700752773}{2^{26} \cdot 3^{14} \cdot 5^6 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19}$$

$$b_6 = \frac{3654712923689}{2^{39} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23}$$

$$b_7 = \frac{53 \cdot 13008618017 \cdot 143537}{2^{50} \cdot 3^{28} \cdot 5^{13} \cdot 7^8 \cdot 11^5 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23}$$

$$b_8 = \frac{41 \cdot 359 \cdot 5505609492791 \cdot 3637}{2^{68} \cdot 3^{35} \cdot 5^{16} \cdot 7^{11} \cdot 11^6 \cdot 13^5 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31}$$

$$b_9 = \frac{757 \cdot 45742439 \cdot 60588179 \cdot 13723}{2^{84} \cdot 3^{42} \cdot 5^{21} \cdot 7^{14} \cdot 11^8 \cdot 13^6 \cdot 17^4 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31}$$

$$b_{10} = \frac{652071900673 \cdot 241845775551409}{2^{105} \cdot 3^{55} \cdot 5^{25} \cdot 7^{17} \cdot 11^{10} \cdot 13^8 \cdot 17^5 \cdot 19^4 \cdot 23^3 \cdot 29 \cdot 37}$$

$$b_{11} = \frac{1318985497 \cdot 5786011415980412140 11811}{2^{121} \cdot 3^{64} \cdot 5^{31} \cdot 7^{19} \cdot 11^{12} \cdot 13^9 \cdot 17^7 \cdot 19^6 \cdot 23^4 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43}$$

$$b_{12} = \frac{113 \cdot 20648963338644792017 5141 \cdot 51839 \cdot 14831}{2^{150} \cdot 3^{75} \cdot 5^{37} \cdot 7^{23} \cdot 11^{15} \cdot 13^{12} \cdot 17^7 \cdot 19^7 \cdot 23^5 \cdot 29^3 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47}$$

$$b_{13} = \frac{46707540694046228719040680670896352548 38677}{2^{174} \cdot 3^{90} \cdot 5^{42} \cdot 7^{28} \cdot 11^{17} \cdot 13^{14} \cdot 17^{10} \cdot 19^9 \cdot 23^6 \cdot 29^3 \cdot 31^3 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47}$$

$$b_{14} = \frac{107 \cdot 1949460466884555953467 79341 \cdot 996075171809335069}{2^{203} \cdot 3^{103} \cdot 5^{50} \cdot 7^{31} \cdot 11^{20} \cdot 13^{17} \cdot 17^{12} \cdot 19^{10} \cdot 23^7 \cdot 29^4 \cdot 31^4 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47 \cdot 53}$$

$$b_{15} = \frac{29547975377 \cdot 3981541 \cdot 1807995588661527603489333681461 \cdot 1584311}{2^{230} \cdot 3^{117} \cdot 5^{57} \cdot 7^{37} \cdot 11^{22} \cdot 13^{19} \cdot 17^{14} \cdot 19^{12} \cdot 23^{9} \cdot 29^{5} \cdot 31^{5} \cdot 37^{3} \cdot 41^{2} \cdot 43^{2} \cdot 47 \cdot 53 \cdot 59}$$

We have the following values for b'_k:

$$b'_1 = \frac{1}{2^2 \cdot 3}$$

$$b'_2 = \frac{1}{2^6 \cdot 3 \cdot 5 \cdot 7}$$

$$b'_3 = \frac{1}{2^{12} \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11}$$

$$b'_4 = \frac{31}{2^{20} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13}$$

$$b'_5 = \frac{227}{2^{30} \cdot 3^{12} \cdot 5^6 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19}$$

$$b'_6 = \frac{67 \cdot 1999}{2^{42} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23}$$

$$b'_7 = \frac{43 \cdot 46663}{2^{56} \cdot 3^{28} \cdot 5^{13} \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23}$$

$$b'_8 = \frac{46743947}{2^{72} \cdot 3^{34} \cdot 5^{16} \cdot 7^{11} \cdot 11^6 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31}$$

$$b'_9 = \frac{19583 \cdot 16249}{2^{90} \cdot 3^{42} \cdot 5^{21} \cdot 7^{14} \cdot 11^8 \cdot 13^6 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31}$$

$$b'_{10} = \frac{3156627824489}{2^{110} \cdot 3^{55} \cdot 5^{25} \cdot 7^{17} \cdot 11^{10} \cdot 13^8 \cdot 17^5 \cdot 19^4 \cdot 23^3 \cdot 29 \cdot 31 \cdot 37}$$

$$b'_{11} = \frac{59 \cdot 11332613 \cdot 33391}{2^{132} \cdot 3^{63} \cdot 5^{31} \cdot 7^{18} \cdot 11^{12} \cdot 13^{10} \cdot 17^5 \cdot 19^5 \cdot 23^4 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43}$$

$$b'_{12} = \frac{241 \cdot 251799899121593}{2^{156} \cdot 3^{75} \cdot 5^{37} \cdot 7^{23} \cdot 11^{15} \cdot 13^{12} \cdot 17^8 \cdot 19^7 \cdot 23^4 \cdot 29^3 \cdot 31^2 \cdot 41 \cdot 43 \cdot 47}$$

$$b'_{13} = \frac{285533 \cdot 37408704134429}{2^{182} \cdot 3^{90} \cdot 5^{42} \cdot 7^{28} \cdot 11^{17} \cdot 13^{14} \cdot 17^{10} \cdot 19^8 \cdot 23^5 \cdot 29^3 \cdot 31^3 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47}$$

$$b'_{14} = \frac{197 \cdot 1462253323 \cdot 6616773091}{2^{210} \cdot 3^{100} \cdot 5^{50} \cdot 7^{31} \cdot 11^{20} \cdot 13^{17} \cdot 17^{12} \cdot 19^{10} \cdot 23^7 \cdot 29^4 \cdot 31^4 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47 \cdot 53}$$

$$b'_{15} = \frac{16255375825174687265195458 37}{2^{240} \cdot 3^{117} \cdot 5^{57} \cdot 7^{37} \cdot 11^{22} \cdot 13^{19} \cdot 17^{14} \cdot 19^{11} \cdot 23^9 \cdot 29^5 \cdot 31^5 \cdot 37^3 \cdot 41^2 \cdot 43^2 \cdot 47 \cdot 53 \cdot 59}$$

Acknowledgements. The authors are grateful to AIM and the Isaac Newton Institute for very generous support and hospitality. JBC was supported by the NSF, MOR by the NSF and NSERC, and NCS by an EPSRC Advanced Research Fellowship.

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Communicated by P. Sarnak