Notes on L-functions and Random Matrix Theory

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1 Introduction

The connection between L-functions and random matrix theory began with the 1972 discovery by Montgomery and Dyson that the zeros of the Riemann zeta-function seem to be distributed on the 1/2-line like the eigenvalues of large random hermitian matrices (the GUE ensemble studied by physicists). Since then there has been a lot of development. In the 1990s Hejhal generalized Montgomery's work to triple correlation and Rudnick and Sarnak to n-correlation for arbitrary n. In the last five years though, the subject has really taken off due to several developments. One was the conjectures of Keating-Snaith and Conrey-Farmer on moments of zeta- and L-functions and another was the development of the notion of symmetry type of families of L-functions by Katz-Sarnak. Added to these was the work of Conrey-Ghosh, Conrey-Gonek, Duke-Friedlander-Iwaniec, Kowalski-Michel-Vanderkam, Jutila, Motohashi, Ivic, Soundararajan, Rubinstein, and others on moments of families of L-functions which provide the evidence for the random matrix conjectures.

We now have at our disposal very accurate models for the behavior of L-functions (i.e. the distribution of values including zeros) and can use these to make arithemtical predictions. Three examples of conjectures on L-functions that have been motivated by random matrix theory will illustrate some of this development.

Conjecture 1 [CFKRS1]: *Let $\zeta(s)$ denote the Riemann zeta-function. Then, for any $\epsilon > 0$,*

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^6\, dt = P_3(\log\frac{T}{2\pi}) + O_\epsilon(T^{\epsilon-1/2})$$

where

$$P_3(x) = 0.00000570852\,x^9 + 0.00040502\,x^8 + 0.011072\,x^7 + 0.148400\,x^6 \\ +1.04592\,x^5 + 3.98438\,x^4 + 8.607319\,x^3 + 10.274330\,x^2 \\ +6.593913\,x + 0.916515.$$

Conjecture 2 [CKRS]: *Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve, where A and B are integers. For a squarefree integer d, let E_d be the twisted elliptic curve which has equation $d^2y = x^3 + Ax + B$. Let r_d denote the rank of the elliptic curve E_d; this is just the number of independent points with rational coordinates required to generate the group of all rational points on the curve. Now restrict attention to those d^* for which the rank is at least 2. Then we conjecture (a) that there exist constants C_E and ν_E which depend only on E such that*

$$\sum_{d^* \le x} 1 \sim C_E x^{3/4} (\log x)^{\nu_E}$$

and (b) *that for any prime p, the limit as $x \to \infty$ of the ratio of $d^* \le x$ with d^* being a square modulo p to the $d^* \le x$ which are non-square modulo p is equal to*

$$\sqrt{\frac{p+1-a_p}{p+1+a_p}}$$

where $a_p = p - N_p$ and N_p is the number of solutions of $y^2 \equiv x^3 + Ax + B$ modulo p.

Conjecture 3 [U]: *There exists a number $c > 0$, a sequence of $N \to \infty$ with an elliptic curve E_N of conductor N such that the rank of E_N is at least $c\frac{\log N}{\log\log N}$.*

The first two of these conjectures have been numerically tested extensively with excellent agreement.

One of the purposes of these notes will be to explain the relationship of conjectures like these to random matrix theory. Another purpose is to gather in one place a collection of useful formulas and examples for researchers working in these fields. We begin with the random matrix side of the story and finish with the L-function side; there are a few brief comments about finite field zeta-functions in the middle. Much of the material is taken from [C], [CFKRS], and [CKRS].

2 Random matrix Theory

2.1 The Classical Groups

In these notes we will be interested in the classical compact matrix groups with their Haar measures. In the case of the unitary group, this ensemble coincides with what is called CUE in the physics literature. However, the

orthogonal and symplectic groups we are interested in are different from COE and CSE in the physics literature. Below we define the groups and give their Haar measures and indicate some of the statistics we are interested in and how to compute them.

- The **unitary group** $\mathrm{U}(N)$ is the group of $N \times N$ matrices U with entries in $\mathbb{C}$ for which $UU^* = I$ where U^* denotes the conjugate transpose of U, i.e. if $U = (u_{i,j})$, then $U^* = (\overline{u_{j,i}})$.
- The **special orthogonal group** $\mathrm{SO}(N)$. This is the subgroup of $\mathrm{U}(N)$ consisting of real matrices with determinant 1. $\mathrm{SO}(2N)$ leads to the symmetry type O^+ and $\mathrm{SO}(2N{+}1)$ leads to the symmetry type O^-. In these notes we will deal only with $\mathrm{SO}(2N)$ and O^+; we will abuse notation somewhat and refer to this as O.
- The **symplectic group** $\mathrm{USp}(2N)$ is the subgroup of $\mathrm{U}(2N)$ of matrices U for which $UZU^t = Z$ where U^t denotes the transpose of U and

$$Z = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

Since all of our matrices are unitary, their eigenvalues all have modulus 1. The $N \times N$ matrix $A \in \mathrm{U}(N)$ has N eigenvalues on the unit circle, $e^{i\theta_1}, \dots e^{i\theta_N}$ with the eigenangles satisfying $0 \le \theta_1 \le \theta_2 \cdots \le \theta_N < 2\pi$. Thus, the average spacing between the eigenangles is $2\pi/N$. In the cases of $\mathrm{Sp}(2N)$ and $\mathrm{SO}(2N)$ the eigenvalues come in pairs $e^{\pm i\theta_j}$ with $0 \le \theta_1 \le \cdots \le \theta_N < \pi$ and the average spacing between the eigenangles is π/N. If we want to scale the eigenangles to have average spacing 1, then we consider $\tilde{\theta_j} = N\theta_j/\pi$ or $N\theta_j/(2\pi)$, whichever is appropriate.

2.2 The Weyl Integration Formula

In order to do analysis on these groups we need to know how to work with the measures. Weyl's formula provides a convenient way to reduce the integrations to ordinary multiple integrals.

The conjugacy classes of $N \times N$ unitary matrices can be parametrized by their N eigenvalues on the unit circle. Any configuration of N points on the unit circle corresponds to a conjugacy class in $\mathrm{U}(N)$. If $f(A) = f(\theta) = f(\theta_1, \dots \theta_N)$ is a symmetric function of N variables, then Weyl's formula for the Haar measure gives

$$\int_{U(N)} f(A) dA_{\mathrm{U}(N)} = \frac{1}{(2\pi)^N N!} \int_{[0,2\pi]^N} f(\theta) \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \, d\theta_1 \dots d\theta_N$$

where $dA_{\mathrm{U}(N)}$ is the Haar measure. Similarly, on $\mathrm{Sp}(2N)$ and $\mathrm{SO}(2N)$ we have respectively

$$\int_{\mathrm{Sp}(N)} f(A) dA_{\mathrm{Sp}(2N)} = \frac{2^{N^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta) \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 \prod_{j=1}^{N} \sin^2 \theta_j \prod_{j=1}^{N} d\theta_j \, ;$$

$$\int_{SO(2N)} f(A) dA_{\mathrm{SO}(2N)} = \frac{2^{(N-1)^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta) \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 \prod_{j=1}^{N} d\theta_j \,.$$

For example, if you wanted to compute the average of the square of the absolute value of the trace of 3×3 unitary matrices you would compute the integral

$$\frac{1}{6(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}|^2 \\ |e^{i\theta_1} - e^{i\theta_2}|^2 |e^{i\theta_1} - e^{i\theta_3}|^2 |e^{i\theta_2} - e^{i\theta_3}|^2 \, d\theta_1 \, d\theta_2 \, d\theta_3;$$

which equals 1.

Alternate formulation of the Haar measure

The Weyl formulas above are especially useful for computing moments of characteristic polynomials, as we shall soon see. For computing "local statistics" such as the neighbor spacing, especially in the large N limit, it is useful to have the Weyl formulas for the Haar measures written in an alternate form.

Let

$$L_{\mathrm{U}(N)}(\theta_j, \theta_k) = \exp\left(\tfrac{-i(N-1)(\theta_j-\theta_k)}{2}\right) \sum_{n=0}^{N-1} e^{in(\theta_j-\theta_k)} = \frac{\sin\frac{N(\theta_j-\theta_k)}{2}}{\sin\frac{\theta_j-\theta_k}{2}}$$

$$L_{\mathrm{Sp}(2N)}(\theta_j, \theta_k) = 2\sum_{n=1}^{N} \sin n\theta_j \sin n\theta_k \\ = \frac{\sin(N+\frac{1}{2})(\theta_j-\theta_k)}{2\sin\frac{\theta_j-\theta_k}{2}} - \frac{\sin(N+\frac{1}{2})(\theta_j+\theta_k)}{2\sin\frac{\theta_j+\theta_k}{2}}$$

and

$$L_{\mathrm{SO}(2N)}(\theta_j, \theta_k) = 1 + 2\sum_{n=1}^{N-1} \cos n\theta_j \cos n\theta_k \\ = \frac{\sin(N-\frac{1}{2})(\theta_k-\theta_k)}{2\sin\frac{\theta_j-\theta_k}{2}} + \frac{\sin(N+\frac{1}{2})(\theta_j+\theta_k)}{2\sin\frac{\theta_j+\theta_k}{2}}.$$

Then

$$dA_{\mathrm{U}(N)} = \frac{1}{(2\pi)^N N!} \det_{N\times N}(L_{\mathrm{U}(N)}(\theta_j,\theta_k)) \prod_{j=1}^{N} d\theta_j$$

$$dA_{\mathrm{Sp}(2N)} = \frac{1}{\pi^N N!} \det_{N\times N}(L_{\mathrm{Sp}(2N)}(\theta_j,\theta_k)) \prod_{j=1}^{N} d\theta_j$$

$$dA_{\mathrm{SO}(2N)} = \frac{1}{\pi^N N!} \det_{N\times N}(L_{\mathrm{SO}(2N)}(\theta_j,\theta_k)) \prod_{j=1}^{N} d\theta_j$$

A key thing to note is that since

$$\lim_{N\to\infty} N \sin\tfrac{x}{N} = \sin x$$

it follows that

$$\lim_{N\to\infty} \frac{1}{N^n} \det_{n\times n} L_{\mathrm{U}(N)}\left(\frac{2\pi x_j}{N}, \frac{2\pi x_k}{N}\right) = \det_{n\times n}\left(\frac{\sin\pi(x_j-x_k)}{\pi(x_j-x_k)}\right).$$

Similarly,

$$\lim_{N\to\infty} \frac{1}{N^n} \det_{n\times n} L_{\mathrm{Sp}(2N)}\left(\frac{\pi x_j}{N}, \frac{\pi x_k}{N}\right) = \det_{n\times n}\left(\frac{\sin\pi(x_j-x_k)}{\pi(x_j-x_k)} - \frac{\sin\pi(x_j+x_k)}{\pi(x_j+x_k)}\right)$$

and

$$\lim_{N\to\infty} \frac{1}{N^n} \det_{n\times n} L_{\mathrm{SO}(2N)}\left(\frac{\pi x_j}{N}, \frac{\pi x_k}{N}\right) = \det_{n\times n}\left(\frac{\sin\pi(x_j-x_k)}{\pi(x_j-x_k)} + \frac{\sin\pi(x_j+x_k)}{\pi(x_j+x_k)}\right).$$

2.3 Four Statistics

In these notes we are particularly interested in theorems and conjectures about moments and values of characteristic polynomials of random matrices and L-functions (these represent a "global" perspective) and we are also interested in things like spacing of consecutive eigenvalues and zeros (these represent a "local" perspective). In this section we describe some local statistics of interest. Note that in the physics literature "n-level density" and "n-correlation" for GUE, GOE, and GSE are not distinguished because the associated measures are rotationally invariant. Here, however, the distinction is important.

There is a simple combinatorial relation (described in section 2.6) between neighbor-spacing statistics and correlation statistics.

For the matrix groups $U(N)$, $SO(2N)$, and $Sp(2N)$ we are interested in the statistics "nearest neighbor" and "n-level correlation" are the same for all three groups whereas "n-level density" and "jth eigenvalue" are different for all three groups. In the physics literature these statistics are different for all three matrix ensembles GUE, GOE, and GSE. Now we describe these statistics.

Suppose we have a sequence $\mathcal{T} = \{T_N\}_{N=1}^{\infty}$ of N-tuples of numbers $T_N = (t_{N,1}, t_{N,2}, \ldots, t_{N,N})$ where $t_{N,1} \le t_{N,2} \le \cdots \le t_{N,N}$ and such that $t_{N,N} - t_{N,1} \sim N$ as $N \to \infty$, so that the average spacing is 1.

- The **n-level density** of $\mathcal{T}$ is $W(x_1, \ldots x_n)$ means that

$$\lim_{N\to\infty} \sum_{\substack{(i_1,\dots i_n), i_j\le N \\ i_j\ne i_k}} f(t_{N,i_1},\dots t_{N,i_n})$$
$$= \int_{\mathbb{R}^n} f(x_1,\dots,x_n) W(x_1,\dots,x_n)\ dx_1,\dots,dx_n\,.$$

for any compactly supported smooth symmetric function $f:\mathbb{R}^n\to\mathbb{C}$.

- The **j-th lowest zero density** is $\nu_j(x)$ means that for any compactly supported smooth function $f:\mathbb{R}\to\mathbb{C}$
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N} f(t_{n,j}) = \int_0^\infty f(x)\nu_j(x)\ dx\,.$$
- The **consecutive spacing density** or **nearest neighbor density** is $\mu(x)$ means that for any compactly supported smooth function $f:\mathbb{R}^n\to\mathbb{C}$ we have
$$\lim_{N\to\infty}\frac{1}{N}\sum_{i\le N-1} f(t_{N,i+1}-t_{N,i}) = \int_0^\infty f(x)\mu(x)\ dx\,.$$
Wigner conjectured that $\mu(x)$ is equal to $\frac{\pi}{2}xe^{-\frac{\pi}{4}x^2}$. This conjecture, which is very accurate for small x, is known as Wigner's surmise.
- The **n-correlation density** is $V(x_1,\dots,x_n)$ means that for any smooth symmetric function f that depends only on the differences of the variables (i.e. $f(x_1+u,\dots,x_n+u)=f(x_1,\dots,x_n)$ for all u), and is rapidly decaying on the hyperplane $P_n:\{(x_1,\dots,x_n:\sum x_i=0\}$, we have, as $N\to\infty$,
$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{t_{i_1},\dots t_{i_n}\in T_N \\ i_j\ne i_k}} f(t_{i_1},\dots t_{i_n})$$
$$= \int_{P_n} f(x_1,\dots,x_n)V(x_1,\dots,x_n)\ dx_1\dots\ dx_{n-1}$$

as $N\to\infty$. The spacing and n-correlation densities are universal, i.e. the same for each of O, Sp, and U, whereas the n-level and j-th lowest zero densities depend on the symmetry type.

2.4 Formulas for the Density Functions

In this section we compile the formulas for the density functions of the statistics described in the last section. In the next two sections we describe some of the computations leading to these formulae.

The 1-level density functions are

$$W(\mathrm{O})(x) = 1+\frac{\sin 2\pi x}{2\pi x}\,,$$
$$W(\mathrm{Sp})(x) = 1-\frac{\sin 2\pi x}{2\pi x}\,,$$
$$W(\mathrm{U})(x) = 1\,.$$

- The n–**level density** is

$$W_\epsilon(x_1,\dots,x_n) = \det_{n\times n}\left(\frac{\sin\pi(x_i-x_j)}{\pi(x_i-x_j)} - \epsilon\frac{\sin\pi(x_i+x_j)}{\pi(x_i+x_j)}\right)$$

  where $\epsilon = 0$ for U; $\epsilon = 1$ for Sp; $\epsilon = -1$ for O^+.
- The **lowest zero density** is $\nu_1(x)$ where

$$\nu_1(x) = -\frac{d}{dx}\prod_{j=0}^{\infty}(1-\lambda_j(x)) \qquad \mathrm{U}\,;$$

$$\nu_1(x) = -\frac{d}{dx}\prod_{j=0}^{\infty}(1-\lambda_{2j+1}(2x)) \qquad \mathrm{Sp}\,;$$

$$\nu_1(x) = -\frac{d}{dx}\prod_{j=0}^{\infty}(1-\lambda_{2j}(2x)) \qquad \mathrm{O}\,,$$

  where $1 \ge \lambda_0(x) \ge \lambda_1(x)\cdots \ge 0$ are the eigenvalues of the integral equation

$$\int_{-x/2}^{x/2}\frac{\sin\pi(t-u)}{\pi(t-u)}f(u)\,du = \lambda(x)f(t)$$

- The **consecutive spacing density** is

$$\mu(x) = \frac{d^2}{dx^2}\prod_{j=0}^{\infty}(1-\lambda_j(x))\,.$$

- The n–**correlation density** is $V(x_1,\dots,x_n) = nW_0(x_1,\dots,x_n)$,

2.5 Gaudin's Lemma

The critical device for calculating the density functions for the "local" statistics is due to Gaudin and is described in this section.

Associated to each $N\times N$ unitary matrix A are its N eigenvalues $\exp(i\theta_j)$ where $0 \le \theta_1 \le \cdots \le \theta_N \le 2\pi$. We let $X(A)$ denote this sequence of θ. We integrate a function $F(A)$ over $\mathrm{U}(N)$ by parametrizing the group by the θ_i and using Weyl's formula to convert the integral into an N-fold integral over the θ_i.

Often one wants to integrate with respect to Haar measure over a group G a function $\tilde{f}(A) = \tilde{f}(\theta_1,\dots,\theta_N)$ of N variables that is "lifted" from a function f of n variables:

$$\tilde{f}(\theta_1,\dots,\theta_N) = \sum_{1\le i_1<i_2<\cdots<i_n\le N} f(\theta_{i_1},\dots,\theta_{i_n})$$

where the sum is over all possible n-tuples $(i_1,\dots,i_n)$ of distinct integers between 1 and N. Gaudin's lemma gives a simplification of this computation

from an N-fold integral to an n-fold integral. Define the measure $dA_{N,\mathrm{U}(n)}$ on $\mathrm{U}(n)$ by

$$dA_{N,\mathrm{U}(n)} = \frac{1}{n!(2\pi)^n} \det_{n\times n}(L_{\mathrm{U}(N)}(\theta_j,\theta_k)) \prod_{j=1}^{n} d\theta_j,$$

the measure $dA_{2N,\mathrm{Sp}(2n)}$ on $\mathrm{Sp}(2n)$ by

$$dA_{2N,\mathrm{Sp}(2n)} = \frac{1}{n!\pi^n} \det_{n\times n}(L_{\mathrm{Sp}(2N)}(\theta_j,\theta_k)) \prod_{j=1}^{n} d\theta_j,$$

and the measure $dA_{2N,\mathrm{SO}(2n)}$ on $\mathrm{SO}(2n)$ by

$$dA_{2N,\mathrm{SO}(2n)} = \frac{1}{n!\pi^n} \det_{n\times n}(L_{\mathrm{SO}(2N)}(\theta_j,\theta_k)) \prod_{j=1}^{n} d\theta_j,$$

Then Gaudin's Lemma asserts that

$$\int_{\mathrm{U}(N)} \tilde{f}(A) dA_{\mathrm{U}(N)} = \int_{\mathrm{U}(n)} f(A) dA_{N,\mathrm{U}(n)}$$

and

$$\int_{\mathrm{Sp}(2N)} \tilde{f}(A) dA_{\mathrm{Sp}(2N)} = \int_{\mathrm{Sp}(2n)} f(A) dA_{2N,\mathrm{Sp}(2n)}$$

and

$$\int_{\mathrm{SO}(2N)} \tilde{f}(A) dA_{\mathrm{SO}(2N)} = \int_{\mathrm{SO}(2n)} f(A) dA_{2N,\mathrm{SO}(2n)}$$

One way to view Gaudin's Lemma is just as a collection of integration formulas; for example

$$\binom{N}{n} \int_{[0,2\pi]^{N-n}} \det_{N\times N}(S_N(\theta_j-\theta_k)) \frac{d\theta_{n+1}\dots d\theta_N}{(2\pi)^N N!} = \det_{n\times n}(S_N(\theta_j-\theta_k)) \frac{1}{(2\pi)^n n!}$$

where $S_N(x) = (\sin Nx/2)/(\sin x/2)$. The identity is proven through induction using the fact that

$$\int_{[0,2\pi]} S_N(x-y) S_N(y-z)\ dy = 2\pi S_N(x-z).$$

We illustrate the utility of Gaudin's formula by computing the n-level density function for $\mathrm{U}(N)$. Let $f(x_1,\dots,x_n)$ be a suitable test function. To compute the n-level density we need to evaluate

$$\lim_{N\to\infty} \int_{U(N)} \sum_{\substack{(i_1,\dots,i_n)\\ i_j\neq i_k}} f\left(\tfrac{N\theta_{i_1}}{2\pi},\dots,\tfrac{N\theta_{i_n}}{2\pi}\right) \prod_{j<k} |e^{i\theta_k} - e^{i\theta_j}|^2\ d\theta_1 \dots\ d\theta_N$$

where we have rescaled the eigenangles of the matrix $A \in U(N)$ so that they have mean spacing 1. The sum over tuples $(i_1, \dots, i_n)$ with the i_j distinct is $n!$ times the sum over ordered tuples $1 \le i_1 < i_2 < \cdots < i_N \le N$. Thus by Gaudin's lemma and after using the new expression for the Haar measure and changing variables $\theta_j \to 2\pi x_j/N$, the above is equal to

$$\lim_{N\to\infty} \frac{1}{N^n} \int_{[0,N]^n} f(x_1, \dots, x_n) \det_{n\times n} L_{\mathrm{U}(N)}(2\pi x_j/N, 2\pi x_k/N) \prod_{j=1}^{n} dx_j.$$

Thus, our integral is

$$= \int_{\mathbb{R}^n} f(x_1, \dots, x_n) \det_{n\times n} \left(\frac{\sin \pi(x_j - x_k)}{\pi(x_j - x_k)} \right) \, dx_1 \dots \, dx_n,$$

so that $W_{U,n}(x_1, \dots, x_n) = \det_{n\times n} \left(\frac{\sin \pi(x_j - x_k)}{\pi(x_j - x_k)} \right)$. Similar calculations lead to the density functions for $\mathrm{Sp}(2N)$ and $\mathrm{SO}(2N)$.

As a second example we sketch the computation of the pair–correlation statistic for each group. Let $\mathrm{G}_\epsilon(N)$ stand for $\mathrm{U}(N)$ if $\epsilon = 0$, $\mathrm{Sp}(2N)$ if $\epsilon = 1$, and $\mathrm{SO}(2N)$ if $\epsilon = -1$. Let $f(x, y) = f(y, x) = g(x - y)$ where g is even, smooth, and compactly supported (or of rapid decay). To compute the pair-correlation statistic for the G_ϵ we need to evaluate

$$\lim_{N\to\infty} \frac{1}{N} \int_{\mathrm{G}(N)} \sum_{\substack{i_1, i_2 \in \{1,\dots,N\} \\ i_1 \neq i_2}} f\left(\tfrac{N\theta_{i_1}}{(2-|\epsilon|)\pi}, \tfrac{N\theta_{i_2}}{(2-|\epsilon|)\pi} \right) dA_{\mathrm{G}(N)}.$$

By Gaudin's lemma and a change of variables, this is

$$= \lim_{N\to\infty} \frac{1}{N^3} \int_0^N \int_0^N g(x_1 - x_2) \det_{2\times 2} \left(L_{\mathrm{G}(N)}(\tfrac{(2-|\epsilon|)\pi x_j}{N}, \tfrac{(2-|\epsilon|)\pi x_k}{N}) \right) \, dx_1 \, dx_2.$$

For large N,

$$\frac{1}{N^2} \det_{2\times 2} \left(L_{\mathrm{G}(N)}(\tfrac{(2-|\epsilon|)\pi x_j}{N}, \tfrac{(2-|\epsilon|)\pi x_k}{N}) \right) \approx$$
$$\det \begin{pmatrix} 1 - \epsilon \frac{\sin 2\pi x_1}{2\pi x_1} & \frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)} - \epsilon \frac{\sin \pi(x_1 + x_2)}{\pi(x_1 + x_2)} \\ \frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)} - \epsilon \frac{\sin \pi(x_1 + x_2)}{\pi(x_1 + x_2)} & 1 - \epsilon \frac{\sin 2\pi x_2}{2\pi x_2} \end{pmatrix}.$$

Now let $u = x_1 - x_2$ and $v = x_2$ in the double integral. Then the double integral is asymptotically

$$\frac{1}{N} \int_{-N}^{N} g(u) \int_{\min\{0,-u\}}^{\max\{N,N-u\}} \det \begin{pmatrix} 1 - \epsilon \frac{\sin 2\pi(u+v)}{2\pi(u+v)} & \frac{\sin \pi u}{\pi u} - \epsilon \frac{\sin \pi(u+2v)}{\pi(u+2v)} \\ \frac{\sin \pi u}{\pi u} - \epsilon \frac{\sin \pi(u+2v)}{\pi(u+2v)} & 1 - \epsilon \frac{\sin 2\pi(u+v)}{2\pi(u+v)} \end{pmatrix} \, dv \, du.$$

The integrals with respect to v of the terms with a $\sin(u + v)$ or $\sin(u + 2v)$ will be bounded. The other terms are constant with respect to v. Since the length of the integration in the v-integral is $N - |u|$, our expression is

$$\sim \int_{-N}^{N} g(u)\left(1-\frac{|u|}{N}\right)\det\begin{pmatrix}1 & \frac{\sin\pi u}{\pi u}\\ \frac{\sin\pi u}{\pi u} & 1\end{pmatrix}\,du$$
$$\to \int_{-\infty}^{\infty} g(u)\det\begin{pmatrix}1 & \frac{\sin\pi u}{\pi u}\\ \frac{\sin\pi u}{\pi u} & 1\end{pmatrix}\,du.$$

Finally, the last expression is

$$= 2\int_{x_1+x_2=0} f(x_1,x_2)\det\begin{pmatrix}1 & \frac{\sin\pi(x_1-x_2)}{\pi(x_1-x_2)}\\ \frac{\sin\pi(x_1-x_2)}{\pi(x_1-x_2)} & 1\end{pmatrix}\,dx_1.$$

Similarly, in computing the n-correlation statistic we are led to consider

$$\frac{1}{N}\int\cdots\int_{[0,N]^n} f(x_1,\dots,x_n)\det_{n\times n}\left(\frac{\sin\pi(x_j-x_k)}{\pi(x_j-x_k)}-\epsilon\frac{\sin\pi(x_j+x_k)}{\pi(x_j+x_k)}\right)dx_1\dots dx_n$$

as $N\to\infty$. Because of the $1/N$ factor, the parts of the determinant involving $\sin(\pi(x+y))/(x+y)$ go to 0 and we wind up with $W_)(x_1,\dots,x_n)$.

2.6 Some notation from Katz-Sarnak and a combinatorial identity

The purpose of this section is to explain some notation from Katz-Sarnak and to give a combinatorial lemma which shows how to pass between neighbor spacing statistics and correlation statistics. Let $a\ge 0$ be an integer and $s\ge 0$ real. Let $G(N)$ denote one of the groups $U(N)$, $SO(2N)$ or $Sp(2N)$ and for $A\in G(N)$ let $X(A)$ be the sequence $X(1)\le X(2)\le\dots X(N)$ of eigenangles in increasing order (from 0 to 2π for $U(N)$ and from 0 to π for $SO(2N)$ or $Sp(2N)$). Define

$$\mathrm{Sep}_a(s) := \#\{(i,j): 1\le i<j\le N, j-i=a+1, X(j)-X(i)=s\}$$

and

$$\mathrm{Clump}_a(s) := \#\{1\le i_0<i_1<\dots<i_{a+1}\le N : X(i_{a+1})-X(i_0)=s\}.$$

Clearly Sep is related to neighbor spacing statistics and Clump is related to correlation statistics (note that Clump is invariant under adding a constant to each eigen-angle). Their relationship is given by

$$\mathrm{Clump}_a(s)=\sum_{b\ge a}\binom{b}{a}\mathrm{Sep}_b(s)$$

and equivalently

$$\mathrm{Sep}_a(s)=\sum_{b\ge a}(-1)^b\binom{b}{a}\mathrm{Clump}_b(s).$$

The case $a = 0$ of this latter formula is just the inclusion-exclusion principle.

If we scale and let $N \to \infty$ we have, for example, for the nearest neighbor spacing that the frequency for which neighbors are in a set S is

$$\int_S \mu_1(s)\, ds = \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \int_S \mathrm{Sep}_0\big(\frac{2\pi s}{N}\big)\, ds\, dA_{U(N)}.$$

Similarly, $\mathrm{Clump}_0(s)$ counts the number of pairs of eigenvalues that are s apart. So, if we scale and take the limit, we get the pair correlation function, that the measure of matrices with the difference of eigenvalues in a set S is

$$\int_S \left(1 - \big(\frac{\sin \pi s}{\pi s}\big)^2\right) ds = \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \int_S \mathrm{Clump}_0\big(\frac{2\pi s}{N}\big)\, ds\, dA_{U(N)}.$$

More generally, for $n \geq 2$,

$$n \int_{S^{n-1}} W_0(s_1, \ldots, s_{n-1}, 0)\, ds = \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \int_S \mathrm{Clump}_{n-2}\big(\frac{2\pi s}{N}\big)\, ds\, dA_{U(N)}.$$

2.7 First eigenvalue and neighbor spacings

We sketch some of the ideas needed to compute the density functions for these statistics. (This is taken from Mehta's book.) Initially, we work with the unitary group, for which the measure is rotationally invariant. Let

$$B_N(\alpha) = \int_{[\frac{\alpha}{2}, 2\pi - \frac{\alpha}{2}]^N} dA_{\mathrm{U}(N)}$$

so that $B_N(\alpha)$ is the measure of the set of unitary $N \times N$ matrices with all eigenangles in $[\alpha/2, 2\pi - \alpha/2]$. Now we use the fact that $dA_{\mathrm{U}(N)}$ is essentially the square of a Vandermonde determinant:

$$\begin{aligned}(2\pi)^N N! dA_{\mathrm{U}(N)} &= \prod_{1\leq j<k\leq N} |e^{i\theta_k} - e^{i\theta_j}|^2 = \prod_{1\leq j<k\leq N} (e^{i\theta_k} - e^{i\theta_j})(e^{-i\theta_k} - e^{-i\theta_j}) \\ &= \left|\det\left(e^{i(j-1)\theta_k}\right)\right|^2.\end{aligned}$$

Now we use Gram's identity in the form: for an interval $\mathcal{S}$ and functions ϕ,

$$\frac{1}{N!} \int_{\mathcal{S}^N} |\det_{N\times N}(\phi_j(x_k))|^2\, dx_1 \ldots dx_N = \det_{N\times N}\left(\int_{\mathcal{S}} \phi_j(x)\overline{\phi_k(x)}\, dx\right).$$

(By the way, Gram's formula with $S = [0, 2\pi]$ and $\phi_j(x) = e^{i(j-1)x}$ gives a quick proof that the Haar measure for the unitary group has total mass 1.) Thus,

$$B_N(\alpha) = \frac{1}{(2\pi)^N} \det_{N\times N}\left(\int_{\frac{\alpha}{2}}^{2\pi-\frac{\alpha}{2}} e^{i(k-j)y}\ dy\right)$$
$$= \det_{N\times N}\left(\delta_{jk} - \frac{1}{2\pi}\int_{-\alpha/2}^{\alpha/2} e^{i(k-j)y}\ dy\right)$$

This determinant is just the characteristic polynomial, evaluated at 1, of the matrix $A = (a_{jk})$ with entries $a_{jk} = \frac{1}{2\pi}\int_{-\alpha/2}^{\alpha/2} e^{i(k-j)y}\ dy$ and so is equal to $\prod_{j=1}^{N}(1-\lambda_{j,N}(\alpha))$ where the $\lambda_{j,N}(\alpha)$ are the eigenvalues of A. It is easy to check that the eigenvalues of A are the same as the eigenvalues of the integral operator

$$(K\psi)(x) = \frac{1}{2\pi}\int_{-\alpha/2}^{\alpha/2}\sum_{j=0}^{N-1} e^{ij(x-y)}\psi(y)\ dy$$

which acts on the N-dimensional vector space of trigonometric polynomials of degree $N-1$; an eigenvector $\mathbf{v} = (v_0, \dots v_{N-1})$ of A with eigenvalue λ corresponds to an eigenfunction $\psi(x) = \sum_{j=0}^{N-1} v_j e^{ijx}$ with the same eigenvalue.

Another version of Gram's identity, namely,

$$\det_{N\times N}(I + z\int_S \phi_j(x)\overline{\phi_k}(x)\ dx) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{S^n}\big(\det_{n\times n}\sum_{h=1}^{N}\phi_h(x)\overline{\phi_h}(x)\ dx\big)$$

together with some estimates on the kernel function allows us to scale and take a limit as $N\to\infty$.

The kernel function

$$\sum_{j=0}^{N-1} e^{ij(x-y)} = L_{\mathrm{U}(N)}(x,y)$$

satisfies

$$\lim_{N\to\infty} L_{\mathrm{U}(N)}(2\pi x/N, 2\pi y/N) = \frac{\sin\pi(x-y)}{\pi(x-y)}.$$

Thus, we are led to consider the eigenvalues of the integral equation with the kernel function $\frac{\sin\pi(x-y)}{\pi(x-y)}$.

In this way it is deduced (see [KS] for some additional details about the limiting process and Fredholm determinants) that

$$B(\alpha) := \lim_{N\to\infty} B_N(N\alpha/(2\pi)) = \prod_{n=1}^{\infty}(1-\lambda_n(\alpha))$$

where the $\lambda_n(\alpha)$ are the eigenvalues of the integral operator with kernel $\frac{\sin\pi(x-y)}{\pi(x-y)}$ on the interval $[-\alpha/2, \alpha/2]$. This formula gives a rapidly converging infinite product for the desired density function. See Mehta [M] for tables, graphs and more information about these functions.

Another way to argue all of this is as follows. Suppose we want the measure $B_N(S)$ of that part of $U(N)$ which has no eigenvalues in some interval S. By the inclusion-exclusion principle this will be the measure of the whole space minus the measure of the space of matrices with at least one eigenvalue in S plus the measure with at least two eigenvalues in S minus the measure with at least three eigenvalues in S and so on. Thus, letting f denote the characteristic function of S,

$$B_N(S) = 1 - \int_S \sum_{n=1}^{N} f(\theta_j) dA_{U(N)} + \int_{S^2} \sum_{j<k} f(\theta_j) f(\theta_k) dA_{U(N)} - + \dots .$$

By Gaudin's lemma, we then have

$$B_N(S) = \sum_{n=0}^{N} \frac{(-1)^n}{(2\pi)^n n!} \int_{S^n} \det_{n\times n} (S_N(\theta_j - \theta_k))\ d\theta_1 \dots d\theta_n.$$

Then by Gram's identity, with $z = \frac{-1}{2\pi}$

$$B_N(S) = \prod_{n=1}^{N} (1 - \lambda_n(S))$$

where the $\lambda_n(S)$ are the eigenvalues of $(\int_S \phi_j(x)\overline{\phi_k}(x)\ dx)$, which, as we've seen, are the same as the eigenvalues of the integral operator with kernel $K(x,y) = \sum_{h=1}^{N} \phi_h(x)\overline{\phi_h}(y)$. This analysis should be compared with the remarks about Clump and Sep in section 2.6. In particular, equating coefficients of T^1 in the identity there we see that the nearest neighbor spacing Sep_0 satisfies

$$\mathrm{Sep}_0(s) = \sum_{a=0}^{\infty} (-1)^a \mathrm{Clump}_a(s);$$

this is just the inclusion-exclusion principle we alluded to. Scaling and taking the limit as $N \to \infty$ leads to

$$\begin{aligned}
\int_S \mu_1(s)\ ds &= \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \mathrm{Sep}_0\big(\frac{2\pi s}{N}\big)\ dA_{U(N)}\ ds \\
&= \int_S \lim_{N\to\infty} \int_{U(N)} \sum_{a=0}^{N} (-1)^a \mathrm{Clump}_a\big(\frac{2\pi s}{N}\big)\ dA_{U(N)}\ ds \\
&= \sum_{a=0}^{\infty} (-1)^a \int_{S^n} W_0(x_1, \dots, x_n)\ dx_1 \dots dx_n
\end{aligned}$$

To deduce the neighbor spacing statistic we argue as follows. The function $B(\alpha)$ is the probability that no rescaled eigenvalues are smaller than α in absolute value. Then it is a simple matter to conclude that the probability of the smallest eigenvalue being α is $-B'(\alpha)$ and the probability density

function for the nearest neighbor spacing is $B''(\alpha)$. These differentiation arguments are special to the unitary group because of the rotational invariance of the measure for the unitary group. However, we have already pointed out that the nearest neighbor spacing can be determined combinatorially from the correlation functions and that the correlation functions are the same for all three of our $G(N)$. Therefore, this calculation gives the nearest neighbor spacing statistic for all three groups.

For the lowest eigenvalue similar calculations can be carried out for all three groups using the kernels $\frac{\sin\pi(x-y)}{\pi(x-y)} - \epsilon\frac{\sin\pi(x+y)}{\pi(x+y)}$.

2.8 The Selberg Integral

For computing global statistics such as moments of characteristic polynomials we require Selberg's integral, which is a generalization of the Euler's beta-integral. There are many versions of Selberg's integral; three of them follow (see [M]).

$$\int_0^1\dots\int_0^1 |\Delta(x)|^{2\gamma}\prod_{j=1}^n x_j^{\alpha-1}(1-x_j)^{\beta-1}\,dx_1\dots dx_n$$
$$= \prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+\gamma j)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(n+j-1)\gamma)}$$

for $\Re\alpha>0, \Re\beta>0, \Re\gamma > -\min(\frac{1}{n}, \frac{\Re\alpha}{n-1}, \frac{\Re\beta}{n-1})$. Here

$$\Delta(x) = \prod_{1\le j<\ell\le n}(x_j - x_\ell).$$

Alternatively,

$$\int_{-1}^1\dots\int_{-1}^1 \prod_{1\le i<j\le N}|x_i-x_j|^{2\gamma}\prod_{j=1}^n(1-x_j)^{\alpha-1}(1+x_j)^{\beta-1}dx_j$$
$$= 2^{\gamma n(n-1)+n(\alpha+\beta-1)}\prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+\gamma(n+j-1))}.$$

Another version has

$$\int_0^\infty\dots\int_0^\infty |\Delta(x)|^{2\gamma}\prod_{j=1}^n x_j^{\alpha-1}(1+x)^{-\alpha-\beta-2\gamma(n-1)}\prod_{j=1}^n dx_j$$
$$= \prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+\gamma j)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(n+j-1)\gamma)}.$$

2.9 Characteristic polynomials of Random Matrices

In this section we describe the properties of characteristic polynomials of unitary matrices, expressing things in a way that highlights the connection with number theory. You may find it helpful to refer to section 4 for the basic properties of L-functions.

Let

$$\Lambda(s) = \Lambda_A(s) = \det(I - As)$$

denote the characteristic polynomial of an $N \times N$ matrix A. We want to compare characteristic polynomials with L-functions, so here we outline some of the relevant properties of these characteristic polynomials.

If we expand $\Lambda(s)$, we obtain

$$\Lambda(s) = \sum_{n=0}^{N} a_n s^n,$$

which is analogous to the Dirichlet series representation for L-functions.

- Analytic continuation: Since $\Lambda(s)$ is a polynomial, it is an entire function.
- Functional equation: Since A is unitary, we have

  $$\Lambda(s) = (-1)^N s^N \det A \det(I - A^\dagger s^{-1}),$$

  and so, writing

  $$\det A = e^{i\phi}$$

  (where unitarity implies that $\phi \in R$), we have

  $$\Lambda(s) = (-1)^N s^N e^{i\phi} \overline{\Lambda}(\tfrac{1}{s}).$$

  This plays the same role for $\Lambda(s)$ as the functional equation for L-functions: it represents a symmetry with respect to the unit circle ($s = re^{i\alpha} \to \frac{1}{s} = \frac{1}{r}e^{-i\alpha}$).
  As just indicated, the critical line corresponds to the unit circle, and furthermore, the half-plane to the right of the critical line corresponds to the interior of the unit circle.
- Critical values: The critical point for $\Lambda(s)$ is $s = 1 = e^{i\cdot 0}$, and $\Lambda(1)$ is the critical value.
- Location of zeros: Since A is unitary, its eigenvalues all have modulus 1, so the zeros of $\Lambda(s)$ lie on the unit circle.
- Average spacing of zeros: Since the $N \times N$ matrix A has N eigenvalues on the unit circle, the average spacing between zeros of $\Lambda_A(s)$ is $2\pi/N$. When modeling a family of L-functions, we choose N as a function of the conductor of $L(s)$ so that the L-function and the characteristic polynomial have the same average spacing between their zeros.

- Approximate functional equation

$$\sum_{n=0}^{N} a_n s^n = (-1)^N e^{i\phi} \sum_{n=0}^{N} \overline{a_n} s^{N-m}$$

and so

$$a_n = (-1)^N e^{i\phi} \overline{a_{N-n}}.$$

Hence, when N is odd, we have

$$\Lambda(s) = \sum_{n=0}^{\frac{N-1}{2}} a_n s^n + (-1)^N e^{i\phi} s^N \sum_{n=0}^{\frac{N-1}{2}} \overline{a_n} s^{-n},$$

which corresponds to the approximate functional equation for L-functions. When N is even, there is an additional term: $a_{\frac{N}{2}} s^{\frac{N}{2}}$.

The above discussion applies to any unitary matrix. We also consider matrices which, in addition to being unitary, are also either symplectic or orthogonal. We use these three ensembles of matrices to model families of L-functions. While the notion of "family of L-functions" has not yet been made precise, we give several natural examples in section 4. Associated to each family is a "symmetry type" which identifies the matrix ensemble which will be used to model the family. This correspondence is most easily seen in terms of the sign of the functional equation, which is analogous to the determinant of the matrix. If A is unitary symplectic, then $\det A = 1$ (i.e. $\phi = 0$), and if A is orthogonal, then $\det A = \pm 1$. Correspondingly, the functional equations for L-functions with unitary symmetry involve a (generally complex) phase factor, whereas for L-functions with symplectic symmetry this phase factor is unity, and in the case of orthogonal symmetry it is either $+1$ or -1.

2.10 Moments of Characteristic Polynomials

Keating and Snaith [KSn1] and [KSn2] calculated exact formulas for the moments of characteristic polynomials of our ensembles for the purpose of comparing with moments of families of L-functions. These are calculated from Weyl's formulas together with appropriate uses of Selberg's integral formula.

$$\begin{aligned} M_{\mathrm{U}(N)}(\lambda) &= \int_{\mathrm{U}(N)} |\det(I - Ae^{it}))|^{2\lambda} \, dA \\ &= \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\lambda)}{\Gamma(j+\lambda)^2}, \end{aligned}$$

$$M_{\mathrm{Sp}(2N)}(\lambda) = \int_{\mathrm{Sp}(2N)} |\det(I-A)|^{\lambda}\, dA$$
$$= 2^{2N\lambda} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(1/2+\lambda+j)}{\Gamma(1/2+j)\Gamma(1+\lambda+N+j)},$$

$$M_{\mathrm{SO}(2N)}(\lambda) = \int_{\mathrm{SO}(2N)} |\det(I-A)|^{\lambda}\, dA$$
$$= 2^{N\lambda} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(\lambda+j-1/2)}{\Gamma(j-1/2)\Gamma(\lambda+j+N-1)}.$$

In the first formula, t is any real number and the right side is independent of t. The right side of this first formula can be re-written in the case λ is an integer k as a polynomila in N:

$$g(k,N) := \frac{(N+1)(N+2)^2 \dots (N+k)^k (N+k+1)^{k-1} \dots (N+2k-1)}{1 \cdot 2^2 \cdot 3^3 \dots k^k (k+1)^{k-1} \dots (2k-1)}.$$

For fixed k, this is asymptotic to

$$\frac{N^{k^2}}{1 \cdot 2^2 \cdot 3^3 \dots k^k (k+1)^{k-1} \dots (2k-1)} = \frac{g_k}{k^2!} N^{k^2}$$

as $N \to \infty$. Keating and Snaith identified the number g_k as the critical constant that was missing from number theorist's attempts to formulate a conjecture for the $2k$th moment of the Riemann zeta-function. We will see that the role of N is played by $\log \frac{T}{2\pi}$ in the case of the zeta-function. Similar remarks apply to the moments for the other groups and their relation to moments of families of L-functions.

2.11 Lower order terms and permutation sums

The polynomial $g(k,N)$ above can be expanded and expressions for the lower order terms in N explicitly displayed; however, it transpires that this is not a good way to approach lower order terms for moments of L-functions. For the more detailed analysis of lower order terms in our moment formulas for L-functions we need a different method than Selberg's integral. Basically, it is necessary to consider shifted moments and to look at a kind of additive refiormulation of $g(k,N)$.

The formulas we arrive at are slightly complicated but can be simplified through some integral formulas which we relate here. Here is an example of one of our formulas. (Recall that $\Lambda_A(s) = \det(I - As)$.)

$$\int_{U(N)} \Lambda_A(e^{-\alpha})\Lambda_A(e^{-\beta})\Lambda_{A^*}(e^{-\gamma})\Lambda_{A^*}(e^{-\delta})\, dA_N$$
$$= Z(\alpha,\beta,\gamma,\delta) + e^{N(\alpha+\gamma)}Z(-\gamma,\beta,-\alpha,\delta) + e^{N(\alpha+\delta)}Z(-\delta,\beta,\gamma,-\alpha)$$
$$+e^{N(\beta+\gamma)}Z(\alpha,-\gamma,-\beta,\delta) + e^{N(\beta+\delta)}Z(\alpha,-\delta,\gamma,-\beta)$$
$$+e^{N(\alpha+\beta+\gamma+\delta)}Z(-\gamma,-\delta,-\alpha,-\beta)$$

where

$$Z(\alpha,\beta,\gamma,\delta) = z(1+\alpha+\gamma)z(1+\alpha+\delta)z(1+\beta+\gamma)z(1+\gamma+\delta)$$

with $z(x) = 1/(1-e^x)$.

We will see that an analogous formula (with error term) holds for the shifted fourth moment of the zeta-function, but with $z(x)$ replaced by $\zeta(1+x)$ and with N replaced by $\log \frac{t}{2\pi}$ and with the inclusion of an arithmetic factor in each term.

Notice that the main term here consists of six terms. The sum of these six terms can be expressed as the residue of a four-fold contour integral as described below.

Suppose $F(a;b) = F(a_1,\dots,a_k;b_1,\dots,b_k)$ is a function of $2k$ variables, which is symmetric with respect to the first k variables and also symmetric with respect to the second set of k variables. Suppose also that F is regular near $(0,\dots,0)$. Suppose further that $f(s)$ has a simple pole of residue 1 at $s=0$ but is otherwise analytic in a neighborhood about $s=0$. Let

$$G(a_1,\dots,a_k;b_1,\dots b_k) = F(a_1,\dots;\dots,b_k)\prod_{i=1}^{k}\prod_{j=1}^{k} f(a_i-b_j).$$

If for all $1 \le i,j \le k$, $\alpha_i - \alpha_{j+k}$ is contained in the region of analyticity of $f(s)$ then

$$\sum_{\sigma\in\Xi} G(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k)};\alpha_{\sigma(k+1)}\dots\alpha_{\sigma(2k)}) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}} \times$$
$$\oint\cdots\oint \frac{G(z_1,\dots,z_k;z_{k+1},\dots,z_{2k})\Delta(z_1,\dots,z_{2k})^2}{\prod_{i=1}^{2k}\prod_{j=1}^{2k}(z_i-\alpha_j)}\, dz_1\dots dz_{2k},$$

where one integrates about small circles enclosing the α_j's, and where Ξ is the set of $\binom{2k}{k}$ permutations $\sigma \in S_{2k}$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(2k)$.

The above applies to the Unitary case. The next formula is useful in the Symplectic and Orthogonal cases.

Suppose F is a symmetric function of k variables, regular near $(0,\dots,0)$, and $f(s)$ has a simple pole of residue 1 at $s=0$ and is otherwise analytic in a neighborhood of $s=0$, and let

$$G(a_1,\dots,a_k) = F(a_1,\dots,a_k)\prod_{1\le i\le j\le k} f(a_i+a_j)$$

or

$$G(a_1,\dots,a_k) = F(a_1,\dots,a_k)\prod_{1\le i< j\le k} f(a_i+a_j).$$

If $\alpha_i+\alpha_j$ are contained in the region of analyticity of $f(s)$ then

$$\sum_{\epsilon_j\in\{-1,1\}} G(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k) = \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k}\frac{2^k}{k!}$$
$$\oint\cdots\oint G(z_1,\dots,z_k)\frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{i=1}^k\prod_{j=1}^k (z_i-\alpha_j)(z_i+\alpha_j)}\,dz_1\cdots dz_k,$$

and

$$\sum_{\epsilon_j\in\{-1,1\}} (\prod_{j=1}^k \epsilon_j) G(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k) = \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k}\frac{2^k}{k!}$$
$$\oint\cdots\oint G(z_1,\dots,z_k)\frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^k \alpha_j}{\prod_{i=1}^k\prod_{j=1}^k (z_i-\alpha_j)(z_i+\alpha_j)}\,dz_1\cdots dz_k,$$

where the path of integration encloses the $\pm\alpha_j$'s.

3 Zeta and L-functions over finite fields

In transition to our discussion of L-functions we mention briefly the finite field analogues. These zeta-and L-functions are polynomials with all roots on a circle and so in fact are characteristic polynomials of matrices (orthogonal and symplectic). Deligne's equidistribution theorem allowed Katz and Sarnak to calculate local statistics for these zeros, which turn out to be the same as the statistics for the matrix groups, after scaling and taking large N limits.

Sarnak and Katz developed their theory of symmetry types of families of L-functions by studying zeta and L-functions over finite fields. In this context they were able to rigorously prove that the zeros of the families obeyed the statistics of the random matrix models; the relevant random matrix group depending on the "geometric monodromy" of the family of L-functions. We give a brief overview of some of their results.

To begin with consider the case of curves over finite fields. Let $\mathbb{F}_q$ be the finite field with q elements. Consider a homogeneous form $F(X,Y,Z)$ in three variables of degree d over $\mathbb{F}_q$. This corresponds to a curve of genus $g = d(d-1)/2$ provided that F and its partial derivatives have no zeros in common. The zeta-function for F is

$$Z(F/\mathbb{F}_q, T) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n}{n} T^n\right)$$

where N_n is the number of solutions of $F(X,Y,Z) = 0$ in $\mathbb{F}_q^3$ (where two solutions (X,Y,Z) and (X',Y',Z') are the same if $(X',Y',Z') = \lambda(X,Y,Z)$ for some λ. Then, Z is a rational function of T of the form $P(T)/(1-T)(1-qT)$. Moreover,

$$P(T) = \prod_{j=1}^{2g}(1-\alpha_j T)$$

where $|\alpha_j| = \sqrt{q}$ – this is the Riemann Hypothesis for Z. Writing $\alpha_j = \sqrt{q}e^{i\phi_j}$ with $0 \le \phi_1 \le \phi_2 \le \cdots \le \phi_{2g} < 2\pi$, we can study the distribution of the angles ϕ_j.

If we suitably normalize the angles and average with respect to all curves of genus g, then when we take the limit as $q \to \infty$ and then as $g \to \infty$ the resulting nearest neighbor distribution is the same as for the unitary group.

The lowest lying zero statistic for this collection is identical to that for Sp.

Another example of symmetry type Sp arises from curves of the form $y^2 = f(x)$ where $f(x)$ is monic and squarefree.

An example where the orthogonal symmetry type arises is for $Dy^2 = x(x-1)(x-t)$ – quadratic twists of an elliptic curve. Here D is a fundamental discriminant and t varies

The critical component to their results is Deligne's equidistribution theorem.

See the book by Katz and Sarnak for more details.

4 L-functions

The definition of L-function which we give below is a slight modification of what has come to be called the "Selberg class" Let $s = \sigma + it$ with σ and t real. An *L-function* is a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n^s},$$

with $\lambda_n \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$, which has three additional properties.

- Analytic continuation: $L(s)$ continues to a meromorphic function of finite order, with at most finitely many poles, and all poles are located on the $\sigma = 1$ line.
- Functional equation: There is a number ε with $|\varepsilon| = 1$, and a function $\gamma_L(s)$ of the form

$$\gamma_L(s) = P(s)Q^s \prod_{j=1}^{k} \Gamma(w_j s + \mu_j)$$

where P is a polynomial whose only zeros in $\sigma > 0$ are at the poles of $L(s)$, $Q > 0$, $w_j > 0$, and $\Re\mu_j \geq 0$, such that

$$\xi_L(s) := \gamma_L(s)L(s)$$

is entire, and

$$\xi_L(s) = \varepsilon\overline{\xi_L}(1-s),$$

where $\overline{\xi_L}(s) = \overline{\xi_L(\overline{s})}$.
It is sometimes convenient to write the functional equation in asymmetric form:

$$L(s) = \varepsilon X_L(s)\overline{L}(1-s),$$

where $X_L(s) = \dfrac{\overline{\gamma_L}(1-s)}{\gamma_L(s)}$.

- Euler product: For $\sigma > 1$ we have

$$L(s) = \prod_p L_p(1/p^s),$$

where the product is over the primes p, and

$$L_p(1/p^s) = \sum_{k=0}^{\infty} \frac{\lambda_{p^k}}{p^{ks}} = \exp\left(\sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}\right),$$

where $b_n \ll n^\theta$ with $\theta < \frac{1}{2}$.
Note that $L(s) \equiv 1$ is the only constant L-function, the set of L-functions is closed under products, and if $L(s)$ is an L-function then so is $L(s+iy)$ for any real y. An L-function is called *primitive* if it cannot be written as a nontrivial product of L-functions, and it can be shown, assuming Selberg's orthonormality conjectures, that any L-function has a unique representation as a product of primitive L-functions. It is believed that L-functions only arise from arithmetic objects, such as characters, automorphic forms, and automorphic representations. Very little is known about L-functions beyond those cases which have been shown to be arithmetic.
There are several interesting consequences of the above properties, and many conjectures which have been established in few (or no) cases. We highlight some of those properties which have random matrix analogues as described in section 2.8.

- Location of zeros: Since $\xi_L(s)$ is entire, $L(s)$ must vanish at the poles of the Γ-functions in the γ_L factor. These are known as the *trivial zeros* of the L-function. By the functional equation and the Euler product, the only other possible zeros of $L(s)$ lie in the *critical strip* $0 \leq \sigma \leq 1$. By the argument principle, the number of nontrivial zeros with $0 < t < T$ is asymptotically $(W/\pi)T\log T$, where $W = \sum w_j$. The *Riemann Hypothesis* for $L(s)$ asserts that the nontrivial zeros of $L(s)$ lie on the *critical line* $\sigma = \frac{1}{2}$. The much weaker (but still deep) assertion that $L(s) \neq 0$ on $\sigma = 1$ has been proven for arithmetic L-functions.

- Average spacing of zeros: By the zero counting result described above, the average gap between zeros of $L(s)$ with imaginary part T is $\pi/W \log T$. In the notation of (1.1.2), the average spacing between the first few zeros of $L(s)$ is $\log Q + W + \sum |\Im(\mu_j)|$.
- Critical values: The value $L(\frac{1}{2})$ is called the *critical value* of the L-function. The significance of $s = \frac{1}{2}$ is that it is the symmetry point of the functional equation. The mean values we study in this paper are averages of (powers of) critical values of L-functions, where the average is taken over a "family" of L-functions.
  Note. If the set $\{\mu_j\}$ is stable under complex conjugation and the λ_n are real, then ε is commonly called the *sign of the functional equation*. If the sign is -1 then $L(s)$ has an odd order zero at $s = \frac{1}{2}$; more generally, if the sign is not 1 then $L(\frac{1}{2}) = 0$. When $L(\frac{1}{2})$ vanishes, it is common to use the term 'critical value' for the first nonzero derivative $L^{(j)}(\frac{1}{2})$, but in this paper we use 'critical value' to mean 'value at the critical point.'
- Approximate functional equation and analytic conductor: A standard tool for studying analytic properties of L-functions is an approximate functional equation for $L(s)$:

$$L(s) = \sum_{1 \le n \le x} \frac{\lambda_n}{n^s} + \varepsilon X_L(s) \sum_{1 \le n \le y} \frac{\overline{\lambda_n}}{n^{1-s}} + error \;\; term$$

  where $xy = (t/2\pi)^{2W}$. The name comes from the fact that the right side looks like $L(s)$ if x is large, and like $\varepsilon X_L(s)\overline{L}(1-s)$ if x is small, which suggests the asymmetric form of the functional equation. The quantity $(t/2\pi)^{2W}$ is called the analytic conductor for this L-function. In general, there is a conductor associated with a family of L-functions; this conductor varies "continuously" with the family. Even though the family is often a discrete family, this is a useful concept; for example, the analytic conductor is the basic parameter which goes to ∞ in our discussion of moments.

Below we give some specific examples. In each example we try to give an up-to-date account of what is known about these L-functions, especially with regard to their moments. In every situation our knowledge of moments is consistent with the conjectures made in [CFKRS] where we give recipes for how to determine all of the main terms for averages of L-functions over families. An amazing fact emerges: when dealing with L-functions, each family has its own basic harmonic detector which on the surface appears to be very different from the detectors from other families, yet in the end somehow functions the same. When trying to prove asymptotic formulas for moments in a family of L-functions, at each step to a higher moment a new aspect or feature of the harmonic detector comes into view and assumes center stage yet somehow conspires to contribute appropriately to the same simple formula that is analogous to the RMT formulas. Moreover, in studying two different families but with the same symmetry type, the end formulas, apart from arithmetic

constants, are always the same. We refer the reader to [CFKRS] for a detailed description of the recipes for these conjectures.

4.1 The Riemann zeta-function

The Riemann zeta-function is given by

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The series converges in the half-plane where the real part of s is larger than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole plane apart from a simple pole at $s = 1$. Moreover, he proved that $\zeta(s)$ satisfies a *functional equation* which in its symmetric form is given by

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$$

where $\Gamma(s)$ is the usual Gamma-function. The zeta-function had been studied previously by Euler and others, but only as a function of a real variable. In particular, Euler proved that

$$\begin{aligned}\zeta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots\right)\left(1 + \frac{1}{5^s} + \dots\right)\dots \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}\end{aligned}$$

where the infinite product (called the *Euler product*) is over all the prime numbers. The product converges when the real part of s is greater than 1 and is an analytic version of the fundamental theorem of arithmetic, which states that every integer can be factored into primes in a unique way. The Euler product implies that there are no zeros of $\zeta(s)$ with real part greater than 1; the functional equation implies that there are no zeros with real parts less than 0, apart from the *trivial zeros* at $s = -2, -4, -6, \dots$. Thus, all of the complex zeros are in the *critical strip* $0 \le \Re s \le 1$. The functional equation shows that the complex zeros are symmetric with respect to the line $\Re s = \frac{1}{2}$. Riemann calculated the first few complex zeros $\frac{1}{2} + i14.134\dots$, $\frac{1}{2} + i21.022\dots$ and proved that the number $N(T)$ of zeros with imaginary parts between 0 and T is

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(1/T)$$

where $S(T) = \frac{1}{\pi}\arg\zeta(1/2 + iT)$ is computed by continuous variation starting from $\arg\zeta(2) = 0$ and proceeding along straight lines, first up to $2 + iT$ and then to $1/2 + iT$. Riemann also proved that $S(T) = O(\log T)$. Note for future reference that at a height T the average gap between zero heights is

$\sim 2\pi/\log T$. Riemann suggested that the number $N_0(T)$ of zeros of $\zeta(1/2+it)$ with $0 < t \le T$ seemed to be about

$$\frac{T}{2\pi}\log\frac{T}{2\pi e};$$

and then made his conjecture that all of the zeros of $\zeta(s)$ in fact lie on the 1/2-line; this is the Riemann Hypothesis.

Weil's explicit formula

André Weil proved the following formula which is a generalization of Riemann's formula mentioned above and which specifically illustrates the dependence between primes and zeros. Let h be an even function which is holomorphic in the strip $|\Im t| \le 1/2+\delta$ and satisfying $h(t) = O((1+|t|)^{-2-\delta})$ for some $\delta > 0$, and let

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-iur}\,dr.$$

Then we have the following duality between primes and zeros:

$$\sum_\gamma h(\gamma) = 2h(\tfrac{i}{2}) - g(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(\tfrac{1}{4}+\tfrac{1}{2}ir)\,dr - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n).$$

In this formula, a zero is written as $\rho = 1/2 + i\gamma$ where $\gamma \in \mathbb{C}$; of course RH is the assertion that all of the γ are real. Also, von Mangoldt's function $\Lambda(n)$ is equal to $\log p$ if n is a power of the prime p and is 0 if n is not a power of a prime ($\Lambda(1) = 0$.) Using this duality Weil gave a criterion for RH: *RH if and only if*

$$\sum_\gamma h(\gamma) > 0$$

for every (admissible) function h of the form $h(r) = h_0(r)\overline{h_0(\overline{r})}$.

Orthogonality

Deriving statistics for families of L-functions are built around orthogonality relations. We can consider the Riemann zeta-function $\zeta(s)$ (or any L-function by itself) as being a "family" in the parameter t where $s = \sigma + it$. Then, the basic orthogonality relation is given by

$$\int_0^T m^{it}n^{-it}\,dt = \begin{cases} T \text{ if } m = n \\ \frac{(m/n)^{iT}-1}{i\log(m/n)} \text{ if } m \ne n \end{cases}$$

The mean-value theorem of Montgomery and Vaughan asserts that

$$\int_0^T \left|\sum_{n=1}^N \lambda_n n^{it}\right|^2 dt = \sum_{n=1}^N (T + O(n)|\lambda_n|^2.$$

Approximate functional equation

Hardy and Littlewood proved what they called an approximate functional equation for $\zeta(s)$:

$$\zeta(s) = \sum_{n \le \sqrt{\frac{|t|}{2\pi}}} \frac{1}{n^s} + \chi(s) \sum_{n \le \sqrt{\frac{|t|}{2\pi}}} \frac{1}{n^{1-s}} + O(|t|^{1/4-\sigma})$$

where $\chi(s) = \chi(1-s)^{-1}$ is the factor from the assymetric form of the functional equation:

$$\chi(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos \pi s/2.$$

This formula is useful for proving moment theorems. The analytic conductor is $t/(2\pi)$.

General remarks on approximate functional equations

In general, an approximate functional equation is obtained as follows. Suppose that we have an L-function $L(s) = \sum_{n=1}^{\infty} \lambda_n n^{-s}$ where the series converges absolutely for $\sigma > 1$ which is entire and satisfies the functional equation $L(s) = X(s)L(1-s)$. Let $G(z)$ be a suitable function which satisfies $G(0) = 1$ and is analytic in $-1-\delta < \Re z < 1+\delta$ for some $\delta > 0$. Consider

$$I(s,Y) = \frac{1}{2\pi i}\int_{(1)} L(s+z)G(z)Y^z \frac{dz}{z}$$

where we are thinking of s having $\Re s \approx 1/2$. On the one hand, we integrate term-by-term (note that $1 + \Re s > 1$) and have

$$I(s,x) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n^s} \tilde{G}(n/Y)$$

where

$$\tilde{G}(r) = \frac{1}{2\pi i}\int_{(1)} G(z) r^{-z} \frac{dz}{z}.$$

The "suitability" of G is supposed to ensure convergence here. On the other hand, we move the path of integration to $\sigma = -1$, crossing the pole at $s = 0$ with residue $L(s)$; then we change variables $z \to -z$ and use the functional equation to obtain

$$I(s,X) = L(s) + \frac{1}{2\pi i}\int_{(1)} L(1-s+z)X(s-z)G(-z)Y^{-z}\frac{dz}{z}.$$

Let $H_s(z) = X(s-z)G(-z)/X(s)$ and assume further that $H(s,z)$ is analytic in the strip $-1-\delta < \Re z < 1+\delta$. (Note that this does not really entail an additional assumption because $X(z) = A^z \prod \Gamma(w_i(1-z)+\mu_i)/\Gamma(w_i z + \mu_i)$

for some $w_i > 0$ and complex μ_i with $\Re\mu_i \geq 0$. Therefore, the leftmost poles of $X(z)$ will have real parts ≥ 1.) Integrating term-by-term again we finally obtain

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n^s} \tilde{G}(n/Y) + X(s) \sum_{n=1}^{\infty} \frac{\lambda_n}{n^{1-s}} \tilde{H}_s(nY).$$

The suitability of G is such that the transforms $\tilde{G}(y)$ and $\tilde{H}_s(y)$ should decay as $y \to \infty$ so that the series converge. Generally speaking these transforms should be about 1 for small y and about 0 for larger y. In this way, we obtain our "approximate" functional equations.

Note that this technique works fine for products

$$L_1(s_1)L_2(s_2)\dots L_k(s_k)$$

too.

Moment theorems

The second moment of $|\zeta(1/2+it)|$ was proven by Hardy and Littlewood and refined by Ingham. It states that

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2 \, dt = \log\frac{T}{2\pi} + 2\gamma - 1 + O(T^{1/2}).$$

The exponent on the error term has been improved to slightly less than 1/3.

If we consider the shifted moment we get a more general statement: Suppose that $|\alpha|, |\beta| \ll 1/\log T$. Then

$$\begin{aligned}\int_1^T &\zeta(1/2+it+\alpha)\zeta(1/2-it-\beta)\, dt \\ &= \int_1^T \left(\zeta(1+\alpha-\beta) + \left(\frac{t}{2\pi}\right)^{\beta-\alpha} \zeta(1+\beta-\alpha)\right)\, dt \\ &\qquad + O(T^{1/2})\end{aligned}$$

The asymptotics of the fourth power moment was first achieved by Ingham:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 \, dt \sim \frac{1}{2\pi^2}\log^4 T.$$

Subsequent works of Atkinson and Heath-Brown revealed that

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 \, dt = \int_0^T P_2\left(\log\frac{t}{2\pi}\right)\, dt + O(T^{7/8+\epsilon})$$

where P_2 is a polynomial of degree 4. This polynomial was computed explicitly by Conrey:

$$
\begin{aligned}
P_2(x) = &\frac{1}{2\pi^2}x^4 + \frac{8}{\pi^4}\left(\gamma\pi^2 - 3\zeta'(2)\right)x^3 \\
&+ \frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2 - 12\zeta''(2)\pi^2 + 7\gamma^2\pi^4 + 144\zeta'(2)^2 - 2\,\gamma_1\pi^4\right)x^2 \\
&+ \frac{12}{\pi^8}\Big(6\gamma^3\pi^6 - 84\gamma^2\zeta'(2)\pi^4 + 24\gamma_1\zeta'(2)\pi^4 - 1728\zeta'(2)^3 + 576\gamma\zeta'(2)^2\pi^2 \\
&\qquad + 288\zeta'(2)\zeta''(2)\pi^2 - 8\zeta'''(2)\pi^4 - 10\gamma_1\gamma\pi^6 - \gamma_2\pi^6 - 48\gamma\zeta''(2)\pi^4\Big)x \\
&+ \frac{4}{\pi^{10}}\Big(-12\zeta''''(2)\pi^6 + 36\gamma_2\zeta'(2)\pi^6 + 9\gamma^4\pi^8 + 21\gamma_1^2\pi^8 + 432\zeta''(2)^2\pi^4 \\
&\qquad + 3456\gamma\zeta'(2)\zeta''(2)\pi^4 + 3024\gamma^2\zeta'(2)^2\pi^4 - 36\gamma^2\gamma_1\pi^8 - 252\gamma^2\zeta''(2)\pi^6 \\
&\qquad + 3\gamma\gamma_2\pi^8 + 72\gamma_1\zeta''(2)\pi^6 + 360\gamma_1\gamma\zeta'(2)\pi^6 - 216\gamma^3\zeta'(2)\pi^6 \\
&\qquad - 864\gamma_1\zeta'(2)^2\pi^4 + 5\gamma_3\pi^8 + 576\zeta'(2)\zeta'''(2)\pi^4 - 20736\gamma\zeta'(2)^3\pi^2 \\
&\qquad - 15552\zeta''(2)\zeta'(2)^2\pi^2 - 96\gamma\zeta'''(2)\pi^6 + 62208\zeta'(2)^4\Big)
\end{aligned}
$$

The point of displaying this formula is for comparison with how simple the formula becomes when recast in terms of shifts below. We note that numerically,

$$P_2(x) = 0.05066\,x^4 + 0.69886\,x^3 + 2.42596\,x^2 + 3.22790\,x + 1.312424$$

Following work of Motohashi [M2] it was discovered how to put the shifted mean value of $\zeta(s)$ into a nice symmetric form. Let $s = 1/2 + it$ and let $\alpha, \beta, \gamma, \delta \ll 1/\log T$. Then

$$
\begin{aligned}
&\int_0^T \zeta(s+\alpha)\zeta(s+\beta)\zeta(1-s+\gamma)\zeta(1-s+\delta)\,dt \\
&\quad = \int_0^T \big(Z(\alpha,\beta,\gamma,\delta) + \tau^{-\alpha-\gamma}Z(-\gamma,\beta,-\alpha,\delta) + \tau^{-\alpha-\delta}Z(-\delta,\beta,\gamma,-\alpha) \\
&\qquad + \tau^{-\beta-\gamma}Z(\alpha,-\gamma,-\beta,\delta) + \tau^{-\beta-\delta}Z(\alpha,-\delta,\gamma,-\beta) \\
&\qquad + \tau^{-\alpha-\beta-\gamma-\delta}Z(-\gamma,-\delta,-\alpha,-\beta)\big)\,dt + O(T^{7/8})
\end{aligned}
$$

where $\tau = \sqrt{\frac{t}{2\pi}}$ and

$$Z(\alpha,\beta,\gamma,\delta) = \frac{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\gamma+\delta)}{\zeta(2+\alpha+\beta+\gamma+\delta)}.$$

This formula should be compared with the analogous formula for the shifted fourth moment of unitary characteristic polynomials presented in section 2.11.

Conrey and Ghosh [CGh] conjectured that

$$\int_0^T |\zeta(1/2+it)|^6\ dt \sim 42 \prod_p \left(1-\frac{1}{p}\right)^4 \left(1+\frac{4}{p}+\frac{1}{p^2}\right) \frac{\log^9 T}{9!}$$

and in general that

$$\int_0^T |\zeta(1/2+it)|^{2k}\ dt \sim g_k a_k \frac{\log^{k^2} T}{k^2!}$$

where

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \binom{k-1}{j}^2 p^{-j}$$

for some integer g_k.

Conrey and Gonek [CGo] conjectured that

$$\int_0^T |\zeta(1/2+it)|^8\ dt \sim 24024 \prod_p \left(1-\frac{1}{p}\right)^9 \left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right) \frac{\log^{16} T}{16!}$$

Keating and Snaith made the key connection with random matrix theory and conjectured that

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS1] have found a way to express all of the lower order terms as well. Their conjecture is

$$\int_0^T |\zeta(1/2+it)|^{2k}\ dt = \int_0^T P_k\left(\log \tfrac{t}{2\pi}\right)\ dt + O(T^{\frac{1}{2}+\epsilon})$$

as $T \to \infty$, where P_k is the polynomial of degree k^2 given by the $2k$-fold residue

$$P_k(x) = \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{G(z_1,\dots,z_{2k})\Delta(z_1,\dots,z_{2k})^2}{\prod_{j=1}^{2k} z_j^{2k}} e^{\frac{x}{2}\sum_{j=1}^k z_j - z_{j+k}}\ dz_1 \dots dz_{2k},$$

where one integrates over small circles about $z_i = 0$, with

$$G(\alpha_1,\dots,\alpha_{2k}) = A_k(\alpha_1,\dots,\alpha_{2k}) \prod_{i=1}^k \prod_{j=1}^k \zeta(1+\alpha_i-\alpha_{j+k}),$$

and A_k is the Euler product which is absolutely convergent for $\sum_{j=1}^k |\alpha_j| + |\beta_j| < 1/2$, defined by $A_k(\alpha) =$

$$\prod_p \prod_{i=1}^{k} \prod_{j=1}^{k} \left(1 - \frac{1}{p^{1+\alpha_i-\alpha_{j+k}}}\right) \int_0^1 \prod_{j=1}^{k} \left(1 - \frac{e(\theta)}{p^{1/2+\alpha_j}}\right)^{-1} \left(1 - \frac{e(-\theta)}{p^{1/2-\alpha_{j+k}}}\right)^{-1} d\theta.$$

More generally, with $s = 1/2 + it$,

$$\int_0^T \zeta(s+\alpha_1)\dots\zeta(s+\alpha_k)\zeta(1-s+\alpha_{k+1})\dots\zeta(1-s+\alpha_{2k})\ dt$$
$$= \int_0^T P_k\left(\alpha, \log\tfrac{t}{2\pi}\right)\ dt + O(T^{\frac{1}{2}+\epsilon}),$$

where $P_k(\alpha, x) =$

$$\frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{G(z_1,\dots,z_{2k})\Delta(z_1,\dots,z_{2k})^2}{\prod_{j=1}^{2k}\prod_{i=1}^{2k}(z_j-\alpha_i)} e^{\frac{x}{2}\sum_{j=1}^{k} z_j - z_{j+k}}\ dz_1\dots dz_{2k},$$

with the path of integration being small circles surrounding the poles α_i. For example,

$$\begin{aligned} P_3(x) = {} & 0.00000570852\,x^9 + 0.00040502\,x^8 + 0.011072\,x^7 + 0.148400\,x^6 \\ & +1.04592\,x^5 + 3.98438\,x^4 + 8.607319\,x^3 + 10.274330\,x^2 \\ & +6.593913\,x + .916515. \end{aligned}$$

When $|\zeta(1/2 + it)|^6$ is integrated numerically from 0 to 2,350,000 the ratio between the actual value and the conjectured value is 1.00017.

Pair Correlation of zeros

In 1972, Hugh Montgomery was investigating the spacings between zeros of the zeta-function in an attempt to solve the class number problem. Montgomery's theorem is that, under the assumption of the Riemann Hypothesis, if $|\alpha| < 1$, then

$$F(\alpha, T) = \frac{1}{N(T)} \int_{-\infty}^{\infty} \left| \sum_{\gamma \le T} T^{i\alpha\gamma} w(t-\gamma) \right|^2 dt = T^{-2\alpha} \log T(1+o(1)) + |\alpha| + o(1)$$

as $T \to \infty$. Here w is a suitable weight function (Montgomery used $w(x) = \sqrt{2/\pi}4/(4 + x^2)$.) Montgomery conjectured, based on the above and on conjectures for the distribution of twin primes and other prime pairs that $F(\alpha, T) = 1 + o(1)$ for $|\alpha| \ge 1$. From this conjecture, he deduced that

$$\sum_{\frac{2\pi\alpha}{\log T} < \gamma - \gamma' \le \frac{2\pi\beta}{\log T}} 1 \sim N(T) \int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right)\ du.$$

The sum on the left counts the number of pairs $0 < \gamma, \gamma' < T$ of ordinates of zeros with normalized spacing between positive numbers $0 < \alpha < \beta$. The integral on the right is the pair-correlation function from random matrix theory. It was this fortuitous discovery, made in a conversation between Montgomery and Freeman Dyson at tea-time at the Institute for Advanced Study, that set in motion the circle of ideas involving L-functions and RMT.

Higher correlations

In 1996 Rudnick and Sarnak [RS] made some interesting progress on the GUE conjecture. To explain their result, number the ordinates of the zeros of $\zeta(s)$: $0 < \gamma_1 \le \gamma_2 \le \dots$. Introduce a scaling $\tilde{\gamma} = \gamma \frac{\log \gamma}{2\pi}$ so that the $\tilde{\gamma}$ have asymptotic mean spacing 1. Then Rudnick and Sarnak proved that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\substack{\gamma_{j_1},\dots,\gamma_{j_n} \le T \\ j_m \ne j_n}} f(\tilde{\gamma}_{j_1}, \dots, \tilde{\gamma}_{j_n}) = \int_{P_n} W_{U,n}(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x}$$

where $W_{U,n}(\mathbf{x}) = W_{U,n}(x_1, \dots, x_n)$ is the n-correlation function for the Gaussian Unitary Ensemble and where f is any function satisfying (1) $f(\mathbf{x} + t(1, \dots, 1)) = f(\mathbf{x})$ for $t \in \mathbb{R}$; (2) f is smooth and symmetric in the variables and decays rapidly as $x \to \infty$ in the hyperplane $P_n := \{(x_1, \dots, x_n) : \sum_{j=1}^n x_j = 0\}$; (3) the Fourier transform $\hat{f}(\mathbf{u})$ of f is supported in $\sum_{j=1}^n |u_j| < 2$. The condition (1) assures that f is a function of the differences of the γ_j. This result agrees with RMT.

4.2 Dirichlet L-functions

In order to see the full analogy between L-functions and Random Matrix Theory, it is necessary to consider a variety of families of L-functions with different symmetry types. The simplest L-function after the ζ-function is the Dirichlet L-function for the non-trivial character of conductor 3:

$$L(s, \chi_{-3}) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + - \dots.$$

This can be written as an Euler product

$$L(s, \chi_{-3}) = \prod_{p \equiv 1 \bmod 3} (1 - p^{-s})^{-1} \prod_{p \equiv 2 \bmod 3} (1 + p^{-s})^{-1},$$

satisfies the functional equation

$$\xi(s, \chi_{-3}) := \left(\tfrac{\pi}{3}\right)^{-\frac{s}{2}} \Gamma(\tfrac{s+1}{2}) L(s, \chi_3) = \xi(1 - s, \chi_{-3}),$$

and is expected to have all of its non-trivial zeros on the 1/2-line. A similar construction works for any primitive Dirichlet character. In general, a Dirichlet

character is a completely multiplicative periodic function $\chi : \mathbb{N} \to \mathbb{C}$; i.e. $\chi(mn) = \chi(m)\chi(n)$ for all m, n and $\chi(m+q) = \chi(m)$ for some integer q. It is the *primitive* characters which lead to the arithmetic L-functions. We now describe how to construct the primitive characters. For each $q \geq 1$ there are precisely

$$\psi(q) = \sum_{d|q} \mu(d)\phi(q/d)$$

primitive characters to the modulus q. If q has the factorization $q = p_1^{e_1} \dots p_r^{e_r}$, then any primitive character χ mod q has a unique representation as a product $\chi = \chi_1 \dots \chi_r$ where χ_j is a primitive character modulo $p_j^{e_j}$. We now describe how to construct the primitive characters modulo p^e. If p is odd, then the number of integers less than or equal to p^e and relatively prime to p^e is given by $\phi(p^e) = p^e - p^{e-1}$. These reduced residues modulo p^e form a multiplicative group which is cyclic; let g be a generator of this group (i.e. a *primitive root* of p^e.) We can specify any character χ modulo p^e by saying what the value of $\chi(g)$ is (clearly this value must be a $\phi(p^e)$ root of unity). The primitive characters are those for which $\chi(g) = \exp(2\pi i a/\phi(p^e))$ where $(a, \phi(p^e)) = 1$. For $p = 2$, the reduced residues modulo 2^e do not form a cyclic group unless $e = 1$ or 2. If $e \geq 3$ then the reduced residues are given by $\pm 5^j$ with $j = 0, 1, \dots 2^{e-2}$. The primitive characters χ modulo 2^e are determined by the value of $\chi(5) = \exp(2\pi i a/2^{e-2})$ with $1 \leq a \leq 2^{e-2}$ odd and by the value of $\chi(-1) = \pm 1$. This describes all primitive characters. For each primitive character χ mod q the *Gauss sum* is given by

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) e(n/q).$$

It satisfies $|\tau(\chi)| = \sqrt{q}$; we write $\tau(\chi) = \epsilon_\chi \sqrt{q}$. The Dirichlet L-function is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

for $\sigma > 1$. Odd characters are those for which $\chi(-1) = -1$; even characters have $\chi(-1) = 1$. The functional equation for an even character is

$$\xi(s, \chi) := (\pi/\sqrt{q})^{-s/2} \Gamma(s/2) L(s, \chi) = \epsilon_q \xi(1 - s, \overline{\chi}).$$

For an odd character, the functional equation is

$$\xi(s, \chi) := (\pi/\sqrt{q})^{-s/2} \Gamma((s+1)/2) L(s, \chi) = \epsilon_q \xi(1 - s, \overline{\chi}).$$

All characters

We have described the primitive characters above. Imprimitive characters arise in two ways. First, the principal character χ_0 modulo p defined by $\chi_0(n) =$

0 if $p \mid n$ and$= +1$ if $p \nmid n$ is an imprimitive character. Second, a primitive character modulo p^e regarded as a character modulo p^f where $f > e$ is an imprimitive character. Finally, the product of a primitive character with an imprimitive character is an imprimitive character. Any character χ (primitive or imprimitive) which satisfies $\chi(m+q) = \chi(m)$ is called a character modulo q. There are $\phi(q)$ characters modulo q. The analytic conductor is $qt/(2\pi)$; in the context of averaging over the characters, the t part is suppressed ao that the conductor is $q/(2\pi)$.

Orthogonality relations

The basic orthogonality relation is then expressed by: if $(mn, q) = 1$, then

$$\sum_{\chi \bmod q} \chi(m)\overline{\chi(n)} = \begin{cases} \phi(q) \text{ if } m = n \bmod q \\ 0 \text{ if } m \neq n \bmod q \end{cases}$$

For primitive characters, this takes the shape: if $(mn, q) = 1$,then

$$\sum_{\chi \bmod q}^{\star} \chi(m)\overline{\chi(n)} = \sum_{d|(q,m-n)} \phi(d)\mu(q/d).$$

The Polya-Vinogradov inequality asserts that

$$|\sum_{n=1}^{N} \chi(n)| \ll q^{1/2} \log q$$

for any non-principal character $\chi \bmod q$.

The large sieve inequality asserts that

$$\sum_{q \le Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^{\star} \left| \sum_{n=1}^{N} a_n \chi(n) \right|^2 \le (Q^2 + N) \sum_{n=1}^{N} |a_n|^2.$$

Pair correlation of zeros

Ali Ozluk, in his thesis [Oz] gave a generalization to Dirichlet L-functions of Montgomery's Pair Correlation results. Let

$$F_K(\alpha, Q) = \frac{1}{N_K(Q)} \sum_{q \le Q} \frac{1}{\phi(q)} \sum_{\chi \bmod q} \left| \sum_{\gamma} K(\frac{1}{2} + i\gamma) Q^{i\alpha\gamma} \right|^2,$$

where the inner sum is over the imaginary parts γ of zeros of $L(s, \chi)$. Here $K(s)$, is a suitable weight function and $N_K(Q) = \sum_\gamma K(1/2 + i\gamma)$ is the normalization factor. The main result of the paper is an asymptotic formula for $F_K(\alpha, Q)$ in the interval $|\alpha| < 2$. For $|\alpha| \le 1$, the result obtained is an analogous to Montgomery's result. For $1 < |\alpha| < 2$ Ozluk shows that $F_k(\alpha, Q) = 1 + O(1/\log Q)$, which supports Montgomery's conjecture.

Moments

For the second moment Heath-Brown [H-B2] showed that

$$\sum_{\chi \bmod p}^{\star} |L(1/2,\chi)|^2 = (p-1)\left(\log\frac{p}{8\pi}+\gamma\right) + 2\zeta(1/2)^2 p^{1/2} + O(1)$$

and for the fourth moment, he proved ([H-B3] that

$$\sum_{\chi \bmod p}^{\star} |L(1/2,\chi)|^4 = \frac{p-1}{2\pi^2}\log^4 p + O(\log^3 p)$$

in analogy with the second and fourth moments of the Riemann zeta-function. Here the * indicates that the sum is over primitive characters modulo the prime p. An analogous formula can be proven in the case of the second moment for primitive characters modulo a composite number q; however, the analogue for the fourth moment for all large moduli q has not yet been proven.

A second moment for shifted L-functions averaged over primitive characters has been proven by Conrey (unpublished); this formula agrees with the analogous formula for the shifted second moment of unitary characteristic polynomials:

$$\begin{aligned}\frac{2}{\psi(q)} &\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{\star} L(1/2+\alpha,\chi)L(1/2+\beta,\overline{\chi}) \\ &= \zeta(1+\alpha+\beta)\prod_{p|q}\big(1-\frac{1}{p^{1+\alpha+\beta}}\big) + \\ &\qquad X^+(q,\alpha,\beta)\zeta(1-\alpha-\beta)\prod_{p|q}\big(1-\frac{1}{p^{1-\alpha-\beta}}\big) + O(q^{\epsilon-1/2}).\end{aligned}$$

Here

$$X^+(q,\alpha,\beta) = \frac{4}{q}\big(\frac{q}{2\pi}\big)^{1-\alpha-\beta}\Gamma(\tfrac{1}{2}-\alpha)\Gamma(\tfrac{1}{2}-\beta)\cos\tfrac{\pi}{2}(\tfrac{1}{2}-\alpha)\cos\tfrac{\pi}{2}(\tfrac{1}{2}-\beta)$$

is the factor from the functional equation, as expected.

4.3 Real primitive characters

A special role is played by the real or quadratic Dirichlet characters. These we denote by χ_d where d is a fundamental discriminant: d can be positive or negative, is either odd, squarefree, and congruent to 1 modulo 4, or is 4 times a squarefree integer congruent to 2 or 3 modulo 4. Thus, the sequence of positive fundamental discriminants begins $d = 1, 5, 8, 12, 13, 17, 21, 24, 28, \ldots$ and the sequence of negative fundamental discriminants begins d=-3, -4, -7, -8, -11, -15, -19,-20, -23, -24, ... The character χ_d only takes on the values $+1, 0, -1$;

it is primitive with the modulus $|d|$. If $d > 0$, then χ_d is an even character and if $d < 0$ it is an odd character. The character χ_d is the character associated with the quadratic field $Q(\sqrt{d})$. In particular, the prime p *splits* or factors in this field if $\chi_d(p) = +1$; it remains prime if $\chi_d(p) = -1$; and it *ramifies* or is a square if $\chi_d(p) = 0$. The real characters χ_d can be decomposed into a product of characters $\chi_{-4}, \chi_8, \chi_{-8}$, and $\chi_{\pm p}$ for odd primes p where $\chi_{\pm p}(n) = \left(\frac{n}{p}\right)$ is the Legendre symbol (= 1 if n is a non-zero square modulo p, and $= -1$ if n is a non-zero non-square modulo p, $= 0$ if $p \mid n$). The character $\chi_{-4}(n)$ is 0 for even n, is $+1$ for n congruent to 1 modulo 4, and is -1 for n congruent to 3 modulo 4. The character $\chi_8(n)$ is 0 for even n, is $+1$ for n congruent to ± 1 modulo 8, and is -1 for n congruent to ± 3 modulo 8. Finally, $\chi_{-8}(n)$ is 0 for even n, is $+1$ for n congruent to 1 or 3 modulo 8, and is -1 for n congruent to 5 or 7 modulo 8.

Orthogonality

First of all, the number $N_q^+(x)$ of fundamental discriminants d with $0 < d \le x$ and $(d, q) = 1$ satisfies $N_q^+(x) \sim \frac{3}{\pi^2}\frac{\phi(q)}{q}x$ and similarly the number $N_q^-(x)$ of negative fundamental discriminants d with $0 < -d < x$ and $(d, q) = 1$ satisfies $N_q^-(x) \sim \frac{3}{\pi^2}\frac{\phi(q)}{q}x$.

By the Polya-Vinogradov inequality,

$$\sum_{0<d\le x} \chi_d(n) = \begin{cases} N_n^+(x) \text{ if } n \text{ is a square} \\ O(n^{1/2+\epsilon}) \text{ if } n \text{ is not a square} \end{cases}$$

Heath-Brown [H-B4] proved a very useful large-sieve type inequality:

$$\sum_{|d|\le Q} \left|\sum_{n=1}^{N} a_n \chi_d(n)\right|^2 \ll_\epsilon (QN)^\epsilon (Q+N) \sum_{n_1 n_2 = \square} |a_{n_1} a_{n_2}|.$$

Soundararajan [So] gave a Poisson summation formula for smooth (real) character sums:

$$\sum_d \chi_d(n) F(d/X) = \frac{X}{n} \sum_{k=-\infty}^{\infty} \hat{F}(kX/n)\tau_k(n)$$

where

$$\tau_k(n) = \sum_{a=1}^{n} \left(\frac{a}{n}\right) e(ak/n)$$

is a Gauss sum.

Approximate functional equations

For integers $j \geq 1$ put $\omega_j(0) = 1$ and for $\xi > 0$ define $\omega_j(\xi)$ by

$$\omega_j(\xi) = \frac{1}{2\pi i} \int_{(c)} \left(\frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \right)^j \xi^{-s} \frac{ds}{s}$$

where c is any positive real number. As usual, $d_j(n)$ will denote the j-th divisor function; that is the coefficient of n^{-s} in the Dirichlet series expansion of $\zeta(s)^j$. For integers $j \geq 1$, we define

$$A_j(d) = \sum_{n=1}^{\infty} \frac{d_j(n)\chi_d(n)}{\sqrt{n}} \omega_j\left(n \left(\frac{\pi}{|d|} \right)^{\frac{j}{2}} \right).$$

Then for all integers $j \geq 1$,

$$L(\tfrac{1}{2}, \chi_d)^j = 2A_j(d).$$

The analytic conductor is $|d|/(2\pi)$.

Moments

Jutila [Jut] proved that

$$\sum_{0<d<X} L(1/2, \chi_d) = \frac{P(1)}{4\zeta(2)} X \left(\log \frac{X}{\pi} + \frac{\Gamma'}{\Gamma}(1/4) + 4\gamma - 1 + 4\frac{P'(1)}{P(1)} \right) + O(X^{3/4+\epsilon})$$

where $P(s) = \prod_p \left(1 - \frac{1}{(p+1)p^s}\right)$ and for the same sum but with negative discriminants $-d < X$, the term $\frac{\Gamma'}{\Gamma}(1/4)$ is replaced by $\frac{\Gamma'}{\Gamma}(3/4)$. In the same paper, he obtains that

$$\sum_{0<d<X} L(1/2, \chi_d)^2 = \frac{c}{\zeta(2)} X \log^3 X + O(X(\log X)^{5/2+\epsilon})$$

and the same for the sum over negative discriminants where

$$c = \frac{1}{48} \prod_p (1 - (4p^2 - 3p + 1)/(p^4 + p^3)).$$

Soundararajan [So] obtains a full degree three polynomial for the mean-square, a full degree six polynomial for the cubic moment, and a conjecture for the fourth moment. These results and conjecture can be summarized, using $D^* = \sum_{|d| \leq D} 1$, as :

$$\frac{1}{D^*}\sum_{|d|\le D} L(\tfrac12,\chi_d) \sim a_1 \log(D^{\frac12});$$

$$\frac{1}{D^*}\sum_{|d|\le D} L^2(\tfrac12,\chi_d) \sim 2\frac{a_2 \log^3(D^{\frac12})}{3!};$$

$$\frac{1}{D^*}\sum_{|d|\le D} L^3(\tfrac12,\chi_d) \sim 16\frac{a_3 \log^6(D^{\frac12})}{6!};$$

$$\frac{1}{D^*}\sum_{|d|\le D} L^4(\tfrac12,\chi_d) \sim 768\frac{a_4 \log^{10}(D^{\frac12})}{10!}.$$

The general conjecture stated by Keating and Snaith using random matrix theory is:

$$\frac{1}{D^*}\sum_{|d|\le D} L^k(\tfrac12,\chi_d) \sim \prod_{\ell=1}^{k}\frac{\ell!}{2\ell!}a_k \log^{k(k+1)/2}(D)$$

where

$$a_k = \prod_p \frac{\left(1-\frac{1}{p}\right)^{\frac{k(k+1)}{2}}}{\left(1+\frac{1}{p}\right)}\left(\frac{\left(1-\frac{1}{\sqrt{p}}\right)^{-k}+\left(1+\frac{1}{\sqrt{p}}\right)^{-k}}{2}+\frac{1}{p}\right).$$

The conjectures all agree.

More generally, we conjecture that

$$\sum_{0<-d<D} L(1/2,\chi_d)^k = \sum_{0<-d<D} Q_k(\log\tfrac{|d|}{2\pi}) + O(D^{1/2+\epsilon})$$

as $D\to\infty$, where Q_k is the polynomial of degree $k(k+1)/2$ given by the k-fold residue

$$Q_k(x) = \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint \frac{G_-(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)^2 e^{\frac{x}{2}\sum_{j=1}^k z_j}}{\prod_{j=1}^k z_j^{2k-1}}\,dz_1\dots dz_k,$$

where

$$G_-(\alpha_1,\dots,\alpha_k) = A_k(\alpha_1,\dots,\alpha_k)\prod_{j=1}^k\left(\frac{\Gamma(\frac34+\frac{\alpha_j}{2})2^{\alpha_j}}{\Gamma(\frac34-\frac{\alpha_j}{2})}\right)^{\frac12}\prod_{1\le i\le j\le k}\zeta(1+\alpha_i+\alpha_j),$$

and A_k is the Euler product which is absolutely convergent for $\sum_{j=1}^k|\alpha_j| < 1/2$, defined by

$$A_k(\alpha_1,\dots,\alpha_k)=\prod_p\prod_{1\le i\le j\le k}\left(1-\frac{1}{p^{1+\alpha_i+\alpha_j}}\right)$$
$$\times\left(\frac{1}{2}\left(\prod_{j=1}^k\left(1-\frac{1}{p^{\frac12+\alpha_j}}\right)^{-1}+\prod_{j=1}^k\left(1+\frac{1}{p^{\frac12+\alpha_j}}\right)^{-1}\right)+\frac1p\right)\left(1+\frac1p\right)^{-1}.$$

Still more generally, we conjecture that

$$\sum_{0<-d<D}\xi(1/2+\alpha_1,\chi_d)\dots\xi(1/2+\alpha_k,\chi_d)=\sum_{0<-d<D}^{\star}Q_k(\alpha,\log\tfrac{|d|}{2\pi})+O(D^{1/2+\epsilon}),$$

where

$$Q_k(\alpha,x)=\frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}$$
$$\times\oint\cdots\oint\frac{G_-(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{\ell=1}^k\prod_{j=1}^k(z_j-\alpha_\ell)(z_j+\alpha_\ell)}e^{\frac{x}{2}\sum_{j=1}^k z_j}\,dz_1\dots dz_k,$$

where the path of integration encloses the $\pm\alpha$'s. There is a similar conjecture for the analogous sum over positive fundamental discriminants. For this conjecture G_- is replaced by G_+, where

$$G_+(\alpha_1,\dots,\alpha_k)=A_k(\alpha_1,\dots,\alpha_k)\prod_{j=1}^k\left(\frac{\Gamma(\frac14+\frac{\alpha_j}{2})2^{\alpha_j}}{\Gamma(\frac14-\frac{\alpha_j}{2})}\right)^{\frac12}\prod_{1\le i\le j\le k}\zeta(1+\alpha_i+\alpha_j),$$

and A_k is as before.

4.4 Modular L-functions

Ramanujan's tau-function, defined implicitly by

$$x\prod_{n=1}^\infty(1-x^n)^{24}=\sum_{n=1}^\infty\tau(n)x^n,$$

also yields an L-function. The associated Fourier series

$$\Delta(z):=\sum_{n=1}^\infty\tau(n)\exp(2\pi inz)$$

satisfies

$$\Delta\left(\frac{az+b}{cz+d}\right)=(cz+d)^{12}\Delta(z)$$

for all integers a, b, c, d with $ad - bc = 1$. A function satisfying these equations is called a *modular form* of weight 12. The associated L-function

$$L_\Delta(s) := \sum_{n=1}^{\infty} \frac{\tau(n)/n^{11/2}}{n^s} = \prod_p \left(1 - \frac{\tau(p)/p^{11/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

satisfies the functional equation

$$\xi_\Delta := (2\pi)^{-s}\Gamma(s + 11/2)L_\Delta(s) = \xi_\Delta(1 - s)$$

and it is expected that all of its complex zeros are on the 1/2-line.In general a cusp form of weight k for the full modular group is a holomorphic function f on the upper half-plane which satisfies

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for all integers a, b, c, d with $ad - bc = 1$ and also has the property that $\lim_{y\to\infty} f(iy) = 0$. Cusp forms for the whole modular group exist only for even integers $k = 12$ and $k \geq 16$. The cusp forms of a given weight k of this form make a complex vector space S_k of dimension $[k/12]$ if $k \neq 2 \bmod 12$ and of dimension $[k/12]-1$ if $k = 2 \bmod 12$. Each such vector space has a special basis H_k of Hecke eigenforms which consist of functions $f(z) = \sum_{n=1}^{\infty} \lambda_f(n)e(nz)$ for which

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} d^{k-1}\lambda_f(mn/d^2).$$

The Fourier coefficients $\lambda_f(n)$ are real algebraic integers of degree equal to the dimension of the vector space $= \#H_k$. Thus, when $k = 12, 16, 18, 20, 22, 26$ the spaces are one dimensional and the coefficients are ordinary integers. We can express these explicitly in terms of the Eisenstein series

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)e(nz)$$

and

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)e(nz)$$

where $\sigma_r(n)$ is the sum of the rth powers of the positive divisors of n:

$$\sigma_r(n) = \sum_{d|n} d^r.$$

Then, $\Delta(z)E_4(z)$ gives the unique Hecke form of weight 16; $\Delta(z)E_6(z)$ gives the unique Hecke form of weight 18; $\Delta(z)E_4(z)^2$ is the Hecke form of weight 20; $\Delta(z)E_4(z)E_6(z)$ is the Hecke form of weight 22; and $\Delta(z)E_4(z)^2E_6(z)$ is the Hecke form of weight 26. The two Hecke forms of weight 24 are given by

$$\Delta(z)E_4(z)^3 + x\Delta(z)^2$$

where $x = -156 \pm 12\sqrt{144169}$. The L-function associated with a Hecke form f of weight k is given by

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n)/n^{(k-1)/2}n^s = \prod_p \left(1 - \frac{\lambda_f(p)/p^{(k-1)/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1}.$$

By Deligne's theorem $\lambda_f(p)/p^{(k-1)/2} = 2\cos\theta_f(p)$ for a real $\theta_f(p)$. It is conjectured (Sato-Tate) that for each f the $\{\theta_f(p) : p \text{ prime}\}$ is uniformly distributed on $[0,\pi)$ with respect to the measure $\frac{2}{\pi}\sin^2\theta\ d\theta$. We write $\cos\theta_f(p) = \alpha_f(p) + \overline{\alpha_f(p)}$ where $\alpha_f(p) = e^{i\theta_f(p)}$; then

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_f(p)}}{p^s}\right)^{-1}.$$

The functional equation satisfied by $L_f(s)$ is

$$\xi_f(s) = (2\pi)^{-s}\Gamma(s + (k-1)/2)L_f(s) = (-1)^{k/2}\xi_f(1-s).$$

Orthogonality relations

The Petersson inner product on the space S_k is defined by

$$\langle f, g\rangle = \iint_{\mathcal{D}} f(z)\overline{g(z)}y^k\frac{dxdy}{y^2}.$$

Here the integration is over the *fundamental domain*

$$\mathcal{D} := \{(x,y) : -1/2 \le x \le 1/2, y \ge \sqrt{1-x^2}\}.$$

Let $\mathcal{F}$ be an orthogonal basis of S_k with respect to this inner product. The Petersson formula tells us that

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}}\sum_{f\in\mathcal{F}}\frac{\lambda_f(m)\overline{\lambda_f(n)}}{\langle f,f\rangle} = \delta_{m,n} + 2\pi i^{-k}\sum_{c=1}^{\infty}\frac{S(m,n,c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where J_{k-1} is the Bessel function of index $k-1$ and $S(m,n,c)$ is the Kloosterman sum

$$S(m,n,c) = \sum_{(x,c)=1} e((mx + n\overline{x})/c)$$

where the sum is over a set of reduced residue classes modulo $c$and where $\overline{x}$ satisfies $x\overline{x} = 1 \bmod c$. By a theorem of Weil, $|S(m,n,c)| \le (m,n,c)^{1/2}d(c)\sqrt{c}$ where $d(c)$ is the number of positive divisors of c.

Higher level modular forms

An example of a higher level modular form is the modular form $\sum_{n=1}^{\infty}\lambda_n e(nz)$ associated to an elliptic curve $E: y^2 = x^3 + Ax + B$ where A, B are integers. The associated L-function, called the Hasse-Weil L-function, is

$$L_E(s) = \sum_{n=1}^{\infty}\frac{\lambda(n)/n^{1/2}}{n^s} = \prod_{p\nmid q}\left(1-\frac{\lambda(p)/p^{1/2}}{p^s}+\frac{1}{p^{2s}}\right)^{-1}\prod_{p|q}\left(1-\frac{\lambda(p)/p^{1/2}}{p^s}\right)^{-1}$$

where q is the conductor of the curve. The coefficients λ_n are constructed easily from λ_p for prime p; in turn the λ_p are given by $\lambda_p = p - N_p$ where N_p is the number of solutions of E when considered modulo p. The work of Wiles and others proved that these L-functions are associated to modular forms of weight 2. This modularity implies the functional equation

$$\xi_E(s) := (2\pi/\sqrt{q})^{-s}\Gamma(s+1/2)L_E(s) = \xi_E(1-s).$$

It is believed that all of the complex zeros of $L_E(s)$ are on the 1/2-line. A similar construction should work for other sets of polynomial equations but so far this has not been proven.

Level q cusp forms with no multiplier system

We let $\Gamma_0(q)$ denote the group of matrices $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ with integers a, b, c, d satisfying $ad - bc = 1$ and $q \mid c$. This group is called the *Hecke congruence group of level q*. A function f holomorphic on the upper half plane satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices in $\Gamma_0(q)$ and $\lim_{y\to 0} f(a/q + iy) = 0$ for all rational numbers a/q is called a cusp form for $\Gamma_0(q)$; the space of these is a finite dimensional vector space $S_k(q)$. The space S_k above is the same as $S_k(1)$. Again, these spaces are empty unless k is an even integer. If k is an even integer, then

$$\dim S_k(q) = \frac{(k-1)}{12}\nu(q) + \left(\left[\frac{k}{4}\right] - \frac{k-1}{4}\right)\nu_2(q) + \left(\left[\frac{k}{3}\right] - \frac{k-1}{3}\right)\nu_3(q) - \frac{\nu_\infty(q)}{2}$$

where $\nu(q)$ is the index of the subgroup $\Gamma_0(q)$ in the full modular group $\Gamma_0(1)$:

$$\nu(q) = q\prod_{p|q}\left(1+\frac{1}{p}\right);$$

$\nu_\infty(q)$ is the number of *cusps* of $\Gamma_0(q)$:

$$\nu_\infty(q) = \sum_{d|q} \phi((d, q/d));$$

$\nu_2(q)$ is the number of inequivalent *elliptic points* of order 2:

$$\nu_2(q) = \begin{cases} 0 \text{ if } 4 \mid q \\ \prod_{p|q}(1 + \chi_{-4}(p)) \text{ otherwise} \end{cases}$$

and $\nu_3(q)$ is the number of inequivalent *elliptic points* of order 3:

$$\nu_3(q) = \begin{cases} 0 \text{ if } 9 \mid q \\ \prod_{p|q}(1 + \chi_{-3}(p)) \text{ otherwise} \end{cases} .$$

It is clear from this formula that the dimension of $S_k(q)$ grows approximately linearly with q and k. For the spaces $S_k(q)$ the issue of primitive forms and imprimitive forms arise, much as the situation with characters. In fact, one should think of the Fourier coefficients of cusp forms as being a generalization of characters. They are not periodic, but they act as harmonic detectors, much as characters do, through their orthogonality relations (below). Imprimitive cusp forms arise in two ways. Firstly , if $f(z) \in S_k(q)$, then $f(z) \in S_k(dq)$ for any integer $d > 1$. Secondly, if $f(z) \in S_k(q)$, then $f(dz) \in S_k(\Gamma_0(dq))$ for any $d > 1$. The dimension of the subspace of primitive forms is given by

$$\dim S_k^{\text{new}}(q) = \sum_{d|q} \mu_2(d) \dim S_k(q/d)$$

where $\mu_2(n)$ is the multiplicative function defined for prime powers by $\mu_2(p^e) = -2$ if $e = 1$, $= 1$ if $e = 2$, and $= 0$ if $e > 2$. The subspace of newforms has a Hecke basis $H_k(q)$ consisting of primitive forms, or newforms, or Hecke forms. These can be identified as those f which have a Fourier series

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) e(nz)$$

where the $\lambda_f(n)$ have the property that the associated L-function has an Euler product

$$\begin{aligned} L_f(s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)/n^{(k-1)/2}}{n^s} \\ &= \prod_{p \nmid q} \left(1 - \frac{\lambda_f(p)/p^{(k-1)/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \prod_{p|q} \left(1 - \frac{\lambda_f(p)/p^{(k-1)/2}}{p^s}\right)^{-1}. \end{aligned}$$

We can express this as

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha'_f(p)}{p^s}\right)^{-1}$$

where if $p \nmid q$ then $\alpha'_f(p) = \overline{\alpha_f(p)}$ whereas if $p \mid q$ then $\alpha'_f(p) = 0$. The functional equation of the L-function is

$$\xi_f(s) := (2\pi/q)^{-s}\Gamma(s + (k-1)/2)L_f(s) = \pm\xi_f(1-s).$$

Now the $\pm$ depends on more than the weight k.

Examples of cuspforms with level > 1

We give a small table of the dimensions of the spaces $S_k(q)$ of cuspforms and of newforms $S_k^{\text{new}}(q)$and give some explicit constructions of some of the elements of $H_k(q)$.

Table 1. Dimensions of spaces $S_k(q)$ of cusp forms

$q\backslash k$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
1	0	0	0	0	0	1	0	1	1	1	1	2	1	2	2
2	0	0	0	1	1	2	2	3	3	4	4	5	5	6	6
3	0	0	1	1	2	3	3	4	5	5	6	7	7	8	9
4	0	0	1	2	3	4	5	6	7	8	9	10	11	12	13
5	0	1	1	3	3	5	5	7	7	9	9	11	11	13	13
6	0	1	3	5	7	9	11	13	15	17	19	21	23	25	27
7	0	1	3	3	5	7	7	9	11	11	13	15	15	17	19
8	0	1	3	5	7	9	11	13	15	17	19	21	23	25	27
9	0	1	3	5	7	9	11	13	15	17	19	21	23	25	27
10	0	3	5	9	11	15	17	21	23	27	29	33	35	39	41
11	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28

Notice that all of the cusp forms of level 1 are primitive. Each such form $f(z)$ leads to $d(q)$ old forms $f(dz)$ of level q. Some of the newforms of low level and weight can be expressed as eta-products. Thus, the unique newform of level 2 and weight 8 is given explicitly by

$$e(z)\prod_{n=1}^{\infty}(1-e(nz))^8(1-e(2nz))^8;$$

the unique newform of weight 6 and level 3 is

$$e(z)\prod_{n=1}^{\infty}(1-e(nz))^6(1-e(3nz))^6;$$

the unique newform of weight 6 and level 4 is

$$e(z)\prod_{n=1}^{\infty}(1-e(2nz))^{12};$$

Table 2. Dimensions of spaces $S_k^{\text{new}}(q)$ of primitive forms

$q\backslash k$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
1	0	0	0	0	0	1	0	1	1	1	1	2	1	2	2
2	0	0	0	1	1	0	2	1	1	2	2	1	3	2	2
3	0	0	1	1	2	1	3	2	3	3	4	3	5	4	5
4	0	0	1	0	1	1	1	1	2	1	2	2	2	2	3
5	0	1	1	3	3	3	5	5	5	7	7	7	9	9	9
6	0	1	1	1	1	3	1	3	3	3	3	5	3	5	5
7	0	1	3	3	5	5	7	7	9	9	11	11	13	13	15
8	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7
9	0	1	1	3	3	4	5	6	6	8	8	9	10	11	11
10	0	1	3	1	3	5	3	5	7	8	7	9	7	9	11
11	1	2	4	6	8	8	12	12	14	16	18	18	22	22	24

the unique newform of weight 4 and level 5 is

$$e(z)\prod_{n=1}^{\infty}(1-e(nz))^4(1-e(5nz))^4;$$

The first cusp form of weight 2 is for level 11. This newform corresponds to the elliptic curve of conductor 11. It can be expressed as a product

$$e(z)\prod_{n=1}^{\infty}(1-e(nz))^2(1-e(11nz))^2.$$

Orthogonality

The space $S_k(q)$ can be made into a Hilbert space by introducing the Petersson inner product, as we did earlier for the case $q = 1$, except that the integral is over the fundamental domain $\mathcal{D}(q)$ of $\Gamma_0(q)$ which can be represented as a union of $\nu(q)$ copies of the fundamental domain $\mathcal{D}$. Let $\mathcal{F}_k(q)$ be an orthogonal basis of $S_k(q)$ with respect to this inner product. The Petersson formula tells us that

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}}\sum_{f\in\mathcal{F}_k(q)}\frac{\lambda_f(m)\overline{\lambda_f(n)}}{\langle f,f\rangle} = \delta_{m,n}+2\pi i^{-k}\sum_{c=0 \bmod q}\frac{S(m,n,c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Let

$$\psi_f(n)=\left(\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\right)^{1/2}\lambda_f(n)$$

and let

$$\mathcal{L}_f(b) = \sum_{n=1}^{N} b_n \psi_f(n).$$

Then for any complex numbers b_n

$$\sum_{f \in \mathcal{F}_k(q)} |\mathcal{L}_f(b)|^2 = \left(1 + O\left(\frac{N}{q}\right)\right) \sum_{n=1}^{N} |b_n|^2.$$

A variant of this involves summing over k for a fixed q. Suppose that g is smooth and supported in $[K, 2K]$. Then

$$2 \sum_{k \text{ even}} g(k-1) \sum_{f \in \mathcal{F}_k(q)} |\mathcal{L}_f(b)|^2 = (\hat{g}(0) + \eta(K,N)) \sum_{n=1}^{N} |b_n|^2$$

where $\hat{g}(0) = \int g(y)\, dy$ and

$$\eta(K,N) \ll \left(\frac{N}{qK^3} + \left(\frac{N}{qK^2}\right)^j\right) N \log 2N$$

for any $j \geq 0$ and the implied constant depends only on j. These results are taken from [Iwa2], Chapter 5.

1-level density or low lying zeros

The average spacing for all the zeros of all the $L(f,s)$ with $f \in H_k(q)$ up to a fixed height t_0 is asymptotic to $2\pi/\log(k^2 q)$. Let ϕ be a test function which is even and rapidly decaying. Iwaniec, Luo, and Sarnak [ILS] proved that if the support of $\hat{\phi}$ is contained in $(-2,2)$, then

$$\lim_{KQ\to\infty} \sum_{k=2}^{K} \sum_{q \leq Q} \frac{1}{|H_k(q)|} \sum_{f \in H_k(q), L_f(1/2+i\gamma_f)=0} \phi\left(\frac{\gamma_f \log k^2 Q}{2\pi}\right) = \int_{-\infty}^{\infty} \phi(x) W(\mathrm{O})(x)\, dx\,.$$

Similar results hold if the summation is restricted to even forms or to odd forms in which case the integral on the right side has $W(\mathrm{O}^{\pm})(x)$.

It should be pointed out that the Fourier transforms of the density functions $W(\mathrm{O})(x)$, $W(\mathrm{O}^+)(x)$, and $W(\mathrm{O}^-)(x)$ all agree in the diagonal range; so it is only when one goes beyond the diagonal that the distinguishing features of these three symmetry types becomes apparent.

Approximate functional equation

Using the functional equation we can represent the central values $L_f(\frac{1}{2})$ by partial sums of lengths about $O(kq)$. To this end we choose a function $G(s)$ which is holomorphic in $|\text{Re } s| \leq A$ such that

$$G(s) = G(-s)$$
$$\Gamma(\tfrac{k}{2})G(0) = 1$$
$$\Gamma(s+\tfrac{k}{2})G(s) \ll (|s|+1)^{-2A}$$

for some $A \geq 1$. Consider the integral

$$I = \tfrac{1}{2\pi i}\int_{(1)} \xi_f(s+\tfrac{1}{2},\chi)G(s)s^{-1}\, ds.$$

Moving the integration to the line $\text{Re } s = -1$ we derive

$$\xi_f(\tfrac{1}{2})G(0) = 2I.$$

On the other hand, integrating termwise we derive

$$I = \sum_1^\infty \lambda_f(n)\chi(n)\left(\frac{q}{2\pi n}\right)^{\frac{1}{2}} V\left(\frac{n}{q}\right)$$

where $V(y)$ is the inverse Mellin transform of $(2\pi)^{-s}\Gamma(s+\frac{k}{2})G(s)s^{-1}$,

$$V(y) = \frac{1}{2\pi i}\int_{(1)} \Gamma(s+\tfrac{k}{2})G(s)(2\pi y)^{-s}s^{-1}\, ds.$$

Hence we get: For any Hecke form $f \in H_k(q)$ we have

$$L_f(\tfrac{1}{2}) = 2\sum_1^\infty \lambda_f(n)\chi(n)n^{-\frac{1}{2}}V(n/q).$$

Observe that $V(y)$ satisfies the following bounds

$$V(y) = 1 + O(y^A),$$
$$V(y) \ll (1+y)^{-A},$$
$$V^{(\ell)}(y) \ll y^A(1+y)^{-2A},$$

for $0 < \ell < A$. One can choose $G(s)$ depending on k so that

$$V(y) \ll k(1+y/k)^{-A},$$

therefore the series dies rapidly as soon as n exceeds kq. If one is not concerned with the dependence of implied constants on the parameter k then one has a simple choice $G(s) = \Gamma(k/2)^{-1}$ getting the incomplete gamma function

$$V(y) = \frac{1}{\Gamma(\frac{k}{2})}\int_{2\pi y}^\infty e^{-x}x^{\frac{k}{2}-1}\, dx.$$

Moments

In this section let q be a prime number. Let $\sum^{\rm h}$ denote the harmonic average over $f \in H_2(q)$, that is we weight the term associated with f in the sum by $1/\langle f, f\rangle$. This leads to simpler results. Then, work of Duke, and of Duke, Friedlander and Iwaniec, and of Kowalski, Michel and Vanderkam, we know

$$\sum_{f\in H_2(q)}^{\rm h} L(1/2,f) \sim a_1$$

$$\sum_{f\in H_2(q)}^{\rm h} L^2(1/2,f) \sim 2a_2 \log q^{\frac12}$$

$$\sum_{f\in H_2(q)}^{\rm h} L^3(1/2,f) \sim 8a_3 \frac{\log^3 q^{\frac12}}{3!}$$

$$\sum_{f\in H_2(q)}^{\rm h} L^4(1/2,f) \sim 128a_4 \frac{\log^6 q^{\frac12}}{6!}$$

where $a_k = A(0,\dots 0)$ with

$$A_k(\alpha_1,\dots,\alpha_k) = \prod_p \prod_{1\le i<j\le k} \left(1-\frac{1}{p^{1+\alpha_i+\alpha_j}}\right) \times \frac{2}{\pi}\int_0^\pi \sin^2\theta \prod_{j=1}^k \frac{e^{i\theta}\left(1-\frac{e^{i\theta}}{p^{\frac12+\alpha_j}}\right)^{-1} - e^{-i\theta}\left(1-\frac{e^{-i\theta}}{p^{\frac12+\alpha_j}}\right)^{-1}}{e^{i\theta}-e^{-i\theta}}\, d\theta.$$

A general conjecture is:

$$\sum_{f\in H_2(q)}^{\rm h} L^k(1/2,f) \sim 2^{k-1}\prod_{\ell=1}^{k-1}\frac{\ell!}{2\ell!} a_k \log^{k(k-1)/2} q\,.$$

A more precise conjecture is

$$\sum_{f\in H_2(q)}^{\rm h} L_f(1/2)^k = R_k\left(\log\tfrac{q}{4\pi^2}\right) + O(q^{-1/2+\epsilon})$$

as $q\to\infty$, where R_k is a polynomial of degree $k(k-1)/2$ given by the k-fold residue

$$R_k(x) = \frac{(-1)^{k(k-1)/2}2^{k-1}}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint \frac{H(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} e^{\frac{x}{2}\sum_{j=1}^k z_j}\, dz_1\dots dz_k,$$

where

$$H(\alpha_1,\dots,\alpha_k) = A_k(\alpha_1,\dots,\alpha_k)\prod_{j=1}^{k}\Gamma(1+\alpha_j)\prod_{1\le i<j\le k}\zeta(1+\alpha_i+\alpha_j)$$

Maass forms

There is another kind of cusp form associated with the group $\Gamma_0(q)$. This is a function $f(z)$ which is real analytic on the upper half-plane. It transforms like a weight 0 cusp form and is an eigenfunction of the Laplace operator:

$$\Delta := y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

It has a Fourier expansion as a linear combination of terms $e(nx)$ in which the dependence on y is expressed through K-Bessel functions. The prototype for these is given by the Eisenstein series (for the full modular group)

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(1)} y(\gamma z)^s = \sum_{(c,d)=1}\frac{y^s}{|cz+d|^{2s}}$$

where $y(z)$ denotes the imaginary part of z and where Γ_∞ is the group which fixes ∞, i.e. the group of matrices $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$ for integer b. This is not a cusp form (because it doesn't vanish at iy as $y\to\infty$.) However, its Fourier expansion is similar to that of the Maass cusp forms for which no explicit construction is known (apart from some forms with eigenvalue 1/4). Let

$$\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s) = \theta(1-s).$$

Then $\theta(s)E(z,s) =$

$$\theta(s)y^s + \theta(1-s)y^{1-s} + 4y^{1/2}\sum_{n=1}^{\infty}\sum_{ab=n}(a/b)^{s-1/2}K_{s-1/2}(2\pi ny)\cos(2\pi nx).$$

Since $\theta(s)$, $\sum_{ab=n}(a/b)^{s-1/2}$ and $K_{s-1/2}(2\pi ny)$ are all invariant under $s\to 1-s$, we see that $\theta(s)E(z,s) = \theta(1-s)E(z,1-s)$.

A Maass form f with eigenvalue $\lambda = 1/2+\kappa^2$ satisfies $(\Delta+\lambda)f = 0$ and has Fourier expansion

$$f(z) = y^{1/2}\sum_{n=1}^{\infty}\lambda_f(n)K_{i\kappa}(2\pi ny)\cos(2\pi nx)$$

for an even Maass form and

$$f(z) = y^{1/2}\sum_{n=1}^{\infty}\lambda_f(n)K_{i\kappa}(2\pi ny)\sin(2\pi nx)$$

for an odd Maass form.

The L-function $L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) N^{-s}$ associated with a Maass from is entire, has an Euler product, and satisfies the functional equation

$$\xi_f(s) := \pi^{-s} \Gamma((s + i\kappa)/2) \Gamma((s - i\kappa)/2) L_f(s) = \xi_f(1 - s)$$

for even Maass forms and

$$\xi_f(s) := \pi^{-s} \Gamma((s + 1 + i\kappa)/2) \Gamma((s + 1 - i\kappa)/2) L_f(s) = \xi_f(1 - s)$$

for odd Maass forms.

Selberg's trace formula provides us with a kind of Weyl law for the number of Maass forms with eigenvalue less than a given quantity.

Ramanujan's conjecture for Maass forms is that $|\lambda_f(p)| \le 2$. However, this has not yet been proven. The best result is $\lambda_f(p) \ll p^{1/9}$.

Moments

Motohashi [M1] and Ivic [Iv] have computed moments of these L-series at the central critical points. These have been achieved for the first through fourth moments and agree with random matrix conjectures.

Quadratic twists of modular L-functions

In this section we give a specific example of what we mean by these quadratic twists. This example should be sufficient to allow the reader to understand a more general situation. Let

$$L_{11}(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n^{1/2+s}}$$

be the L-function of conductor 11 of the elliptic curve

$$y^2 + y = x^3 - x^2.$$

The coefficients λ_n are obtained from cusp form of weight two and level 11 given by

$$\sum_{n=1}^{\infty} \lambda_n q^n = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2.$$

Expanding the right hand side using Euler's pentagonal theorem provides an efficient means to compute the λ_n's.

$L_{11}(s)$ satisfies an even functional equation (i.e. its sign is $+1$)

$$\left(\frac{11^{1/2}}{2\pi}\right)^s \Gamma(s + 1/2) L_{11}(s) = \left(\frac{11^{1/2}}{2\pi}\right)^{1-s} \Gamma(3/2 - s) L_{11}(1 - s),$$

and $L_{11}(s)$ may be written as a product over primes

$$L_{11}(s) = \frac{1}{1-11^{-s-1/2}} \prod_{p \neq 11} \frac{1}{1-\lambda_p p^{-s-1/2}+p^{-2s}}.$$

Consider now quadratic twists of $L_{11}(s)$,

$$L_{11}(s,\chi_d) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n^{1/2+s}} \chi_d(n).$$

with $(d,11)=1$. $L_{11}(s,\chi_d)$ satisfies the functional equation

$$L_{11}(s,\chi_d) = \chi_d(-11) \frac{\Gamma(3/2-s)}{\Gamma(s+1/2)} \left(\frac{2\pi}{N^{1/2}}\right)^{2s-1} |d|^{2(1/2-s)} L_{11}(1-s,\chi_d).$$

We wish to look at moments of $L_{11}(1/2,\chi_d)$ but only for those $L(s,\chi_d)$ that have an even functional equation, i.e. $\chi_d(-11)=1$. We further only look at $d<0$ since in that case a theorem of Kohnen and Zagier enables us to easily gather numerical data for $L_{11}(1/2,\chi_d)$ with which to check our conjecture.

When $d<0$, $\chi_d(-1)=-1$, hence, in order to have an even functional equation, we require that $\chi_d(11)=-1$, i.e. $d = 2,6,7,8,10 \mod 11$. The sum over fundamental discriminants is

$$\sum_{\substack{-D<d<0 \\ d=2,6,7,8,10 \mod 11}}^{\star} L_{11}(1/2,\chi_d)^k$$
$$= \sum_{\substack{-D<d<0 \\ d=2,6,7,8,10 \mod 11}}^{\star} \mathcal{D}_k\left(\log\left(\frac{|d|11^{1/2}}{2\pi}\right)\right) + O(D^{\frac{1}{2}+\epsilon})$$

where $\mathcal{D}_k$ is the polynomial of degree $k(k-1)/2$ given by the k-fold residue

$$\mathcal{D}_k(x) = \frac{(-1)^{k(k-1)/2}2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{R_{11}(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2}{\prod_{j=1}^{k} z_j^{2k-1}} e^{x\sum_{j=1}^{k} z_j}\, dz_1 \ldots dz_k,$$

where

$$R_{11}(z_1,\ldots,z_k) = A_k(z_1,\ldots,z_k) \prod_{j=1}^{k} \left(\frac{\Gamma(1+z_j)}{\Gamma(1-z_j)}\right)^{\frac{1}{2}} \prod_{1\leq i<j\leq k} \zeta(1+z_i+z_j),$$

and A_k is the Euler product which is absolutely convergent for $\sum_{j=1}^{k}|z_j| < 1/2$,

$$A_k(z_1,\dots,z_k)=\prod_p R_{11,p}(z_1,\dots,z_k)\prod_{1\le i<j\le k}\left(1-\frac{1}{p^{1+z_i+z_j}}\right)$$

with, for $p\neq 11$, $R_{11,p}=\left(1+\frac{1}{p}\right)^{-1}\times$

$$\left(\frac{1}{p}+\frac{1}{2}\left(\prod_{j=1}^k\frac{1}{1-\lambda_p p^{-1-z_j}+p^{-1-2z_j}}+\prod_{j=1}^k\frac{1}{1+\lambda_p p^{-1-z_j}+p^{-1-2z_j}}\right)\right)$$

and

$$R_{11,11}=\prod_{j=1}^k\frac{1}{1+11^{-1-z_j}}.$$

The local factor $R_{11,11}$ differs slightly from (4.3.13) because we are restricting ourselves to $\chi_d(11)=-1$, therefore only one term appears.

In [CFKRS] we compare moments computed numerically with moments estimated by our conjecture. The two agree to within the accuracy we have for the moment polynomial coefficients. We believe that if one were to compute the coefficients to higher accuracy, one would see an even better agreement with the data.

While one can compute $L_{11}(1/2,\chi_d)$ using standard techniques, one can in our case exploit a theorem of Kohnen and Zagier which relates $L_{11}(1/2,\chi_d)$, for fundamental discriminants $d<0$, $d=2,6,7,8,10 \mod 11$, to the coefficients $c_{11}(|d|)$ of a weight $3/2$ modular form

$$L_{11}(1/2,\chi_d)=\kappa_{11}c_{11}(|d|)^2/\sqrt{d}$$

where κ_{11} is a constant. The weight $3/2$ form in question was determined by Rodriguez-Villegas (private communication)

$$\begin{aligned}\sum_{n=1}^\infty c_{11}(n)q^n&=(\theta_1(q)-\theta_2(q))/2\\&=-q^3+q^4+q^{11}+q^{12}-q^{15}-2q^{16}-q^{20}\dots\end{aligned}$$

where

$$\theta_1(q)=\sum_{\substack{(x,y,z)\in\mathbb{Z}^3\\ x=y \mod 2}}q^{x^2+11y^2+11z^2}=1+2q^4+2q^{11}+4q^{12}+4q^{15}+2q^{16}+\dots$$

and

$$\theta_2(q)=\sum_{\substack{(x,y,z)\in\mathbb{Z}^3\\ x=y \mod 3\\ y=z \mod 2}}q^{(x^2+11y^2+33z^2)/3}=1+2q^3+2q^{12}+6q^{15}+6q^{16}+\dots.$$

This was used to compute the $c_{11}(|d|)$'s for $d<85,000,000$; the numerical evidence supports the conjectures.

Moments

The first moment of these twists, for a general element of $H_k(q)$ has been evaluated by Murty and Murty [MM] and by Bump, Friedberg, and Hoffstein [BFH]. Iwaniec [Iwa] has given a particularly elegant proof of the first moment.

4.5 Symmetric square L-functions

Recall that the Euler product for a level q modular form has the shape

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha'_f(p)}{p^s}\right)^{-1}.$$

We can form the symmetric square L-function associated to f as

$$L_f(\mathrm{sym}^2, s) = \prod_p \left(1 - \frac{\alpha_f^2(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)\alpha'_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha'_f(p)^2}{p^s}\right)^{-1}.$$

Note that this L-function has a degree three Euler product associated with it. Shimura proved that this is an entire function which satisfies the functional equation

$$\begin{aligned}\xi_f(\mathrm{sym}^2, s) :&= \pi^{-3s/2} q^s \Gamma(s/2)\Gamma((s+k-1)/2)\Gamma((s+k)/2) L_f(\mathrm{sym}^2, s) \\ &= \xi_f(\mathrm{sym}^2, 1-s).\end{aligned}$$

1-level density

The average spacing for all the zeros of all the $L_f(\mathrm{sym}^2, s)$ with $f \in H_k(1)$ up to a fixed height t_0 is asymptotic to $2\pi/\log(k^2)$. Let ϕ be a test function which is even and rapidly decaying. Iwaniec, Luo, and Sarnak [ILS] proved that if the support of $\hat{\phi}$ is contained in $(-3/2, 3/2)$, then (for fixed q)

$$\begin{aligned}\lim_{K\to\infty} \sum_{k=2}^{K} \frac{1}{|H_k(q)|} \sum_{f\in H_k(q), L_f(\mathrm{sym}^2, 1/2+i\gamma_f)=0} \phi\left(\frac{\gamma_f \log k^2 q^2}{2\pi}\right) \\ = \int_{-\infty}^{\infty} \phi(x) W(\mathrm{Sp})(x)\, dx\,.\end{aligned}$$

Moments

The first moment for symmetric square L-functions can be evaluated asymptotically, but so far not the second, see Iwaniec and Michel [IM]. The symmetry type of this family is symplectic and the difficulty of achieving the second moment over this family is like the fourth moment of quadratic L-functions.

4.6 Convolution L-functions

Given two cuspidal L-functions

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha'_f(p)}{p^s}\right)^{-1}$$

where $f \in H_k(q_1)$ and

$$L_g(s) = \prod_p \left(1 - \frac{\beta_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta'_g(p)}{p^s}\right)^{-1}$$

where $g \in H_\ell(q_2)$ with $(q_1, q_2) = 1$ we form the convolution L-function

$$L_{f\times g}(s) = \prod_p \left(1 - \frac{\alpha_f(p)\beta_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)\beta'_g(p)}{p^s}\right)^{-1} \times \left(1 - \frac{\alpha'_f(p)\beta_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha'_f(p)\beta'_g(p)}{p^s}\right)^{-1}.$$

If $f \neq g$, then this L-function is entire – an Euler product of degree 4 – and satisfies the functional equation

$$\begin{aligned}\xi_{f\times g}(s) :&= (2\pi)^{-2s}(q_1q_2)^s \Gamma(s + (|k-\ell|)/2)\Gamma(s - 1 + (k+\ell-1)/2)L_{f\times g}(s)\\ &= \pm\xi_{f\times g}(1-s).\end{aligned}$$

Moments

Kowalski, Michel, and Vanderkam have successfully computed all of the main terms (see (http://gauss.math.univ-montp2.fr/ michel/publi.html [22]) in the second moment of the convolution L-function at the central point for a fixed g and f varying over $H_k(q)$ for large prime q. The leading term is of size $q \log^3 q$ in accordance with the expectation that this is an orthogonal family.

5 Other directions

Here we describe a few things that there weren't room for in the notes; or things that are still being developed.

5.1 Integrals of ratios of zeta-functions

In 1994 Farmer [F] made the conjecture that

$$\frac{1}{T}\int_0^T \frac{\zeta(s+a)\zeta(1-s+b)}{\zeta(s+u)\zeta(1-s+v)}\,dt \sim 1 + \frac{T^{-u+v}-1}{(u+v)(a+b)}.$$

Here, $s = 1/2 + ir$ and the real parts of u and v are positive. Farmer showed that this conjecture implies Montgomery's pair-correlation conjecture. He later made a conjecture for an integral of a ratio of three zetas over 3 zetas.

After a lecture at MSRI in June 1999 Martin Zirnbauer and Stephen Nonnemacher proposed to Farmer and colleagues a way to generalize this conjecture to the integral of a ratio with any number of zeta-factors in the numerator and denominator. Their suggestion was based on an analogous moment for characteristic polynomials.

They have a method to compute exact formulas for the average over $\mathrm{U}(N)$, $\mathrm{Sp}(N)$, or $\mathrm{SO}(2N)$ of such a ratio. The method involves representation theory – supersymmetry and Weyl's formula for the characters of representations.

5.2 Mollifiers

For many applications in number theory one needs to compute moments of 'mollified' L-functions. Examples include

$$\int_0^T |\zeta(1/2+it)|^2 \left| \sum_{n \le y} \frac{\mu(n) \log y/n}{n^{1/2+it}} \right|^2 dt.$$

This example is relevant to proofs that a positive proportion of zeros of $\zeta(s)$ are on the one-half line. Levinson used an asymptotic formula for this with the length y of the mollifier taken to be $y = T^{1/2-\epsilon}$ to show that at least one-third of the zeros are on the critical line; Conrey used such a moment with $y = T^{4/7-\epsilon}$ to prove that at least two-fifths of the zeros are on the critical line.

Farmer's conjecture from the previous section can be used to prove an asymptotic formula for this mollified mean-square with an arbitrary length. Such a formula would imply that almost all of the zeros are on the critical line.

In [CF] formulas are given for the mollified mean square of L-functions from three different families. The asymptotic formula are conjectured to depend only on the symmetry type of the family and not on the family itself.

The more general conjectures for ratios could be used to conjecture formulas for any moment of an L-function times a mollifier (not just the second moment).

5.3 Connections with primes in short intervals

Montgomery and Goldston showed that Montgomery's Pair Correlation conjecture is equivalent to an assertion about a second moment of primes in short intervals.

The method relates both quantities to an asymptotic formula for

$$\int_0^T \left| \frac{\zeta'(1/2+a+it)}{\zeta(1/2+a+it)} \right|^2 \, dt$$

with small a with positive real part.

It should be mentioned that Bogolmony and Keating showed how a heuristic beginning with the Hardy-Littlewood conjectures for pairs of primes leads to all of the CUE n-correlation statistics for the zeros of the Riemann zeta-function.

5.4 Distribution of zeros of derivatives

The Riemann Hypothesis is equivalent to the assertion that all of the zeros of $\zeta'(s)$ have real parts greater than or equal to $1/2$. The question of the distribution of the real parts of these zeros arises in Levinson's method. The proper scaling at a height T is $1/\log T$. So the question is to determine the distribution function

$$d(\alpha) := \lim_{T\to\infty} \#\{\rho' = \beta' + i\gamma' : 0 < \gamma' < T, \beta' < 1/2 + \alpha/\log T\}.$$

The analogous question for CUE has to do with the distribution of the zeros of $Z'(U,s)$ inside the unit circle on the scale of $1/N$. Francesco Mezzadri [Mez] has made some progress on this question but has not solved it completely.

A similar question arises about the distribution of zeros of $\xi'(s)$ on the critical line.

5.5 Moments of derivatives

The asymptotics of the moments

$$\int_0^T |\zeta'(1/2+it)|^{2k} \, dt$$

can be conjectured for integral k (see [Hu]) but so far not for non-integral k and, of course, the analogous question for moments of derivatives of characteristic polynomials.

5.6 Lower order terms for non-integral moments of L-functions

We [CFKRS1] have conjectures for all of the main terms for the $2k$-th moment of L-functions in families. The leading main term is an analytic function of k; similarly for the second main term, third, and so on for any fixed term. However, we don't yet have an analytic expression for all of the terms taken together.

5.7 Extremely large values

We would like to know the size of the largest values of $|\zeta(1/2+it)|$. These could presumably be deduced from our conjectured formula for the $2k$-th moment by taking k extremely large. However, we have been unable so far to determine the large k asymptotics for the conjecture when we include all of the lower order terms. See [CGo], [Hu], and [U].

5.8 Distribution of small values

Random Matrix Theory does seem to predict very well the small and intermediate size values of L-functions. These models can be used [CKRS] together with an appropriate discretization to predict the frequency of vanishing to order two within certain families of the central values of the L-functions in the family. This prediction is especially of interest with regard to elliptic curves and their ranks. However, we don't seem to be able to use Random matrix Theory to predict vanishing to order three.

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