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REMARKS ON THE GENERALIZED LINDELÖF HYPOTHESIS

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Abstract: Within the study of arithmetical Dirichlet series, those that have a functional equation and Euler product are of particular interest. In 1989 Selberg described a class $\mathcal{S}$ of Dirichlet series through a set of four axioms which possibly contain all of these interesting Dirichlet series and made a number of interesting conjectures. In particular, he conjectured the Riemann Hypothesis for this class. We prove that one consequence of the Riemann Hypothesis for functions in $\mathcal{S}$ is the generalized Lindelöf Hypothesis. Moreover, we give an example of a function D which satisfies the first three of Selberg's axioms but fails the Lindelöf Hypothesis in the Q aspect.
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Within the study of arithmetical Dirichlet series, those that have a functional equation and Euler product of a type resembling that of the Riemann zeta-function, or of the Dirichlet L-functions for primitive Dirichlet characters, or of the L-functions associated with cusp forms that are eigenfunctions of the Hecke operators are of particular interest. Selberg [3] has described a class of Dirichlet series through a set of four axioms which possibly contain all of these interesting Dirichlet series. We recall his axioms for a series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

1. Analyticity: $(s-1)^m F(s)$ is entire of finite order for some non-negative integer m. 2. Ramanujan Bound: For any $\varepsilon > 0$, $a_n \ll_\varepsilon n^\varepsilon$. 3. Functional Equation: There exist $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$, $Q > 0$, $w_1, \dots, w_j > 0$, $\mu_1, \dots, \mu_k \in \mathbb{C}$ with $\Re\mu_j \geqslant 0$ for each j, such that

$$\varepsilon Q^s \prod_{j=1}^{k} \Gamma(w_j s + \mu_j) F(s) = \Phi(s) = \overline{\Phi(1-\overline{s})}.$$

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4. Euler Product: $a_1 = 1$, and there exists $\theta < 1/2$ such that if

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

then $b_n = 0$ unless n is a prime power, and $b_n \ll n^\theta$. Selberg has made a number of interesting conjectures about the elements of this class, which we call the Selberg class $\mathcal{S}$. In particular, he has conjectured the Riemann Hypothesis for this class, that is, that all of the non-trivial zeros of any element of $\mathcal{S}$ have real part equal to 1/2. (See the paper of Conrey and Ghosh [1] for further comments on the Selberg class.) It is interesting to study Dirichlet series which are not in $\mathcal{S}$, but which satisfy some of the axioms of $\mathcal{S}$, and to see to what extent they obey or disobey some of Selberg's conjectures. One consequence of the Riemann Hypothesis for the functions in $\mathcal{S}$ is the generalized Lindelöf Hypothesis.

Theorem 1. *Suppose that $F \in \mathcal{S}$ is entire and F satisfies the Riemann Hypothesis. Suppose further that $w_j = 1/2$ for each j and that the Euler product condition is changed to the stronger condition:*

$$F(s) = \prod_p \prod_{j=1}^{k} (1 - \alpha_{p,j}/p^{-s})^{-1}$$

where for all p and j, either $|\alpha_{p,j}| = 1$ or $\alpha_{p,j} = 0$. (The first condition can be expressed as $F \in \mathcal{S}^*$ in the notation of [1] while the second condition is believed to hold for functions of degree k in this class. Then, for any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon, k)$ such that

$$(LH) \qquad |F(1/2 + it)| \leqslant c \left(Q(1 + |t|)^{k/2} \prod_{j=1}^{k} (1 + |\mu_j|) \right)^{\varepsilon}.$$

Proof. We follow Littlewood's proof that RH implies LH in the case of the Riemann zeta-function (see Titchmarsh [4].) The idea is to use the Phragmen- Lindelöf Theorem to get a bound for F in the critical strip from bounds for F on the edge. This gives a bound for $\log |F(s)|$ for $\sigma > 1/2$. Then by the Borel-Carathéodory Theorem we can give a bound for $|\log F(s)|$ for $\sigma > 1/2$. Then, by the Hadamard Three-Circles Theorem, we can deduce bounds for $\log F(s)$ for $\sigma > 1/2$. This gives the bound of the Theorem for $\sigma > 1/2$. Then, by the Phragmen-Lindelöf Theorem again we get the desired bound on the 1/2-line (and throughout the critical strip). Proceeding to details, we need bounds for F on the edges of the critical strip. We begin with a lemma.

Lemma. *If $-1/2 \leqslant \sigma \leqslant 1/2$, then*

$$\left| \frac{\Gamma\left(\frac{1-s}{2} + \overline{\mu}\right)}{\Gamma\left(\frac{s}{2} + \mu\right)} \right| \leqslant \begin{cases} \left|\frac{s}{2} + \mu\right|^{\frac{1}{2} - \sigma} & \text{if } \kappa \geqslant \frac{1}{4} \\ \left|\frac{s+1}{2} - \overline{\mu}\right|^{\frac{1}{2} - \sigma} & \text{if } 0 \leqslant \kappa \leqslant \frac{1}{4} \end{cases}$$

where $\mu = \kappa + i\lambda$.

Proof. We use the generalized Phragmen-Lindelöf Theorem in the form that appears in Rademacher's book [2]. To this end, let

$$f(s) = \frac{\Gamma\left(\frac{1-s}{2} + \overline{\mu}\right)}{\Gamma\left(\frac{s}{2} + \mu\right)}.$$

Then

$$|f(1/2 + it)| = 1$$

and

$$\begin{aligned} |f(-1/2 + it)| &= \left| \frac{\Gamma\left(\frac{3}{4} + \kappa + i(\lambda + \frac{t}{2})\right)}{\Gamma\left(-\frac{1}{4} + \kappa + i(\lambda + \frac{t}{2})\right)} \right| = \left| \kappa - \frac{1}{4} + i(\lambda + \frac{t}{2}) \right| \\ &= \begin{cases} \left|\frac{s}{2} + \mu\right| & \text{if } \kappa \geqslant \frac{1}{4} \\ \left|\frac{s+1}{2} - \overline{\mu}\right| & \text{if } 0 \leqslant \kappa \leqslant \frac{1}{4} \end{cases} \end{aligned}$$

Now the hypotheses of the generalized Phragmen-Lindelöf Theorem are satisfied, and the lemma follows. ∎

Now the Euler product hypothesis implies that $|a_n| \leqslant d_k(n)$, the k-fold divisor function. Therefore,

$$|F(1 + \eta + it)| \leqslant \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{1+\eta}} = \zeta(1 + \eta)^k$$

for any $\eta > 0$. By the functional equation and the lemma, we see that

$$|F(-\eta + it)| \leqslant Q^{1+2\eta} \prod_{\kappa_j \leqslant \frac{1}{4}} \left| \frac{1 - \eta + it}{2} - \overline{\mu_j} \right|^{\frac{1}{2}+\eta} \prod_{\kappa_j > \frac{1}{4}} \left| \frac{-\eta + it}{2} + \mu_j \right|^{\frac{1}{2}+\eta} \zeta(1 + \eta)^k$$

for $0 < \eta < 1/2$. Now we apply a generalization (to a product of terms which have the shape $|Q+s|^{\alpha}$ in Rademacher's notation) of the generalized Phragmen-Lindelöf Theorem to F to obtain

$$\begin{aligned} &|F(\sigma + it) \\ &\quad \leqslant \zeta(1 + \eta)^k Q^{1+\eta-\sigma} \prod_{\kappa_j \leqslant \frac{1}{4}} \left| \frac{\sigma + 1 + it}{2} - \overline{\mu_j} \right|^{\frac{1+\eta-\sigma}{2}} \prod_{\kappa_j > \frac{1}{4}} \left| \frac{\sigma + it}{2} - \mu_j \right|^{\frac{1+\eta-\sigma}{2}} \end{aligned}$$

for $-\eta \leqslant \sigma \leqslant 1 + \eta$. Taking the logarithm of both sides of this inequality, we find that $\log|F(\sigma + it)|$ is

$$\leqslant k \log \zeta(1+\eta) + \frac{1 + \eta - \sigma}{2} \left(2 \log Q + \sum_{\kappa_j \leqslant \frac{1}{4}} \log \left| \frac{s + 1}{2} - \overline{\mu_j} \right| + \sum_{\kappa_j > \frac{1}{4}} \log \left| \frac{s}{2} + \mu_j \right| \right).$$

Now take $\sigma = 1/2 + \eta/2$. After some simplification we have

$$\log |F((1+\eta)/2 + it)| \leqslant k \log \zeta(1+\eta) + \log Q + k \log(1+|t|) + \log \Lambda$$

where

$$\Lambda = \prod_{j=1}^{k} (1 + |\mu_j|).$$

(We have used $\eta < 1/2$ and $(1+|a|+|b|) \leqslant (1+|a|)(1+|b|)$ to obtain this simpler form.) Now we apply the Borel-Carathéodory Lemma to bound $|\log F|$. We use it on the circles centered at $2+it$ with radii $R = 3/2 - \eta/2$ and $r = 3/2 - \eta$. The Lemma asserts that

$$M \leqslant \frac{2r}{R-r} A + \frac{R+r}{R-r} |\log F(2+it)|$$

where M is the maximum of $|\log F|$ on the smaller circle and A is the max of $\log |F|$ on the big circle, which is given by the above. Note that for $\sigma > 1$,

$$\begin{aligned} |\log F(s)| &= \left| -\sum_p \sum_{j=1}^{k} \log(1 - \alpha_{p,j} p^{-s}) \right| \\ &= \left| \sum_p \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{\alpha_{p,j}^n}{p^n s} \right| \\ &\leqslant \sum_p \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{p^{n\sigma}} = k \log \zeta(\sigma). \end{aligned}$$

Also, $2r/(R-r) \leqslant 6/\eta$ and $(R+r)/(R-r) \leqslant 6/\eta$. Thus, for any positive $\eta < 1/2$,

$$|\log F(1/2 + \eta + it)| \leqslant \frac{6}{\eta} \left(2k \log \zeta(1+\eta) + \log Q + k \log(1+|t|) + \log \Lambda \right).$$

Now we apply the Hadamard Three-Circles Theorem to $\log F$ on the circles centered at $\sigma_1 + it$ which pass through $1+\eta+it$, $\sigma + it$, and $1/2+\eta+it$. These have radii $r_1 = \sigma_1 - 1 - \eta$, $r_2 = \sigma_1 - \sigma$, and $r_3 = \sigma_1 - \frac{1}{2} - \eta$, respectively. Let M_1, M_2, and M_3 be the maxima of $\log F$ on these three circles, respectively. The Theorem then asserts that

$$M_2 \leqslant M_1^{1-a} M_3^a$$

where

$$a = \frac{\log \frac{r_2}{r_1}}{\log \frac{r_3}{r_1}}.$$

Here σ_1 will be fairly large,

$$\sigma_1 = \log \log D$$

where

$$D = \max\{Q, \Lambda, (1+|t|)^k\}.$$

We obtain $M_1 \leqslant k \log \zeta(1+\eta)$ and

$$\begin{aligned} M_3 &\leqslant |\log F(1/2+\eta+it)| \\ &\leqslant \frac{6}{\eta}\left(2k\log\zeta(1+\eta) + \log Q + k\log(1+|t|+\sigma_1) + \log\Lambda\right). \end{aligned}$$

For a we have

$$a = \frac{\log(1+x)}{\log(1+y)}$$

where $x = (1+\eta-\sigma)/(\sigma_1 - 1 - \eta)$ and $y = (1/2)/(\sigma_1 - 1 - \eta)$. It is not difficult to show that for $0 \leqslant x \leqslant y$,

$$\frac{\log(1+x)}{\log(1+y)} \leqslant \frac{x}{y} + y.$$

Therefore,

$$a \leqslant 2 + 2\eta - 2\sigma + \frac{1}{\sigma_1}$$

and

$$\begin{aligned} M_2 &\leqslant M_1^{1-a} M_3^a \leqslant M_1 M_3^a \\ &\leqslant k\log\zeta(1+\eta)\left(\frac{6}{\eta}(2k\log\zeta(1+\eta) + 4\log D)\right)^{2+2\eta-2\sigma+1/\log\log D}. \end{aligned}$$

We take $\eta = 1/\sigma_1$. Then we have

$$\log F(\sigma+it) \ll_k \log\log D(\log D)^{2-2\sigma}$$

for $1/2 + \frac{1}{\log\log D} \leqslant \sigma \leqslant 1$. It follows that for any $\varepsilon > 0$,

$$F(\sigma+it) \ll_{\varepsilon,k} D^{\varepsilon}.$$

Then by the Phragmen-Lindelöf Theorem,

$$F(1/2+it) \ll_{k,\varepsilon} D^{\varepsilon}$$

for any $\varepsilon > 0$. ■

It is clear that finite linear combinations of elements of $\mathcal{S}$ which satisfy the Riemann Hypothesis will also satisfy the Lindelöf Hypothesis. In fact, it is thought by some people that the Lindelöf Hypothesis would be very difficult to prove without using the Riemann Hypothesis. We give an example of a function D which satisfies the first three of Selberg's axioms but fails the Lindelöf Hypothesis in the Q aspect rather badly. In fact, the bound for $D(1/2+it)$ which follows from the trivial bound for $\sigma > 1$, the functional equation, and convexity turns out to be the correct bound. For a prime number q let

$$D(s) = \sum_{n=1}^{\infty} \frac{d(n)\cos(2\pi n/q)}{n^s}$$

where $d(.)$ is the divisor function.

Theorem 2. *With $D(s)$ as above, $(s-1)^2D(s)$ is entire. Moreover,*

$$\left(\frac{q}{\pi}\right)^s \Gamma\left(\frac{s}{2}\right)^2 D(s) = \Phi(s) = \Phi(1-s).$$

Also, for $s = 1/2 + it$ and prime q,

$$D(s) = \frac{2q^{1-s}X(s)}{\phi(q)}\left(\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=+1}} |L(s,\chi)|^2 + \left(\frac{\phi(q)-\frac{1}{2}}{q^{1-s}} + \frac{2-\phi(q)}{2q} - \frac{1}{q^{1+s}}\right)|\zeta(s)|^2\right)$$

where $X(s) = 2(2\pi)^{1-s}\Gamma(1-s)\sin\frac{\pi s}{2}$.

Corollary 1. *As $q \to \infty$ through primes,*

$$D(1/2) \sim 2\pi(1+i)\sqrt{q}\log q.$$

Corollary 2. *For any fixed t,*

$$D(1/2+it) \gg q^{1/2}\log q$$

for large prime q.

Corollary 3. *If q is prime and $s = 1/2+it$, then $D(s) = 0$ implies that $\zeta(s) = 0$ and $L(s,\chi) = 0$ for all $\chi \bmod q$ with $\chi(-1) = +1$ or that $t = \frac{2\pi n}{\log q}$ for some integer n.*

Proof of Theorem 2. The analytic continuation and functional equation are simple consequences of Estermann's study of

$$D(s,h/k) = \sum_{n=1}^{\infty} \frac{d(n)e(nh/k)}{n^s}.$$

Estermann proved that if $(h,k) = 1$ then D has a double pole at $s = 1$ and is analytic everywhere else in the s-plane. Moreover,

$$D(s,h/k) = 2G(s)^2k^{1-2s}(\cos\pi s D(1-s-\overline{h}/k) - D(1-s,\overline{h}/k)$$

where

$$G(s) = -i(2\pi)^{s-1}\Gamma(1-s).$$

Therefore,

$$\begin{aligned} &D(s)\\ &= \frac{1}{2}(D(s,\frac{1}{q}) + D(s,-\frac{1}{q})) = G(s)^2q^{1-2s}(\cos\pi s - 1)(D(1-s,\frac{1}{q}) + D(1-s,-\frac{1}{q}))\\ &= q^{1-2s}(2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2})^2D(1-s)\\ &= q^{1-2s}X(s)^2D(1-s) \end{aligned}$$

where $X(s)$ is the usual factor in the functional equation for the Riemann zeta function $\zeta(s)$, i.e.

$$\zeta(s) = X(s)\zeta(1-s).$$

But the symmetric form of this functional equation for ζ is given by

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \Phi(s) = \Phi(1-s).$$

Now suppose that q is prime. If $(q,n)=1$, then

$$e(n/q) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \tau(\overline{\chi})\chi(n)$$

where

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a)e(a/q)$$

is the Gauss sum associated with the character χ. Therefore, for $\sigma > 1$,

$$D(s,1/q) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \tau(\overline{\chi})L(s,\chi) + \sum_{n=1}^{\infty} \frac{d(nq)}{(nq)^s}.$$

Next we note that for prime q,

$$\sum_{n=1}^{\infty} \frac{d(nq)}{(nq)^s} = \frac{2-q^{-s}}{q^s}\zeta(s)^2$$

(see Titchmarsh [4], equation (1.4.2), for example). For the principal character χ_0 we have

$$L(s,\chi_0)^2 = (1-q^{-s})^2\zeta(s)^2$$

and $\tau(\chi_0) = \mu(q) = -1$. For $\chi \neq \chi_0$, we apply the functional equation

$$L(s,\chi) = \tau(\chi)q^{-s}G(s)(e^{\pi is/2} - \chi(-1)e^{-\pi is/2})L(1-s,\overline{\chi})$$

to one of the $L(s,\chi)$. Using the relation

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)q$$

we obtain

$$\begin{aligned} &D(s,1/q) \\ &\quad = G(s)\Bigg(\frac{q^{1-s}}{\phi(q)} \sum_{\chi\neq\chi_0} (\chi(-1)e^{\pi is/2} - e^{-\pi is/2})L(s,\chi)L(1-s,\overline{\chi}) \\ &\qquad\qquad + (e^{\pi is/2} - e^{-\pi is/2})\left(\frac{2-q^{-s}}{q^s} - \frac{(1-q^{-s})^2}{\phi(q)}\right)\zeta(s)\zeta(1-s)\Bigg). \end{aligned}$$

Note that if $s = 1/2 + it$, then $L(s,\chi)L(1-s,\overline{\chi}) = |L(s,\chi)|^2$. Combining this formula with a similar one for $D(s,-1/q)$, we obtain the last statement of Theorem 2. ■

The corollaries are all straightforward.

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