ON MEAN VALUES OF THE ZETA-FUNCTION

J. B. CONREY AND A. GHOSH

§1.   Let

$$I_k(T) = \int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt, \qquad k \geqslant 0.$$ (1)

It is conjectured that for any k,

$$I_k(T) \sim c_k\, T(\log T)^{k^2}$$ (2)

for some constant c_k. It is well known that (2) holds for $k = 0$, 1 and 2 with $c_0 = 1, c_1 = 1$, and $c_2 = (2\pi^2)^{-1}$, but there is not even a conjectural value of c_k for any other k. However, it is known that the Riemann hypothesis implies

$$I_k(T) \gg T(\log T)^{k^2},$$ (3)

for all $k \geqslant 0$ (see Ramachandra [2] and Heath-Brown [1]).
Here we give a simple proof that the Riemann hypothesis, RH for short, implies

$$I_k(T) \geqslant \big(c_k' + o(1)\big) T(\log T)^{k^2}$$ (4)

with an explicit value of c_k'. If (2) is correct, then, on RH, $c_k' \leqslant c_k$. Since $c_0 = c_0'$ and $c_1 = c_1'$ it could be that $c_k = c_k'$ for $0 < k < 1$.

§2.   Let $y = y_k = o(T)$ and

$$A_k(s) = \sum_{n \leqslant y_k} d_k(n) n^{-s},$$ (5)

where, as usual, $d_k(n)$ is the n-th coefficient in the Dirichlet series for $\zeta(s)^k$. Then

$$0 \leqslant \int_1^T |\zeta(\tfrac{1}{2}+it)^k - A_k(\tfrac{1}{2}+it)|^2 dt$$

$$= I_k(T) + \int_1^T |A_k(\tfrac{1}{2}+it)|^2 dt - 2 \operatorname{Re} \int_1^T \zeta(\tfrac{1}{2}+it)^k A_k(\tfrac{1}{2}-it) dt.$$ (6)

By the mean value theorem for Dirichlet polynomials,

$$\int_1^T |A_k(\tfrac{1}{2}+it)|^2\,dt = (T+O(y_k)) \sum_{n \le y_k} d_k(n)^2/n. \tag{7}$$

Also, by Cauchy's theorem, with $a = 1+1/\log T$,

$$\int_1^T \zeta(\tfrac{1}{2}+it)^k A_k(\tfrac{1}{2}-it)\,dt = \frac{1}{i}\int_{a+i}^{a+iT} \zeta(s)^k A_k(1-s)\,ds + O\left(\sum_{n \le y_k} d_k(n) \exp\left(\frac{kB\log T}{\log\log T}\right)\right) \tag{8}$$

since, on RH, there exists a $B > 0$ such that (see [3, p. 300])

$$\zeta(s) \ll \exp\left(\frac{B\log T}{\log\log T}\right) \quad \text{for} \quad \sigma \ge \tfrac{1}{2}, \quad 1 \le t \le T, \tag{9}$$

and since

$$A_k(1-s) \ll A_k(1-a) = \sum_{n \le y_k} d_k(n)n^{1-a} \ll \sum_{n \le y_k} d_k(n). \tag{10}$$

Now

$$\frac{1}{i}\int_{a+i}^{a+iT} \zeta(s)^k A_k(1-s)\,ds = \sum_{m=1}^{\infty} d_k(m) \sum_{n \le y_k} \frac{d_k(n)}{n}\frac{1}{i}\int_{a+i}^{a+iT} (m/n)^{-s}\,ds$$

$$= T \sum_{n \le y_k} \frac{d_k(n)^2}{n} + O\left(\sum_{\substack{m \ne n \\ n \le y_k}} \frac{d_k(m)d_k(n)}{|\log m/n|m^a n^{1-a}}\right). \tag{11}$$

The error term here is

$$\ll \sum_{m=1}^{\infty} \frac{d_k(m)^2}{m^a} \sum_{\substack{n \le y_k \\ n \ne m}} \frac{1}{|\log m/n|} + \sum_{n \le y_k} d_k(n)^2 \sum_{\substack{m=1 \\ m \ne n}}^{\infty} \frac{m^{-a}}{|\log m/n|}$$

$$\ll \sum_{m=1}^{\infty} \frac{d_k(m)^2}{m^a} y_k \log T + \sum_{n \le y_k} d_k(n)^2 \log^2 T. \tag{12}$$

Thus, by (6), (7), (8), (11) and (12),

$$I_k(T) \ge T \sum_{n \le y_k} \frac{d_k(n)^2}{n} + O\left(y_k\left(\sum_{n \le y_k} \frac{d_k(n)^2}{n} + \sum_{m=1}^{\infty} \frac{d_k(m)^2}{m^a}\log T\right)\right.$$

$$\left. + \sum_{n \le y_k} d_k(n)e^{Bk\log T/\log\log T} + \sum_{n \le y_k} d_k(n)^2 \log^2 T\right). \tag{13}$$

Finally, it is easy to show by standard methods that

$$\sum_{n \le y_k} \frac{d_k(n)^2}{n} - \sim c_k'(\log y_k)^{k^2},$$

where

$$c'_k = \left(\Gamma(k^2+1)\right)^{-1} \prod_p \left((1-1/p)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(k+m)}{\Gamma(k)m!}\right)^2 p^{-m}\right),$$

$$\sum_{m=1}^{\infty} \frac{d_k(m)^2}{m^a} \ll \zeta(a)^{k^2} \ll (\log T)^{k^2},$$

$$\sum_{n \leqslant y_k} d_k(n)^2 \ll y_k(\log y_k)^{k^2-1},$$

and

$$\sum_{n \leqslant y_k} d_k(n) \ll y_k(\log y_k)^{k-1}.$$

Therefore, if we choose

$$y_k = Te^{-B_1 k \log T/\log\log T}$$

where $B_1 > B$, then

$$\log y_k = \left(1+o(1)\right) \log T$$

and, by (13),

$$I_k(T) \geqslant \left(c'_k + o(1)\right) T(\log T)^{k^2}.$$

References

1. D. R. Heath-Brown. Fractional moments of the Riemann zeta-function. *J. London Math. Soc.* (2), 24 (1981), 65–78.
2. K. Ramachandra. Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series, I. *Hardy–Ramanujan J.*, 1 (1978), 1–15.
3. E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function* (Clarendon Press, Oxford, 1951).

Prof. J. B. Conrey,
Department of Mathematics,
Oklahoma State University,
Stillwater, OK 74078,
U.S.A.

Prof. A. Ghosh,
Department of Mathematics,
Oklahoma State University,
Stillwater, OK 74078,
U.S.A.