



A converse theorem for $\Gamma_0(13)$ [☆]

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Abstract

We prove that a Dirichlet series with a functional equation and Euler product of a particular form can only arise from a holomorphic cusp form on the Hecke congruence group $\Gamma_0(13)$. The proof does not assume a functional equation for the twists of the Dirichlet series. The main new ingredient is a generalization of the familiar Weil's lemma that played a prominent role in previous converse theorems.

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1. Introduction and statement of theorem

An important question in the theory of L -functions, is whether a Dirichlet series with functional equation and Euler product of appropriate type can arise only from some kind of a transform of a related automorphic form. An affirmative answer to this question has been given for the simplest types of Dirichlet series—those with ‘degree one’ functional equations and arbitrary conductor, and degree two functional equations with small conductors; see the work of

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Hamburger, Kaczorowski–Perelli, and Hecke [2–5]. In each of these cases, the main ingredient was the functional equation, the Euler product playing at most a small role. Conrey and Farmer [1] investigated this question in the setting of Dirichlet series with degree two functional equations and slightly larger conductors. For these, it can be shown that some assumption beyond a functional equation is absolutely necessary. Weil [6], in his converse theorem, imposed the extra-assumption that twists of the given Dirichlet series also had functional equations. In [1], the more natural condition that the Dirichlet series has an Euler product—of the type that one finds associated to holomorphic modular forms—is assumed. They prove that for conductors 5 through 17 (conductors 1 through 4 having been settled by Hecke as mentioned above), with the possible exception of 13, that all such Dirichlet series are, in fact, transforms of modular forms.

In this paper, we introduce a new idea that allows us to fill the gap at 13 in the theorem of [1]. The new ingredient (which is in Section 5) may be regarded as a generalization of Weil’s lemma, that holomorphic functions which transform in a certain way under elliptic transformations of infinite order are identically zero, which played an important role in [1,6].

Here is a statement of our theorem. Though the notation is standard, an explanation of it is given later. Also, this paper is almost completely self-contained; some standard arguments are repeated here for the convenience of the reader. Below we use the notation $e(z) = e^{2\pi iz}$.

Theorem 1. *Suppose*

$$f(z) := \sum_{n=1}^{\infty} a_n e(nz)$$

is holomorphic in $\Re z > 0$. Suppose further that we have a positive even integer k such that

$$L_f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges in some half-plane $\Re s > c$ and that

$$L_f(s) = \left(1 - \frac{a_2}{2^s} + \frac{2^{k-1}}{2^{2s}}\right)^{-1} \left(1 - \frac{a_3}{3^s} + \frac{3^{k-1}}{3^{2s}}\right)^{-1} \sum_{(n,6)=1} \frac{a_n}{n^s}. \tag{1.1}$$

In other words, we are assuming that the sequence (a_n) does not grow too fast and that it is (degree 2) multiplicative with respect to the primes 2 and 3 and weight k . Suppose finally that

$$\Lambda(s) = \left(\frac{\sqrt{13}}{2\pi}\right)^s \Gamma(s) L_f(s)$$

is an entire function which is bounded in any fixed vertical strip, and that it satisfies the functional equation

$$\Lambda(s) = \epsilon \Lambda(k - s) \tag{1.2}$$

where $\epsilon = \pm 1$. Then f is a cusp form of weight k and level 13; i.e. $f \in S_k(\Gamma_0(13))$.

2. Some notation

For the convenience of the reader we recall some notation, beginning with the notion of the “stroke” operator. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real 2×2 matrix with positive determinant. Then

$$f(z)|_k\gamma = (\det \gamma)^{k/2}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

Since k is fixed throughout the paper, we will suppress the dependence on k in this stroke notation. Also, we will assume that all matrices have positive determinants and real entries. It is easy to verify that

$$f(z) | (\gamma_1\gamma_2) = (f(z) | \gamma_1) | \gamma_2$$

and that

$$f(z) \left| \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} \right. = f(z) \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right.$$

for any real number $r \neq 0$.

To prove Theorem 1, we need to show $f(z) | \gamma = f(z)$ for all

$$\gamma \in \Gamma_0(13) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 \pmod{13} \right\}$$

and that $f(z)$ vanishes at all of the cusps of $\Gamma_0(13)$; this is what is meant by $f \in S_k(\Gamma_0(13))$. It is convenient to work in the group ring $G = \mathbb{C}[\text{GL}_2^+(\mathbb{R})]$ of formal linear combinations of matrices with real entries and positive determinants. We extend the stroke notation linearly so that

$$f(z) | (a_1\gamma_1 + a_2\gamma_2) = a_1 f(z) | \gamma_1 + a_2 f(z) | \gamma_2$$

for complex numbers a_1 and a_2 and real matrices γ_1 and γ_2 with positive determinants. Let $\Omega = \Omega_f = \{\omega \in G : f | \omega = 0\}$. Then Ω is a right ideal. It is convenient to work with congruences modulo Ω : thus we write

$$\omega_1 \equiv \omega_2 \pmod{\Omega_f}$$

to mean that

$$f(z) | \omega_1 = f(z) | \omega_2.$$

To simplify the notation we will usually omit the $\pmod{\Omega_f}$ from what we write. So to prove Theorem 1 we need to verify that $\gamma \equiv 1$ for all $\gamma \in \Gamma_0(13)$.

Since Ω_f is a right ideal one can multiply on the right a given congruence by anything: thus, $\omega_1 \equiv \omega_2$ implies $\omega_1\omega \equiv \omega_2\omega$ for any $\omega \in G$. Also $\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

It is not difficult to check that $\Gamma_0(13)$ is generated by four matrices:

$$\Gamma_0(13) = \left\langle P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 \\ 13 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 2 & -1 \\ 13 & -6 \end{pmatrix}, g_3 = \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \right\rangle.$$

So the main step to prove Theorem 1 is to show $P \equiv W \equiv g_2 \equiv g_3 \equiv 1$. The vanishing at the cusps of $\Gamma_0(13)$ will follow easily, as described near the beginning of the next section.

3. Invariance under $P, W,$ and g_2

Now $P \equiv 1$ asserts exactly the same thing as $f(z + 1) = f(z)$, which follows from the definition of $f(z)$ as a Fourier series.

By Hecke’s work, the functional equation (1.2) is equivalent to $H \equiv \epsilon$ where

$$H := \begin{pmatrix} 0 & -1 \\ 13 & 0 \end{pmatrix}.$$

Since

$$H \cdot P^{-1} \cdot H = \begin{pmatrix} -13 & 0 \\ -169 & -13 \end{pmatrix} \equiv W$$

and $\epsilon^2 = 1$, we have $W \equiv 1$. That takes care of two of the four generators of $\Gamma_0(13)$.

Now we can address the vanishing of $f(z)$ at the cusps. By the Fourier series, $f(z)$ vanishes at the cusp ∞ . Since $f(z) | H = \epsilon f(z)$, and H switches 0 and ∞ , we see that $f(z)$ also vanishes at 0. But 0 and ∞ are the only cusps of $\Gamma_0(13)$, so from the Fourier expansion and the matrix H , if $f(z)$ is invariant under $\Gamma_0(13)$ then $f(z)$ must actually be a cusp form on $\Gamma_0(13)$.

To prove $g_2 \equiv 1$ we need the multiplicativity of a_n at the prime 2. The following lemma is well known.

Lemma 1. *We have*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \left(1 - \frac{a_p}{p^s} + \frac{p^{k-1}}{p^{2s}}\right)^{-1} \sum_{(n,p)=1} \frac{a_n}{n^s},$$

if and only if

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{p-1} \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} \equiv a_p p^{1-k/2}. \tag{3.1}$$

Proof. It is convenient to adopt the convention that $a_x = 0$ if x is not a positive integer. Equating the coefficient of $(pn)^{-s}$ on both sides of the equation

$$\left(1 - \frac{a_p}{p^s} + \frac{p^{k-1}}{p^{2s}}\right) \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{(n,p)=1} \frac{a_n}{n^s},$$

we have

$$a_{pn} - a_p a_n + p^{k-1} a_{n/p} = 0. \tag{3.2}$$

A brief calculation shows that

$$f(z) \left| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{p-1} \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} = p^{k/2} f(pz) + p^{1-k/2} \sum_{n=1}^{\infty} a_{np} e(nz). \right.$$

Thus, equating the coefficient of $e(nz)$ on both sides of (3.1), we find that

$$p^{k/2} a_{n/p} + p^{1-k/2} a_{np} = a_p a_n p^{1-k/2}$$

which is equivalent to (3.2). \square

Thus, hypothesis (1.1) is equivalent to

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \equiv 2^{1-k/2} a_2 \tag{3.3}$$

and

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \equiv 3^{1-k/2} a_3. \tag{3.4}$$

We multiply each of these equivalences on the left and right by H . (We can multiply on the left by H because $H \equiv \pm 1$.) Using $H \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot H \equiv \begin{pmatrix} d & -c/13 \\ -13b & a \end{pmatrix}$ we find that

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -13 & 1 \end{pmatrix} \equiv 2^{1-k/2} a_2 \tag{3.5}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ -13 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ -26 & 1 \end{pmatrix} \equiv 3^{1-k/2} a_3. \tag{3.6}$$

We subtract (3.3) from (3.5) to obtain

$$\begin{pmatrix} 2 & 0 \\ -13 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix},$$

from which we deduce that

$$g_2 = W \cdot \begin{pmatrix} 2 & 0 \\ -13 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{-1} \equiv \begin{pmatrix} 2 & 0 \\ -13 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{-1} \equiv 1. \tag{3.7}$$

To complete the proof of Theorem 1 we need only show that $g_3 \equiv 1$.

4. Three expressions for $f(z) \mid (1 - g_3)$

Invariance under g_3 is more difficult, and requires an analytic argument. We wish to show $f(z) \mid (1 - g_3) = 0$, so first we develop some identities for $f(z) \mid (1 - g_3)$.

We subtract (3.4) from (3.6) to obtain

$$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \equiv \begin{pmatrix} 3 & 0 \\ -13 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ -26 & 1 \end{pmatrix}. \tag{4.1}$$

In this expression, we replace $\begin{pmatrix} 3 & 0 \\ -26 & 1 \end{pmatrix}$ by the equivalent matrix

$$W \cdot \begin{pmatrix} 3 & 0 \\ -26 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 13 & 1 \end{pmatrix};$$

we replace $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ by the equivalent matrix

$$\epsilon H \cdot P^{-1} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & -3 \\ 13 & -13 \end{pmatrix};$$

and we replace $\begin{pmatrix} 3 & 0 \\ -13 & 1 \end{pmatrix}$ by the equivalent matrix

$$\epsilon H \cdot \begin{pmatrix} 3 & 0 \\ -13 & 1 \end{pmatrix} = \epsilon \begin{pmatrix} 13 & -1 \\ 39 & 0 \end{pmatrix}.$$

Thus, (4.1) can be rewritten as

$$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & -3 \\ 13 & -13 \end{pmatrix} - \epsilon \begin{pmatrix} 13 & -1 \\ 39 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 13 & 1 \end{pmatrix} \equiv 0.$$

Now, multiply on the right by the inverse of the first matrix to obtain

$$1 + \epsilon \begin{pmatrix} 0 & -3 \\ 39 & -26 \end{pmatrix} - \epsilon \begin{pmatrix} 39 & -14 \\ 117 & -39 \end{pmatrix} - g_3 \equiv 0. \tag{4.2}$$

This expression factors as

$$(1 - g_3) \cdot \left(1 - \epsilon \begin{pmatrix} 39 & -14 \\ 117 & -39 \end{pmatrix} \right) \equiv 0. \tag{4.3}$$

This expression is the first of three similar factorizations we will find involving g_3 .

To obtain the second such expression, we first show that

$$H \cdot \left(\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \right) \cdot H \equiv \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}. \tag{4.4}$$

We derive this expression by first squaring (3.3) to obtain

$$2 + P + \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \equiv 2^{2-k} a_2^2.$$

We can replace the terms $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$ here by using (3.3) twice: once multiplied on the right by $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and once multiplied on the right by $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. In this way we obtain, after some rearrangement,

$$\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \equiv 2^{2-k} a_2^2 - P - 2^{1-k/2} a_2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - 2^{1-k/2} a_2 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The right-hand side is unchanged when multiplied on the left and right by H which verifies (4.4).

Now (4.4) can be rewritten as

$$\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ -13 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ -39 & 1 \end{pmatrix} \equiv 0. \tag{4.5}$$

In this expression, we replace $\begin{pmatrix} 4 & 0 \\ -39 & 1 \end{pmatrix}$ by the equivalent matrix

$$W \cdot \begin{pmatrix} 4 & 0 \\ -39 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 13 & 1 \end{pmatrix};$$

we replace $\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$ by the equivalent matrix

$$\epsilon H \cdot P^{-1} \cdot \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & -4 \\ 13 & -13 \end{pmatrix};$$

and we replace $\begin{pmatrix} 4 & 0 \\ -13 & 1 \end{pmatrix}$ by the equivalent matrix

$$\epsilon H \cdot \begin{pmatrix} 4 & 0 \\ -13 & 1 \end{pmatrix} = \epsilon \begin{pmatrix} 13 & -1 \\ 52 & 0 \end{pmatrix};$$

thus, (4.5) can be rewritten as

$$\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 4 \\ -13 & 13 \end{pmatrix} - \epsilon \begin{pmatrix} 13 & -1 \\ 52 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 13 & 1 \end{pmatrix} \equiv 0. \tag{4.6}$$

Now we multiply on the right by $\begin{pmatrix} 1 & 0 \\ -13 & 4 \end{pmatrix}$; this yields

$$g_3 + \epsilon \begin{pmatrix} -26 & 8 \\ -91 & 26 \end{pmatrix} - \epsilon \begin{pmatrix} 13 & -2 \\ 26 & 0 \end{pmatrix} - 1 \equiv 0. \tag{4.7}$$

This expression factors as

$$-(1 - g_3) \cdot \left(1 - \epsilon \begin{pmatrix} -26 & 8 \\ -91 & 26 \end{pmatrix} \right) \equiv 0 \tag{4.8}$$

and gives our second relation of this sort.

The third uses the relation $g_2g_3^{-1} \equiv g_3g_2^{-1}$. This is true because $g_3^{-1}g_2 = \begin{pmatrix} 5 & -2 \\ 13 & -5 \end{pmatrix}$ has order 2, so $g_2g_3^{-1} \equiv (g_2g_3^{-1})^{-1} = g_3g_2^{-1}$. Using this relation and $g_2 \equiv 1$ we have

$$1 - g_3 \equiv 1 - g_2g_3^{-1}g_2 \equiv 1 - g_3^{-1}g_2 \equiv g_2 - g_3^{-1}g_2 = -(1 - g_3)g_3^{-1}g_2$$

so

$$(1 - g_3)(1 + g_3^{-1}g_2) \equiv 0. \tag{4.9}$$

5. In variance under g_3

In this section, we give an analytic argument to show that f is invariant under g_3 . Let $g(z) = f(z) | (1 - g_3)$ and let

$$\delta_1 = \begin{pmatrix} \sqrt{13} & \frac{-14}{3\sqrt{13}} \\ 3\sqrt{13} & -\sqrt{13} \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 5 & -2 \\ 13 & -5 \end{pmatrix} \quad \text{and} \quad \delta_3 = \begin{pmatrix} -\sqrt{13} & \frac{4}{\sqrt{13}} \\ \frac{-7\sqrt{13}}{2} & \sqrt{13} \end{pmatrix}.$$

Then by (4.3), (4.9), and (4.8) we have shown that

$$g(z) | \delta_1 = \epsilon g(z), \quad g(z) | \delta_2 = -g(z), \quad g(z) | \delta_3 = \epsilon g(z). \tag{5.1}$$

We will now prove that these relations, *and the fact that g_3 is elliptic* imply that $g(z)$ is 0. The key fact we will use about δ_1, δ_2 and δ_3 is that $h_2 := \delta_2\delta_1$ and $h_3 := \delta_3\delta_1$ are irrational powers of each other. Therefore h_2 and h_3 generate a nondiscrete subgroup of $SL(2, \mathbb{R})$, and $g(z)$ is invariant under stroking by the elements of that group. It would be nice if this implied that $g(z)$ is identically zero. Unfortunately, this is not quite true, as the following example shows: if $p(z) = z^{-k/2}$ then $p(z) | \begin{pmatrix} X & \\ & 1/X \end{pmatrix} = p(z)$, for any $X \in \mathbb{R}$. This is essentially the only counterexample, as described in the following lemma.

Lemma 2. *If $p(z)$ is an analytic function and*

$$p(z) \left| \begin{pmatrix} X & 0 \\ 0 & 1/X \end{pmatrix} \right. = p(z) \tag{5.2}$$

for all X in a dense subset of \mathbb{R}_+ , then $p(z) = Cz^{-k/2}$ for some constant C .

Proof. Let $\ell(z) = z^{k/2}p(z)$. A calculation verifies that $\ell(z) = \ell(X^2z)$, so $\ell(z)$ is constant. \square

We now put h_2 and h_3 in a form where we can apply the lemma. One can check that h_2 and h_3 commute, so they are simultaneously diagonalizable. We have

$$h_2 := \delta_2\delta_1 = \begin{pmatrix} -\sqrt{13} & \frac{8}{3\sqrt{13}} \\ -2\sqrt{13} & \frac{\sqrt{13}}{2} \end{pmatrix} = A \begin{pmatrix} \frac{-2-\sqrt{13}}{3} & 0 \\ 0 & \frac{2-\sqrt{13}}{3} \end{pmatrix} A^{-1}$$

and

$$h_3 := \delta_3 \delta_1 = \begin{pmatrix} -1 & \frac{2}{3} \\ -\frac{13}{2} & \frac{10}{3} \end{pmatrix} = A \begin{pmatrix} \frac{7-\sqrt{13}}{6} & 0 \\ 0 & \frac{7+\sqrt{13}}{6} \end{pmatrix} A^{-1},$$

where

$$A = \begin{pmatrix} \frac{13+\sqrt{13}}{39} & \frac{13-\sqrt{13}}{39} \\ 1 & 1 \end{pmatrix}.$$

Thus

$$h_2^m h_3^n = (-1)^m A \begin{pmatrix} \left(\frac{2+\sqrt{13}}{3}\right)^m & 0 \\ 0 & \left(\frac{-2+\sqrt{13}}{3}\right)^m \end{pmatrix} \begin{pmatrix} \left(\frac{7-\sqrt{13}}{6}\right)^n & 0 \\ 0 & \left(\frac{7+\sqrt{13}}{6}\right)^n \end{pmatrix} A^{-1}.$$

Let $\lambda = -0.91177\dots$ be the real number such that

$$\left(\frac{2 + \sqrt{13}}{3}\right)^\lambda = \frac{7 - \sqrt{13}}{6}$$

and let $Y = \frac{2+\sqrt{13}}{3}$. Then

$$h_2^m h_3^n = (-1)^m A \begin{pmatrix} Y^{m+n\lambda} & 0 \\ 0 & 1/Y^{m+n\lambda} \end{pmatrix} A^{-1}.$$

We have $g(z) \mid h_2^m h_3^n = (-\epsilon)^m g(z)$ since the number of (-1) 's that we get is the same as the number of times that δ_2 appears in $h_2^m h_3^n$ and the number of ϵ 's is the combined number of times that δ_1 and δ_3 appear, which is $m + 2n$.

Replacing m by $2m$ we have $g(z) \mid h_2^{2m} h_3^n = g(z)$ for all integers m, n . Thus $p(z) := g(z) \mid A$ satisfies

$$p(z) \mid \begin{pmatrix} X & 0 \\ 0 & 1/X \end{pmatrix} = p(z) \tag{5.3}$$

for all X of the form $Y^{2m+n\lambda}$ for some integers m, n . Since Y and λ are irrational, the set of such X is dense in \mathbb{R}_+ and we apply Lemma 5.2 to conclude that $p(z) = Cz^{-k/2}$ for some constant C . We must show that $C = 0$.

At this point we must use more information about the function $g(z)$. Indeed, if we let $\tilde{g}(z) := Cz^{-k/2} \mid A^{-1}$ then a direct calculation shows for any C that

$$\tilde{g}(z) \mid \delta_1 = (-1)^{-k/2} \tilde{g}(z), \quad \tilde{g}(z) \mid \delta_2 = (-1)^{-k/2} \tilde{g}(z), \quad \tilde{g}(z) \mid \delta_3 = (-1)^{-3k/2} \tilde{g}(z). \tag{5.4}$$

Thus, if $k \equiv 2 \pmod{4}$ and $\epsilon = -1$ then $\tilde{g}(z)$ satisfies (5.1), so (5.1) is not sufficient by itself to imply that $g(z)$ is zero. We must use the fact that $g(z) = f(z) \mid (1 - g_3)$ and g_3 is elliptic.

Since $g_3^3 = I$ we have $(1 - g_3)(1 + g_3 + g_3^2) \equiv 0$. In particular, $f(z) \mid (1 - g_3)(1 + g_3 + g_3^2) = 0$ so $g(z) \mid (1 + g_3 + g_3^2) = 0$. Combining this with the fact that $g(z) = Cz^{-k/2} \mid A^{-1}$ we obtain

$$\begin{aligned}
 0 &= Cz^{-k/2} \mid (1 + A^{-1}g_3A + A^{-1}g_3^2A) \\
 &= Cz^{-k/2} + Cz^{-k/2} \left| \left(\begin{array}{cc} -\frac{1}{2} & \frac{5-2\sqrt{13}}{6} \\ \frac{5+2\sqrt{13}}{6} & -\frac{1}{2} \end{array} \right) + Cz^{-k/2} \left| \left(\begin{array}{cc} \frac{1}{2} & \frac{5-2\sqrt{13}}{6} \\ \frac{5+2\sqrt{13}}{6} & \frac{1}{2} \end{array} \right) \right. \\
 &= C(z^{-k/2} + 6^k(-3z - 2\sqrt{13} + 5)^{-k/2}((5 + 2\sqrt{13})z - 3)^{-k/2} \\
 &\quad + 6^k(3z - 2\sqrt{13} + 5)^{-k/2}((5 + 2\sqrt{13})z + 3)^{-k/2}).
 \end{aligned}$$

The final expression above must be identically 0. Since we assumed k was a positive integer, if $C \neq 0$ the final expression above blows up as $z \rightarrow 0$. Thus $C = 0$, so $g(z) = 0$, so $f(z) \mid (1 - g_3) = 0$, giving invariance under the final generator of $\Gamma_0(13)$ and completing the proof of Theorem 1.

It is curious that if $k = -2$ then the final displayed equation above actually is identically zero. So the assumption that the weight k is positive is necessary for the final step of our proof.

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