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APPLICATIONS OF THE L-FUNCTIONS RATIOS CONJECTURES

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Abstract

In upcoming papers by Conrey, Farmer and Zirnbauer there appear conjectural formulas for averages, over a family, of ratios of products of shifted L-functions. In this paper we will present various applications of these ratios conjectures to a wide variety of problems that are of interest in number theory, such as lower order terms in the zero statistics of L-functions, mollified moments of L-functions and discrete averages over zeros of the Riemann zeta function. In particular, using the ratios conjectures we easily derive the answers to a number of notoriously difficult computations.

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1. *Introduction*

Applications of random matrix theory in number theory began with Montgomery's pair correlation conjecture [**36**]. In this paper Montgomery conjectured that, in the limit for large height up the critical line, any local statistic of the zeros of the Riemann zeta function is given by the corresponding statistic for eigenvalues from the Gaussian Unitary Ensemble (GUE) of random matrix theory [**34**]. A local statistic is one that involves only correlations between zeros separated on a scale of a few mean spacings. Odlyzko checked the statistics numerically for the pair correlation and the nearest neighbour spacing distribution and found spectacular agreement [**37**]. At leading order the zero statistics and eigenvalues statistics are identical; asymptotically no factors of an arithmetical nature appear. However, it is clear that arithmetical contributions play a role in lower order terms, and Bogomolny and Keating [**3**] identified these in the case of the pair correlation function.

Katz and Sarnak [**27**] proposed that local statistics of zeros of families of L-functions could be modelled by the eigenvalues of matrices from the classical compact groups with Haar measure. In this way each family of L-functions is believed to have a symmetry type: unitary, symplectic or orthogonal. Iwaniec, Luo and Sarnak [**24**] calculated the leading asymptotics for the one-level

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densities (using test functions whose Fourier transforms have limited support) for families of L-functions with each symmetry type and found agreement with random matrix theory. Again these leading terms had no arithmetic part.

More recently, random matrix theory has been applied to the moments of L-functions averaged over a family. These are global, rather than local, statistics. A characteristic feature of a global statistic is that an arithmetic factor appears in the leading order term. In the original papers [**29**, **30**] the leading term was a product of the corresponding moment of a characteristic polynomial from random matrix theory and a seemingly independent Euler product.

Often in random matrix theory one can calculate such global statistics exactly for any finite matrix size N. In particular, when evaluating moments of characteristic polynomials one obtains an exact asymptotic expansion in N as $N \to \infty$. We now understand conjecturally the analogue for moments of L-functions and, in particular, how the arithmetic and random matrix factors interact in the lower order terms. For any family of L-functions we can conjecture [**10**] an asymptotic expansion for any moment which we believe is accurate essentially to the square root of the size of the family.

A natural way to generalize these moment formulae is to consider averages of ratios of products of L-functions or characteristic polynomials. In two forthcoming papers [**11**, **12**] there appear conjectural formulas for averages, over a family, of ratios of products of shifted L-functions. Those papers contain several applications of these conjectures, as well as theorems proving the random matrix analogues of these conjectures. In [**13**] and [**5**], different proofs of the random matrix theorems are given, although not for the full range of the main parameter, the dimension of the matrix.

The point is that these ratios conjectures are useful for calculating both local and global statistics. In fact, quoting from [**4**], ‘The averages of products and ratios of characteristic polynomials are more fundamental characteristics of random matrix models than the correlation functions.’ We would argue that the same can be said for L-functions. From the ratios conjectures not only can you obtain all n-level correlations, but also essentially any local or global statistic. An important feature on the number theory side is that this includes all lower order terms; in particular it shows the arithmetic contribution present in local statistics.

In this paper we will present various applications of these ratios conjectures. In Section 2 we give the precise statement and sketch the derivations of some examples of the ratios conjecture for each of the three symmetry types: unitary, symplectic and orthogonal. These examples, which have one or two L-functions in the numerator and denominator, cover most of the cases that we need in the applications in this paper, but the conjectures are more general in that they can involve any number of L-functions [**11**]. Theorems 2.7 and 2.10 give auxiliary formulae useful in calculating the most basic local statistic, the one-level density. In Section 3 we then show how the ratios conjecture can be used to compute the one-level density of the simplest family of L-functions with symplectic symmetry, namely Dirichlet L-functions with real quadratic characters. We state a similar result for the orthogonal family associated with quadratic twists of the Ramanujan τ-function. In the following section we consider lower order terms in the pair correlation of the zeros of the Riemann zeta function. As mentioned above, Bogomolny and Keating were the first to find these lower order terms; their heuristic method involved a careful analysis of the Hardy–Littlewood conjectures for prime pairs. The strength of our method is that it allows us to avoid such detailed considerations.

The next two sections consider averages of mollified L-functions. Mollifiers are used to obtain information about small values of L-functions, in particular zeros. Mollifiers were first introduced in the context of the Riemann zeta function to bound the number of zeros in a vertical strip to the right of the half-line (that is, zero density results). Subsequently Selberg, and then Levinson, obtained lower bounds for the proportion of zeros satisfying the Riemann Hypothesis by mollifying zeta in the neighbourhood of the critical line. Recent uses have focused on obtaining non-vanishing results at the central point for families of L-functions. All of these results involve complicated analysis, for example Levinson's asymptotic evaluation of the mollified second moment of zeta takes nearly fifty pages. Before embarking on such a calculation it would be useful to know ahead of time what the answer is. In Section 5 we show how to obtain these answers quickly. For each of the families that we have introduced we calculate the mollified second moment of arbitrary linear combinations of derivatives and reveal the simple structure of the result. In all cases where these have been rigourously calculated (only accomplished when the mollifier is sufficiently short), these results are in agreement. In Section 6 we show how to mollify any moment of the Riemann zeta function and give detailed expressions in the case of the fourth moment; none of these, apart from the second moment, have been calculated without using the ratios conjecture. It is interesting to note that unlike other averages considered in this paper, there does not seem to be a random matrix analogue of mollifying as there is nothing that naturally corresponds to a partial Dirichlet series.

Another kind of average which gives useful information about the distribution of zeros is a discrete moment summing the zeta function, or its derivatives, at or near the zeros. In Section 7 we consider moments of $|\zeta'(\rho)|$ and $|\zeta(\rho+a)|$. Using the ratios conjecture we show how to obtain all of the lower order terms for these averages. While the leading order terms had previously been conjectured or proved, it was not known how to obtain these lower order terms.

In Section 8 we show how to use the ratios conjecture to reproduce the asymptotic formulae used to obtain non-vanishing results for various families. In addition we sketch how one should go about proving that the proportion of non-vanishing for the kth derivative $L^{(k)}(1/2,\chi)$ approaches 100% as $k \to \infty$ for the family of all Dirichlet L-functions; by contrast, in [**35**] a convincing argument is made that one cannot do better than 2/3 non-vanishing for $\Lambda^{(k)}(1/2,\chi)$, where Λ is the completed L-function, without mollifying a higher power than the second.

In short, there are a number of difficult computations which the ratios conjectures simplify significantly. A few of these computations have the property that they could be made into theorems by proceeding alternatively; some are purely conjectural. However, even for those that could be proved by other methods, knowing the answer ahead of time is useful as a guide along the way, a check at the end and even in deciding whether to commence what could be a painful calculation.

Throughout this paper we assume the Riemann Hypothesis for all the L-functions that arise.

2. *Ratios conjectures*

2.1. *A unitary example*

An example of a basic conjecture for the zeta function follows. This was the example that Farmer first considered when formulating his initial conjecture about averages of ratios of zeta functions with shifts. With $s = 1/2 + it$, let

$$R_\zeta(\alpha,\beta,\gamma,\delta) := \int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt. \tag{2.1}$$

Farmer [**16**] conjectured that for $\alpha,\beta,\gamma,\delta \ll 1/\log T$,

$$R_\zeta(\alpha,\beta,\gamma,\delta) \sim T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}, \tag{2.2}$$

as $T \to \infty$, provided that $\Re\gamma, \Re\delta > 0$. Our ratios conjecture gives us a recipe for computing a more precise conjecture for R_ζ.

Briefly, we use the approximate functional equation

$$\zeta(s) = \sum_{n\leqslant X}\frac{1}{n^s} + \chi(s)\sum_{n\leqslant Y}\frac{1}{n^{1-s}} + \text{remainder}, \tag{2.3}$$

where $s = \sigma + it$, $\chi(s) = 2^s \pi^{s-1} \sin(s\pi/2)\Gamma(1-s)$ and $XY = t/(2\pi)$, for the zeta functions in the numerator and ordinary Dirichlet series expansions for those in the denominator:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \tag{2.4}$$

We only use the pieces which have the same number of $\chi(s)$ as $\chi(1-s)$ and we integrate term-by-term, retaining only the diagonal pieces. We then complete all of the sums that we arrive at.

Thus, the term from the 'first' part of the two approximate functional equations gives T times

$$\sum_{hm=kn} \frac{\mu(h)\mu(k)}{m^{1/2+\alpha} n^{1/2+\beta} h^{1/2+\gamma} k^{1/2+\delta}} = \prod_p \sum_{h+m=k+n} \frac{\mu(p^h)\mu(p^k)}{p^{(1/2+\alpha)m+(1/2+\beta)n+(1/2+\gamma)h+(1/2+\delta)k}}. \tag{2.5}$$

The only possibilities for h and k here are 0 and 1. Thus, we easily find that the right-hand sum above is equal to

$$\frac{1}{\left(1-\frac{1}{p^{1+\alpha+\beta}}\right)} \left(1 - \frac{1}{p^{1+\beta+\gamma}} - \frac{1}{p^{1+\alpha+\delta}} + \frac{1}{p^{1+\gamma+\delta}}\right); \tag{2.6}$$

thus, the product over primes is

$$\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)} A_\zeta(\alpha,\beta,\gamma,\delta), \tag{2.7}$$

where

$$A_\zeta(\alpha,\beta,\gamma,\delta) = \prod_p \frac{\left(1-\frac{1}{p^{1+\gamma+\delta}}\right)\left(1 - \frac{1}{p^{1+\beta+\gamma}} - \frac{1}{p^{1+\alpha+\delta}} + \frac{1}{p^{1+\gamma+\delta}}\right)}{\left(1-\frac{1}{p^{1+\beta+\gamma}}\right)\left(1-\frac{1}{p^{1+\alpha+\delta}}\right)}. \tag{2.8}$$

The other term comes from the second piece of each approximate functional equation and is similar to the first piece except that α is replaced by $-\beta$, and β is replaced by $-\alpha$. Also, because of the χ-factors in the functional equation, we have an extra factor of

$$\chi(s+\alpha)\chi(1-s+\beta) = \left(\frac{t}{2\pi}\right)^{-\alpha-\beta}\left(1 + O\left(\frac{1}{|t|}\right)\right). \tag{2.9}$$

Thus, the more precise ratios conjecture gives the following conjecture.

CONJECTURE 2.1 (Conrey, Farmer and Zirnbauer [**11**]). With constraints on α, β, γ and δ as described below at (2.11), we have

$$\begin{aligned} R_\zeta(\alpha,\beta,\gamma,\delta) &= \int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt \\ &= \int_0^T \left(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)} A_\zeta(\alpha,\beta,\gamma,\delta)\right. \\ &\quad \left. + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)} A_\zeta(-\beta,-\alpha,\gamma,\delta)\right) dt + O(T^{1/2+\epsilon}), \end{aligned} \tag{2.10}$$

where A_ζ is defined at (2.8).

In the following sections we have similar conjecture for ratios of L-functions averaged over various families. In these families the L-functions are indexed by an integer d and we consider averages for $d < X$. In all of these examples we constrain the shifts as follows. For α a generic shift in the numerator (α and β in the above example) and δ a generic shift in the denominator, we require

$$-\frac{1}{4} < \Re\alpha < \frac{1}{4}, \tag{2.11a}$$

$$\frac{1}{\log C} \ll \Re\delta < \frac{1}{4}, \tag{2.11b}$$

$$\Im\alpha, \Im\delta \ll_\epsilon C^{1-\epsilon} \qquad \text{(for every } \epsilon > 0), \tag{2.11c}$$

where $C = T$ in the above example and $C = X$ in the case of discrete families of L-functions. In conjectures that refer to these conditions the error terms are believed to be uniform in the above range of parameters.

REMARK 2.2. Equation (2.11b) can be relaxed if for each shift in the denominator going to zero there is a corresponding shift in the numerator going to zero at the same rate.

REMARK 2.3. The bound of 1/4 on the absolute values of the real parts of the shifts are to prevent divergence of the Euler products that appear in the ratios conjectures.

REMARK 2.4. Because of the uniformity in the parameters α, β, γ and δ we can differentiate our conjectural formulas with respect to these parameters and the results are valid with the same range and error terms.

For obtaining lower order terms in pair correlation in Section 4, we need the following.

THEOREM 2.5. *Assuming Conjecture* 2.1, *we have*

$$\begin{aligned}
&\int_0^T \frac{\zeta'}{\zeta}(s+\alpha)\frac{\zeta'}{\zeta}(1-s+\beta)\,dt \\
&\quad = \int_0^T \Bigg(\left(\frac{\zeta'}{\zeta}\right)'(1+\alpha+\beta) \\
&\qquad + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)\prod_p \frac{(1-\frac{1}{p^{1+\alpha+\beta}})(1-\frac{2}{p}+\frac{1}{p^{1+\alpha+\beta}})}{(1-\frac{1}{p})^2} \\
&\qquad - \sum_p \left(\frac{\log p}{(p^{1+\alpha+\beta}-1)}\right)^2 \Bigg)\,dt + O(T^{1/2+\epsilon}),
\end{aligned} \tag{2.12}$$

provided that $1/\log T \ll \Re\alpha, \Re\beta < \frac{1}{4}$.

This theorem follows from (2.10) by differentiating with respect to α and β and setting $\gamma = \alpha$ and $\delta = \beta$. To perform this calculation, it is helpful to observe that $A(\alpha,\beta,\alpha,\beta) = 1$. Also, when differentiating the second term on the right side of (2.10) it is useful to observe that for a function $f(z,w)$ which is analytic at $(z,w) = (\alpha,\alpha)$,

$$\frac{d}{d\alpha}\frac{f(\alpha,\gamma)}{\zeta(1-\alpha+\gamma)}\bigg|_{\gamma=\alpha} = -f(\alpha,\alpha). \tag{2.13}$$

2.2. *Symplectic examples*

As a second example we consider the family of Dirichlet L-functions $L(s,\chi_d)$ associated with real, even, Dirichlet characters χ_d. Let

$$R_D(\alpha,\beta;\gamma,\delta) := \sum_{d\leqslant X} \frac{L(1/2+\alpha,\chi_d)L(1/2+\beta,\chi_d)}{L(1/2+\gamma,\chi_d)L(1/2+\delta,\chi_d)}, \tag{2.14}$$

with the usual conditions (2.11) on the shifts α, β, γ and δ. Let us also consider the simpler example

$$R_D(\alpha;\gamma) := \sum_{d\leqslant X} \frac{L(1/2+\alpha,\chi_d)}{L(1/2+\gamma,\chi_d)}. \tag{2.15}$$

As part of our recipe, we replace the $L(s,\chi_d)$ in the numerator by the approximate functional equation

$$L(\tfrac{1}{2}+\alpha,\chi_d) = \sum_{m<x} \frac{\chi_d(m)}{m^{1/2+\alpha}} + \left(\frac{d}{\pi}\right)^{-\alpha} \frac{\Gamma(1/4-\alpha/2)}{\Gamma(1/4+\alpha/2)} \sum_{n<y} \frac{\chi_d(n)}{n^{1/2-\alpha}} + \text{remainder}, \tag{2.16}$$

where $xy = d/(2\pi)$, and we replace the $L(s,\chi_d)$ in the denominator by their infinite series:

$$\frac{1}{L(s,\chi_d)} = \sum_{h=1}^{\infty} \frac{\mu(h)\chi_d(h)}{h^s}. \tag{2.17}$$

We consider each of the 2^λ (if there are λ factors in the numerator) pieces separately and average term-by-term within those pieces. We only retain the terms where we are averaging over squares; in other words we use the main part of the formula

$$\sum_{d\leqslant X} \chi_d(n) = \begin{cases} a(n)X^* + \text{ small} & \text{if } n \text{ is a square,} \\ \text{small} & \text{if } n \text{ is not a square,} \end{cases} \tag{2.18}$$

where $X^* = \sum_{d\leqslant X} 1$ is the number of fundamental discriminants below X and where

$$a(n) = \prod_{p|n} \frac{p}{p+1}. \tag{2.19}$$

After computing these 'diagonal' terms, we complete the sums by extending to infinity the ranges of the summation variables; we identify these terms as ratios of products of zeta functions multiplied by absolutely convergent Euler products. The sum of these expressions, one for each product of pieces of the approximate functional equations, forms our conjectural answer.

Proceeding to details, let us first consider the simpler example $R_D(\alpha;\gamma)$. We restrict attention to the 'first' piece of the approximate functional equation. Thus, we consider

$$\sum_{d\leqslant X} \sum_{h,m} \frac{\mu(h)\chi_d(hm)}{h^{1/2+\gamma}m^{1/2+\alpha}}. \tag{2.20}$$

Retaining only the terms for which hm is square, leads us to

$$X^* \sum_{hm=\square} \frac{\mu(h)a(hm)}{h^{1/2+\gamma}m^{1/2+\alpha}}. \tag{2.21}$$

We express this sum as an Euler product (to 'save' variables we now replace h by p^h and m by p^m):

$$\prod_p \sum_{\substack{h+m \\ \text{even}}} \frac{\mu(p^h)a(p^{h+m})}{p^{h(1/2+\gamma)+m(1/2+\alpha)}}. \tag{2.22}$$

The effect of $\mu(p^h)$ is to limit the choices for h to 0 or 1. When $h=0$ we have

$$\sum_{\substack{m\\ \text{even}}} \frac{a(p^m)}{p^{m(1/2+\alpha)}} = 1 + \frac{p}{p+1}\sum_{m=1}^{\infty}\frac{1}{p^{m(1+2\alpha)}} = 1 + \frac{p}{(p+1)}\frac{1}{p^{1+2\alpha}}\frac{1}{(1-\frac{1}{p^{1+2\alpha}})}, \tag{2.23}$$

and when $h=1$ there is a contribution of

$$\sum_{\substack{m\\ \text{odd}}} -\frac{a(p^{m+1})}{p^{1/2+\gamma}p^{m(1/2+\alpha)}} = -\frac{p}{(p+1)}\frac{1}{p^{1+\alpha+\gamma}}\frac{1}{(1-\frac{1}{p^{1+2\alpha}})}. \tag{2.24}$$

Thus, the Euler product simplifies to

$$\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\gamma)}\prod_p \left(1-\frac{1}{p^{1+\alpha+\gamma}}\right)^{-1}\left(1-\frac{1}{(p+1)p^{1+2\alpha}}-\frac{1}{(p+1)p^{\alpha+\gamma}}\right). \tag{2.25}$$

The product over primes is absolutely convergent as long as $\Re\alpha, \Re\gamma > -1/4$.

The other piece can be determined by recalling the functional equation

$$L(1/2+\alpha,\chi_d) = \left(\frac{d}{\pi}\right)^{-\alpha}\frac{\Gamma(1/4-\alpha/2)}{\Gamma(1/4+\alpha/2)}L(1/2-\alpha,\chi_d). \tag{2.26}$$

Thus, in total we expect that the following conjecture is true.

CONJECTURE 2.6 (Conrey, Farmer and Zirnbauer [**11**]). With constraints on α and γ as described at (2.11), we have

$$\begin{aligned} R_D(\alpha;\gamma) &= \sum_{d\leqslant X}\frac{L(1/2+\alpha,\chi_d)}{L(1/2+\gamma,\chi_d)} \\ &= \sum_{d\leqslant X}\left(\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\gamma)}A_D(\alpha;\gamma)\right. \\ &\quad \left. + \left(\frac{d}{\pi}\right)^{-\alpha}\frac{\Gamma(1/4-\alpha/2)}{\Gamma(1/4+\alpha/2)}\frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)}A_D(-\alpha;\gamma)\right) + O(X^{1/2+\epsilon}), \end{aligned} \tag{2.27}$$

where

$$A_D(\alpha;\gamma) = \prod_p \left(1-\frac{1}{p^{1+\alpha+\gamma}}\right)^{-1}\left(1-\frac{1}{(p+1)p^{1+2\alpha}}-\frac{1}{(p+1)p^{\alpha+\gamma}}\right). \tag{2.28}$$

For applications to the one-level density in the next section, we note that

$$\sum_{d\leqslant X}\frac{L'(1/2+r,\chi_d)}{L(1/2+r,\chi_d)} = \frac{d}{d\alpha}R_D(\alpha;\gamma)\Big|_{\alpha=\gamma=r}. \tag{2.29}$$

Now

$$\frac{d}{d\alpha}\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\gamma)}A_D(\alpha;\gamma)\Big|_{\alpha=\gamma=r} = \frac{\zeta'(1+2r)}{\zeta(1+2r)}A_D(r;r) + A_D'(r;r) \tag{2.30}$$

and

$$\begin{aligned} &\frac{d}{d\alpha}\left(\frac{d}{\pi}\right)^{-\alpha}\frac{\Gamma(1/4-\alpha/2)}{\Gamma(1/4+\alpha/2)}\frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)}A_D(-\alpha;\gamma)\Big|_{\alpha=\gamma=r} \\ &\qquad = -\left(\frac{d}{\pi}\right)^{-r}\frac{\Gamma(1/4-r/2)}{\Gamma(1/4+r/2)}\zeta(1-2r)A_D(-r;r). \end{aligned} \tag{2.31}$$

Also, $A_D(r;r)=1$,

$$A_D(-r;r)=\prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\left(1-\frac{1}{p}\right)^{-1}, \tag{2.32}$$

and

$$A_D'(r;r)=\sum_p\frac{\log p}{(p+1)(p^{1+2r}-1)}. \tag{2.33}$$

Thus, the ratios conjecture implies (see Remark 2.4) that the following holds.

THEOREM 2.7. *Assuming Conjecture* 2.6, $1/\log X\ll \Re r<\frac14$ *and* $\Im r\ll_\epsilon X^{1-\epsilon}$ *we have*

$$\begin{aligned}\sum_{d\leqslant X}\frac{L'(1/2+r,\chi_d)}{L(1/2+r,\chi_d)} &= \sum_{d\leqslant X}\left(\frac{\zeta'(1+2r)}{\zeta(1+2r)}+A_D'(r;r)-\left(\frac{d}{\pi}\right)^{-r}\frac{\Gamma(1/4-r/2)}{\Gamma(1/4+r/2)}\zeta(1-2r)A_D(-r;r)\right)\\ &\quad+O(X^{1/2+\epsilon}),\end{aligned} \tag{2.34}$$

where $A_D(\alpha;\gamma)$ *is defined in* (2.28).

Now we look at the case of two L-functions in the numerator and denominator. Here we will only work to keep the first main terms when the shifts α, β, γ and δ are $\ll 1/\log X$ and $X\to\infty$. We consider, from the first part of the functional equation for each of the L-functions,

$$\sum_{d\leqslant X}\sum_{h,k,m,n}\frac{\mu(h)\mu(k)\chi_d(hkmn)}{h^{1/2+\gamma}k^{1/2+\delta}m^{1/2+\alpha}n^{1/2+\beta}}. \tag{2.35}$$

Retaining only the terms for which $hkmn$ is square, leads us to

$$X^*\sum_{hkmn=\square}\frac{\mu(h)\mu(k)a(hkmn)}{h^{1/2+\gamma}k^{1/2+\delta}m^{1/2+\alpha}n^{1/2+\beta}}. \tag{2.36}$$

We express this sum as an Euler product (to 'save' variables we now replace h by p^h, etc.)

$$\prod_p\sum_{\substack{h+k+m+n\\ \text{even}}}\frac{\mu(p^h)\mu(p^k)a(p^{h+k+m+n})}{p^{h(1/2+\gamma)+k(1/2+\delta)+m(1/2+\alpha)+n(1/2+\beta)}}. \tag{2.37}$$

We analyze the inner sum by dividing it into the four cases according to $h=0,1$ and $k=0,1$; also it is helpful to note that

$$\sum_{\substack{m+n\\ \text{even}}}x^my^n=\frac{1+xy}{(1-x^2)(1-y^2)}\qquad\text{and}\qquad\sum_{\substack{m+n\\ \text{odd}}}x^my^n=\frac{x+y}{(1-x^2)(1-y^2)}.$$

It is more complicated to write down the exact formula for this, complete with the arithmetic factor $A_D(\alpha,\beta;\gamma,\delta)$. This factor is asymptotically 1 for small values of the parameters. Since we are interested in the first main terms here, we record that the relevant zeta factors in the expression above are

$$\begin{aligned}&\frac{\zeta(1+2\alpha)\zeta(1+2\beta)\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)}\\ &\qquad=\frac{(\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta)}{4\alpha\beta(\alpha+\beta)(\gamma+\delta)}+O(1/\log X).\end{aligned} \tag{2.38}$$

Thus we have, from the remaining parts of the functional equation the following conjecture.

CONJECTURE 2.8 (Conrey, Farmer and Zirnbauer [**11**]). With $\alpha,\beta,\gamma,\delta \ll 1/\log X$, we have

$$\begin{aligned}\frac{1}{X^*}R_D(\alpha,\beta;\gamma,\delta) &= \frac{(\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta)}{4\alpha\beta(\alpha+\beta)(\gamma+\delta)} - X^{-\alpha}\frac{(-\alpha+\gamma)(-\alpha+\delta)(\beta+\gamma)(\beta+\delta)}{4\alpha\beta(-\alpha+\beta)(\gamma+\delta)}\\ &\quad - X^{-\beta}\frac{(\alpha+\gamma)(\alpha+\delta)(-\beta+\gamma)(-\beta+\delta)}{4\alpha\beta(\alpha-\beta)(\gamma+\delta)}\\ &\quad - X^{-\alpha-\beta}\frac{(-\alpha+\gamma)(-\alpha+\delta)(-\beta+\gamma)(-\beta+\delta)}{4\alpha\beta(\alpha+\beta)(\gamma+\delta)} + O(1/\log X), \qquad (2.39)\end{aligned}$$

as $X \to \infty$.

2.3. *Orthogonal examples*

As a third example, we consider the orthogonal family of quadratic twists of the L-function L_Δ associated with the unique weight 12 cusp form for the full modular group:

$$L_\Delta(s,\chi_d) = \sum_{n=1}^{\infty}\frac{\chi_d(n)\tau^*(n)}{n^s} = \prod_p\left(1-\frac{\tau^*(p)\chi_d(p)}{p^s}+\frac{\chi_d(p^2)}{p^{2s}}\right)^{-1}, \qquad (2.40)$$

where $\tau^*(n)=\tau(n)/n^{11/2}$ and $\tau(n)$ is Ramanujan's tau-function. For $d>0$, this has functional equation

$$\xi_\Delta(s,\chi_d) := \left(\frac{d}{2\pi}\right)^s\Gamma(s+11/2)L_\Delta(s,\chi_d) = \xi_\Delta(1-s,\chi_d). \qquad (2.41)$$

Let

$$R_\Delta(\alpha;\gamma) := \sum_{d\leqslant X}\frac{L_\Delta(1/2+\alpha,\chi_d)}{L_\Delta(1/2+\gamma,\chi_d)} \qquad (2.42)$$

and let

$$R_\Delta(\alpha,\beta;\gamma,\delta) := \sum_{d\leqslant X}\frac{L_\Delta(1/2+\alpha,\chi_d)L_\Delta(1/2+\beta,\chi_d)}{L_\Delta(1/2+\gamma,\chi_d)L_\Delta(1/2+\delta,\chi_d)}. \qquad (2.43)$$

As in the symplectic example we will calculate the full expression for $R_\Delta(\alpha;\gamma)$ and only the leading main terms for $R_\Delta(\alpha,\beta;\gamma,\delta)$.

Note that

$$\frac{1}{L_\Delta(s,\chi_d)} = \prod_p\left(1-\frac{\tau^*(p)\chi_d(p)}{p^s}+\frac{\chi_d(p^2)}{p^{2s}}\right) =: \sum_{n=1}^{\infty}\frac{\mu_\Delta(n)\chi_d(n)}{n^s}. \qquad (2.44)$$

To commence the calculation of $R_\Delta(\alpha;\gamma)$ we replace each L-function in the numerator by the first half of the approximate functional equation

$$L_\Delta(1/2+\alpha,\chi_d) = \sum_{m<x}\frac{\chi_d(m)\tau^*(m)}{m^s} + \left(\frac{d}{2\pi}\right)^{-2\alpha}\frac{\Gamma(6-\alpha)}{\Gamma(6+\alpha)}\sum_{n<y}\frac{\chi_d(n)\tau^*(n)}{n^{1-s}} + \text{remainder}, \qquad (2.45)$$

where $xy = d^2/(2\pi)$. We must then consider

$$\sum_{d\leqslant X}\sum_{h,m}\frac{\mu_\Delta(h)\tau^*(m)\chi_d(hm)}{h^{1/2+\gamma}k^{1/2+\alpha}}, \qquad (2.46)$$

which leads to

$$X^* \sum_{hm=\square} \frac{\mu_\Delta(h)\tau^*(m)a(hm)}{h^{1/2+\gamma}k^{1/2+\alpha}} = X^* \prod_p \left(1 + \frac{p}{p+1} \sum_{\substack{h+m>0 \\ \text{even}}} \frac{\mu_\Delta(p^h)\tau^*(p^m)}{p^{h(1/2+\gamma)+m(1/2+\alpha)}}\right). \quad (2.47)$$

We note that $\mu_\Delta(p) = -\tau^*(p)$, $\mu_\Delta(p^2) = 1$, and $\mu_\Delta(p^m) = 0$ for $m > 2$, so that the product over primes here is

$$\prod_p \left(1 + \frac{p}{p+1}\left(\sum_{m=1}^\infty \frac{\tau^*(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\tau^*(p)}{p^{1+\alpha+\gamma}} \sum_{m=0}^\infty \frac{\tau^*(p^{2m+1})}{p^{m(1+2\alpha)}} + \frac{1}{p^{1+2\gamma}} \sum_{m=0}^\infty \frac{\tau^*(p^{2m})}{p^{m(1+2\alpha)}}\right)\right). \quad (2.48)$$

We note that

$$\sum_{m=0}^\infty \tau^*(p^{2m})x^{2m} = \frac{1}{2}\left\{\left(1 - \tau^*(p)x + x^2\right)^{-1} + \left(1 + \tau^*(p)x + x^2\right)^{-1}\right\} \quad (2.49)$$

and

$$\sum_{m=0}^\infty \tau^*(p^{2m+1})x^{2m+1} = \frac{1}{2}\left\{\left(1 - \tau^*(p)x + x^2\right)^{-1} - \left(1 + \tau^*(p)x + x^2\right)^{-1}\right\}. \quad (2.50)$$

The 'polar' part of the product (2.48) is $\zeta(1+2\gamma)/\zeta(1+\alpha+\gamma)$; we can factor these terms out and be left with a convergent Euler product. However, we prefer at this point to factor out some other L-functions present here with values near the 1-line and to be left with an Euler product which is more rapidly convergent. To this end, we recall the Rankin–Selberg convolution of L_Δ and the symmetric square L-function associated with L_Δ. We can write the Euler product for L_Δ as

$$L_\Delta(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_p}}{p^s}\right)^{-1}, \quad (2.51)$$

where $\alpha_p + \overline{\alpha_p} = \tau^*(p)$ and $\alpha_p\overline{\alpha_p} = |\alpha_p|^2 = 1$. The Rankin–Selberg L-function is

$$L(\tau \otimes \tau, s) = \sum_{n=1}^\infty \frac{\tau^*(n)^2}{n^s} = \zeta(s)L_\Delta(\mathrm{sym}^2, s)\zeta(2s)^{-1}, \quad (2.52)$$

where the symmetric square L-function is given by

$$L_\Delta(\mathrm{sym}^2, s) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_p^2}}{p^s}\right)^{-1} \quad (2.53)$$

and is an entire function of s. As a Dirichlet series, we can write

$$L_\Delta(\mathrm{sym}^2, s) = \zeta(2s)^{-1} \sum_{n=1}^\infty \frac{\tau^*(n^2)}{n^s}. \quad (2.54)$$

Thus, the product (2.48) can be expressed as

$$\frac{\zeta(1+2\gamma)L_\Delta(\mathrm{sym}^2, 1+2\alpha)}{\zeta(1+\alpha+\gamma)L_\Delta(\mathrm{sym}^2, 1+\alpha+\gamma)} B_\Delta(\alpha;\gamma), \quad (2.55)$$

where

$$\begin{aligned} &B_\Delta(\alpha;\gamma) \\ &= \prod_p \left(1+\frac{p}{p+1}\left(\sum_{m=1}^\infty \frac{\tau^*(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\tau^*(p)}{p^{1+\alpha+\gamma}}\sum_{m=0}^\infty \frac{\tau^*(p^{2m+1})}{p^{m(1+2\alpha)}} + \frac{1}{p^{1+2\gamma}}\sum_{m=0}^\infty \frac{\tau^*(p^{2m})}{p^{m(1+2\alpha)}}\right)\right) \\ &\quad\times \frac{\left(1-\frac{\tau^*(p^2)}{p^{1+2\alpha}}+\frac{\tau^*(p^2)}{p^{2+4\alpha}}-\frac{1}{p^{3+6\alpha}}\right)\left(1-\frac{1}{p^{1+2\gamma}}\right)}{\left(1-\frac{\tau^*(p^2)}{p^{1+\alpha+\gamma}}+\frac{\tau^*(p^2)}{p^{2+2\alpha+2\gamma}}-\frac{1}{p^{3+3\alpha+3\gamma}}\right)\left(1-\frac{1}{p^{1+\alpha+\gamma}}\right)}. \end{aligned} \tag{2.56}$$

Note that $B(r;r)=1$; this follows from the fact that $\tau^*(p^{2m+1})\tau^*(p)=\tau^*(p^{2m+2})+\tau^*(p^{2m})$. Thus, the ratios conjecture gives the following.

CONJECTURE 2.9 (Conrey, Farmer and Zirnbauer [**11**]). With constraints on α and γ as described at (2.11), we have

$$\begin{aligned} R_\Delta(\alpha;\gamma) &= \sum_{d\leqslant X} \frac{L_\Delta(1/2+\alpha,\chi_d)}{L_\Delta(1/2+\gamma,\chi_d)} \\ &= \sum_{d\leqslant X}\Bigg(\frac{\zeta(1+2\gamma)L_\Delta(\mathrm{sym}^2,1+2\alpha)}{\zeta(1+\alpha+\gamma)L_\Delta(\mathrm{sym}^2,1+\alpha+\gamma)}B_\Delta(\alpha;\gamma) \\ &\quad + \left(\frac{d}{2\pi}\right)^{-2\alpha}\frac{\Gamma(6-\alpha)}{\Gamma(6+\alpha)}\frac{\zeta(1+2\gamma)L_\Delta(\mathrm{sym}^2,1-2\alpha)}{\zeta(1-\alpha+\gamma)L_\Delta(\mathrm{sym}^2,1-\alpha+\gamma)}B_\Delta(-\alpha;\gamma)\Bigg) \\ &\quad + O(X^{1/2+\epsilon}), \end{aligned} \tag{2.57}$$

where $B_\Delta(\alpha,\gamma)$ is defined in (2.56).

For application to the one-level density, we note that

$$\sum_{d\leqslant X}\frac{L'_\Delta(1/2+r,\chi_d)}{L_\Delta(1/2+r,\chi_d)} = \frac{d}{d\alpha}R_\Delta(\alpha;\gamma)\Big|_{\alpha=\gamma=r}. \tag{2.58}$$

Now

$$\begin{aligned} &\frac{d}{d\alpha}\frac{\zeta(1+2\gamma)L_\Delta(\mathrm{sym}^2,1+2\alpha)}{\zeta(1+\alpha+\gamma)L_\Delta(\mathrm{sym}^2,1+\alpha+\gamma)}B_\Delta(\alpha;\gamma)\Big|_{\alpha=\gamma=r} \\ &\qquad = -\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \frac{L'_\Delta(\mathrm{sym}^2,1+2r)}{L_\Delta(\mathrm{sym}^2,1+2r)} + B'_\Delta(r;r) \end{aligned} \tag{2.59}$$

and

$$\begin{aligned} &\frac{d}{d\alpha}\left(\frac{d}{2\pi}\right)^{-2\alpha}\frac{\Gamma(6-\alpha)}{\Gamma(6+\alpha)}\frac{\zeta(1+2\gamma)L_\Delta(\mathrm{sym}^2,1-2\alpha)}{\zeta(1-\alpha+\gamma)L_\Delta(\mathrm{sym}^2,1-\alpha+\gamma)}B_\Delta(-\alpha;\gamma)\Big|_{\alpha=\gamma=r} \\ &\qquad = -\left(\frac{d}{2\pi}\right)^{-2r}\frac{\Gamma(6-r)}{\Gamma(6+r)}\frac{\zeta(1+2r)L_\Delta(\mathrm{sym}^2,1-2r)}{L_\Delta(\mathrm{sym}^2,1)}B_\Delta(-r;r). \end{aligned} \tag{2.60}$$

Thus, the ratios conjecture implies the following.

THEOREM 2.10. *Assuming Conjecture* 2.9*, if* $1/\log X \ll \Re r < \frac{1}{4}$ *and* $\Im r \ll_\epsilon X^{1-\epsilon}$*, then*

$$\sum_{d\leqslant X}\frac{L_\Delta'(1/2+r,\chi_d)}{L_\Delta(1/2+r,\chi_d)} = \sum_{d\leqslant X}\Bigg(-\frac{\zeta'(1+2r)}{\zeta(1+2r)}+\frac{L_\Delta'(\mathrm{sym}^2,1+2r)}{L_\Delta(\mathrm{sym}^2,1+2r)}+B_\Delta'(r;r) - \left(\frac{d}{2\pi}\right)^{-2r}\frac{\Gamma(6-r)}{\Gamma(6+r)}\frac{\zeta(1+2r)L_\Delta(\mathrm{sym}^2,1-2r)}{L_\Delta(\mathrm{sym}^2,1)}B_\Delta(-r;r)\Bigg) + O(X^{1/2+\epsilon}) \tag{2.61}$$

where $B_\Delta(\alpha,\gamma)$ *is defined in* (2.56).

We now determine the main terms when $\alpha,\beta,\gamma,\delta \ll 1/\log X$ and $X\to\infty$ for the average over this family of the ratio $R_\Delta(\alpha,\beta;\gamma,\delta)$ of two L-functions over two L-functions. We are quickly led to consider

$$\sum_{hkmn=\square}\frac{\mu_\Delta(h)\mu_\Delta(k)\tau^*(m)\tau^*(n)a(hkmn)}{h^{1/2+\gamma}k^{1/2+\delta}m^{1/2+\alpha}n^{1/2+\beta}}. \tag{2.62}$$

When we go to Euler products, we find that this expression evaluates to

$$\frac{\zeta(1+\alpha+\beta)\zeta(1+2\gamma)\zeta(1+\gamma+\delta)\zeta(1+2\delta)}{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)}A_\Delta(\alpha,\beta;\gamma,\delta), \tag{2.63}$$

where A is analytic if the real parts of α, β, γ and δ are smaller than $1/4$ in absolute value; moreover $A_\Delta(0,0;0,0)=1$. Thus, this part is equal to

$$\frac{(\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta)}{(\alpha+\beta)(2\gamma)(\gamma+\delta)(2\delta)}+O(1/\log X). \tag{2.64}$$

Taking the symmetric sum of four of these terms, arising from the product of the approximate functional equations of the two L-functions in the numerator, we find that the following holds.

CONJECTURE 2.11 (Conrey, Farmer and Zirbauer [**11**]). With $\alpha,\beta,\gamma,\delta \ll 1/\log X$, we have

$$\begin{aligned}\frac{1}{X^*}R_\Delta(\alpha,\beta;\gamma,\delta) &= \frac{1}{X^*}\sum_{d\leqslant X}\frac{L_\Delta(1/2+\alpha,\chi_d)L_\Delta(1/2+\beta,\chi_d)}{L_\Delta(1/2+\gamma,\chi_d)L_\Delta(1/2+\delta,\chi_d)}\\ &= \frac{(\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta)}{(\alpha+\beta)(2\gamma)(\gamma+\delta)(2\delta)}\\ &\quad + X^{-2\alpha}\frac{(-\alpha+\gamma)(-\alpha+\delta)(\beta+\gamma)(\beta+\delta)}{(-\alpha+\beta)(2\gamma)(\gamma+\delta)(2\delta)}\\ &\quad + X^{-2\beta}\frac{(\alpha+\gamma)(\alpha+\delta)(-\beta+\gamma)(-\beta+\delta)}{(\alpha-\beta)(2\gamma)(\gamma+\delta)(2\delta)}\\ &\quad - X^{-2\alpha-2\beta}\frac{(-\alpha+\gamma)(-\alpha+\delta)(-\beta+\gamma)(-\beta+\delta)}{(\alpha+\beta)(2\gamma)(\gamma+\delta)(2\delta)}+O(1/\log X),\end{aligned} \tag{2.65}$$

as $X\to\infty$.

3. *One-level density*

In this section we use the ratios conjecture to compute the one-level density function for zeros of quadratic Dirichlet L-functions, complete with lower order terms. Özlük and Snyder [**38**] have proven such results (assuming the generalized Riemann Hypothesis) for test functions f for which the support of $\hat{f}$ is limited. The ratios conjectures imply a result consistent with [**38**] but with no constraint on the support of the Fourier transform of the test function.

For simplicity, we assume that

$$\begin{aligned} &f(z) \text{ is holomorphic throughout the strip } |\Im z| < 2,\\ &\text{is real on the real line and even,} \\ &\text{and that } f(x) \ll 1/(1+x^2) \text{ as } x \to \infty. \end{aligned} \tag{3.1}$$

We consider

$$S_1(f) := \sum_{d \leqslant X} \sum_{\gamma_d} f(\gamma_d), \tag{3.2}$$

where γ_d denotes the ordinate of a generic zero of $L(s, \chi_d)$ on the half-line (we are assuming that all of the complex zeros are on the half-line).

We have

$$S_1(f) = \sum_{d \leqslant X} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \frac{L'(s, \chi_d)}{L(s, \chi_d)} f(-i(s - 1/2))\, ds, \tag{3.3}$$

where (c) denotes a vertical line from $c - i\infty$ to $c + i\infty$ and $3/4 > c > 1/2 + 1/\log X$. The integral on the c-line is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t - i(c - 1/2)) \sum_{d \leqslant X} \frac{L'(1/2 + (c - 1/2 + it), \chi_d)}{L(1/2 + (c - 1/2 + it), \chi_d)}\, dt. \tag{3.4}$$

It follows by the Riemann Hypothesis that on the path of integration (c)

$$\frac{L'(s, \chi_d)}{L(s, \chi_d)} \ll \log^2(|s|d). \tag{3.5}$$

For $|t| > X^{1-\epsilon}$ we estimate the integral using (3.5) and (3.1) and the result is $\ll X^{\epsilon}$. By the ratios conjecture (2.34), if $|t| < X^{1-\epsilon}$ then the sum over d in (3.4) is

$$\begin{aligned} \sum_{d \leqslant X} \Bigg(&\frac{\zeta'(1+2r)}{\zeta(1+2r)} + A_D'(r; r) \\ &- \left(\frac{d}{\pi}\right)^{-r} \frac{\Gamma(1/4 - r/2)}{\Gamma(1/4 + r/2)} \zeta(1 - 2r) A_D(-r; r) \Bigg) \Bigg|_{r = c - 1/2 + it} + O(X^{1/2+\epsilon}). \end{aligned} \tag{3.6}$$

Since the quantity in (3.6) is $\ll X^{1+\epsilon}$ for $|t| < X^{1-\epsilon}$ and $f(t) \ll 1/t^2$, we can extend the integration in t to infinity. Finally, since the integrand is regular at $r = 0$, we can move the path of integration to $c = 1/2$ and so obtain

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \sum_{d \leqslant X} \Bigg(&\frac{\zeta'(1+2it)}{\zeta(1+2it)} + A_D'(it; it) \\ &- \left(\frac{d}{\pi}\right)^{-it} \frac{\Gamma(1/4 - it/2)}{\Gamma(1/4 + it/2)} \zeta(1 - 2it) A_D(-it; it) \Bigg)\, dt + O(X^{1/2+\epsilon}). \end{aligned} \tag{3.7}$$

For the integral on the $1 - c$ line, we change variables, letting $s \to 1 - s$, and we use the functional equation

$$\frac{L'(1-s, \chi_d)}{L(1-s, \chi_d)} = \frac{X'(s, \chi_d)}{X(s, \chi_d)} - \frac{L'(s, \chi_d)}{L(s, \chi_d)} \tag{3.8}$$

where

$$\frac{X'(s, \chi_d)}{X(s, \chi_d)} = -\log \frac{d}{\pi} - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right). \tag{3.9}$$

The contribution from the L'/L term is now exactly as before, since f is even. Thus, we obtain the following result.

THEOREM 3.1. *Assuming Conjecture* 2.6 *and* f *satisfying* (3.1), *we have*

$$\sum_{d\leqslant X}\sum_{\gamma_d} f(\gamma_d) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\sum_{d\leqslant X}\Bigg(\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4+it/2)+\frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4-it/2)$$
$$+2\bigg(\frac{\zeta'(1+2it)}{\zeta(1+2it)}+A_D'(it;it)$$
$$-\left(\frac{d}{\pi}\right)^{-it}\frac{\Gamma(1/4-it/2)}{\Gamma(1/4+it/2)}\zeta(1-2it)A_D(-it;it)\bigg)\Bigg)\,dt+O(X^{1/2+\epsilon}), \tag{3.10}$$

where

$$A_D(-r;r)=\prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\left(1-\frac{1}{p}\right)^{-1}, \tag{3.11}$$

and

$$A_D'(r;r)=\sum_p\frac{\log p}{(p+1)(p^{1+2r}-1)}. \tag{3.12}$$

The low-lying zeros of this family of L-functions are expected to display the same statistics as the eigenvalues of the matrices from $USp(2N)$ chosen with respect to Haar measure. Thus in the large X limit, the one-level density of the scaled zeros will have the form, as proved by Özlük and Snyder [**38**],

$$\lim_{X\to\infty}\frac{1}{X^*}\sum_{d\leqslant X}\sum_{\gamma_d} f\left(\gamma_d\frac{\log(d/\pi)}{2\pi}\right)=\int_{-\infty}^{\infty}f(x)\left(1-\frac{\sin(2\pi x)}{2\pi x}\right)dx, \tag{3.13}$$

where X^* is the number of terms in the sum (and is proportional to X).

Defining

$$f(t)=g\left(\frac{t\log X}{2\pi}\right)$$

and scaling the variable t from Theorem 3.1 as $\tau=(t\log X)/(2\pi)$, we have

$$\sum_{d\leqslant X}\sum_{\gamma_d} g\left(\frac{\gamma_d\log X}{2\pi}\right)$$
$$=\frac{1}{\log X}\int_{-\infty}^{\infty}g(\tau)\sum_{d\leqslant X}\Bigg(\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1/4+\frac{i\pi\tau}{\log X}\right)$$
$$+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1/4-\frac{i\pi\tau}{\log X}\right)+2\bigg(\frac{\zeta'(1+\frac{4i\pi\tau}{\log X})}{\zeta(1+\frac{4i\pi\tau}{\log X})}+A_D'\left(\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)$$
$$-e^{-(2\pi i\tau/\log X)\log(d/\pi)}\frac{\Gamma(1/4-\frac{i\pi\tau}{\log X})}{\Gamma(1/4+\frac{i\pi\tau}{\log X})}\zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\bigg)\Bigg)\,d\tau$$
$$+O(X^{1/2+\epsilon}). \tag{3.14}$$

For large X only the $\log(d/\pi)$ term, the ζ'/ζ term and the final term in the integral contribute, yielding the asymptotic

$$\sum_{d\leqslant X}\sum_{\gamma_d} g\left(\frac{\gamma_d \log X}{2\pi}\right) \sim \frac{1}{\log X}\int_{-\infty}^{\infty} g(\tau)\left(X^*\log X - X^*\frac{\log X}{2\pi i\tau} + X^*\frac{e^{-2\pi i\tau}}{2\pi i\tau}\log X\right)d\tau. \quad (3.15)$$

However, since g is an even function, the middle term above drops out and the last term can be duplicated with a change of sign of τ, leaving

$$\lim_{X\to\infty}\frac{1}{X^*}\sum_{d\leqslant X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log(d/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty} g(\tau)\left(1+\frac{e^{-2\pi i\tau}}{4\pi i\tau}+\frac{e^{2\pi i\tau}}{-4\pi i\tau}\right)d\tau, \quad (3.16)$$

and resulting in exactly the answer expected.

In much the same way as for Theorem 3.1 we can compute the lower order terms in the one-level density for the zeros of the functions from the orthogonal family $L_\Delta(s,\chi_d)$ by using (2.61).

THEOREM 3.2. *Assuming Conjecture* 2.9 *and with f satisfying* (3.1), *we have*

$$\begin{aligned}\sum_{d\leqslant X}\sum_{\gamma_{\Delta,d}} f(\gamma_{\Delta,d}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\sum_{d\leqslant X}\Bigg(&2\log\frac{d}{2\pi}+\frac{\Gamma'}{\Gamma}(6+it)+\frac{\Gamma'}{\Gamma}(6-it)\\ &+2\bigg(-\frac{\zeta'(1+2it)}{\zeta(1+2it)}+\frac{L'_\Delta(\mathrm{sym}^2,1+2it)}{L_\Delta(\mathrm{sym}^2,1+2it)}+B'_\Delta(it;it)\\ &-\left(\frac{d}{2\pi}\right)^{-2it}\frac{\Gamma(6-it)}{\Gamma(6+it)}\frac{\zeta(1+2it)L_\Delta(\mathrm{sym}^2,1-2it)}{L_\Delta(\mathrm{sym}^2,1)}B_\Delta(-it;it)\bigg)\Bigg)\,dt\\ &+O(X^{1/2+\epsilon})\end{aligned} \quad (3.17)$$

where B_Δ is defined in (2.56).

In the same way as above, the main terms here give the one-level density of eigenvalues of matrices from the group $\mathrm{SO}(2N)$, which in the limit of large N is $1+(\sin 2\pi x)/(2\pi x)$.

4. *Pair-correlation*

We show how to use the ratios conjecture to compute the pair-correlation of the zeros of the Riemann zeta function, originally conjectured by Montgomery [**36**], together with lower order (arithmetic) terms that have been found heuristically by Bogolmony and Keating [**3**] (see [**28**] for a more expository description, [**2**] for numerical calculation of these lower order terms and [**19**] for related rigorous results). When Farmer formulated his original ratio conjecture (2.2) he observed in [**16**] that it implied the leading order terms of Montgomery's pair correlation conjecture. Farmer's method is completely different from what we present below.

We want to evaluate the sum

$$S(f) = \sum_{0<\gamma,\gamma'<T} f(\gamma-\gamma') \quad (4.1)$$

for a test function f satisfying (3.1). We rewrite the sum in question in terms of contour integrals. Let $1/2+1/\log T<a<b<3/4$ and let $\mathcal{C}_1$ be the positively oriented rectangular contour with corners a, $a+iT$, $1-a+iT$ and $1-a$ and let $\mathcal{C}_2$ be the rectangular contour with corners b, $b+iT$, $1-b+iT$ and $1-b$. Then

$$S(f) = \frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2}\frac{\zeta'}{\zeta}(z)\frac{\zeta'}{\zeta}(w)f(-i(z-w))\,dw\,dz; \quad (4.2)$$

the point, of course, is that the poles inside the contours are simple poles with residue 1 at the zeros $z = 1/2 + i\gamma$ and $w = 1/2 + i\gamma'$ of the zeta function. The integrals along the horizontal sides are small and may be ignored. Thus, we consider four double integrals. We consider each of the four double integrals separately; call them $I_1, \dots, I_4$, where I_1 has vertical parts a and b, I_2 has vertical parts $1-a$ and $1-b$, I_3 has vertical parts a and $1-b$, and I_4 has vertical parts $1-a$ and b.

It is easy to see using the Riemann Hypothesis that $I_1 = O(T^\epsilon)$, just by moving the contours to the right of 1 and integrating term-by-term.

For I_2, we use the functional equation

$$\frac{\zeta'}{\zeta}(s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(1-s)$$

for $s = w$ and $s = z$ and find similarly that

$$I_2 = \frac{1}{(2\pi)^2}\int_0^T\int_0^T \frac{\chi'}{\chi}(1/2+iu)\frac{\chi'}{\chi}(1/2+iv) f(u-v)\,du\,dv + O(T^\epsilon). \tag{4.3}$$

Using the fact that

$$\frac{\chi'}{\chi}(1/2+it) = -\log\frac{|t|}{2\pi}\left(1 + O\left(\frac{1}{|t|}\right)\right) \tag{4.4}$$

and that f is even, we see, after the substitution $u = v + \eta$, that

$$\begin{aligned} I_2 &= \frac{2}{(2\pi)^2}\int_0^T\int_v^T \log\frac{v}{2\pi}\log\frac{u}{2\pi} f(u-v)\,du\,dv + O(T^\epsilon) \\ &= \frac{2}{(2\pi)^2}\int_0^T f(\eta)\int_0^{T-\eta}\log\frac{v}{2\pi}\log\frac{v+\eta}{2\pi}\,dv\,d\eta + O(T^\epsilon). \end{aligned} \tag{4.5}$$

Recall that f satisfies

$$f(x) \ll \frac{1}{1+x^2} \tag{4.6}$$

for real x. Letting $v \to vT$ in the inner integral above, we have

$$I_2 = \frac{2}{(2\pi)^2}T\int_0^T f(\eta)\int_0^{1-\eta/T}\log\frac{vT}{2\pi}\log\frac{vT+\eta}{2\pi}\,dv\,d\eta + O(T^\epsilon). \tag{4.7}$$

We may extend the upper limit of the inner integral to $v = 1$, introducing an error term of size $\ll \int_0^T \eta f(\eta)\log^2 T\,d\eta \ll \log^3 T$. We can also replace $\log(vT+\eta)$ by $\log vT$ with the same error term. Thus,

$$\begin{aligned} I_2 &= \frac{2}{(2\pi)^2}T\int_0^T f(\eta)\int_0^1 \log^2\frac{vT}{2\pi}\,dv\,d\eta + O(T^\epsilon) \\ &= \frac{1}{(2\pi)^2}\int_{-T}^T f(\eta)\int_0^T \log^2\frac{v}{2\pi}\,dv\,d\eta + O(T^\epsilon). \end{aligned} \tag{4.8}$$

Next we consider I_3. Letting $z = w + \eta$, we see that it is

$$\begin{aligned} I_3 &= \frac{-1}{(2\pi i)^2}\int_{1-b}^{1-b+iT}\int_a^{a+iT}\frac{\zeta'}{\zeta}(w)\frac{\zeta'}{\zeta}(z) f(-i(z-w))\,dw\,dz \\ &= \frac{-1}{(2\pi)^2 i}\int_{1-a-b-iT}^{1-a-b+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{\zeta'}{\zeta}(a+it)\frac{\zeta'}{\zeta}(a+it+\eta)\,dt\,d\eta, \end{aligned} \tag{4.9}$$

where $T_1 = \max\{0, -\Im\eta\}$ and $T_2 = \min\{T - \Im\eta, T\}$. We use the functional equation

$$\frac{\zeta'}{\zeta}(a+\eta+it) = \frac{\chi'}{\chi}(a+\eta+it) - \frac{\zeta'}{\zeta}(1-a-\eta-it). \tag{4.10}$$

The term with the χ'/χ is small as is seen by moving the contour to the right. Thus, we see that

$$I_3 = \frac{1}{(2\pi)^2 i}\int_{1-a-b-iT}^{1-a-b+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{\zeta'}{\zeta}(a+it)\frac{\zeta'}{\zeta}(1-a-it-\eta)\,dt\,d\eta + O(T^\epsilon)$$
$$= \frac{1}{(2\pi)^2 i}\int_{1-a-b-iT}^{1-a-b+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{\zeta'}{\zeta}(s+(a-1/2))\frac{\zeta'}{\zeta}(1-s+(1/2-a-\eta))\,dt\,d\eta$$
$$+ O(T^\epsilon), \tag{4.11}$$

where $s = 1/2+it$. By Theorem 2.5, we have

$$I_3 = \frac{1}{(2\pi)^2 i}\int_{1-a-b-iT}^{1-a-b+iT} f(-i\eta)\int_{T_1}^{T_2}\Bigg(\left(\frac{\zeta'}{\zeta}\right)'(1-\eta)$$
$$+\left(\frac{t}{2\pi}\right)^{\eta}\zeta(1-\eta)\zeta(1+\eta)\prod_p\frac{(1-\frac{1}{p^{1-\eta}})(1-\frac{2}{p}+\frac{1}{p^{1-\eta}})}{(1-\frac{1}{p})^2}$$
$$-\sum_p\left(\frac{\log p}{(p^{1-\eta}-1)}\right)^2\Bigg)\,dt\,d\eta + O(T^{1/2+\epsilon}). \tag{4.12}$$

Let $\delta = a+b-1$ and let $g(-\eta,t)$ be the integrand in the second integral above. We can extend the range of the inner integration, much as we did for the I_2 integral, to the interval $[0,T]$ with an error term of size $\ll T^\epsilon\int_\eta |\eta||f(\eta)|\,d\eta \ll T^\epsilon$. Thus, we obtain

$$I_3 = \frac{1}{(2\pi)^2 i}\int_0^T\int_{-\delta-iT}^{-\delta+iT} f(-i\eta)g(-\eta,t)\,d\eta\,dt + O(T^{1/2+\epsilon}). \tag{4.13}$$

Now we consider I_4. Again letting $z = w+\eta$, we have

$$I_4 = \frac{1}{(2\pi i)^2}\int_{1-a}^{1-a+iT}\int_b^{b+iT}\frac{\zeta'}{\zeta}(w)\frac{\zeta'}{\zeta}(z)f(-i(z-w))\,dz\,dw$$
$$= \frac{1}{(2\pi)^2 i}\int_{a+b-1-iT}^{a+b-1+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{\zeta'}{\zeta}(1-a+it)\frac{\zeta'}{\zeta}(1-a+it+\eta)\,dt\,d\eta. \tag{4.14}$$

We use the functional equation

$$\frac{\zeta'}{\zeta}(1-a+it) = \frac{\chi'}{\chi}(1-a+it) - \frac{\zeta'}{\zeta}(a-it). \tag{4.15}$$

Again, the contribution of the χ'/χ term is negligible. Thus,

$$I_4 = \frac{1}{(2\pi)^2 i}\int_{a+b-1-iT}^{a+b-1+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{\zeta'}{\zeta}(a-it)\frac{\zeta'}{\zeta}(1-a+it+\eta)\,dt\,d\eta + O(T^\epsilon)$$
$$= \frac{1}{(2\pi)^2 i}\int_{a+b-1-iT}^{a+b-1+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{\zeta'}{\zeta}(1-s+(a-1/2))\frac{\zeta'}{\zeta}(s+(1/2-a+\eta))\,dt\,d\eta$$
$$+ O(T^\epsilon). \tag{4.16}$$

Now, by Theorem 2.5,

$$I_4 = \frac{1}{(2\pi)^2 i} \int_{a+b-1-iT}^{a+b-1+iT} f(-i\eta) \int_{T_1}^{T_2} \Bigg(\left(\frac{\zeta'}{\zeta}\right)' (1+\eta) + \left(\frac{t}{2\pi}\right)^{-\eta} \zeta(1-\eta)\zeta(1+\eta) \prod_p \frac{(1-\frac{1}{p^{1+\eta}})(1-\frac{2}{p}+\frac{1}{p^{1+\eta}})}{(1-\frac{1}{p})^2} - \sum_p \left(\frac{\log p}{(p^{1+\eta}-1)}\right)^2 \Bigg) dt\, d\eta + O(T^{1/2+\epsilon}). \tag{4.17}$$

Using the notation introduced after the calculation of I_3, and again extending the range of the integration in the inner integral, we can write the expression for I_4 as

$$I_4 = \frac{1}{(2\pi)^2 i} \int_0^T \int_{\delta-iT}^{\delta+iT} f(-i\eta) g(\eta,t)\, d\eta\, dt + O(T^{1/2+\epsilon}). \tag{4.18}$$

Combining this with what we found for I_3 we have, after a change of variables,

$$I_3 + I_4 = \frac{2}{(2\pi)^2 i} \int_0^T \int_{\delta-iT}^{\delta+iT} f(i\eta) g(\eta,t)\, d\eta\, dt + O(T^{1/2+\epsilon}). \tag{4.19}$$

Now let

$$A(\eta) = \prod_p \frac{(1-\frac{1}{p^{1+\eta}})(1-\frac{2}{p}+\frac{1}{p^{1+\eta}})}{(1-\frac{1}{p})^2} \tag{4.20}$$

and

$$B(\eta) = \sum_p \left(\frac{\log p}{(p^{1+\eta}-1)}\right)^2 \tag{4.21}$$

so that

$$g(\eta,t) = \left(\frac{\zeta'}{\zeta}\right)'(1+\eta) + \left(\frac{t}{2\pi}\right)^{-\eta} \zeta(1-\eta)\zeta(1+\eta)A(\eta) - B(\eta). \tag{4.22}$$

Near 0, we see that (note that $A'(0)=0$),

$$g(\eta,t) = \frac{\log(t/2\pi)}{\eta} + O(1). \tag{4.23}$$

We move the path of integration in η to the imaginary axis from $-T$ to T with a principal value as we pass through 0; the contribution from half of the residue from the pole of g at $\eta = 0$ is

$$\pi \int_0^T f(0) \log \frac{t}{2\pi}\, dt. \tag{4.24}$$

Combining our expressions for $I_1, \dots, I_4$, and changing η into ir we have the following.

Theorem 4.1. *Assuming Conjecture 2.1, and with f satisfying (3.1), we have*

$$\sum_{\gamma,\gamma' \leqslant T} f(\gamma-\gamma') = \frac{1}{(2\pi)^2} \int_0^T \Bigg(2\pi f(0) \log \frac{t}{2\pi} + \int_{-T}^{T} f(r) \Bigg(\log^2 \frac{t}{2\pi} + 2\Bigg(\left(\frac{\zeta'}{\zeta}\right)'(1+ir) + \left(\frac{t}{2\pi}\right)^{-ir} \zeta(1-ir)\zeta(1+ir)A(ir) - B(ir)\Bigg)\Bigg) dr \Bigg) dt + O(T^{1/2+\epsilon}); \tag{4.25}$$

here the integral is to be regarded as a principal value near $r = 0$,

$$A(\eta) = \prod_p \frac{(1 - \frac{1}{p^{1+\eta}})(1 - \frac{2}{p} + \frac{1}{p^{1+\eta}})}{(1 - \frac{1}{p})^2}, \tag{4.26}$$

and

$$B(\eta) = \sum_p \left(\frac{\log p}{(p^{1+\eta} - 1)} \right)^2. \tag{4.27}$$

We believe that this formula, originally found by Bogomolny and Keating [**3**], is very accurate, indeed, down to a square root error term. It includes all of the lower order terms that arise from arithmetical considerations and should include all of the fluctuations found in any of the extensive numerical experiments that have been done. We have not scaled any of the terms here so that terms of different scales are shown all at once.

To see the leading order term from Montgomery's pair-correlation conjecture, let

$$L = \log \frac{T}{2\pi} \quad \text{and} \quad g\left(x \frac{L}{2\pi} \right) = f(x),$$

and scale the variable r in the inner integral in Theorem 4.1 as $y = r(L/2\pi)$:

$$\begin{aligned}
&\sum_{\gamma,\gamma' \leqslant T} g\left((\gamma - \gamma') \frac{L}{2\pi} \right) \\
&\quad = \frac{1}{(2\pi)^2} \int_0^T \left(2\pi g(0) \log \frac{t}{2\pi} + \frac{2\pi}{L} \int_{-T(L/2\pi)}^{T(L/2\pi)} g(y) \Bigg(\log^2 \frac{t}{2\pi} + 2\Bigg(\left(\frac{\zeta'}{\zeta} \right)' \left(1 + \frac{2\pi i y}{L} \right) \right. \\
&\qquad \left. + e^{-2\pi i y (\log(t/2\pi)/L)} \zeta\left(1 - \frac{2\pi i y}{L} \right) \zeta\left(1 + \frac{2\pi i y}{L} \right) A\left(\frac{2\pi i y}{L} \right) - B\left(\frac{2\pi i y}{L} \right) \Bigg) \Bigg) \, dy \right) dt \\
&\qquad + O(T^{1/2+\epsilon}).
\end{aligned} \tag{4.28}$$

For large T, only the $\log^2(t/2\pi)$ and the two terms containing zeta functions contribute, so we have the asymptotic

$$\begin{aligned}
\sum_{\gamma,\gamma' \leqslant T} g\left((\gamma - \gamma') \frac{L}{2\pi} \right) &\sim \frac{1}{(2\pi)^2} \int_0^T \left(2\pi g(0) \log \frac{t}{2\pi} + \frac{2\pi}{L} \int_{-T(L/2\pi)}^{T(L/2\pi)} g(y) \Bigg(\log^2 \frac{t}{2\pi} - \frac{L^2}{2\pi^2 y^2} \right. \\
&\qquad \left. + e^{-2\pi i y (\log(t/2\pi)/L)} \left(\frac{L^2}{2\pi^2 y^2} \right) \Bigg) \right) dy \, dt.
\end{aligned} \tag{4.29}$$

Integrating over t, we find that

$$\begin{aligned}
\sum_{\gamma,\gamma' \leqslant T} g\left((\gamma - \gamma') \frac{L}{2\pi} \right) &\sim \frac{T}{2\pi} \log \frac{T}{2\pi} \left(g(0) + \int_{-\infty}^{\infty} g(y) \left(1 - \frac{1}{2\pi^2 y^2} + \frac{\cos(2\pi y)}{2\pi^2 y^2} \right) dy \right) \\
&= \frac{T}{2\pi} \log \frac{T}{2\pi} \left(g(0) + \int_{-\infty}^{\infty} g(y) \left(1 - \left(\frac{\sin \pi y}{\pi y} \right)^2 \right) dy \right).
\end{aligned} \tag{4.30}$$

The expression $1 - (\sin^2 \pi y)/(\pi y)^2$ is exactly the limiting two-point correlation function predicted by Montgomery [**36**].

5. *Mollifying second moments*

The technique of mollifying is used for computing information about zeros in families of L-functions, for example for obtaining lower bounds for the proportion of zeros on the critical line or for showing that not many L-functions in a family vanish at the central point. The general set up is that we have a family of L-functions to average over. Before performing the average we multiply by a Dirichlet polynomial whose coefficients arise from the inverses of the members of the family, multiplied by a smoothing function. We will compute one example arising from each of the three basic symmetry types.

As we discussed in the introduction, mollifier calculations are in general quite complicated. The ratios conjectures give a relatively easy way to obtain the relevant asymptotic formula. Thus, they can serve as a guide as to whether to embark on a calculation and a check as to whether a calculation is correct. They also provide evidence that mean-value formulas which can be proven for short mollifiers remain correct for long mollifiers. So, these calculations are valuable even though we assume the Riemann Hypothesis.

5.1. *A unitary example*

We start with the Riemann zeta function in t-aspect as a prototype of a unitary family. So, let

$$M(s,P)=\sum_{n\leqslant y}\frac{\mu(n)P\left(\log(y/n)/\log y\right)}{n^s} \tag{5.1}$$

where $\mu(n)$ is the Möbius function,

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}, \tag{5.2}$$

and P is a polynomial satisfying $P(0)=0$. Also,

$$y=T^{\theta} \tag{5.3}$$

where, classically, the following results have been proven for $\theta<1/2$, and, with a more modern treatment, for $\theta<4/7$ [**7**]. Conjecturally, the asymptotic formula we obtain should be valid for any fixed θ, no matter how large. We want to consider

$$I=\int_0^T|\zeta(1/2+it)|^2|M(1/2+it,P)|^2\,dt, \tag{5.4}$$

and more generally

$$I_\zeta(\alpha,\beta,P_1,P_2)=\int_0^T\zeta(s+\alpha)\zeta(1-s+\beta)M(s,P_1)M(1-s,P_2)\,dt, \tag{5.5}$$

where $s=1/2+it$. Also, it is useful to discuss the scaled and differentiated form of this quantity, namely,

$$I_\zeta(Q_1,Q_2,P_1,P_2):=Q_1\left(\frac{-1}{\log T}\frac{d}{d\alpha}\right)Q_2\left(\frac{-1}{\log T}\frac{d}{d\beta}\right)I_\zeta(\alpha,\beta,P_1,P_2)\Big|_{\alpha=\beta=0}, \tag{5.6}$$

for polynomials Q_1 and Q_2.

To relate this to our ratios conjecture we note that by Perron's formula

$$\frac{1}{2\pi i}\int_{(c)}x^z\frac{dz}{z^{m+1}}=\begin{cases}(\log^m x)/m! & \text{if } x>1,\\ 0 & \text{if } 0<x<1,\end{cases} \tag{5.7}$$

where $c>0$. Therefore, if $P(x)=\sum_{m\geqslant 1}p_m x^m$, then

$$M(s,P)=\sum_{m\geqslant 1}\frac{p_m m!}{\log^m y}\frac{1}{2\pi i}\int_{(c)}\frac{y^z}{z^{m+1}}\frac{1}{\zeta(s+z)}\,dz. \tag{5.8}$$

This expression leads us to

$$I_\zeta(\alpha,\beta,P_1,P_2) = \sum_{m,n} \frac{p_{1,m}m!p_{2,n}n!}{\log^{m+n} y} \frac{1}{(2\pi i)^2} \int_{(c_1)}\int_{(c_2)} \frac{y^{w+z}}{w^{m+1}z^{n+1}} R_\zeta(\alpha,\beta,w,z)\,dw\,dz, \quad (5.9)$$

where $c_1 = c_2 = 1/\log y$, and R_ζ is defined at (2.1). Using the ratios conjecture, Conjecture 2.1, we see that the double integral above is equal to

$$\begin{aligned}&\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)} \frac{y^{w+z}}{w^{m+1}z^{n+1}} \int_0^T \left(\frac{\zeta(1+\alpha+\beta)\zeta(1+w+z)}{\zeta(1+\alpha+z)\zeta(1+\beta+w)} A_\zeta(\alpha,\beta,w,z) \right.\\ &\left. + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta)\zeta(1+w+z)}{\zeta(1-\beta+z)\zeta(1-\alpha+w)} A_\zeta(-\beta,-\alpha,w,z)\right) dt\,dw\,dz + O(T^{1/2+\epsilon}). \quad (5.10)\end{aligned}$$

From this formula we could work out a precise conjecture with all lower order terms included. However, we are mainly interested in the leading order term when $\alpha,\beta \approx 1/\log T$. The leading order terms come from the residues of the poles in w and z at zero; to obtain these we use arguments similar to the proof of the Prime Number Theorem to move the paths of integration slightly to the left of zero, allowing us to replace the contours of z and w with circles of radius $1/\log T$ and $2/\log T$ respectively. The error term is then certainly $1/\log T$ smaller than the main term. Also we use $A = 1 + O(1/\log T)$ and $\zeta(1+x) = 1/x + O(1)$ for small x and large T. Then we have

$$\begin{aligned}I_\zeta(\alpha,\beta,P_1,P_2) &= \sum_{m,n} \frac{p_{1,m}m!p_{2,n}n!}{\log^{m+n} y} \frac{1}{(2\pi i)^2} \oint\oint \frac{y^{w+z}}{w^{m+1}z^{n+1}}\\ &\times \int_0^T \left(\frac{(\alpha+z)(\beta+w)}{(\alpha+\beta)(w+z)} + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{(-\beta+z)(-\alpha+w)}{(-\alpha-\beta)(w+z)} \right) dt\,dw\,dz\\ &+ O(T/\log T). \quad (5.11)\end{aligned}$$

It is convenient to write, for $\Re(w+z) > 0$,

$$\frac{y^{w+z}}{w+z} = \int_0^y u^{w+z} \frac{du}{u} \quad (5.12)$$

so that the above becomes

$$\begin{aligned}I_\zeta(\alpha,\beta,P_1,P_2) &= \frac{1}{\alpha+\beta} \sum_{m,n} \frac{p_{1,m}m!p_{2,n}n!}{\log^{m+n} y} \int_0^T \int_1^y \frac{1}{(2\pi i)^2} \oint\oint \frac{u^{w+z}}{w^{m+1}z^{n+1}}\\ &\times \left((\alpha+z)(\beta+w) - \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} (-\beta+z)(-\alpha+w) \right) dw\,dz\,\frac{du}{u}\,dt\\ &+ O(T/\log T); \quad (5.13)\end{aligned}$$

note that the integration in u is for $u \geqslant 1$ since for $u < 1$ the integrals in z and w are 0.

Now

$$\sum_m \frac{p_{1,m}m!}{\log^m y} \frac{1}{2\pi i} \oint \frac{u^w}{w^{m+1}}\,dw = P_1\left(\frac{\log u}{\log y}\right) \quad (5.14)$$

and

$$\sum_m \frac{p_{1,m}m!}{\log^m y} \frac{1}{2\pi i} \oint \frac{u^w}{w^m}\,dw = \frac{1}{\log y} P_1'\left(\frac{\log u}{\log y}\right). \quad (5.15)$$

Therefore,

$$I_\zeta(\alpha,\beta,P_1,P_2)=\frac{T}{\alpha+\beta}\int_1^y\bigg(\Big(\alpha+\frac{d}{dz}\Big)\Big(\beta+\frac{d}{dw}\Big)-T^{-\alpha-\beta}\Big(-\beta+\frac{d}{dz}\Big)\Big(-\alpha+\frac{d}{dw}\Big)\bigg)$$
$$\times P_1\Big(\frac{w+\log u}{\log y}\Big)P_2\Big(\frac{z+\log u}{\log y}\Big)\bigg|_{w=z=0}\frac{du}{u}+O(T/\log T). \tag{5.16}$$

Letting $u=y^r$, we deduce that

$$I_\zeta(\alpha,\beta,P_1,P_2)=\frac{T\log y}{\alpha+\beta}\bigg(\Big(\alpha+\frac{d}{dz}\Big)\Big(\beta+\frac{d}{dw}\Big)-T^{-\alpha-\beta}\Big(-\beta+\frac{d}{dz}\Big)\Big(-\alpha+\frac{d}{dw}\Big)\bigg)$$
$$\times\int_0^1 P_1\Big(\frac{w}{\log y}+r\Big)P_2\Big(\frac{z}{\log y}+r\Big)\,dr\bigg|_{w=z=0}+O(T/\log T). \tag{5.17}$$

It is useful to rewrite the main term of this as

$$\frac{T\log y(1-T^{-\alpha-\beta})}{\alpha+\beta}\Big(-\beta+\frac{d}{dz}\Big)\Big(-\alpha+\frac{d}{dw}\Big)\int_0^1 P_1\Big(\frac{w}{\log y}+r\Big)P_2\Big(\frac{z}{\log y}+r\Big)\,dr\bigg|_{w=z=0}$$
$$+\frac{T\log y}{\alpha+\beta}(\alpha+\beta)\Big(\frac{d}{dw}+\frac{d}{dz}\Big)\int_0^1 P_1\Big(\frac{w}{\log y}+r\Big)P_2\Big(\frac{z}{\log y}+r\Big)\,dr\bigg|_{w=z=0}. \tag{5.18}$$

The second term here is equal to $TP_1(1)P_2(1)$. For the first term, we write

$$\frac{1-T^{-\alpha-\beta}}{\alpha+\beta}=\log T\int_0^1 T^{-u(\alpha+\beta)}\,du \tag{5.19}$$

and note, for example, that $\log y(-\alpha+\frac{d}{dw})P_1(w/\log y+r)|_{w=0}=\frac{d}{dw}y^{-\alpha w}P_1(w+r)|_{w=0}$. Finally, recalling that $y=T^\theta$, we have

$$I_\zeta(\alpha,\beta,P_1,P_2)=TP_1(1)P_2(1)$$
$$+\frac{T}{\theta}\frac{d}{dw}\frac{d}{dz}y^{-\alpha w-\beta z}\int_0^1\int_0^1 T^{-(\alpha+\beta)u}P_1(w+r)P_2(z+r)\,dr\,du\bigg|_{w=z=0}$$
$$+O(T/\log T). \tag{5.20}$$

This formula appears in [**7**, p. 11]. To compute $I_\zeta(Q_1,Q_2,P_1,P_2)$ we observe, for example, that

$$Q_1\Big(\frac{-1}{\log T}\frac{d}{d\alpha}\Big)y^{-\alpha w}T^{-\alpha u}\bigg|_{\alpha=0}=Q_1(w\theta+u). \tag{5.21}$$

Thus, we have the following result.

THEOREM 5.1. *Let P_1, P_2, Q_1 and Q_2 be polynomials, with $P_1(0)=P_2(0)=0$. Assuming the ratios conjecture 2.1, for any fixed $\theta>0$, we have (using $s=1/2+it$)*

$$\frac{1}{T}\int_0^T Q_1\Big(\frac{-1}{\log T}\frac{d}{d\alpha}\Big)Q_2\Big(\frac{-1}{\log T}\frac{d}{d\beta}\Big)\zeta(s+\alpha)\zeta(1-s+\beta)M(s,P_1)M(1-s,P_2)\,dt\bigg|_{\alpha=\beta=0}$$
$$=P_1(1)P_2(1)Q_1(0)Q_2(0)$$
$$+\frac{d}{dw}\frac{d}{dz}\frac{1}{\theta}\int_0^1\int_0^1 P_1(w+r)P_2(z+r)Q_1(w\theta+u)Q_2(z\theta+u)\,dr\,du\bigg|_{w=z=0}+O(1/\log T)$$
$$=P_1(1)P_2(1)Q_1(0)Q_2(0)$$
$$+\frac{1}{\theta}\int_0^1\int_0^1\big(P_1'(r)Q_1(u)+\theta P_1(r)Q_1'(u)\big)\big(P_2'(r)Q_2(u)+\theta P_2(r)Q_2'(u)\big)\,dr\,du$$
$$+O(1/\log T). \tag{5.22}$$

As remarked earlier, if $\theta < 4/7$, then this is a theorem of [**7**] which generalizes work of Levinson [**33**]. Farmer [**16**] was the first to propose that this formula should hold for any fixed value of $\theta > 0$; he calls this the 'long mollifiers' conjecture. Other examples of mollifying a second moment in a unitary family are in [**17**, **25**, **35**].

5.2. *A symplectic example*

We consider mollifying in the family of L-functions $L(s,\chi_d)$ associated with real Dirichlet characters. Let

$$M(\chi_d,P) = \sum_{n\leqslant y} \frac{\mu(n)\chi_d(n)P\left(\frac{\log(y/n)}{\log y}\right)}{n^{1/2}}, \tag{5.23}$$

where P is a polynomial satisfying $P(0)=P'(0)=0$ and $y=X^\theta$. Consider the second mollified moment

$$\mathcal{M}(\alpha,\beta,P_1,P_2) = \sum_{d\leqslant X} L(1/2+\alpha,\chi_d)L(1/2+\beta,\chi_d)M(\chi_d,P_1)M(\chi_d,P_2). \tag{5.24}$$

As in our previous example, we can express

$$M(\chi_d,P) = \sum_n \frac{p_n n!}{\log^n y}\frac{1}{2\pi i}\int_{(c)} \frac{y^w}{L(1/2+w,\chi_d)w^{n+1}}\,dw, \tag{5.25}$$

where the p_n are the coefficients of the polynomial P. So, letting $p_{m,1}$ and $p_{n,2}$ be the coefficients of P_1 and P_2 we have

$$\begin{aligned}\mathcal{M}(\alpha,\beta,P_1,P_2) = \sum_{m,n} \frac{p_{m,1}m!p_{n,2}n!}{\log^{m+n} y}&\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)} \frac{y^{w+z}}{w^{m+1}z^{n+1}}\\ &\times \sum_{d\leqslant X}\frac{L(1/2+\alpha,\chi_d)L(1/2+\beta,\chi_d)}{L(1/2+w,\chi_d)L(1/2+z,\chi_d)}\,dw\,dz. \end{aligned}\tag{5.26}$$

For the sum over d we substitute from (2.39); we find that

$$\begin{aligned}&\frac{1}{X^*}\mathcal{M}(\alpha,\beta,P_1,P_2)\\ &\quad\sim \sum_{m,n}\frac{p_{m,1}m!p_{n,2}n!}{\log^{m+n} y}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}\frac{y^{w+z}}{w^{m+1}z^{n+1}}\\ &\quad\times\left(\frac{(\alpha+w)(\alpha+z)(\beta+w)(\beta+z)}{4\alpha\beta(\alpha+\beta)(w+z)} - X^{-\alpha}\frac{(-\alpha+w)(-\alpha+z)(\beta+w)(\beta+z)}{4\alpha\beta(-\alpha+\beta)(w+z)}\right.\\ &\quad - X^{-\beta}\frac{(\alpha+w)(\alpha+z)(-\beta+w)(-\beta+z)}{4\alpha\beta(\alpha-\beta)(w+z)}\\ &\quad\left. - X^{-\alpha-\beta}\frac{(-\alpha+w)(-\alpha+z)(-\beta+w)(-\beta+z)}{4\alpha\beta(\alpha+\beta)(w+z)}\right)dw\,dz. \end{aligned}\tag{5.27}$$

For simplicity from now on we write asymptotic formulas but, as in the previous section, they could all be replaced by equality with an error term that is one log smaller than the main term.

As before, we replace $y^{w+z}/(w+z)$ by $\int_1^y u^{w+z}\frac{du}{u}$. Then the poles are all at $w=0$ and $z=0$ and only the numerators in the last set of brackets depend on w and z. Removing the factor $(w+z)$ from the denominator, we expand this bracket into an expression that is a polynomial of total degree 4 in w and z with maximum degree 2 in each variable:

$$\begin{aligned}
&\frac{1}{4\alpha\beta}\left(\frac{1-X^{-\alpha-\beta}}{\alpha+\beta}+X^{-\alpha}\frac{1-X^{\alpha-\beta}}{\alpha-\beta}\right)w^2z^2+\frac{(1-X^{-\alpha})(1-X^{-\beta})}{4\alpha\beta}(w^2z+wz^2)\\
&\quad+\left(\frac{1-X^{-\alpha-\beta}}{4(\alpha+\beta)}-X^{-\alpha}\frac{1-X^{\alpha-\beta}}{4(\alpha-\beta)}\right)(w^2+z^2)\\
&\quad+\left(\frac{(\alpha+\beta)(1-X^{-\alpha-\beta})}{4\alpha\beta}+\frac{(\alpha-\beta)(X^{-\alpha}-X^{-\beta})}{4\alpha\beta}\right)wz+\frac{1}{4}(1+X^{-\alpha})(1+X^{-\beta})(w+z)\\
&\quad+\frac{\alpha\beta(1-X^{-\alpha-\beta})}{4(\alpha+\beta)}+X^{-\alpha}\frac{\alpha\beta(1-X^{\alpha-\beta})}{4(\alpha-\beta)}. \qquad (5.28)
\end{aligned}$$

Using the analogue of formulas (5.14) and (5.15), we see that we now should replace w^2z^2 in this expression by

$$\frac{1}{\log^4 y}\int_1^y P_1''\left(\frac{\log u}{\log y}\right)P_2''\left(\frac{\log u}{\log y}\right)\frac{du}{u}=\frac{1}{\log^3 y}\int_0^1 P_1''(r)P_2''(r)\,dr. \qquad (5.29)$$

Likewise, w^2z+wz^2 should be replaced by

$$\frac{1}{\log^2 y}\int_0^1\big(P_1''(r)P_2'(r)+P_1'(r)P_2''(r)\big)\,dr, \qquad (5.30)$$

w^2+z^2 by

$$\frac{1}{\log y}\int_0^1\big(P_1''(r)P_2(r)+P_1(r)P_2''(r)\big)\,dr, \qquad (5.31)$$

wz by

$$\frac{\int_0^1 P_1'(r)P_2'(r)\,dr}{\log y}, \qquad (5.32)$$

$w+z$ by

$$\int_0^1\big(P_1'(r)P_2(r)+P_1(r)P_2'(r)\big)\,dr, \qquad (5.33)$$

and the constant term by

$$\log y\int_0^1 P_1(r)P_2(r)\,dr. \qquad (5.34)$$

In this way, we find that

$$\begin{aligned}
\frac{4}{X^*}\mathcal{M}(\alpha,\beta,P_1,P_2)\sim{}&\frac{1}{\alpha\beta}\left(\frac{1-X^{-\alpha-\beta}}{\alpha+\beta}+X^{-\alpha}\frac{1-X^{\alpha-\beta}}{\alpha-\beta}\right)\frac{\int_0^1 P_1''(r)P_2''(r)\,dr}{\log^3 y}\\
&+\frac{(1-X^{-\alpha})(1-X^{-\beta})}{\alpha\beta}\frac{\int_0^1\big(P_1''(r)P_2'(r)+P_1'(r)P_2''(r)\big)\,dr}{\log^2 y}\\
&+\left(\frac{1-X^{-\alpha-\beta}}{(\alpha+\beta)}-X^{-\alpha}\frac{1-X^{\alpha-\beta}}{(\alpha-\beta)}\right)\frac{\int_0^1\big(P_1''(r)P_2(r)+P_1(r)P_2''(r)\big)\,dr}{\log y}\\
&+\left((1+X^{-\alpha})\frac{1-X^{-\beta}}{\beta}+(1+X^{-\beta})\frac{1-X^{-\alpha}}{\alpha}\right)\frac{\int_0^1 P_1'(r)P_2'(r)\,dr}{\log y}\\
&+(1+X^{-\alpha})(1+X^{-\beta})\int_0^1\big(P_1'(r)P_2(r)+P_1(r)P_2'(r)\big)\,dr\\
&+\left(\frac{\alpha\beta(1-X^{-\alpha-\beta})}{\alpha+\beta}+X^{-\alpha}\frac{\alpha\beta(1-X^{\alpha-\beta})}{\alpha-\beta}\right)\log y\int_0^1 P_1(r)P_2(r)\,dr. \qquad (5.35)
\end{aligned}$$

This gives our final formula for $\mathcal{M}(\alpha,\beta,P_1,P_2)$.

If, instead, we consider the mollified second moment of $\xi(1/2,\chi_d)$, we can put our answer into a more symmetric form. Recall that by the functional equation (2.26) we have

$$\xi(1/2+\alpha,\chi_d) := \left(\frac{d}{\pi}\right)^{\alpha/2}\Gamma\left(\frac{1}{4}+\frac{\alpha}{2}\right)L(1/2+\alpha,\chi_d) = \xi(1/2-\alpha,\chi_d). \tag{5.36}$$

Therefore, if we multiply (5.35) by $X^{(\alpha+\beta)/2}$, we will obtain the asymptotic formula for the mollified second moment of ξ:

$$\mathcal{N}(\alpha,\beta,P_1,P_2) := \sum_{d\leqslant X}\xi(1/2+\alpha,\chi_d)\xi(1/2+\beta,\chi_d)M(\chi_d,P_1)M(\chi_d,P_2). \tag{5.37}$$

We have

$$\begin{aligned}
&\frac{4}{X^*}\mathcal{N}(\alpha,\beta,P_1,P_2)\\
&\quad\sim \frac{1}{\alpha\beta}\left(\frac{X^{(\alpha+\beta)/2}-X^{-(\alpha+\beta)/2}}{\alpha+\beta}-\frac{X^{(\alpha-\beta)/2}-X^{(\beta-\alpha)/2}}{\alpha-\beta}\right)\frac{\int_0^1 P_1''(r)P_2''(r)\,dr}{\log^3 y}\\
&\qquad+\frac{(X^{\alpha/2}-X^{-\alpha/2})(X^{\beta/2}-X^{-\beta/2})}{\alpha\beta}\frac{\int_0^1\big(P_1''(r)P_2'(r)+P_1'(r)P_2''(r)\big)\,dr}{\log^2 y}\\
&\qquad+\left(\frac{X^{(\alpha+\beta)/2}-X^{-(\alpha+\beta)/2}}{(\alpha+\beta)}+\frac{X^{(\alpha-\beta)/2}-X^{(\beta-\alpha)/2}}{(\alpha-\beta)}\right)\frac{\int_0^1\big(P_1''(r)P_2(r)+P_1(r)P_2''(r)\big)\,dr}{\log y}\\
&\qquad+\left((X^{\alpha/2}+X^{-\alpha/2})\frac{X^{\beta/2}-X^{-\beta/2}}{\beta}+(X^{\beta/2}+X^{-\beta/2})\frac{X^{\alpha/2}-X^{-\alpha/2}}{\alpha}\right)\frac{\int_0^1 P_1'(r)P_2'(r)\,dr}{\log y}\\
&\qquad+(X^{\alpha/2}+X^{-\alpha/2})(X^{\beta/2}+X^{-\beta/2})\int_0^1\big(P_1'(r)P_2(r)+P_1(r)P_2'(r)\big)\,dr\\
&\qquad+\left(\frac{\alpha\beta(X^{(\alpha+\beta)/2}-X^{-(\alpha+\beta)/2})}{\alpha+\beta}-\frac{\alpha\beta(X^{(\alpha-\beta)/2}-X^{(\beta-\alpha)/2})}{\alpha-\beta}\right)\log y\int_0^1 P_1(r)P_2(r)\,dr.
\end{aligned} \tag{5.38}$$

We introduce a scaling, writing $\alpha=2a/\log X$ and $\beta=2b/\log X$. Then it is not difficult, remembering that $y=X^\theta$, to see that the above can be rewritten as

$$\begin{aligned}
\frac{4}{X^*}\mathcal{N}(\alpha,\beta,P_1,P_2) &\sim \frac{1}{2\theta^3}\int_0^1\frac{\sinh au}{a}\frac{\sinh bu}{b}\,du\int_0^1 P_1''(r)P_2''(r)\,dr\\
&\quad+\frac{1}{\theta^2}\frac{\sinh a}{a}\frac{\sinh b}{b}\int_0^1\big(P_1''(r)P_2'(r)+P_1'(r)P_2''(r)\big)\,dr\\
&\quad+\frac{2}{\theta}\int_0^1\cosh au\cosh bu\,du\int_0^1\big(P_1''(r)P_2(r)+P_1(r)P_2''(r)\big)\,dr\\
&\quad+\frac{2}{\theta}\left(\frac{\cosh a\sinh b}{b}+\frac{\cosh b\sinh a}{a}\right)\int_0^1 P_1'(r)P_2'(r)\,dr\\
&\quad+4\cosh a\cosh b\int_0^1\big(P_1'(r)P_2(r)+P_1(r)P_2'(r)\big)\,dr\\
&\quad+8\theta ab\int_0^1\sinh au\sinh bu\,du\int_0^1 P_1(r)P_2(r)\,dr.
\end{aligned} \tag{5.39}$$

We now apply $Q_1\left(\frac{d}{da}\right)Q_2\left(\frac{d}{db}\right)$ to this expression to obtain

$$\mathcal{N}(Q_1,Q_2,P_1,P_2) := Q_1\left(\frac{d}{da}\right)Q_2\left(\frac{d}{db}\right)\mathcal{N}\left(\frac{2a}{\log X},\frac{2b}{\log X},P_1,P_2\right)\Big|_{a=b=0}. \tag{5.40}$$

We may assume that Q_1 and Q_2 are even functions, since for an odd number r we have $\xi^{(r)}(1/2,\chi_d)=0$. To perform this calculation, we observe, for example, that

$$\begin{aligned}
&Q_1\left(\frac{d}{da}\right)Q_2\left(\frac{d}{db}\right)\int_0^1 \frac{\sinh au}{a}\frac{\sinh bu}{b}\,du\Big|_{a=b=0}\\
&= Q_1\left(\frac{d}{da}\right)Q_2\left(\frac{d}{db}\right)\int_0^1\int_0^u \cosh at_1\,dt_1\int_0^u \cosh bt_2\,dt_2\,du\Big|_{a=b=0}\\
&= \frac{1}{4}\int_0^1\int_0^u (Q_1(t_1)+Q_1(-t_1))\,dt_1\int_0^u (Q_2(t_2)+Q_2(-t_2))\,dt_2\,du\\
&= \int_0^1 \widetilde{Q_1}(u)\widetilde{Q_2}(u)\,du,
\end{aligned} \tag{5.41}$$

where we have used the notation

$$\widetilde{Q}(u) = \int_0^u Q(t)\,dt. \tag{5.42}$$

By similar, but easier, calculations we find that

$$\begin{aligned}
\frac{4}{X^*}\mathcal{N}(Q_1,Q_2,P_1,P_2) \sim{}& \frac{1}{2\theta^3}\int_0^1 \widetilde{Q_1}(u)\widetilde{Q_2}(u)\,du\int_0^1 P_1''P_2''(r)\,dr\\
&+\frac{1}{\theta^2}\widetilde{Q_1}(1)\widetilde{Q_2}(1)\int_0^1 \left(P_1''(r)P_2'(r)+P_1'(r)P_2''(r)\right)dr\\
&+\frac{2}{\theta}\int_0^1 Q_1(u)Q_2(u)\,du\int_0^1 \left(P_1''(r)P_2(r)+P_1(r)P_2''(r)\right)dr\\
&+\frac{2}{\theta}\left(Q_1(1)\widetilde{Q_2}(1)+\widetilde{Q_1}(1)Q_2(1)\right)\int_0^1 P_1'(r)P_2'(r)\,dr\\
&+4Q_1(1)Q_2(1)\int_0^1 \left(P_1'(r)P_2(r)+P_1(r)P_2'(r)\right)dr\\
&+8\theta\int_0^1 Q_1'(u)Q_2'(u)\,du\int_0^1 P_1(r)P_2(r)\,dr.
\end{aligned} \tag{5.43}$$

The right-hand side here can be written in a more compact form as

$$\begin{aligned}
&\frac{1}{2\theta}\int_0^1\int_0^1 \left(\frac{1}{\theta}P_1''(r)\widetilde{Q_1}(u)-4\theta P_1(r)Q_1'(u)\right)\left(\frac{1}{\theta}P_2''(r)\widetilde{Q_2}(u)-4\theta P_2(r)Q_2'(u)\right)du\,dr\\
&\qquad+\left(\frac{1}{\theta}P_1'(1)\widetilde{Q_1}(1)+2P_1(1)Q_1(1)\right)\left(\frac{1}{\theta}P_2'(1)\widetilde{Q_2}(1)+2P_2(1)Q_2(1)\right).
\end{aligned} \tag{5.44}$$

To verify this assertion we need to use identities which follow from integration-by-parts, such as

$$\int_0^1 (P_1'(r)P_2(r) + P_1(r)P_2'(r))\, dr = P_1(1)P_2(1), \tag{5.45}$$

$$\int_0^1 (P_1''(r)P_2'(r) + P_1'(r)P_2''(r))\, dr = P_1'(1)P_2'(1), \tag{5.46}$$

$$\int_0^1 P_1''(r)P_2(r)\, dr = P_1'(1)P_2(1) - \int_0^1 P_1'(r)P_2'(r)\, dr, \tag{5.47}$$

and

$$\int_0^1 \widetilde{Q_1}(u)Q_2'(u)\, du = \widetilde{Q_1}(1)Q_2(1) - \int_0^1 Q_1(u)Q_2(u)\, du. \tag{5.48}$$

In the last equation note that we have used $\widetilde{Q}(0) = 0$.

THEOREM 5.2. *Assuming Conjecture* 2.8, *we have for even polynomials Q_1 and Q_2, and polynomials P_1 and P_2 satisfying $P_1(0) = P_1'(0) = P_2(0) = P_2'(0) = 0$, and $y = X^\theta$ with any $\theta > 0$,*

$$\begin{aligned} &Q_1\left(\frac{2}{\log X}\frac{d}{d\alpha}\right) Q_2\left(\frac{2}{\log X}\frac{d}{d\beta}\right) \sum_{d\leqslant X} \xi(1/2+\alpha,\chi_d)\xi(1/2+\beta,\chi_d)M(\chi_d,P_1)M(\chi_d,P_2)\Big|_{\alpha=\beta=0} \\ &\quad = X^*\Bigg(\frac{1}{8\theta}\int_0^1\int_0^1 \left(\frac{1}{\theta}P_1''(r)\widetilde{Q_1}(u) - 4\theta P_1(r)Q_1'(u)\right)\left(\frac{1}{\theta}P_2''(r)\widetilde{Q_2}(u) - 4\theta P_2(r)Q_2'(u)\right) du\, dr \\ &\qquad + \frac{1}{4}\left(\frac{1}{\theta}P_1'(1)\widetilde{Q_1}(1) + 2P_1(1)Q_1(1)\right)\left(\frac{1}{\theta}P_2'(1)\widetilde{Q_2}(1) + 2P_2(1)Q_2(1)\right) \\ &\qquad + O(1/\log X)\Bigg). \end{aligned} \tag{5.49}$$

Examples of second moment mollifying in a symplectic family occur in [**40**] and [**15**].

5.3. *An orthogonal example*

Here we compute

$$\mathcal{M}_\Delta(\alpha,\beta;P_1,P_2) := \sum_{d\leqslant X} L_\Delta(1/2+\alpha,\chi_d)L_\Delta(1/2+\beta,\chi_d)M_\Delta(\chi_d,P_1)M_\Delta(\chi_d,P_2), \tag{5.50}$$

where

$$M_\Delta(\chi_d,P) := \sum_{m\leqslant y} \frac{\mu_\Delta(m)\chi_d(m)P\left(\frac{\log(y/m)}{\log y}\right)}{m^{1/2}}. \tag{5.51}$$

As in equation (5.26), we have

$$\begin{aligned} \mathcal{M}_\Delta(\alpha,\beta,P_1,P_2) = \sum_{m,n} \frac{p_{m,1}m!p_{n,2}n!}{\log^{m+n} y} \frac{1}{(2\pi i)^2} \int_{(c_1)}\int_{(c_2)} \frac{y^{w+z}}{w^{m+1}z^{n+1}} \\ \times \sum_{d\leqslant X} \frac{L_\Delta(1/2+\alpha,\chi_d)L_\Delta(1/2+\beta,\chi_d)}{L_\Delta(1/2+w,\chi_d)L_\Delta(1/2+z,\chi_d)}\, dw\, dz. \end{aligned} \tag{5.52}$$

Using (2.65) leads to

$$\begin{aligned}\mathcal{M}_\Delta(\alpha,\beta,P_1,P_2) \sim \frac{1}{X^*}\sum_{m,n}\frac{p_{m,1}m!p_{n,2}n!}{\log^{m+n} y}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}\frac{y^{w+z}}{w^{m+1}z^{n+1}4wz(w+z)}\\ \times\bigg(\frac{(\alpha+w)(\alpha+z)(\beta+w)(\beta+z)}{(\alpha+\beta)}\\ +X^{-2\alpha}\frac{(-\alpha+w)(-\alpha+z)(\beta+w)(\beta+z)}{(-\alpha+\beta)}\\ +X^{-2\beta}\frac{(\alpha+w)(\alpha+z)(-\beta+w)(-\beta+z)}{(\alpha-\beta)}\\ -X^{-2\alpha-2\beta}\frac{(-\alpha+w)(-\alpha+z)(-\beta+w)(-\beta+z)}{(\alpha+\beta)}\bigg)\,dw\,dz. \qquad (5.53)\end{aligned}$$

We expand the brackets into powers of w and z yielding

$$\begin{aligned}&\left(\frac{1-X^{-2\alpha-2\beta}}{\alpha+\beta}-\frac{X^{-2\alpha}-X^{-2\beta}}{\alpha-\beta}\right)w^2z^2\\ &+(1+X^{-2\alpha})(1+X^{-2\beta})(w^2z+wz^2)\\ &+\left(\frac{\alpha\beta(1-X^{-2\alpha-2\beta})}{\alpha+\beta}+\frac{\alpha\beta(X^{-2\alpha}-X^{-2\beta})}{\alpha-\beta}\right)(w^2+z^2)\\ &+\Big((\alpha+\beta)(1-X^{-2\alpha-2\beta})-(\alpha-\beta)(X^{-2\alpha}-X^{-2\beta})\Big)wz\\ &+\alpha\beta(1-X^{-2\alpha})(1-X^{-2\beta})(w+z)\\ &+\left(\frac{\alpha^2\beta^2(1-X^{-2\alpha-2\beta})}{\alpha+\beta}-\frac{\alpha^2\beta^2(X^{-2\alpha}-X^{-2\beta})}{\alpha-\beta}\right). \qquad (5.54)\end{aligned}$$

As we did in the other cases, we replace $y^{w+z}/(w+z)$ by $\int_1^y u^{w+z}\frac{du}{u}$. In a similar manner to (5.29), we evaluate the sums over m and n using

$$\begin{aligned}\sum_{m,n}\frac{p_{m,1}m!p_{n,2}n!}{\log^{m+n} y}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}\int_1^y\frac{u^{w+z}}{w^{m+1}z^{n+1}wz}\,\frac{du}{u}\,dw\,dz\\ =\log^2 y\int_1^y\widetilde{P_1}\left(\frac{\log u}{\log y}\right)\widetilde{P_2}\left(\frac{\log u}{\log y}\right)\frac{du}{u}, \qquad (5.55)\end{aligned}$$

$$\begin{aligned}\sum_{m,n}\frac{p_{m,1}m!p_{n,2}n!}{\log^{m+n} y}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}\int_1^y\frac{u^{w+z}}{w^{m+1}z^{n+1}}\,\frac{du}{u}\,dw\,dz\\ =\int_1^y P_1\left(\frac{\log u}{\log y}\right)P_2\left(\frac{\log u}{\log y}\right)\frac{du}{u}, \qquad (5.56)\end{aligned}$$

and, in general,

$$\begin{aligned}\sum_{m,n}\frac{p_{m,1}m!p_{n,2}n!}{\log^{m+n} y}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}\int_1^y\frac{u^{w+z}w^az^b}{w^{m+1}z^{n+1}wz}\,\frac{du}{u}\,dw\,dz\\ =(\log y)^{2-a-b}\int_1^y P_1^{(a-1)}\left(\frac{\log u}{\log y}\right)P_2^{(b-1)}\left(\frac{\log u}{\log y}\right)\frac{du}{u}\\ =(\log y)^{3-a-b}\int_0^1 P_1^{(a-1)}(t)P_2^{(b-1)}(t)\,dt, \qquad (5.57)\end{aligned}$$

where $P^{(a)}$ means the ath derivative of P; if $a<0$ then it means the $(-a)$th integral of P, so that, for example, $P^{(-1)}=\widetilde{P}$. Inputting this into our expression for $\mathcal{M}_\Delta$ leads to

$$\begin{aligned}
&\frac{4}{X^*}\mathcal{M}_\Delta(\alpha,\beta,P_1,P_2)\\
&\quad\sim \frac{1}{\log y}\left(\frac{1-X^{-2\alpha-2\beta}}{\alpha+\beta}-\frac{X^{-2\alpha}-X^{-2\beta}}{\alpha-\beta}\right)\int_0^1 P_1'(t)P_2'(t)\,dt\\
&\qquad+(1+X^{-2\alpha})(1+X^{-2\beta})\int_0^1 (P_1(t)P_2'(t)+P_1'(t)P_2(t))\,dt\\
&\qquad+\log y\left(\frac{\alpha\beta(1-X^{-2\alpha-2\beta})}{\alpha+\beta}+\frac{\alpha\beta(X^{-2\alpha}-X^{-2\beta})}{\alpha-\beta}\right)\int_0^1\left(P_1'(t)\widetilde{P_2}(t)+\widetilde{P_1}(t)P_2'(t)\right)\,dt\\
&\qquad+\log y\left((\alpha+\beta)(1-X^{-2\alpha-2\beta})-(\alpha-\beta)(X^{-2\alpha}-X^{-2\beta})\right)\int_0^1 P_1(t)P_2(t)\,dt\\
&\qquad+\log^2 y\;\alpha\beta(1-X^{-2\alpha})(1-X^{-2\beta})\int_0^1\left(\widetilde{P_1}(t)P_2(t)+P_1(t)\widetilde{P_2}(t)\right)\,dt\\
&\qquad+\log^3 y\;\left(\frac{\alpha^2\beta^2(1-X^{-2\alpha-2\beta})}{\alpha+\beta}-\frac{\alpha^2\beta^2(X^{-2\alpha}-X^{-2\beta})}{\alpha-\beta}\right)\int_0^1\widetilde{P_1}(t)\widetilde{P_2}(t)\,dt. \qquad (5.58)
\end{aligned}$$

We want to compare mollifying in an orthogonal family with that in a symplectic family. To this end, we consider, as we did for the symplectic family, mollifying the xi-functions. In this situation it just means multiplying the above result by $X^{\alpha+\beta}$. This gives

$$\begin{aligned}
&\frac{4}{X^*}\sum_{d\leqslant X}\xi_\Delta(1/2+\alpha,\chi_d)\xi_\Delta(1/2+\beta,\chi_d)M_\Delta(\chi_d,P_1)M_\Delta(\chi_d,P_2)\\
&\quad\sim \frac{1}{\log y}\left(\frac{X^{\alpha+\beta}-X^{-\alpha-\beta}}{\alpha+\beta}+\frac{X^{\alpha-\beta}-X^{\beta-\alpha}}{\alpha-\beta}\right)\int_0^1 P_1'(t)P_2'(t)\,dt\\
&\qquad+(X^{\alpha}+X^{-\alpha})(X^{\beta}+X^{-\beta})\int_0^1 (P_1(t)P_2'(t)+P_1'(t)P_2(t))\,dt\\
&\qquad+\log y\left(\frac{\alpha\beta(X^{\alpha+\beta}-X^{-\alpha-\beta})}{\alpha+\beta}-\frac{\alpha\beta(X^{\alpha-\beta}-X^{\beta-\alpha})}{\alpha-\beta}\right)\int_0^1\left(P_1'(t)\widetilde{P_2}(t)+\widetilde{P_1}(t)P_2'(t)\right)dt\\
&\qquad+\log y\left((\alpha+\beta)(X^{\alpha+\beta}-X^{-\alpha-\beta})+(\alpha-\beta)(X^{\alpha-\beta}-X^{\beta-\alpha})\right)\int_0^1 P_1(t)P_2(t)\,dt\\
&\qquad+\log^2 y\;\alpha\beta(X^{\alpha}-X^{-\alpha})(X^{\beta}-X^{-\beta})\int_0^1\left(\widetilde{P_1}(t)P_2(t)+P_1(t)\widetilde{P_2}(t)\right)dt\\
&\qquad+\log^3 y\;\left(\frac{\alpha^2\beta^2(X^{\alpha+\beta}-X^{-\alpha-\beta})}{\alpha+\beta}+\frac{\alpha^2\beta^2(X^{\alpha-\beta}-X^{\beta-\alpha})}{\alpha-\beta}\right)\int_0^1\widetilde{P_1}(t)\widetilde{P_2}(t)\,dt. \qquad (5.59)
\end{aligned}$$

If we now scale, letting $\alpha=a/\log X$ and $\beta=b/\log X$, and continuing our $y=X^\theta$ convention, then we can rewrite the above as

$$\begin{aligned}
&\frac{4}{X^*}\sum_{d\leqslant X}\xi_\Delta\left(\frac12+\frac{a}{\log X},\chi_d\right)\xi_\Delta\left(\frac12+\frac{b}{\log X},\chi_d\right)M_\Delta(\chi_d,P_1)M_\Delta(\chi_d,P_2)\\
&\quad\sim\frac{2}{\theta}\left(\frac{\sinh(a+b)}{a+b}+\frac{\sinh(a-b)}{a-b}\right)\int_0^1 P_1'(t)P_2'(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ 4\cosh a\cosh b\int_0^1 (P_1(t)P_2'(t) + P_1'(t)P_2(t))\,dt\\
&+ \theta\biggl(\frac{2ab\sinh(a+b)}{a+b} - \frac{2ab\sinh(a-b)}{a-b}\biggr)\int_0^1 \Bigl(P_1'(t)\widetilde{P_2}(t) + \widetilde{P_1}(t)P_2'(t)\Bigr)\,dt\\
&+ \theta\Bigl(2(a+b)\sinh(a+b) + 2(a-b)\sinh(a-b)\Bigr)\int_0^1 P_1(t)P_2(t)\,dt\\
&+ \theta^2\, 4ab\sinh a\sinh b\int_0^1 \Bigl(\widetilde{P_1}(t)P_2(t) + P_1(t)\widetilde{P_2}(t)\Bigr)\,dt\\
&+ \theta^3\left(\frac{2a^2b^2\sinh(a+b)}{a+b} + \frac{2a^2b^2\sinh(a-b)}{a-b}\right)\int_0^1 \widetilde{P_1}(t)\widetilde{P_2}(t)\,dt. \qquad (5.60)
\end{aligned}
$$

We now apply $Q_1(\frac{d}{da})Q_2(\frac{d}{db})|_{a=b=0}$ to both sides of this expression; we assume that Q_1 and Q_2 are even. We use the notation $\mathcal{M}_\Delta(Q_1,Q_2,P_1,P_2)$ as in the symplectic example. Thus,

$$
\begin{aligned}
\frac{4}{X^*}\mathcal{M}_\Delta(Q_1,Q_2,P_1,P_2) \sim{}& \frac{4}{\theta}\int_0^1 Q_1(u)Q_2(u)\,du\int_0^1 P_1'(t)P_2'(t)\,dt\\
&+ 4Q_1(1)Q_2(1)\int_0^1 (P_1(t)P_2'(t) + P_1'(t)P_2(t))\,dt\\
&+ 4\theta\int_0^1 Q_1'(u)Q_2'(u)\,du\int_0^1 \Bigl(P_1'(t)\widetilde{P_2}(t) + \widetilde{P_1}(t)P_2'(t)\Bigr)\,dt\\
&+ 4\theta(Q_1(1)Q_2'(1) + Q_1'(1)Q_2(1))\int_0^1 P_1(t)P_2(t)\,dt\\
&+ 4\theta^2\, Q_1'(1)Q_2'(1)\int_0^1 \Bigl(\widetilde{P_1}(t)P_2(t) + P_1(t)\widetilde{P_2}(t)\Bigr)\,dt\\
&+ 4\theta^3\int_0^1 Q_1''(u)Q_2''(u)\,du\int_0^1 \widetilde{P_1}(t)\widetilde{P_2}(t)\,dt. \qquad (5.61)
\end{aligned}
$$

This expression can be simplified to obtain the following theorem.

THEOREM 5.3. *Assuming Conjecture* 2.11, *with even polynomials* Q_1, Q_2, *and polynomials* P_1, P_2, *satisfying* $P_1(0) = P_2(0) = 0$, *and using* $y = X^\theta$, *we have for arbitrary* θ,

$$
\begin{aligned}
&\frac{1}{X^*}Q_1\left(\frac{1}{\log X}\frac{d}{d\alpha}\right)Q_2\left(\frac{1}{\log X}\frac{d}{d\beta}\right)\\
&\quad\times\sum_{d\leqslant X}\xi_\Delta(1/2+\alpha,\chi_d)\xi_\Delta(1/2+\beta,\chi_d)M_\Delta(\chi_d,P_1)M_\Delta(\chi_d,P_2)\Big|_{\alpha=\beta=0}\\
&\quad= \frac{1}{\theta}\int_0^1\int_0^1 \Bigl(P_1'(t)Q_1(u) - \theta^2\widetilde{P_1}(t)Q_1''(u)\Bigr)\Bigl(P_2'(t)Q_2(u) - \theta^2\widetilde{P_2}(t)Q_2''(u)\Bigr)\,dt\,du\\
&\qquad+ \Bigl(P_1(1)Q_1(1) + \theta\widetilde{P_1}(1)Q_1'(1)\Bigr)\Bigl(P_2(1)Q_2(1) + \theta\widetilde{P_2}(1)Q_2'(1)\Bigr)\\
&\qquad+ \theta\biggl(Q_1'(0)Q_2(0)\int_0^1 \widetilde{P}_1(t)P_2'(t)\,dt + Q_1(0)Q_2'(0)\int_0^1 P_1'(t)\widetilde{P}_2(t)\,dt\biggr)\\
&\qquad+ O(1/\log X). \qquad (5.62)
\end{aligned}
$$

Examples of second moment mollifying in an orthogonal family occur in [**26**] and [**31**]; in [**32**] a fourth moment mollification is performed.

6. *Mollifying the kth moment of* $\zeta(s)$

Chris Hughes has unpublished notes giving an asymptotic formula for

$$\int_0^T |\zeta(1/2+it)|^4 |A(1/2+it)|^2\,dt \tag{6.1}$$

where

$$A(s) = \sum_{n\leqslant y} \frac{a_n}{n^s} \tag{6.2}$$

is an arbitrary Dirichlet polynomial and where $y = T^\theta$ with $\theta < 5/27$. For applications to zeros of $\zeta(s)$ it would be extremely useful to specialize this formula to the case that $A(s) = M(s)$ is a mollifying polynomial, but this would still involve a lot of work. Via ratios we produce a conjectural formula which can serve as a check against the more complicated rigorous proof via Hughes' formula. There are (at least) two obvious choices for a mollifying polynomial $M(s)$. One is $M(s) = M_1(s,P)^2$ where

$$M_1(s,P) = \sum_{n\leqslant y} \frac{\mu(n) P\left(\frac{\log(y/n)}{\log y}\right)}{n^s} \tag{6.3}$$

with $y = T^\theta$. The other is $M(s) = M_2(s,P)$ with

$$M_2(s,P) = \sum_{n\leqslant y} \frac{\mu_2(n) P\left(\frac{\log(y/n)}{\log y}\right)}{n^s} \tag{6.4}$$

where $y = T^\theta$ and μ_2 is the coefficient in the generating function for $1/\zeta(s)^2$.

Here we will compute what the ratios conjecture tells us about the asymptotics for the kth mollified moments in the case where we mollify with $M_k(s,P)$, where $P(x) = \sum_m p_m x^m$ is a polynomial satisfying

$$P(0) = P'(0) = \ldots = P^{(k^2-1)}(0) = 0. \tag{6.5}$$

These conditions on $P(x)$ ensure that we have a smooth cut-off at $n = y$. It is only in the course of the calculation that we see why we need $k^2 - 1$ derivatives to be zero.

We note that

$$M_k(s,P) = \sum_{n\leqslant y} \frac{\mu_k(n) P\left(\frac{\log(y/n)}{\log y}\right)}{n^s} = \sum_m \frac{p_m m!}{(\log y)^m} \frac{1}{2\pi i} \int_{(c)} \frac{y^w}{\zeta^k(s+w) w^{m+1}}\,dw, \tag{6.6}$$

where μ_k is the coefficient in the generating function for $1/\zeta(s)^k$, $y = T^\theta$ and $c > 0$.

Thus, using $s = 1/2 + it$, we have

$$\begin{aligned} \mathcal{M}_k(\alpha,\beta) :=& \frac{1}{T}\int_0^T \zeta(s+\alpha_1)\ldots\zeta(s+\alpha_k)\zeta(1-s-\beta_1)\ldots \\ & \times \zeta(1-s-\beta_k) M_k(s,Q) M_k(1-s,P)\,dt \\ =& \sum_{m,n} \frac{q_m m! p_n n!}{\log^{m+n} y} \frac{1}{(2\pi i)^2} \int_{(c_1)}\int_{(c_2)} \frac{y^{w+z}}{w^{m+1} z^{n+1}} \\ & \times \frac{1}{T}\int_0^T \frac{\zeta(s+\alpha_1)\ldots\zeta(s+\alpha_k)\zeta(1-s-\beta_1)\ldots\zeta(1-s-\beta_k)}{\zeta(s+w)^k \zeta(1-s+z)^k}\,dt\,dw\,dz. \end{aligned} \tag{6.7}$$

Using the contour integral form of the ratios conjectures (see, for example, [**9**, Lemma 2.1] or [**10**, Lemma 2.5.1]; the sum of residues of this integral equals the $\binom{2k}{k}$ terms in the ratio conjectures as we have previously been writing them, in for example (2.10)), we see that the integral over t is asymptotic to

$$\frac{T^{(-\sum_{j=1}^{k}(\alpha_j-\beta_j))/2}}{(w+z)^{k^2}}\,\frac{(-1)^{2k(2k-1)/2}}{(2\pi i)^{2k}k!k!}\oint\cdots\oint\frac{G(v_1,\ldots,v_{2k})\Delta(v_1,\ldots,v_{2k})^2}{\prod_{i=1}^{2k}\prod_{j=1}^{k}(v_i-\alpha_j)(v_i-\beta_j)}\,dv_1\ldots dv_{2k} \qquad (6.8)$$

where

$$G(v_1,\ldots,v_k,v_{k+1},\ldots,v_{2k})=\frac{\prod_{j=1}^{k}(v_j+z)^k\ \prod_{j=1}^{k}(w-v_{j+k})^k}{D(v_{1+k},\ldots,v_{2k};v_1,\ldots,v_k)}T^{(\sum_{j=1}^{k}(v_j-v_{j+k}))/2}. \qquad (6.9)$$

Here $D(v_{k+1},\ldots,v_{2k};v_1,\ldots,v_k)=\prod_{j=1}^{k}\prod_{i=1}^{k}(v_j-v_{i+k})$.

Noting the identity

$$\frac{y^{w+z}}{(w+z)^A}=\frac{1}{(A-1)!}\int_0^y u^{w+z}\left(\log\frac{y}{u}\right)^{A-1}\frac{du}{u}, \qquad (6.10)$$

with $A=k^2$, we have

$$\begin{aligned}\mathcal{M}_k(\alpha,\beta)\sim\sum_{m,n}\frac{q_m m!p_n n!}{\log^{m+n}y}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}&\frac{T^{(-\sum_{j=1}^{k}(\alpha_j-\beta_j))/2}}{(k^2-1)!}\frac{(-1)^{2k(2k-1)/2}}{(2\pi i)^{2k}k!k!}\\&\times\int_0^y\oint\cdots\oint\frac{G(v_1,\ldots,v_{2k})\Delta(v_1,\ldots,v_{2k})^2}{\prod_{i=1}^{2k}\prod_{j=1}^{k}(v_i-\alpha_j)(v_i-\beta_j)}\,dv_1\ldots dv_{2k}\\&\times u^{w+z}\left(\log\frac{y}{u}\right)^{k^2-1}\frac{du}{u}\frac{dw\,dz}{w^{m+1}z^{n+1}}. \end{aligned} \qquad (6.11)$$

So, focusing on just the integrals over u, w and z, we have

$$\begin{aligned}&\int_{(c_1)}\int_{(c_2)}\int_0^y\prod_{j=1}^{k}(v_j+z)^k\ \prod_{j=1}^{k}(w-v_{j+k})^k\frac{u^{w+z}(\log(y/u))^{k^2-1}}{w^{m+1}z^{n+1}}\frac{du}{u}\,dw\,dz\\&\sim\int_{(c_1)}\int_{(c_2)}\log^{k^2}y\int_0^1 e^{\eta(w+z)\log y}(1-\eta)^{k^2-1}\,d\eta\\&\quad\times\frac{\prod_{j=1}^{k}(v_j+z)^k\ \prod_{j=1}^{k}(w-v_{j+k})^k}{w^{m+1}z^{n+1}}\,dw\,dz,\end{aligned} \qquad (6.12)$$

where the substitution was $\log u/\log y=\eta$ and we note that the part of the integral with $u<1$ will not contribute since for these values of u we can move the path of integration in w and z as far to the right as we like. Now let $y=T^\theta$, $\alpha_j=a_j/\log T=a_j\theta/\log y$, and similarly for b_j

and β_j, as well as making the replacements $w \to w/\log y$, $z \to z/\log y$ and $v_j \to v_j/\log y$. So we end up with

$$\begin{aligned}\mathcal{M}_k(a/\log T, b/\log T) \sim \sum_{m,n} q_m m! p_n n! \frac{1}{(2\pi i)^2} \frac{e^{(-\sum_{j=1}^k (a_j-b_j))/2}}{(k^2-1)!} \frac{(-1)^{2k(2k-1)/2}}{(2\pi i)^{2k} k!k!} \\ \times \oint \cdots \oint \int_{(c_1)} \int_{(c_2)} \int_0^1 e^{\eta(w+z)} (1-\eta)^{k^2-1}\, d\eta \\ \times \frac{\prod_{j=1}^k (v_j+z)^k \quad \prod_{j=1}^k (w-v_{j+k})^k}{D(v_{k+1},\ldots,v_{2k};v_1,\ldots,v_k) w^{m+1} z^{n+1}}\, dw\, dz \\ \times \frac{e^{(\sum_{j=1}^k (v_j-v_{j+k}))/2\theta} \Delta(v_1,\ldots,v_{2k})^2}{\prod_{i=1}^{2k}\prod_{j=1}^k (v_i-\theta a_j)(v_i-\theta b_j)} dv_1\ldots dv_{2k}. \qquad (6.13)\end{aligned}$$

Now we note that

$$\begin{aligned}&\prod_{j=1}^k (v_j+z)^k \quad \prod_{j=1}^k (w-v_{j+k})^k \qquad (6.14)\\ &\quad = \frac{d^k}{du_1^k} \cdots \frac{d^k}{du_{2k}^k} e^{u_1(v_1+z)+\ldots+u_k(v_k+z)+u_{k+1}(w-v_{k+1})+\ldots+u_{2k}(w-v_{2k})}\Big|_{u_1=\ldots=u_{2k}=0},\end{aligned}$$

and that

$$\frac{1}{2\pi i}\oint \frac{e^{aw}}{w^{m+1}}\, dw = \frac{a^m}{m!}, \qquad (6.15)$$

and use these to write

$$\begin{aligned}&\mathcal{M}_k(a/\log T, b/\log T)\\ &\quad \sim \frac{e^{(-\sum_{j=1}^k (a_j-b_j))/2}}{(k^2-1)!} \frac{(-1)^{2k(2k-1)/2}}{(2\pi i)^{2k} k!k!} \frac{d^k}{du_1^k} \cdots \frac{d^k}{du_{2k}^k} \int_0^1 (1-\eta)^{k^2-1}\\ &\quad \times Q(\eta+u_{k+1}+\ldots+u_{2k}) P(\eta+u_1+\ldots+u_k) \oint \cdots \oint e^{(\sum_{j=1}^k (v_j-v_{j+k}))/2\theta}\\ &\quad \times \frac{e^{u_1v_1+\ldots+u_kv_k-u_{k+1}v_{k+1}-\ldots-u_{2k}v_{2k}} \Delta(v_1,\ldots,v_{2k})^2}{D(v_{k+1},\ldots,v_{2k};v_1,\ldots,v_k)\prod_{i=1}^{2k}\prod_{j=1}^k (v_i-\theta a_j)(v_i-\theta b_j)}\\ &\quad \times dv_1\ldots dv_{2k}\, d\eta\Big|_{u_1=\ldots=u_{2k}=0}. \qquad (6.16)\end{aligned}$$

Now we concentrate on the contour integral over the v_j variables:

$$\begin{aligned}&I_v(u_1,\ldots,u_{2k})\\ &:= \frac{1}{(2\pi i)^{2k} k!k!} \oint \cdots \oint e^{(\frac{1}{2\theta}+u_1)v_1+(\frac{1}{2\theta}+u_2)v_2+\ldots+(\frac{1}{2\theta}+u_k)v_k-(\frac{1}{2\theta}+u_{k+1})v_{k+1}-\ldots-(\frac{1}{2\theta}+u_{2k})v_{2k}}\\ &\quad \times \frac{\Delta(v_1,\ldots,v_{2k})\Delta(v_1,\ldots,v_k)\Delta(v_{k+1},\ldots,v_{2k})}{D(\theta a_1,\ldots,\theta a_k;v_1,\ldots,v_{2k})D(\theta b_1,\ldots,\theta b_k;v_1,\ldots,v_{2k})}\, dv_1\ldots dv_{2k}. \qquad (6.17)\end{aligned}$$

Expanding the determinants $\Delta(z_1,\dots,z_k)=\det[z_j^{m-1}]_{1\leqslant j,m\leqslant k}$, we obtain

$$\begin{aligned}I_v=&\frac{1}{(2\pi i)^{2k}k!k!}\oint\dots\oint\frac{e^{(\frac{1}{2\theta}+u_1)v_1+(\frac{1}{2\theta}+u_2)v_2+\dots+(\frac{1}{2\theta}+u_k)v_k-(\frac{1}{2\theta}+u_{k+1})v_{k+1}-\dots-(\frac{1}{2\theta}+u_{2k})v_{2k}}}{D(\theta a_1,\dots,\theta a_k;v_1,\dots,v_{2k})D(\theta b_1,\dots,\theta b_k;v_1,\dots,v_{2k})}\\&\times\left(\sum_S\operatorname{sgn}(S)v_1^{S_0}v_2^{S_1}\dots v_k^{S_{k-1}}v_{k+1}^{S_k}\dots v_{2k}^{S_{2k-1}}\right)\left(\sum_Q\operatorname{sgn}(Q)v_1^{Q_0}\dots v_k^{Q_{k-1}}\right)\\&\times\left(\sum_R\operatorname{sgn}(R)v_{k+1}^{R_0}\dots v_{2k}^{R_{k-1}}\right)dv_1\dots dv_{2k}.\end{aligned}\tag{6.18}$$

Here Q and R are permutations of $\{0,1,\dots,k-1\}$ and S is a permutation of $\{0,1,\dots,2k-1\}$.

Since the integrand is symmetric amongst $v_1,\dots,v_k$ and also amongst $v_{k+1},\dots,v_{2k}$, in each term of the sum over Q we permute the variables $v_1,\dots,v_k$ so that v_j appears with the exponent $j-1$, for $j=1,\dots,k$. In the sum over S the effect is to redefine the permutations, and the additional sign involved with this exactly cancels $\operatorname{sgn}(Q)$. We do the same with the sum over R, and as a result we are left with $k!^2$ copies of the sum over the permutation S:

$$\begin{aligned}I_v=&\frac{1}{(2\pi i)^{2k}}\oint\dots\oint\frac{e^{(\frac{1}{2\theta}+u_1)v_1+(\frac{1}{2\theta}+u_2)v_2+\dots+(\frac{1}{2\theta}+u_k)v_k-(\frac{1}{2\theta}+u_{k+1})v_{k+1}-\dots-(\frac{1}{2\theta}+u_{2k})v_{2k}}}{D(\theta a_1,\dots,\theta a_k;v_1,\dots,v_{2k})D(\theta b_1,\dots,\theta b_k;v_1,\dots,v_{2k})}\\&\times\sum_S\operatorname{sgn}(S)v_1^{S_0}v_2^{S_1+1}\dots v_k^{S_{k-1}+(k-1)}v_{k+1}^{S_k}v_{k+2}^{S_{k+1}+1}\dots v_{2k}^{S_{2k-1}+(k-1)}\,dv_1\dots dv_{2k}.\end{aligned}\tag{6.19}$$

This can then be written as the following determinant, where we have written out the jth row, with $j=1,\dots,2k$:

$$\begin{aligned}I_v=\det\Bigg\{&\frac{1}{2\pi i}\oint\frac{e^{(\frac{1}{2\theta}+u_1)v_1}v_1^{j-1}}{\prod_{i=1}^k(v_1-\theta a_i)(v_1-\theta b_i)}dv_1,\ \frac{1}{2\pi i}\oint\frac{e^{(\frac{1}{2\theta}+u_2)v_2}v_2^{j}}{\prod_{i=1}^k(v_2-\theta a_i)(v_2-\theta b_i)}dv_2,\dots,\\&\frac{1}{2\pi i}\oint\frac{e^{(\frac{1}{2\theta}+u_k)v_k}v_k^{j+k-2}}{\prod_{i=1}^k(v_k-\theta a_i)(v_k-\theta b_i)}dv_k,\\&\frac{1}{2\pi i}\oint\frac{e^{-(\frac{1}{2\theta}+u_{k+1})v_{k+1}}v_{k+1}^{j-1}}{\prod_{i=1}^k(v_{k+1}-\theta a_i)(v_{k+1}-\theta b_i)}dv_{k+1},\\&\frac{1}{2\pi i}\oint\frac{e^{-(\frac{1}{2\theta}+u_{k+2})v_{k+2}}v_{k+2}^{j}}{\prod_{i=1}^k(v_{k+2}-\theta a_i)(v_{k+2}-\theta b_i)}dv_{k+2},\dots,\\&\frac{1}{2\pi i}\oint\frac{e^{-(\frac{1}{2\theta}+u_{2k})v_{2k}}v_{2k}^{j+k-2}}{\prod_{i=1}^k(v_{2k}-\theta a_i)(v_{2k}-\theta b_i)}dv_{2k}\Bigg\}.\end{aligned}\tag{6.20}$$

Setting the a and b variables equal to zero and performing the integration, we have (for integer n)

$$\frac{1}{2\pi i}\oint e^{bv}v^n\,dv=\begin{cases}0 & \text{for } n\geqslant 0,\\ \dfrac{b^{-n-1}}{(-n-1)!} & \text{for } n<0,\end{cases}\tag{6.21}$$

giving us

$$I_v(\vec{u}) = \det \begin{pmatrix} \frac{(\frac{1}{2\theta}+u_1)^{2k-1}}{(2k-1)!} & \cdots & \frac{(\frac{1}{2\theta}+u_k)^{k}}{k!} & \frac{-(\frac{1}{2\theta}+u_{k+1})^{2k-1}}{(2k-1)!} & \cdots & \frac{(-1)^k(\frac{1}{2\theta}+u_{2k})^{k}}{k!} \\ \frac{(\frac{1}{2\theta}+u_1)^{2k-2}}{(2k-2)!} & \cdots & \frac{(\frac{1}{2\theta}+u_k)^{k-1}}{(k-1)!} & \frac{(\frac{1}{2\theta}+u_{k+1})^{2k-2}}{(2k-2)!} & \cdots & \frac{(-1)^{k-1}(\frac{1}{2\theta}+u_{2k})^{k-1}}{(k-1)!} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{(\frac{1}{2\theta}+u_1)^{0}}{0!} & \cdots & \frac{(\frac{1}{2\theta}+u_k)^{1-k}}{(1-k)!} & \frac{(\frac{1}{2\theta}+u_{k+1})^{0}}{0!} & \cdots & \frac{(-1)^{k-1}(\frac{1}{2\theta}+u_{2k})^{1-k}}{(1-k)!} \end{pmatrix}. \tag{6.22}$$

Note that many of the lower matrix entries are zero as we apply the convention that $1/n! = 0$ for integer $n < 0$.

So, the mollified kth moment with the α and β variables set to zero now has the form

$$\begin{aligned} \mathcal{M}_k(0,0) \sim \frac{(-1)^{2k(2k-1)/2}}{(k^2-1)!} \frac{d^k}{du_1^k} \cdots \frac{d^k}{du_{2k}^k} \int_0^1 (1-\eta)^{k^2-1} I_v(\vec{u}) \\ \times Q(\eta + u_{k+1} + \ldots + u_{2k}) P(\eta + u_1 + \ldots + u_k)\, d\eta \Big|_{u_1=\ldots=u_{2k}=0}. \end{aligned} \tag{6.23}$$

The differentiation with respect to the u variables is not difficult, but we will now restrict ourselves to the case $k = 2$ where we can write the result fairly concisely. In this case we have

$$\begin{aligned} \mathcal{M}_2(0,0) \sim \frac{1}{3!} \frac{d^2}{du_1^2} \cdots \frac{d^2}{du_4^2} \int_0^1 (1-\eta)^3 Q(\eta + u_3 + u_4) P(\eta + u_1 + u_2)\, d\eta \\ \times \det \begin{pmatrix} \frac{(\frac{1}{2\theta}+u_1)^3}{3!} & \frac{(\frac{1}{2\theta}+u_2)^2}{2!} & \frac{-(\frac{1}{2\theta}+u_3)^3}{3!} & \frac{(\frac{1}{2\theta}+u_4)^2}{2!} \\ \frac{(\frac{1}{2\theta}+u_1)^2}{2!} & (\frac{1}{2\theta}+u_2) & \frac{(\frac{1}{2\theta}+u_3)^2}{2!} & -(\frac{1}{2\theta}+u_4) \\ (\frac{1}{2\theta}+u_1) & 1 & -(\frac{1}{2\theta}+u_3) & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \Bigg|_{u_1=\ldots=u_4=0}. \end{aligned} \tag{6.24}$$

Performing the differentiation leads to

$$\begin{aligned} \mathcal{M}_2(0,0) \sim \int_0^1 \frac{(1-\eta)^3}{6} \Big(& Q(\eta)P^{(4)}(\eta) + Q^{(4)}(\eta)P(\eta) + 4(Q^{(3)}(\eta)P'(\eta) + Q'(\eta)P^{(3)}(\eta)) \\ & + 6Q''(\eta)P''(\eta) + \frac{2}{\theta}(Q^{(4)}(\eta)P'(\eta) + Q'(\eta)P^{(4)}(\eta)) \\ & + \frac{8}{\theta}(Q^{(3)}(\eta)P''(\eta) + Q''(\eta)P^{(3)}(\eta)) \\ & + \frac{2}{\theta^2}(Q^{(4)}(\eta)P''(\eta) + Q''(\eta)P^{(4)}(\eta)) + \frac{4}{\theta^2}Q^{(3)}(\eta)P^{(3)}(\eta) \\ & + \frac{2}{3\theta^3}(Q^{(4)}(\eta)P^{(3)}(\eta) + Q^{(3)}(\eta)P^{(4)}(\eta)) + \frac{1}{12\theta^4}Q^{(4)}(\eta)P^{(4)}(\eta) \Big)\, d\eta. \end{aligned} \tag{6.25}$$

Integration by parts gives the following.

THEOREM 6.1. *Assuming the ratios conjecture as indicated in* (6.8), *if Q and P are polynomials which vanish at* 0 *and whose first three derivatives vanish at* 0, *then for any* $\theta > 0$ *we have*

$$\begin{aligned}
&\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 M_2(1/2+it,Q)M_2(1/2-it,P)\,dt\\
&= P(1)Q(1) + \frac{1}{\theta}\int_0^1 \frac{(1-\eta)^3}{6}\bigg(2(Q^{(4)}(\eta)P'(\eta)+Q'(\eta)P^{(4)}(\eta))\\
&\quad + 8(Q^{(3)}(\eta)P''(\eta)+Q''(\eta)P^{(3)}(\eta)) + \frac{2}{\theta}(Q^{(4)}(\eta)P''(\eta)+Q''(\eta)P^{(4)}(\eta))\\
&\quad + \frac{4}{\theta}Q^{(3)}(\eta)P^{(3)}(\eta) + \frac{2}{3\theta^2}(Q^{(4)}(\eta)P^{(3)}(\eta)+Q^{(3)}(\eta)P^{(4)}(\eta))\\
&\quad + \frac{1}{12\theta^3}Q^{(4)}(\eta)P^{(4)}(\eta)\bigg)\,d\eta + O(1/\log T).
\end{aligned} \tag{6.26}$$

REMARK 6.2. While we do not know which are the minimizing polynomials, with the choice that $P(x) = Q(x) = x^4$, the right side of (6.26) is equal to

$$1 + \frac{208}{35\theta} + \frac{48}{5\theta^2} + \frac{32}{5\theta^3} + \frac{2}{\theta^4}. \tag{6.27}$$

By a similar calculation one can show that with polynomials P_i and Q_j satisfying $P_i(0) = P_i'(0) = Q_j(0) = Q_j'(0) = 0$, we have

$$\begin{aligned}
&\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 M_1(s,P_1)M_1(s,P_2)M_1(1-s,Q_1)M_1(1-s,Q_2)\,dt\\
&= \frac{1}{16}\frac{d}{du_1}\cdots\frac{d}{du_4}\frac{d}{dU_1}\cdots\frac{d}{dU_4}\iiiint_{\mathcal{R}} P_1\left(\frac{\eta_1}{2}+\frac{\eta_2}{2}+u_3+u_4\right)P_2\left(\frac{\eta_3}{2}+\frac{\eta_4}{2}+U_3+U_4\right)\\
&\quad\times Q_1\left(\frac{\eta_1}{2}+\frac{\eta_3}{2}+u_1+u_2\right)Q_2\left(\frac{\eta_2}{2}+\frac{\eta_4}{2}+U_1+U_2\right)d\eta_1\ldots d\eta_4\\
&\quad\times I_v(u_1+U_1,u_2+U_2,u_3+U_3,u_4+U_4)\big|_{u_1=\ldots=u_4=U_1=\ldots=U_4=0}\\
&\quad + O(1/\log T),
\end{aligned} \tag{6.28}$$

where $\mathcal{R}$ is the subset of $[-1,1]^4$ for which $\eta_1+\eta_2 \geqslant 0$, $\eta_3+\eta_4 \geqslant 0$, $\eta_1+\eta_3 \geqslant 0$ and $\eta_2+\eta_4 \geqslant 0$.

REMARK 6.3. While we do not know which are the minimizing polynomials, with the choice that $P_1(x) = P_2(x) = Q_1(x) = Q_2(x) = x^2$, the right side of (6.28) is equal to

$$1 + \frac{68}{21\theta} + \frac{10}{3\theta^2} + \frac{64}{45\theta^3} + \frac{2}{9\theta^4}. \tag{6.29}$$

6.1. *A third power mollification*

In the work of Hughes alluded to earlier it is likely that, in applications to zeros of $\zeta(s)$ on the critical line, the moment

$$\int_0^T |\zeta(1/2+it)|^2 |M_1(1/2+it,P_1) + \zeta(1/2+it)M_2(1/2+it,P_2)|^2\,dt \tag{6.30}$$

will need to be evaluated. Here, as at (6.3) and (6.4), M_1 is a mollifier with arithmetic coefficients $\mu(m)$ smoothed by P_1 and

$$M_2(s,P_2) := \sum_{m\leqslant y_2} \frac{\mu_2(m)P_2\left(\frac{y_2/m}{\log y_2}\right)}{m^s}, \tag{6.31}$$

where the μ_2 are the coefficients of $1/\zeta(s)^2$. In order to evaluate this integral we can use the results (5.20) and Theorem 6.1; in addition to these we need to evaluate

$$I_3(\alpha,\beta,\gamma;P_1,P_2) := \int_0^T \zeta(s+\alpha)\zeta(s+\beta)\zeta(1-s+\gamma)M_1(1-s,P_1)M_2(s,P_2)\,dt. \tag{6.32}$$

Proceeding as usual, we have

$$\begin{aligned} I_3 = \sum_{m,n} &\frac{p_{1,m}m!p_{2,n}n!}{\log^m y_1 \log^n y_2}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)} \frac{y^{w+z}}{w^{m+1}z^{n+1}} \\ &\times \int_0^T \frac{\zeta(s+\alpha)\zeta(s+\beta)\zeta(1-s+\gamma)}{\zeta(s+w)^2\zeta(1-s+z)}\,dt\,dw\,dz. \end{aligned} \tag{6.33}$$

By the ratios conjecture,

$$\begin{aligned} I_3 \sim T\sum_{m,n} &\frac{p_{1,m}m!p_{2,n}n!}{\log^m y_1 \log^n y_2}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)} \frac{y_1^w y_2^z}{w^{m+1}z^{n+1}(w+z)^2} \\ &\times\left(\frac{(\alpha+z)(\beta+z)(\gamma+w)^2}{(\alpha+\gamma)(\beta+\gamma)} - T^{-\alpha-\gamma}\frac{(-\gamma+z)(\beta+z)(-\alpha+w)^2}{(\alpha+\gamma)(\beta-\alpha)}\right. \\ &\left. - T^{-\beta-\gamma}\frac{(\alpha+z)(-\gamma+z)(-\beta+w)^2}{(\alpha-\beta)(\beta+\gamma)}\right)dw\,dz. \end{aligned} \tag{6.34}$$

Taking the limit (MATHEMATICA can be helpful here) as $\alpha,\beta,\gamma\to 0$, we have

$$\begin{aligned} I_3 \sim T\sum_{m,n} &\frac{p_{1,m}m!p_{2,n}n!}{\log^m y_1 \log^n y_2}\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)} \frac{y_1^z y_2^w}{z^{m+1}w^{n+1}(w+z)^2} \\ &\times \left((w+z)^2 + w^2zL + 2wz^2L + w^2z^2L^2/2\right)dw\,dz. \end{aligned} \tag{6.35}$$

The contribution of the $(w+z)^2$ term at the beginning of the brackets above is $P_1(1)P_2(1)$, essentially using (5.57). For the rest of the terms, we write, using (6.10),

$$\frac{y_1^z y_2^w}{(w+z)^2} = \frac{y_2^{w+z}(y_1/y_2)^z}{(w+z)^2} = (y_1/y_2)^z\int_0^{y_2} u^{w+z}\log\frac{y_2}{u}\frac{du}{u}. \tag{6.36}$$

The integral over w is 0 unless $u\geqslant 1$; the integral over z is 0 unless $uy_1/y_2 > 1$; this inequality is weaker than the requirement that $u>1$, since, in general $y_1 > y_2$. In this way, we see that

$$\begin{aligned} I_3 \sim T\Bigg(P_1(1)P_2(1) + \int_1^{y_2}\Bigg(&\frac{L}{\log y_1\log^2 y_2}P_1'\left(\frac{\log(y_1u/y_2)}{\log y_1}\right)P_2''\left(\frac{\log u}{\log y_2}\right) \\ &+ \frac{2L}{\log^2 y_1\log y_2}P_1''\left(\frac{\log(y_1u/y_2)}{\log y_1}\right)P_2'\left(\frac{\log u}{\log y_2}\right) \\ &+ \frac{L^2}{2\log^2 y_1\log^2 y_2}P_1''\left(\frac{\log(y_1u/y_2)}{\log y_1}\right)P_2''\left(\frac{\log u}{\log y_2}\right)\Bigg)\log\frac{y_2}{u}\,du\Bigg). \end{aligned} \tag{6.37}$$

With a change of variables $u=y_2^\eta$, and with $y_1 = T^{\theta_1}$ and $y_2 = T^{\theta_2}$, we see that the following holds.

THEOREM 6.4. *Let I_3 be as defined in* (6.32). *Assuming the ratios conjecture as indicated in* (6.8), *if P_1 and P_2 are polynomials which vanish at 0 and whose first derivatives vanish at 0, then for any $\theta > 0$ we have*

$$I_3(0,0,0;P_1,P_2) = T\bigg(P_1(1)P_2(1) + \int_0^1 \bigg(\frac{1}{\theta_1}P_1'\big(1+(1-\eta)\theta_2/\theta_1\big)P_2''(\eta) + \frac{2\theta_2}{\theta_1^2}P_1''\big(1+(1-\eta)\theta_2/\theta_1\big)P_2'(\eta) + \frac{1}{2\theta_1^2}P_1''\big(1+(1-\eta)\theta_2/\theta_1\big)P_2''(\eta)\bigg)(1-\eta)\,d\eta + O(1/\log T)\bigg). \tag{6.38}$$

7. *Discrete moments of the Riemann zeta function and its derivatives*

So far in this paper we have considered integer moments. Another kind of average which gives useful information about the distribution of zeros is a discrete moment summing the zeta function, or its derivatives, at or near the zeros.

In the 1980s Gonek [**20**], assuming the Riemann Hypothesis, proved, amongst much more general results, that

$$\sum_{1\leqslant\gamma\leqslant T} |\zeta'(\rho)|^2 = \frac{T}{24\pi}\log^4 T + O(T\log^3 T), \tag{7.1}$$

where $\rho = 1/2 + i\gamma$ is a zero of the Riemann zeta function.

Hughes, Keating and O'Connell used the analogy with random matrix theory to propose the following conjecture.

CONJECTURE 7.1 (Hughes, Keating and O'Connell [**23**]). For $k > -3/2$ and bounded,

$$\sum_{0<\gamma_n\leqslant T} |\zeta'(\tfrac12 + i\gamma_n)|^{2k} \sim \frac{T}{2\pi}\frac{G^2(k+2)}{G(2k+3)}a(k)(\log T)^{k(k+2)+1} \tag{7.2}$$

as $T\to\infty$, with

$$a(k) = \prod_p \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty}\left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}. \tag{7.3}$$

Here $G(k)$ is the Barnes G-function.

While the above conjecture produces the leading order terms, it was previously not known how to obtain lower order terms for the moments of $|\zeta'(\rho)|$. In the first two subsections we consider the second and fourth moments of $|\zeta'(\rho)|$, and in the last subsection the second moment of $|\zeta(\rho+a)|$. Using the ratios conjecture we show how to obtain all of the lower order terms for these averages.

Note that Hughes [**22**] has conjectured the following, using random matrix theory.

CONJECTURE 7.2. One has

$$\sum_{0<\gamma_n\leqslant T}\left|\zeta\left(\rho + 2\pi i\alpha\log^{-1}\frac{T}{2\pi}\right)\right|^{2k} \sim \frac{T}{2\pi}G^2(k+1)/G(2k+1)a(k)F_k(2\pi\alpha)(\log T)^{k^2+1}, \tag{7.4}$$

where

$$F_k(2x) = \frac{\pi}{2}(xJ_{k+1/2}(x)^2 + xJ_{k-1/2}(x)^2 - 2kJ_{k+1/2}(x)J_{k-1/2}(x)). \tag{7.5}$$

Here J_k is the usual Bessel function.

7.1. *Second moment of the derivative:* $\sum_{1\leqslant\gamma\leqslant T}|\zeta'(\rho)|^2$

In this section we will show how to use the ratios conjecture to reproduce the result of Gonek (7.1) and to derive all the lower order terms.

The first step is to write the sum over zeros as a contour integral

$$\sum_{\gamma<T}|\zeta'(\rho)|^2=\frac{1}{2\pi i}\int_C\frac{\zeta'(z)}{\zeta(z)}\zeta'(z)\zeta'(1-z)\,dz, \tag{7.6}$$

where the contour C has corners c, $c+iT$, $1-c+iT$ and $1-c$, with $1/2<c<1$. The integrals along the horizontal sides of this rectangle can be neglected, and so we look first at

$$\begin{aligned}I_r&=\frac{1}{2\pi i}\int_c^{c+iT}\frac{\zeta'(z)}{\zeta(z)}\zeta'(z)\zeta'(1-z)\,dz\\&=\frac{1}{2\pi}\int_0^T\frac{\zeta'(c+it)}{\zeta(c+it)}\zeta'(c+it)\zeta'(1-c-it)\,dt\\&=\frac{d}{d\beta}\frac{d}{d\gamma}\frac{d}{d\delta}\frac{1}{2\pi}\int_0^T\frac{\zeta(c+it+\beta)}{\zeta(c+it)}\zeta(c+it+\gamma)\zeta(1-c-it+\delta)\,dt\Bigg|_{\beta=\gamma=\delta=0}.\end{aligned} \tag{7.7}$$

Now we follow the recipe for computing the ratio of zeta functions in the integrand. As in Section 1, we replace the zeta functions in the numerator by

$$\zeta(s)\sim\sum_{n\leqslant\sqrt{t/2\pi}}\frac{1}{n^s}+\chi(s)\sum_{n\leqslant\sqrt{t/2\pi}}\frac{1}{n^{1-s}} \tag{7.8}$$

and the zeta functions in the denominator by

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}. \tag{7.9}$$

Since each of the zeta functions in the numerator is replaced by two sums, when multiplied out we have a total of eight terms in the integral. Considering the term involving the first term from each approximate functional equation, the resulting sum is

$$\begin{aligned}\sum_{hmn=\ell}\frac{\mu(h)}{m^{1/2+\beta}n^{1/2+\gamma}h^{1/2}\ell^{1/2+\delta}}&=\sum_{h,m,n}\frac{\mu(h)}{m^{1+\beta+\delta}n^{1+\gamma+\delta}h^{1+\delta}}\\&=\frac{\zeta(1+\beta+\delta)\zeta(1+\gamma+\delta)}{\zeta(1+\delta)}.\end{aligned} \tag{7.10}$$

Of the eight terms in the integrand, only those with the same number of χ factors resulting from $\zeta(z)$ as from $\zeta(1-z)$ will survive. This means that the recipe implies two further terms, and using (2.9), we have

$$\begin{aligned}I_r=\frac{d}{d\beta}\frac{d}{d\gamma}\frac{d}{d\delta}\frac{1}{2\pi}\int_0^T&\Bigg(\frac{\zeta(1+\beta+\delta)\zeta(1+\gamma+\delta)}{\zeta(1+\delta)}+\left(\frac{t}{2\pi}\right)^{-\beta-\delta}\frac{\zeta(1-\delta-\beta)\zeta(1+\gamma-\beta)}{\zeta(1-\beta)}\\&+\left(\frac{t}{2\pi}\right)^{-\gamma-\delta}\frac{\zeta(1+\beta-\gamma)\zeta(1-\delta-\gamma)}{\zeta(1-\gamma)}\Bigg)(1+O(t^{-1/2+\epsilon}))\,dt\Bigg|_{\beta=\gamma=\delta=0}.\end{aligned} \tag{7.11}$$

We now consider the contribution from the other side of the contour of integration:

$$I_\ell=\frac{1}{2\pi i}\int_{1-c+iT}^{1-c}\frac{\zeta'(z)}{\zeta(z)}\zeta'(z)\zeta'(1-z)\,dz. \tag{7.12}$$

Replacing z with $1-z$, we have

$$I_\ell = \frac{-1}{2\pi i}\int_{c-iT}^{c} \frac{\zeta'(1-z)}{\zeta(1-z)}\zeta'(1-z)\zeta'(z)\,dz. \tag{7.13}$$

Differentiating the functional equation gives us

$$\frac{\zeta'}{\zeta}(1-z) = \frac{\chi'}{\chi}(1-z) - \frac{\zeta'}{\zeta}(z), \tag{7.14}$$

and so

$$I_\ell = \frac{-1}{2\pi i}\int_{c-iT}^{c} \frac{\chi'(1-z)}{\chi(1-z)}\zeta'(1-z)\zeta'(z)\,dz + \frac{1}{2\pi i}\int_{c-iT}^{c} \frac{\zeta'(z)}{\zeta(z)}\zeta'(z)\zeta'(1-z)\,dz. \tag{7.15}$$

Here the first integral can be shifted over to the half-line, while the second one is just the complex conjugate of I_r, already calculated in (7.11). In addition, we can use

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{1/2-s} e^{it+\pi i/4}\left(1+O\left(\frac{1}{t}\right)\right) \tag{7.16}$$

to approximate $\chi'/\chi(1-z)$ with $-\log t/2\pi$. Thus we have

$$\begin{aligned}
\sum_{\gamma<T} |\zeta'(\rho)|^2 &= 2I_r + \frac{1}{2\pi}\int_0^T \log\frac{t}{2\pi}\, |\zeta'(1/2+it)|^2(1+O(t^{-1}))\,dt \\
&= 2I_r + \frac{d}{d\alpha}\frac{d}{d\beta}\frac{1}{2\pi}\int_0^T \log\frac{t}{2\pi}\,\zeta(1/2+it+\alpha)\zeta(1/2-it+\beta)(1+O(t^{-1}))\,dt\Big|_{\alpha=\beta=0} \\
&= 2I_r + \frac{d}{d\alpha}\frac{d}{d\beta}\frac{1}{2\pi}\int_0^T \log\frac{t}{2\pi}\,\left(\zeta(1+\alpha+\beta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta}\zeta(1-\alpha-\beta)\right) \\
&\qquad\qquad \times (1+O(t^{-1/2+\epsilon}))\,dt\Big|_{\alpha=\beta=0},
\end{aligned} \tag{7.17}$$

where the last line is a further application of the ratios conjecture (or in this case the simpler moment conjecture [**10**]) similar to that in Section 1.

Using (7.11) for I_r, it is now necessary to carry out the differentiation and take the limits as α, β, γ and δ tend to zero. This results in a polynomial in $\log(t/2\pi)$. If we write

$$\zeta(1+s) = \frac{1}{s} + \gamma - \gamma_1 s + \frac{\gamma_2}{2!}s^2 - \frac{\gamma_3}{3!}s^3 \ldots, \tag{7.18}$$

then the final result is as follows.

THEOREM 7.3. *Assuming the ratio conjecture as indicated in* (7.11), *we have*

$$\begin{aligned}
\sum_{\gamma<T} |\zeta'(\rho)|^2 &= \int_0^T \left(\frac{1}{24\pi}\log^4\frac{t}{2\pi} + \frac{\gamma}{3\pi}\log^3\frac{t}{2\pi} + \left(\frac{\gamma^2}{2\pi} - \frac{\gamma_1}{\pi}\right)\log^2\frac{t}{2\pi}\right. \\
&\qquad - \left(\frac{\gamma^3}{\pi} + \frac{5\gamma\gamma_1}{\pi} + \frac{\gamma_2}{2\pi}\right)\log\frac{t}{2\pi} \\
&\qquad \left. + \frac{\gamma^4}{\pi} + \frac{6\gamma^2\gamma_1}{\pi} + \frac{7\gamma_1^2}{\pi} + \frac{4\gamma\gamma_2}{\pi} + \frac{5\gamma_3}{3\pi}\right)(1+O(t^{-1/2+\epsilon}))\,dt \\
&= \frac{T}{24\pi}\log^4 T + O(T\log^3 T).
\end{aligned} \tag{7.19}$$

Remark 7.4. The leading order term of the above agrees with Gonek's result (7.1). It is possible that Gonek's methods could prove the theorem conditional only on the Riemann Hypothesis. Also, Pokharel and Rubinstein have checked this numerically.

Remark 7.5. Since the original version of this paper appeared on the archive, Milinovich has used Gonek's method to verify all the main terms above. He also remarks that this result can probably be obtained from a theorem of Fujii.

7.2. *Fourth moment of the derivative:* $\sum_{\gamma<T}|\zeta'(\rho)|^4$

Higher moments are more difficult because of complicated arithmetic contributions. Unlike the case of the fourth moment of the zeta function itself,

$$\begin{aligned}
&\frac{1}{T}\int_0^T |\zeta(\tfrac12+it)|^4\,dt\\
&\quad=\frac{1}{T}\int_0^T \frac{1}{2\pi^2}\log^4\frac{t}{2\pi}+\frac{8}{\pi^4}\left(\gamma\pi^2-3\zeta'(2)\right)\log^3\frac{t}{2\pi}\\
&\qquad+\frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2-12\zeta''(2)\pi^2+7\gamma^2\pi^4+144\zeta'(2)^2-2\,\gamma_1\pi^4\right)\log^2\frac{t}{2\pi}\\
&\qquad+\frac{12}{\pi^8}\Big(6\gamma^3\pi^6-84\gamma^2\zeta'(2)\pi^4+24\gamma_1\zeta'(2)\pi^4-1728\zeta'(2)^3+576\gamma\zeta'(2)^2\pi^2\\
&\qquad\qquad+288\zeta'(2)\zeta''(2)\pi^2-8\zeta'''(2)\pi^4-10\gamma_1\gamma\pi^6-\gamma_2\pi^6-48\gamma\zeta''(2)\pi^4\Big)\log\frac{t}{2\pi}\\
&\qquad+\frac{4}{\pi^{10}}\Big(-12\zeta''''(2)\pi^6+36\gamma_2\zeta'(2)\pi^6+9\gamma^4\pi^8+21\gamma_1^2\pi^8+432\zeta''(2)^2\pi^4\\
&\qquad\qquad+3456\gamma\zeta'(2)\zeta''(2)\pi^4+3024\gamma^2\zeta'(2)^2\pi^4-36\gamma^2\gamma_1\pi^8-252\gamma^2\zeta''(2)\pi^6\\
&\qquad\qquad+3\gamma\gamma_2\pi^8+72\gamma_1\zeta''(2)\pi^6+360\gamma_1\gamma\zeta'(2)\pi^6-216\gamma^3\zeta'(2)\pi^6\\
&\qquad\qquad-864\gamma_1\zeta'(2)^2\pi^4+5\gamma_3\pi^8+576\zeta'(2)\zeta'''(2)\pi^4-20736\gamma\zeta'(2)^3\pi^2\\
&\qquad\qquad-15552\zeta''(2)\zeta'(2)^2\pi^2-96\gamma\zeta'''(2)\pi^6+62208\zeta'(2)^4\Big)\,dt+o(1)\\
&\quad=\frac{1}{12\zeta(2)}\log^4T+O(\log^3T)
\end{aligned}\tag{7.21}$$

(from the moment conjecture formula of [**10**] and also implied by [**21**]), the fourth moment of the modulus of the derivative involves arithmetic factors that are more complicated than derivatives of the zeta function at 2. In the following we calculate the first four leading order terms, demonstrating where the first of these new arithmetic factors appears.

As for the second moment, the first step is to write the sum over zeros as a contour integral

$$\sum_{\gamma<T}|\zeta'(\rho)|^4=\frac{1}{2\pi i}\int_C\frac{\zeta'(z)}{\zeta(z)}\zeta'(z)\zeta'(1-z)\zeta'(z)\zeta'(1-z)\,dz,\tag{7.22}$$

with the contour C running from c to $c+iT$, $1-c+iT$ and $1-c$. The horizontal integrals do not contribute significantly, and so we define

$$\begin{aligned}
I_R=&\frac{d}{d\alpha}\frac{d}{d\beta}\frac{d}{d\gamma}\frac{d}{d\delta}\frac{d}{d\epsilon}\frac{1}{2\pi}\int_0^T\frac{\zeta(c+it+\alpha)}{\zeta(c+it)}\\
&\times\zeta(c+it+\beta)\zeta(c+it+\gamma)\zeta(1-c-it+\delta)\zeta(1-c-it+\epsilon)\,dt\Big|_{\alpha=\beta=\gamma=\delta=\epsilon=0}.
\end{aligned}\tag{7.23}$$

The sum resulting from taking the first half of each approximate functional equation is

$$\sum_{m_1m_2m_3h=n_1n_2}\frac{\mu(h)}{m_1^{1/2+\alpha}m_2^{1/2+\beta}m_3^{1/2+\gamma}h^{1/2}n_1^{1/2+\delta}n_2^{1/2+\epsilon}}. \tag{7.24}$$

Here we note that if we let $\gamma=0$ then we obtain the sum

$$\prod_p\sum_{m_1+m_2+c=n_1+n_2}\frac{1}{p^{(1/2+\alpha)m_1+(1/2+\beta)m_2+(1/2)c+(1/2+\delta)n_1+(1/2+\epsilon)n_2}}\sum_{m_3+h=c}\mu(p^h). \tag{7.25}$$

The final sum is zero unless $c=0$, and thus the whole expression reduces to the corresponding sum that results from applying the ratios (or moments) conjecture to the fourth moment of zeta,

$$\frac{1}{T}\int_0^T\zeta(1/2+it+\alpha)\zeta(1/2+it+\beta)\zeta(1/2-it+\delta)\zeta(1/2-it+\epsilon)\,dt,$$

which itself produces the arithmetic contribution $1/\zeta(2+\alpha+\beta+\delta+\epsilon)$ observed as the factor $1/\zeta(2)$ in the leading order term of (7.21) (see for example [**10**]).

We keep this in mind as we continue with the sum in (7.24). This sum can be written as an Euler product, and we can pull out the divergent terms in the form of zeta functions:

$$\begin{aligned}
&T(\alpha,\beta,\gamma,\delta,\epsilon)\\
&\quad:=\frac{\zeta(1+\alpha+\delta)\zeta(1+\alpha+\epsilon)\zeta(1+\beta+\delta)\zeta(1+\beta+\epsilon)\zeta(1+\gamma+\delta)\zeta(1+\gamma+\epsilon)}{\zeta(1+\delta)\zeta(1+\epsilon)}\\
&\qquad\times\prod_p\Bigg[\frac{(1-\frac{1}{p^{1+\alpha+\delta}})(1-\frac{1}{p^{1+\alpha+\epsilon}})(1-\frac{1}{p^{1+\beta+\delta}})(1-\frac{1}{p^{1+\beta+\epsilon}})(1-\frac{1}{p^{1+\gamma+\delta}})(1-\frac{1}{p^{1+\gamma+\epsilon}})}{(1-\frac{1}{p^{1+\delta}})(1-\frac{1}{p^{1+\epsilon}})}\\
&\qquad\times\sum_{m_1+m_2+m_3+h=n_1+n_2}\frac{\mu(p^h)}{p^{(1/2+\alpha)m_1+(1/2+\beta)m_2+(1/2+\gamma)m_3+h/2+(1/2+\delta)n_1+(1/2+\epsilon)n_2}}\Bigg].
\end{aligned}\tag{7.26}$$

With the remaining terms from the approximate functional equations we have

$$\begin{aligned}
I_R&=\frac{d}{d\alpha}\frac{d}{d\beta}\frac{d}{d\gamma}\frac{d}{d\delta}\frac{d}{d\epsilon}\frac{1}{2\pi}\int_0^T T(\alpha,\beta,\gamma,\delta,\epsilon)+\left(\frac{t}{2\pi}\right)^{-\alpha-\delta}T(-\delta,\beta,\gamma,-\alpha,\epsilon)\\
&\quad+\left(\frac{t}{2\pi}\right)^{-\alpha-\epsilon}T(-\epsilon,\beta,\gamma,\delta,-\alpha)+\left(\frac{t}{2\pi}\right)^{-\beta-\delta}T(\alpha,-\delta,\gamma,-\beta,\epsilon)\\
&\quad+\left(\frac{t}{2\pi}\right)^{-\beta-\epsilon}T(\alpha,-\epsilon,\gamma,\delta,-\beta)\\
&\quad+\left(\frac{t}{2\pi}\right)^{-\gamma-\delta}T(\alpha,\beta,-\delta,-\gamma,\epsilon)+\left(\frac{t}{2\pi}\right)^{-\gamma-\epsilon}T(\alpha,\beta,-\epsilon,\delta,-\gamma)\\
&\quad+\left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\delta-\epsilon}T(-\delta,-\epsilon,\gamma,-\alpha,-\beta)+\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma-\delta-\epsilon}T(-\delta,\beta,-\epsilon,-\alpha,-\gamma)\\
&\quad+\left(\frac{t}{2\pi}\right)^{-\beta-\gamma-\delta-\epsilon}T(\alpha,-\delta,-\epsilon,-\beta,-\gamma)\,dt\,\Bigg|_{\alpha=\beta=\gamma=\delta=\epsilon=0}+O(T^{1/2+\epsilon}).
\end{aligned}\tag{7.27}$$

The most concise way to write the ten terms in the integrand above is as a contour integral (as described in [**10**]). So, we have

$$\begin{aligned} I_R = {}& \frac{d}{d\alpha}\frac{d}{d\beta}\frac{d}{d\gamma}\frac{d}{d\delta}\frac{d}{d\epsilon} e^{((\log t/(2\pi))/2)(-\alpha-\beta-\gamma-\delta-\epsilon)} \frac{1}{2\pi}\int_0^T \frac{1}{3!\ 2!(2\pi i)^5} \\ & \times \oint \cdots \oint e^{((\log t/(2\pi))/2)(z_1+z_2+z_3-z_4-z_5)} \\ & \times \frac{T(z_1,z_2,z_3,-z_4,-z_5)\Delta(z_1,\ldots,z_5)^2}{\prod_{j=1}^5 (z_j-\alpha)(z_j-\beta)(z_j-\gamma)(z_j+\delta)(z_j+\epsilon)} dz_1 \ldots dz_5\, dt + O(T^{1/2+\epsilon}). \end{aligned} \tag{7.28}$$

With the formulae

$$\frac{d}{d\alpha}\frac{e^{-a\alpha}}{\prod_{j=1}^n (z_j-\alpha)}\bigg|_{\alpha=0} = \frac{1}{\prod_{j=1}^n z_j}\left(\sum_{j=1}^n \frac{1}{z_j} - a\right) \tag{7.29}$$

and

$$\frac{d}{d\delta}\frac{e^{-a\delta}}{\prod_{j=1}^n (z_j+\delta)}\bigg|_{\delta=0} = \frac{1}{\prod_{j=1}^n z_j}\left(-\sum_{j=1}^n \frac{1}{z_j} - a\right), \tag{7.30}$$

the differentiation can easily be performed, to yield

$$\begin{aligned} I_R = {}& \frac{1}{2\pi}\int_0^T \frac{1}{3!\ 2!(2\pi i)^5} \oint \cdots \oint \frac{T(z_1,z_2,z_3,-z_4,-z_5)\Delta(z_1,\ldots,z_5)^2}{\prod_{j=1}^5 z_j^5} \\ & \times \left(-\frac{\log t/(2\pi)}{2} - \sum_{j=1}^5 \frac{1}{z_j}\right)^2 \left(-\frac{\log t/(2\pi)}{2} + \sum_{j=1}^5 \frac{1}{z_j}\right)^3 \\ & \times e^{((\log t/(2\pi))/2)(z_1+z_2+z_3-z_4-z_5)}\, dz_1 \ldots dz_5\, dt + O(T^{1/2+\epsilon}). \end{aligned} \tag{7.31}$$

For the contribution from the other side of the contour of integration we have, following the same method as for the second moment,

$$\begin{aligned} I_L &= \frac{1}{2\pi i}\int_{1-c+iT}^{1-c} \frac{\zeta'(z)}{\zeta(z)}\zeta'(z)\zeta'(1-z)\zeta'(z)\zeta'(1-z)\, dz \\ &= \frac{1}{2\pi}\int_0^T \left(\log\frac{t}{2\pi} + O(1/(t+1))\right) |\zeta'(\tfrac{1}{2}+it)|^4\, dt + I_R \\ &= I_R + \frac{d}{d\alpha}\frac{d}{d\beta}\frac{d}{d\gamma}\frac{d}{d\delta}\frac{1}{2\pi}\int_0^T \left(\log\frac{t}{2\pi} + O(1/(t+1))\right)\zeta(1/2+it+\alpha)\zeta(1/2+it+\beta) \\ &\qquad \times \zeta(1/2-it+\gamma)\zeta(1/2-it+\delta)\, dt\bigg|_{\alpha=\beta=\gamma=\delta=0}. \end{aligned} \tag{7.32}$$

The most concise way to write the six terms that will result from applying the ratio conjecture (actually in this case there is no denominator so it is just a moment conjecture) to the fourth moment of ζ in the last line above is as a contour integral similar to (7.28) (as described in [**10**]). So, we have

$$\begin{aligned}I_L = I_R &+ \frac{d}{d\alpha}\frac{d}{d\beta}\frac{d}{d\gamma}\frac{d}{d\delta}e^{((\log t/(2\pi))/2)(-\alpha-\beta-\gamma-\delta)}\frac{1}{2\pi}\int_0^T \log\frac{t}{2\pi}\frac{1}{2!\,2!(2\pi i)^4}\\ &\times\oint\cdots\oint e^{((\log t/(2\pi))/2)(z_1+z_2-z_3-z_4)}\\ &\times\frac{\zeta(1+z_1-z_3)\zeta(1+z_1-z_4)\zeta(1+z_2-z_3)\zeta(1+z_2-z_4)}{\zeta(2+z_1+z_2-z_3-z_4)}\\ &\times\frac{\Delta(z_1,\ldots,z_4)^2}{\prod_{j=1}^4(z_j-\alpha)(z_j-\beta)(z_j+\gamma)(z_j+\delta)}\,dz_1\ldots dz_4+O(T^{1/2+\epsilon}).\end{aligned}\tag{7.33}$$

Now we use the formulae (7.29) and (7.30) and arrive at

$$\begin{aligned}I_L = I_R &+ \frac{1}{2\pi}\int_0^T \log\frac{t}{2\pi}\frac{1}{2!\,2!(2\pi i)^4}\\ &\times\oint\cdots\oint\frac{\zeta(1+z_1-z_3)\zeta(1+z_1-z_4)\zeta(1+z_2-z_3)\zeta(1+z_2-z_4)}{\zeta(2+z_1+z_2-z_3-z_4)}\\ &\times\frac{\Delta(z_1,\ldots,z_4)^2}{\prod_{j=1}^4 z_j^4}\left(-\frac{\log t/(2\pi)}{2}-\sum_{j=1}^4\frac{1}{z_j}\right)^2\left(-\frac{\log t/(2\pi)}{2}+\sum_{j=1}^4\frac{1}{z_j}\right)^2\\ &\times e^{((\log t/(2\pi))/2)(z_1+z_2-z_3-z_4)}\,dz_1\ldots dz_4+O(T^{1/2+\epsilon}).\end{aligned}\tag{7.34}$$

We now compute the residues at $z_1=z_2=z_3=z_4=z_5=0$ of the contour integrals in (7.31) and (7.34) (using MATHEMATICA). If we write

$$\zeta(1+s)=\frac{1}{s}+\gamma-\gamma_1 s+\frac{\gamma_2}{2!}s^2-\frac{\gamma_3}{3!}s^3\ldots,\tag{7.35}$$

then the final result is as follows.

THEOREM 7.6. *Assuming the ratios conjecture as indicated in* (7.28), *we have constants* $C_0,\ldots,C_6$ *such that*

$$\begin{aligned}\sum_{\gamma<T}|\zeta'(\rho)|^4 &= \frac{1}{2\pi}\int_0^T\Bigg(\frac{\log^9(t/2\pi)}{8640\zeta(2)}-\frac{(-2\gamma\zeta(2)+\zeta'(2))\log^8(t/2\pi)}{480\zeta^2(2)}\\ &\quad+\frac{(7\gamma^2\zeta^2(2)-2\gamma_1\zeta^2(2)-8\gamma\zeta(2)\zeta'(2)+4(\zeta'(2))^2-2\zeta(2)\zeta''(2))\log^7(t/2\pi)}{120\zeta^3(2)}\\ &\quad+C_6\log^6(t/2\pi)+\ldots+C_0\Bigg)\,dt+O(T^{1/2+\epsilon})\\ &=\frac{T}{2\pi}\frac{\log^9 T}{8640\zeta(2)}+O(T\log^8 T).\end{aligned}\tag{7.36}$$

This leading term agrees with Conjecture 7.1 of Hughes, Keating and O'Connell [**23**]; since $G^2(4)/G(7)=8640$ and $a(2)=1/\zeta(2)$, we see that there is agreement between the leading order term of (7.36) and the conjecture in the case that $k=2$.

The first three terms in decreasing powers of $\log T$ shown in (7.36) contain only arithmetic factors that are derivatives of the Riemann zeta function evaluated at 2. This is not true of the $(\log T)^6$ term. We can see that this will be the case by the comment after (7.25).

The arithmetic factor which forms part of $T(\alpha,\beta,\gamma,\delta,\epsilon)$ in (7.26),

$$\begin{aligned} &A(\alpha,\beta,\gamma,\delta,\epsilon) \\ &\quad = \prod_p \Bigg[\frac{(1-\frac{1}{p^{1+\alpha+\delta}})(1-\frac{1}{p^{1+\alpha+\epsilon}})(1-\frac{1}{p^{1+\beta+\delta}})(1-\frac{1}{p^{1+\beta+\epsilon}})(1-\frac{1}{p^{1+\gamma+\delta}})(1-\frac{1}{p^{1+\gamma+\epsilon}})}{(1-\frac{1}{p^{1+\delta}})(1-\frac{1}{p^{1+\epsilon}})} \\ &\qquad \times \sum_{m_1+m_2+m_3+h=n_1+n_2} \frac{\mu(p^h)}{p^{(1/2+\alpha)m_1+(1/2+\beta)m_2+(1/2+\gamma)m_3+h/2+(1/2+\delta)n_1+(1/2+\epsilon)n_2}} \Bigg], \end{aligned} \tag{7.37}$$

expands as a Taylor series

$$\begin{aligned} &A(z_1,z_2,z_3,-z_4,-z_5) \\ &\quad = A_0 + A_1(z_1+z_2+z_3-z_4-z_5) \\ &\qquad + A_{12}(-z_1z_4-z_2z_4-z_3z_4-z_1z_5-z_2z_5-z_3z_5+z_1z_2+z_1z_3+z_2z_3+z_4z_5) \\ &\qquad + \frac{A_{11}}{2}(z_1^2+z_2^2+z_3^2+z_4^2+z_5^2) \\ &\qquad + A_{124}(-z_1z_2z_4-z_1z_3z_4-z_2z_3z_4-z_1z_2z_5-z_1z_3z_5-z_2z_3z_5+(z_1+z_2+z_3)z_4z_5) \\ &\qquad + A_{123}z_1z_2z_3 \\ &\qquad + \frac{A_{112}}{2}(z_4^2(z_1+z_2+z_3-z_5)+z_3^2(z_1+z_2-z_4-z_5)+z_2^2(z_1+z_3-z_4-z_5) \\ &\qquad\qquad + z_1^2(z_2+z_3-z_4-z_5)+(z_1+z_2+z_3-z_4)z_5^2) \\ &\qquad + \frac{A_{111}}{6}(z_1^3+z_2^3+z_3^3-z_4^3-z_5^3)+\dots, \end{aligned} \tag{7.38}$$

where A_j is the partial derivative, evaluated at zero, of $A(z_1,z_2,z_3,z_4,z_5)$ with respect to the jth variable. Note that A is symmetric amongst the first three variables, and amongst the final two variables, so, for example,

$$A_{12} = \frac{\partial A(z_1,z_2,z_3,z_4,z_5)}{\partial z_1 \partial z_2}\Big|_{z_1=z_2=z_3=z_4=z_5=0} = A_{13} = A_{23}.$$

In addition, we noted at (7.25) that $A(\alpha,\beta,0,\delta,\gamma)$ is just the same as the arithmetic factor from the fourth moment of zeta,

$$\frac{1}{T}\int_0^T \zeta(1/2+it+\alpha)\zeta(1/2+it+\beta)\zeta(1/2-it+\delta)\zeta(1/2-it+\epsilon)\,dt,$$

that is, $1/\zeta(2+\alpha+\beta+\delta+\epsilon)$. Therefore all the partial derivatives of A in (7.38) can be computed by taking partial derivatives of $1/\zeta(2+z_1+z_2+z_4+z_5)$ except for the derivative

$$A_{123} = \frac{\partial^3 A}{\partial z_1 \partial z_2 \partial z_3},$$

which involves all of the first three variables and gives a contribution that is not expressed as a derivative of $\zeta(2)$. This contribution shows up first in the $\log^6 T$ term.

7.3. *A second moment:* $\sum_{0<\gamma<T} |\zeta(\rho+a)|^2$

In [**18**] A. Fujii generalizes work of [**20**] and proves, under the assumption of the Riemann Hypothesis, the following theorem.

THEOREM 7.7 (Fujii [**18**]). *Assume the Riemann Hypothesis is true. If T is sufficiently large and α is a real number such that $|\alpha| \ll \log T$, then*

$$\begin{aligned}
&\sum_{1\leqslant\gamma\leqslant T}\left|\zeta\left(\frac{1}{2}+i\left(\gamma+2\pi\alpha/\log\frac{T}{2\pi}\right)\right)\right|^2\\
&=\left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T\\
&\quad+2\left(-1+\gamma+(1-2\gamma)\frac{\sin 2\pi\alpha}{2\pi\alpha}+\Re\left(\frac{\zeta'}{\zeta}\left(1+i\frac{2\pi\alpha}{\log(T/2\pi)}\right)\right)\right)\frac{T}{2\pi}\log\frac{T}{2\pi}\\
&\quad+G(T,\alpha)+O(\sqrt{T}\log^3 T),
\end{aligned}\tag{7.39}$$

where γ is Euler's constant and $G(T,\alpha)=O(T)$ is explicitly given.

The ratios conjecture can reproduce this result in a straightforward way.

First we write the quantity we want to calculate in a form in which we can apply the ratios conjecture:

$$\sum_{0<\gamma<T}|\zeta(\rho+a)|^2=\lim_{a_1,a_2\to a}\frac{1}{2\pi i}\int_C\frac{\zeta'(z)}{\zeta(z)}\zeta(z+a_1)\zeta(1-z-a_2)\,dz.\tag{7.40}$$

Now we follow the identical method to that described in Section 7.1 and so obtain

$$\begin{aligned}
\sum_{0<\gamma<T}|\zeta(\rho+a)|^2=\lim_{a_1,a_2\to a}\Bigg[&\frac{d}{d\alpha}\frac{1}{2\pi}\int_0^T\frac{\zeta(1+\alpha-a_2)\zeta(1+a_1-a_2)}{\zeta(1-a_2)}\\
&\qquad+\left(\frac{t}{2\pi}\right)^{-\alpha+a_2}\frac{\zeta(1+a_2-\alpha)\zeta(1+a_1-\alpha)}{\zeta(1-\alpha)}\\
&\qquad+\left(\frac{t}{2\pi}\right)^{-a_1+a_2}\frac{\zeta(1+\alpha-a_1)\zeta(1+a_2-a_1)}{\zeta(1-a_1)}\,dt\Big|_{\alpha=0}\\
&+\frac{d}{d\alpha}\frac{1}{2\pi}\int_0^T\frac{\zeta(1+\alpha+a_1)\zeta(1-a_2+a_1)}{\zeta(1+a_1)}\\
&\qquad+\left(\frac{t}{2\pi}\right)^{-\alpha-a_1}\frac{\zeta(1-a_1-\alpha)\zeta(1-a_2-\alpha)}{\zeta(1-\alpha)}\\
&\qquad+\left(\frac{t}{2\pi}\right)^{a_2-a_1}\frac{\zeta(1+\alpha+a_2)\zeta(1-a_1+a_2)}{\zeta(1+a_2)}\,dt\Big|_{\alpha=0}\\
&+\frac{1}{2\pi}\int_0^T\log\frac{t}{2\pi}\left(\zeta(1-a_2+a_1)+\left(\frac{t}{2\pi}\right)^{-a_1+a_2}\zeta(1-a_1+a_2)\right)dt\Bigg]\\
&+O(T^{1/2+\epsilon}).
\end{aligned}\tag{7.41}$$

Performing the differentiation and setting $\alpha = 0$, we have

$$\begin{aligned}\sum_{0<\gamma<T} |\zeta(\rho+a)|^2 = \lim_{a_1,a_2\to a} \frac{1}{2\pi}\int_0^T & \frac{\zeta'(1-a_2)\zeta(1+a_1-a_2)}{\zeta(1-a_2)} - \left(\frac{t}{2\pi}\right)^{a_2} \zeta(1+a_2)\zeta(1+a_1) \\ &+ \left(\frac{t}{2\pi}\right)^{-a_1+a_2} \frac{\zeta'(1-a_1)\zeta(1+a_2-a_1)}{\zeta(1-a_1)} \\ &+ \frac{\zeta'(1+a_1)\zeta(1-a_2+a_1)}{\zeta(1+a_1)} - \left(\frac{t}{2\pi}\right)^{-a_1} \zeta(1-a_1)\zeta(1-a_2) \\ &+ \left(\frac{t}{2\pi}\right)^{a_2-a_1} \frac{\zeta'(1+a_2)\zeta(1-a_1+a_2)}{\zeta(1+a_2)} \\ &+ \log\frac{t}{2\pi}\left(\zeta(1-a_2+a_1) + \left(\frac{t}{2\pi}\right)^{-a_1+a_2}\zeta(1-a_1+a_2)\right)dt \\ &+ O(T^{1/2+\epsilon}). \end{aligned} \tag{7.42}$$

To perform the limit, let $a_1 = a$ and $a_2 = a+s$. Then

$$\lim_{s\to 0} \zeta(1-s) + \left(\frac{t}{2\pi}\right)^{s} \zeta(1+s) = \log\frac{t}{2\pi} + 2\gamma, \tag{7.43}$$

and

$$\begin{aligned}\lim_{s\to 0} \frac{\zeta'(1-a-s)\zeta(1-s)}{\zeta(1-a-s)} + \left(\frac{t}{2\pi}\right)^{s} \frac{\zeta'(1-a)\zeta(1+s)}{\zeta(1-a)} \\ = \left(\log\frac{t}{2\pi} + 2\gamma\right)\frac{\zeta'(1-a)}{\zeta(1-a)} + \frac{\zeta''(1-a)}{\zeta(1-a)} - \frac{(\zeta'(1-a))^2}{\zeta^2(1-a)}. \end{aligned} \tag{7.44}$$

Thus, assuming the ratios conjecture as indicated in (7.11), we have

$$\begin{aligned}\sum_{0<\gamma<T} |\zeta(\rho+a)|^2 = \frac{1}{2\pi}\int_0^T & \left(\log\frac{t}{2\pi} + 2\gamma\right)\left(\log\frac{t}{2\pi} + \frac{\zeta'(1-a)}{\zeta(1-a)} + \frac{\zeta'(1+a)}{\zeta(1+a)}\right) \\ &+ \frac{\zeta''(1-a)}{\zeta(1-a)} + \frac{\zeta''(1+a)}{\zeta(1+a)} - \left(\frac{\zeta'(1-a)}{\zeta(1-a)}\right)^2 - \left(\frac{\zeta'(1+a)}{\zeta(1+a)}\right)^2 \\ &- \left(\frac{t}{2\pi}\right)^{a} \zeta(1+a)\zeta(1+a) - \left(\frac{t}{2\pi}\right)^{a} \zeta(1-a)\zeta(1-a)\,dt \\ &+ O(T^{1/2+\epsilon}). \end{aligned} \tag{7.45}$$

This result matches up exactly with Theorem 7.7; see in particular the bottom of page 66 in [**18**].

If we now let $a = 2\pi i\alpha \log^{-1}(T/2\pi)$, then for large T we can use the first few terms of the series

$$\zeta(1+s) = \frac{1}{s} + \gamma - \gamma_1 s + \gamma_2 \frac{s^2}{2} + \dots,$$

as well as the similar expressions for the derivatives and inverse of $\zeta(1+s)$, and perform the integration over t to obtain a more standard expression for the leading terms:

$$\sum_{0<\gamma<T}\left|\zeta\left(\rho+2\pi i\alpha\log^{-1}\frac{T}{2\pi}\right)\right|^2$$
$$=\left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2\frac{T}{2\pi}+\frac{T}{2\pi}\log\frac{T}{2\pi}\left(\frac{\sin 2\pi\alpha}{\pi\alpha}-2\gamma\frac{\sin 2\pi\alpha}{\pi\alpha}+4\gamma-2\right)$$
$$+\frac{T}{2\pi}(4\gamma\cos 2\pi\alpha-2\cos 2\pi\alpha-2\gamma^2\cos 2\pi\alpha-4\gamma_1\cos 2\pi\alpha+2\gamma^2-4\gamma+2)+O(T). \quad (7.46)$$

8. *Further connections with the literature*

8.1. *Non-vanishing of Dirichlet L-functions*

Michel and VanderKam's paper 'Non-vanishing of high derivatives of Dirichlet L-functions at the central point' [**35**] actually is concerned with non-vanishing of derivatives of $\Lambda(s,\chi)$, the completed Dirichlet L-functions:

$$\Lambda(s,\chi)=\left(\frac{q}{\pi}\right)^{s/2}\Gamma(s/2)L(s,\chi), \quad (8.1)$$

which satisfies

$$\Lambda(s,\chi)=\epsilon_\chi\Lambda(1-s,\overline{\chi}). \quad (8.2)$$

They use mollifying techniques to give a lower bound for the frequency of $\Lambda^{(k)}(1/2,\chi)\neq 0$ as χ ranges over primitive characters modulo q. They find, first, that using a mollifier with two pieces

$$M_k(1/2,\chi):=\sum_{m\leqslant y}\frac{\mu(m)\left(\chi(m)+(-1)^k\overline{\epsilon}_\chi(\chi)\overline{\chi}(m)\right)P\left(\frac{\log(y/m)}{\log y}\right)}{m^{1/2}} \quad (8.3)$$

is more efficient than the conventional mollifier

$$M_k^*(1/2,\chi):=\sum_{m\leqslant y}\frac{\mu(m)\chi(m)P\left(\frac{\log(y/m)}{\log y}\right)}{m^{1/2}}; \quad (8.4)$$

in fact for $\Lambda(1/2,\chi)$ (no derivatives) they find that asymptotically half do not vanish, improving work of Iwaniec and Sarnak [**25**] who had a third in place of a half here. They also prove that when mollifying the high derivatives of Λ, the proportion of non-vanishing of the kth derivative can be shown to approach $2/3$ as $k\to\infty$, and that to obtain this result it is critical to use the general mollifier.

In this section we use the ratios conjecture to reproduce the asymptotic formulae of [**35**]. In addition, we indicate how one can show that the proportion of non-vanishing for $L^{(k)}(1/2,\chi)$ does approach 100% as $k\to\infty$ for this family.

Rather than work with the Dirichlet L-functions, we find it convenient to work with the Riemann zeta function in t aspect; they are both unitary families, so the results will be identical. For the analogue of the $\Lambda^{(k)}(1/2,\chi)$ function we will use the function

$$\chi(s)^{1/2}\mathcal{Z}^{(k)}(s), \quad (8.5)$$

where

$$\mathcal{Z}(s)=\chi(s)^{-1/2}\zeta(s) \quad (8.6)$$

is a complex analogue of Hardy's $Z(t)$ function. This is appropriate because the $\mathcal{Z}$-function associated with $L(s,\chi)$ is

$$\mathcal{Z}(s,\chi) = \epsilon_\chi^{-1/2}\Lambda(s,\chi), \tag{8.7}$$

so that

$$\Lambda^{(k)}(s,\chi) = \epsilon_\chi^{1/2}\mathcal{Z}^{(k)}(s,\chi). \tag{8.8}$$

Note that χ is used in two different roles here; recall that $\chi(s)$ is the factor from the functional equation of the zeta function (see (2.3)), and it plays the role of ϵ_χ.

The analogue of the quantity considered in [**35**, Section 7] is

$$\frac{\left(\int_0^T \chi(s)^{1/2}\mathcal{Z}^{(k)}(s)M(s)\,dt\right)^2}{\int_0^T \mathcal{Z}^{(k)}(s)\mathcal{Z}^{(k)}(1-s)M(s)M(1-s)\,dt}, \tag{8.9}$$

for a two-piece mollifier. However, here we will illustrate the calculation with the conventional mollifier

$$M(s) = M(s,P) = \sum_{m\leqslant y}\mu(m)P(\log(y/m)/\log y)/m^s. \tag{8.10}$$

The object is to choose P in such a way that this ratio is minimized.

Since

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{1/2-s}(1+O(1/t)) \tag{8.11}$$

for $t>1$, we have

$$\frac{d}{ds}\left(\chi(s)^{-1/2}\right) = \frac{\ell}{2}\chi(s)^{-1/2}(1+O(1/t)), \tag{8.12}$$

with $\ell = \log(t/2\pi)$. Hence,

$$\chi(s)^{1/2}\mathcal{Z}^{(k)}(s) = \left(\frac{d}{d\alpha}\right)^k e^{\alpha\ell/2}\zeta(s+\alpha)(1+O(1/t))\Big|_{\alpha=0}. \tag{8.13}$$

Thus the integral in the numerator can be evaluated by considering

$$\left(\frac{d}{d\alpha}\right)^k\int_0^T e^{\alpha\ell/2}\zeta(s+\alpha)M(s,P)\,dt\Big|_{\alpha=0} \sim \left(\frac{d}{d\alpha}\right)^k Te^{\alpha L/2}P(1)\Big|_{\alpha=0} = TP(1)2^{-k}L^k \tag{8.14}$$

where $L=\log T$. (The ratios conjecture gives $\int_0^T(\zeta(s+\alpha)/\zeta(s+w))\,dt \sim T$.) The denominator is, by (5.20) and after rescaling $\alpha = a/L$ and $\beta = b/L$,

$$\begin{aligned}
&\sim TL^{2k}\left(\frac{d}{da}\right)^k\left(\frac{d}{db}\right)^k e^{(a+b)/2}\\
&\quad\times\left(P(1)^2 + \frac{1}{\theta}\frac{d}{dw}\frac{d}{dz}e^{-a\theta w - b\theta z}\int_0^1\int_0^1 e^{-(a+b)u}P(w+r)P(z+r)\,dr\,du\Big|_{\substack{w=z=0\\a=b=0}}\right)\\
&= TL^{2k}\left(\frac{P(1)^2}{2^{2k}}\right.\\
&\quad\left.+\frac{1}{\theta}\frac{d}{dw}\frac{d}{dz}\int_0^1\int_0^1 P(r+w)(1/2-u-\theta w)^k P(r+z)(1/2-u-\theta z)^k\,dr\,du\right)\Big|_{w=z=0}\\
&= TL^{2k}\left(\frac{P(1)^2}{2^{2k}} + \frac{1}{\theta}\int_0^1\int_0^1\left(P'(r)(1/2-u)^k - k\theta P(r)(1/2-u)^{k-1}\right)^2 dr\,du\right)\\
&= \frac{TL^{2k}}{2^{2k}}\left(P(1)^2 + \frac{1}{\theta}\int_0^1\frac{P'(r)^2}{2k+1}\,dr + 4\theta\int_0^1\frac{k^2P(r)^2}{2k-1}\,dr\right).
\end{aligned} \tag{8.15}$$

This corresponds to the evaluation of $\mathcal{Q}_1$, accomplished in [**35**, equation (17)]. Note that $\Delta = 2\theta$.

Thus, the ratio (8.9) is

$$\frac{P(1)^2}{P(1)^2 + \frac{1}{\theta}\int_0^1 \frac{P'(r)^2}{2k+1}\,dr + 4\theta\int_0^1 \frac{k^2 P(r)^2}{2k-1}\,dr}. \tag{8.16}$$

If $k = 0$, we take $P(r) = r$ and $\theta = 1/2$ and deduce that at least one third of L-functions do not vanish at 1/2. This is the result of Iwaniec and Sarnak [**25**]. For large k if we take $P(r) = r^k$, we see that this ratio is

$$\frac{1}{1 + \left(4\theta + \frac{1}{\theta}\right)\frac{k^2}{4k^2-1}} \sim \frac{1}{1 + \left(\theta + \frac{1}{4\theta}\right)} \tag{8.17}$$

which is 1/2 when $\theta = 1/2$.

In general if $A, B > 0$, the minimum of $A\int_0^1 P'(x)^2\,dx + B\int_0^1 P(x)^2\,dx$ over smooth functions P satisfying $P(0) = 0$ and $P(1) = 1$ is $AP'(1)$ and is achieved by

$$P(x) = (\sinh\sqrt{B/A}x)/(\sinh\sqrt{B/A}).$$

So the optimal choice for (8.16) is

$$P(r) = \frac{\sinh(\Lambda r)}{\sinh\Lambda}, \qquad \Lambda = 2\theta k\sqrt{\frac{2k+1}{2k-1}} \tag{8.18}$$

as in [**35**]; however, this still gives that the ratio is $1/2 + O(1/k^2)$.

Next we explain the use of the two part mollifier in the case that $k = 0$. For this, we consider a mollifier of the form

$$M(s, P, a) := \sum_{n\leqslant y}\mu(n)P\left(\frac{\log(y/n)}{\log y}\right)(n^{-s} + a\chi(1-s)n^{s-1}) \tag{8.19}$$

and we want to maximize the ratio

$$\frac{\left(\int_0^T \zeta(s)M(s,P,a)\,dt\right)^2}{\int_0^T \zeta(s)\zeta(1-s)M(s,P,a)M(1-s,P,a)\,dt}. \tag{8.20}$$

The key things to observe here are that, with $M(s, P)$ as in (8.10),

$$\int_0^T \zeta(s)\chi(1-s)M(1-s,P)\,dt = \int_0^T \zeta(1-s)M(1-s,P)\,dt \sim TP(1) \tag{8.21}$$

and

$$\begin{aligned}\int_0^T \zeta(s)\zeta(1-s)M(1-s,P)\chi(1-s)M(1-s,P)\,dt &= \int_0^T \zeta(1-s)^2 M(1-s,P)^2\,dt \\ &\sim P(1)^2 T.\end{aligned} \tag{8.22}$$

Thus, the ratio is

$$\sim T\frac{(1+a)^2 P(1)^2}{(1+a^2)\left(P(1)^2 + \frac{1}{\theta}\int_0^1 P'(t)^2\,dt\right) + 2aP(1)^2}. \tag{8.23}$$

The optimal choices here are $P(r) = r$ and $a = 1$; for $\theta = 1/2$ this gives a ratio of 1/2 as claimed in [**35**].

To handle high derivatives of $L(s, \chi)$ at $s = 1/2$ we consider, by analogy, high derivatives of $\zeta(s)$. The trick is to insert a factor of $\chi(s)$ and to ask about the non-vanishing of

$$\chi(s)\zeta^{(k)}(1-s). \tag{8.24}$$

Thus, we want to maximize the ratio of

$$\frac{\left(\int_0^T \chi(s)\zeta^{(k)}(1-s)M(s,P)\,ds\right)^2}{\int_0^T |\chi(s)\zeta^{(k)}(1-s)M(s,P)|^2\,dt}. \tag{8.25}$$

Now,

$$\begin{aligned}\zeta^{(k)}(1-s) = \left(\frac{d}{ds}\right)^k \chi(1-s)\zeta(s) &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j\chi^{(j)}(1-s)\zeta^{(k-j)}(s)\\ &= \chi(1-s)\left(\frac{d}{d\alpha}\right)^k e^{\alpha\ell}\zeta(s+\alpha)\Big|_{\alpha=0}.\end{aligned} \tag{8.26}$$

Thus, the numerator is

$$\sim (TP(1)L^k)^2. \tag{8.27}$$

The denominator is evaluated optimally in [**14**] and is $T|P(1)|^2L^{2k}(1-O(1/k^2))$. Applying this method to $L(s,\chi)$, one can deduce that there is a constant $C>0$ such that the proportion of the $L^{(k)}(1/2,\chi)$ which vanish is smaller than C/k^2.

Remark 8.1. Michel and VanderKam [**35**] give a nice explanation at the end of Section 2 of why one cannot expect to do better than 1/2 non-vanishing of $\Lambda^{(k)}(1/2,\chi)$ using a conventional mollifier and 2/3 using a two-piece mollifier. The reason relies on the symmetry of the approximate functional equation for $\Lambda^{(k)}(1/2,\chi)$ and the uniform distribution of ϵ_χ. That we can get a proportion of non-vanishing of $L^{(k)}(1/2,\chi)$ approaching 1 as $k\to\infty$ does not contradict their argument because of the lack of symmetry of the approximate functional equation for $L^{(k)}(1/2,\chi)$. For applications to bounding multiplicities of central zeros, information about non-vanishing of $L^{(k)}(1/2,\chi)$ is equally as good as for $\Lambda^{(k)}(1/2,\chi)$.

8.2. *Non-vanishing of automorphic L-functions*

The main theorem of the paper 'Non-vanishing of high derivatives of automorphic L-functions at the center of the critical strip' [**31**] by Kowalski, Michel, and VanderKam is a mollification of the second moment of weight 2 primitive cusp forms of a prime level. The formula achieved is slightly different to the result we mention above (Theorem 5.3) for mollifying the second moment in an orthogonal family. The reason for this is that they use a slightly different mollifier. Instead of choosing a smoothed sum of the coefficients of the inverses of the Dirichlet series in question they choose a mollifier of the shape

$$\sum_{m\leqslant y}\frac{\lambda_f(n)\mu(n)P\left(\frac{\log(y/n)}{\log y}\right)}{\psi(n)n^{1/2}}, \tag{8.28}$$

where $\lambda_f(n)$ are the coefficients of the L-function which is to be mollified and where $\psi(n)=\prod_{p|n}(1+1/p)$. The analogue of our Theorem 5.3 has the right side replaced by

$$\frac{1}{\theta^2}\left((Q(1)P'(1)+\theta Q'(1)P(1))^2+\frac{1}{\theta}\int_0^1\int_0^1 (P''(x)Q(y)-\theta^2P(x)Q''(y))^2\,dx\,dy\right). \tag{8.29}$$

This result, which is not deducible from our ratios conjecture, was reported in the paper of Conrey and Farmer [**8**] as the general result one would obtain from mollifying a second moment in an orthogonal family. We wish to correct that statement and replace it with the statement of Theorem 5.3.

8.3. *Non-vanishing of quadratic L-functions*

The papers of Soundararajan, and Conrey and Soundararajan, deal with non-vanishing of Dirichlet L-functions for real quadratic characters, at the central point and on the real axis. The results of Theorem 5.2 are consistent with the results of these papers [**40**, **15**].

9. *Conclusion*

The purpose of this paper was to illustrate the use of the ratios conjectures by deriving from them a number of important results from the theory of L-functions. The variety of applications is by no mean exhausted by what we have presented. Other calculations that might be valuable include lower order terms in moments of $S(t)$, $\log|\zeta(1/2+it)|$ and $S(t+h)-S(t)$. For the second moment of $S(t)$ the lower order terms have already been computed by Tsz Ho Chan [**6**], while lower order terms of the second moment of $S(t+h)-S(t)$ have been considered in [**1**]. Precise evaluations of n-level correlations might be combined to obtain the secondary terms in the nearest neighbour spacing distribution for the zeros of the Riemann zeta function. In [**39**], the leading term in the n-correlation function is calculated for a restricted space of test functions, but for essentially any L-function. Ratios conjectures could also be used to evaluate possible schemes to improve lower bounds for proportions of zeros on the critical line.

References

1. M. V. Berry, 'Semiclassical formula for the number variance of the Riemann zeros', *Nonlinearity* 1 (1988) 399–407.
2. M. V. Berry and J. P. Keating, 'The Riemann zeros and eigenvalue asymptotics', *SIAM Rev.* 41 (1999) 236–266.
3. E. B. Bogomolny and J. P. Keating, 'Gutzwiller's trace formula and spectral statistics: beyond the diagonal approximation', *Phys. Rev. Lett.* 77 (1996) 1472–1475.
4. A. Borodin and E. Strahov, 'Averages of characteristic polynomials in random matrix theory', *Comm. Pure Appl. Math.* 59 (2006) 161–253.
5. D. Bump and A. Gamburd, 'On the averages of characteristic polynomials from classical groups', *Comm. Math. Phys.* 265 (2006) 227–274.
6. T. H. Chan, 'Lower order terms of the second moment of $S(t)$', Preprint, AIM 2004 – 24, arXiv:math.NT/0411501.
7. J. B. Conrey, 'More than two fifths of the zeros of the Riemann zeta function are on the critical line', *J. reine angew. Math.* 399 (1989) 1–26.
8. J. B. Conrey and D. W. Farmer, 'Mean values of L-functions and symmetry', *Internat. Math. Res. Notices* 17 (2000) 883–908.
9. J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, 'Autocorrelation of random matrix polynomials', *Comm. Math. Phys.* 237 (2003) 365–395.
10. J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, 'Integral moments of L-functions', *Proc. London Math. Soc.* (3) 91 (2005) 33–104.
11. J. B. Conrey, D. W. Farmer and M. R. Zirnbauer, 'Autocorrelation of ratios of L-functions', Preprint, American Institute of Mathematics, 2006.
12. J. B. Conrey, D. W. Farmer and M. R. Zirnbauer, 'Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the classical compact groups', Preprint, 2005, arXiv:math-ph/0511024.
13. J. B. Conrey, P. J. Forrester and N. C. Snaith, 'Averages of ratios of characteristic polynomials for the compact classical groups', *Internat. Math. Res. Notices* 7 (2005) 397–431.
14. J. B. Conrey and A. Ghosh, 'Zeros of derivatives of the Riemann zeta-function near the critical line', *Analytic number theory*, Allerton Park, IL, 1989 (ed. B. C. Berndt, H. G. Diamond, H. Halberstam and A. J. Hildebrand), Progress in Mathematics 85 (Birkhäuser, Boston, MA, 1990) 95–110.
15. J. B. Conrey and K. Soundararajan, 'Real zeros of quadratic Dirichlet L-functions', *Invent. Math.* 150 (2002) 1–44.
16. D. W. Farmer, 'Long mollifiers of the Riemann zeta-function', *Mathematika* 40 (1993) 71–87.
17. D. W. Farmer, 'Mean value of Dirichlet series associated with holomorphic cusp forms', *J. Number Theory* 49 (1994) 209–245.
18. A. Fujii, 'On a mean value theorem in the theory of the Riemann zeta function', *Comment. Math. Univ. St. Paul.* 44 (1995) 59–67.
19. A. Fujii, 'On the Berry conjecture', *J. Math. Kyoto Univ.* 37 (1997) 55–98.
20. S. M. Gonek, 'Mean values of the Riemann zeta function and its derivatives', *Invent. Math.* 75 (1984) 123–141.

21. D. R. Heath-Brown, 'The fourth power moment of the Riemann zeta-function', *Proc. London Math. Soc.* (3) 38 (1979) 385–422.
22. C. P. Hughes, 'Random matrix theory and discrete moments of the Riemann zeta function', *Random matrix theory, J. Phys. A* 36 (2003) 2907–2917.
23. C. P. Hughes, J. P. Keating and N. O'Connell, 'Random matrix theory and the derivative of the Riemann zeta function', *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 456 (2000) 2611–2627.
24. H. Iwaniec, W. Luo and P. Sarnak, 'Low lying zeros of families of L-functions', *Inst. Hautes Études Sci. Publ. Math.* 91 (2000) 55–131.
25. H. Iwaniec and P. Sarnak, 'Dirichlet L-functions at the central point', *Number theory in progress* 2, Zakopane-Kościelisko, 1997 (ed. K. Gyory, H. Iwaniec and J. Urbanowicz; de Gruyter, Berlin, 1999) 941–952.
26. H. Iwaniec and P. Sarnak, 'The non-vanishing of central values of automorphic L-functions and Landau–Siegel zeros', *Israel J. Math.* 120 (2000) 155–177.
27. N. M. Katz and P. Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, American Mathematical Society Colloquium Publications 45 (AMS, Providence, RI, 1999).
28. J. P. Keating, 'Statistics of quantum eigenvalues and the Riemann zeros', *Supersymmetry and trace formulae: chaos and disorder* (ed. I. V. Lerner, J. P. Keating and D. E. Khmelnitskii; Plenum Press, New York, 1999) 1–15.
29. J. P. Keating and N. C. Snaith, 'Random matrix theory and $\zeta(1/2 + it)$', *Comm. Math. Phys.* 214 (2000) 57–89.
30. J. P. Keating and N. C. Snaith, 'Random matrix theory and L-functions at $s = 1/2$', *Comm. Math. Phys.* 214 (2000) 91–110.
31. E. Kowalski, P. Michel and J. VanderKam, 'Non-vanishing of high derivatives of automorphic L-functions at the center of the critical strip', *J. reine angew. Math.* 526 (2000) 1–34.
32. E. Kowalski, P. Michel and J. VanderKam, 'Mollification of the fourth moment of automorphic L-functions and arithmetic applications', *Invent. Math.* 142 (2000) 95–151.
33. N. Levinson, 'More than one third of zeros of Riemann's zeta-function are on $\sigma = 1/2$', *Adv. Math.* 13 (1974) 383–436.
34. M. L. Mehta, *Random matrices*, 2nd edn (Academic Press, Boston, MA, 1991).
35. P. Michel and J. VanderKam, 'Non-vanishing of high derivatives of Dirichlet L-functions at the central point', *J. Number Theory* 81 (2000) 130–148.
36. H. L. Montgomery, 'The pair correlation of zeros of the zeta function', *Analytic number theory*, St. Louis, MO, 1972 (ed. H. G. Diamond), Proceedings of Symposia in Pure Mathematics 24 (American Mathematical Society, Providence, RI, 1973) 181–193.
37. A. M. Odlyzko, 'On the distribution of spacings between zeros of the zeta function', *Math. Comp.* 48 (1987) 273–308.
38. A. E. Özlük and C. Snyder, 'On the distribution of the nontrivial zeros of quadratic L-functions close to the real axis', *Acta Arith.* 91 (1999) 209–228.
39. Z. Rudnick and P. Sarnak, 'Zeros of principal L-functions and random matrix theory', A celebration of John F. Nash Jr, *Duke Math. J.* 81 (1996) 269–322.
40. K. Soundararajan, 'Nonvanishing of quadratic Dirichlet L-functions at $s = \frac{1}{2}$', *Ann. of Math.* (2) 152 (2000) 447–488.

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