Math. Proc. Camb. Phil. Soc. (1985), **97**. 385
Printed in Great Britain

A simpler proof of Levinson's theorem

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(*Received* 18 *June* 1984)

1. *Introduction*

In this paper we present a proof of the mean-value theorem required by Levinson to show that at least one-third of the zeros of $\zeta(s)$ are on the critical line. As in Levinson [**3**], let

$$G(s) = \zeta(s) + \frac{\zeta'(s)}{-\chi'/\chi(s)}, \tag{1}$$

where $\chi(s) = \zeta(s)/\zeta(1-s)$ is the usual factor from the functional equation, and let

$$\psi(s) = \sum_{k \leqslant y} \frac{b(k)}{k^s}, \tag{2}$$

where
$$b(k) = \frac{\mu(k) \log y/k}{k^{\frac{1}{2}-a} \log y} \quad (k \leqslant y) \tag{3}$$

and
$$y = T^{\frac{1}{2}} \mathscr{L}^{-20} \tag{4}$$

where
$$\mathscr{L} = \log \frac{T}{2\pi}. \tag{5}$$

The vast majority of Levinson's paper is devoted to the evaluation of

$$I = \int_T^{T+U} |\psi G(a+it)|^2 \, dt, \tag{6}$$

where
$$U = T\mathscr{L}^{-10} \tag{7}$$

and
$$a = \frac{1}{2} - \frac{R}{\mathscr{L}} \tag{8}$$

where R is a certain constant (which is eventually chosen to be 1·3). Levinson's approach is to develop an approximate functional equation for G, which splits it into finite sums from which arise three different integrals upon considering $|G|^2$; the integral arising from the 'cross' term causes the most difficulty. Even the easier integrals are somewhat bothersome, the difficulty being of an arithmetic nature. When the results from the three integrals are added together a massive cancellation occurs. Then some arithmetic sums involving the Möbius function have to be evaluated to give the final result.

The portion of the proof we shall consider is that which precedes the evaluation of the arithmetic sums. We give a completely different treatment which avoids the approximate functional equation. The only 'hard' part is the use of large sieve estimates to bound some error terms, but what is required here is already well-known. Moreover, the main terms arise essentially at the outset; one can proceed almost immediately to the arithmetic part of Levinson's argument which should properly be regarded as the difficult part of his proof.

Briefly, what we do is write the integral as a complex integral and then move the line of integration to the right of $\sigma = 1$ where the integrand can be expressed as a Dirichlet series multiplied by $\chi(1-s)$. We integrate term-by-term using something like the relation:

$$\frac{1}{2\pi i}\int_{c+iT}^{c+i(T+U)} \chi(1-s)\, r^{-s}\, ds \sim e(-r)$$

for $T \leqslant 2\pi r \leqslant T+U$, where $e(x) = \exp(2\pi i x)$. Then we have to estimate a coefficient sum, for which we use Perron's formula. The main terms then arise from the residues of the poles of the generating function. Since the coefficients have oscillating terms from the exponential function at imaginary arguments, we have to do some work to determine the principal parts of the generating function. However, this is not difficult; it is analogous to the same determination for Estermann's function

$$\sum_{n=1}^{\infty} d(n)\, e(nh/k)\, n^{-s},$$

and this is classical (see Estermann [2]).

For ease of reference we state the precise result we prove.

THEOREM. *With I as in* (6) *and $\delta = R/\mathscr{L}$ we have*

$$I = U(\alpha_0 S_0 + (\alpha_1+\alpha_2) S_1 + \alpha_3 S_2) + U\left(\frac{T}{2\pi}\right)^{2\delta} (\beta_0 K_0 + (\beta_1+\beta_2) K_1 + \beta_3 K_2) + O(U\mathscr{L}^{-5}),$$

where (*just as in Levinson* ([3]; § 10))

$$\alpha_0 = \frac{-1}{1-2a} + C_1 - \frac{2}{\mathscr{L}^2(1-2a)^3} + \frac{C_4}{\mathscr{L}^2} - \frac{2}{\mathscr{L}(1-2a)^2} - \frac{C_9}{\mathscr{L}},$$

$$\alpha_1+\alpha_2 = \frac{2}{\mathscr{L}^2(1-2a)^2} + \frac{C_3}{\mathscr{L}^2} + \frac{2}{\mathscr{L}(1-2a)} - \frac{C_8}{\mathscr{L}},$$

$$\alpha_3 = \frac{-1}{\mathscr{L}^2(1-2a)} + \frac{C_2}{\mathscr{L}^2},$$

$$\beta_0 = \frac{2}{\mathscr{L}^2(1-2a)^3} + \frac{C_7}{\mathscr{L}^2}, \quad \beta_1+\beta_2 = \frac{2}{\mathscr{L}^2(1-2a)^2} + \frac{C_6}{\mathscr{L}^2},$$

and

$$\beta_3 = \frac{1}{\mathscr{L}^2(1-2a)} + \frac{C_5}{\mathscr{L}^2},$$

for certain bounded functions C_i which depend only on T and R, and

$$S_0 = \Sigma \frac{b(j)\,b(k)}{j^{2a}k^{2a}} (j,k)^{2a},$$

$$S_1 = \Sigma \frac{b(j)\,b(k)}{j^{2a}k^{2a}} (j,k)^{2a} \log \frac{j}{(j,k)},$$

$$S_2 = \Sigma \frac{b(j)\,b(k)}{j^{2a}k^{2a}}(j,k)^{2a}\log\frac{j}{(j,k)}\log\frac{k}{(j,k)},$$

$$K_0 = \Sigma \frac{b(j)\,b(k)}{jk}(j,k)^{2-2a},$$

$$K_1 = \Sigma \frac{b(j)\,b(k)}{jk}(j,k)^{2-2a}\log\frac{j}{(j,k)},$$

$$K_2 = \Sigma \frac{b(j)\,b(k)}{jk}(j,k)^{2-2a}\log\frac{j}{(j,k)}\log\frac{k}{(j,k)}.$$

2. *Auxiliary Lemmas*

LEMMA 1. *Suppose that* $A(s) = \sum_{n=1}^{\infty} a(n)\,n^{-s}$ *for* $\sigma = \mathrm{Re}\,s > 1$, *where*

$$a(n) \ll \tau_k(n)\,(\log n)^l\dagger,$$

and that $B(s) = \sum_{n\leqslant y} b(n)\,n^{-s}$ *where* $b(n) \ll 1$ *and* $T^{\frac{1}{2}-1/k} \ll y \ll T^{\frac{1}{2}}$. *Then*

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s)\,B(1-s)\,A(s)\,ds = \sum_{n\leqslant y}\frac{b(n)}{n}\sum_{m\leqslant nT/2\pi} a(m)\,e(-m/n) + O(T^{c-\frac{1}{2}}y^c(c-1)^{-k-l} + T^{\frac{1}{2}}y^{2-c}(\log T)^{k+l}).$$

This is lemma 1 of Conrey, Ghosh, and Gonek [**1**].

LEMMA 2. *Suppose that* $\mathscr{U} \geqslant 2$. *Then for any complex numbers* a_n *we have*

$$\sum_{q\leqslant Q}\frac{1}{\phi(q)}\sum_{\psi}{}^{*}\int_{-\mathscr{U}}^{\mathscr{U}} \Big|\sum_n a_n\,\psi(n)\,n^{-it}\Big|^2\frac{dt}{\frac{1}{2}+|t|} \ll \sum_n (n+Q\log\mathscr{U})\,|a_n|^2$$

where Σ^* *indicates that the sum is over primitive characters* mod q.

This is a well-known consequence of the large sieve.

LEMMA 3. *Suppose that* $\mathscr{U} \geqslant 2$ *and* $|\sigma-\frac{1}{2}| \ll (\log q\mathscr{U})^{-1}$. *Then*

$$\sum_{\psi}{}^{*}\int_{-\mathscr{U}}^{\mathscr{U}} |L(\sigma+it,\psi)|^4\frac{dt}{\frac{1}{2}+|t|} \ll \phi(q)\,(\log q\mathscr{U})^5$$

and

$$\sum_{\psi}{}^{*}\int_{-\mathscr{U}}^{\mathscr{U}} |L'(\sigma+it,\psi)|^4\frac{dt}{\frac{1}{2}+|t|} \ll \phi(q)(\log q\mathscr{U})^9.$$

This is well-known, too (see Vaughan [**4**], for example).

LEMMA 4. *Suppose that* $A_j(s) = \Sigma a_j(n)\,n^{-s}$ *for* $\sigma > 1$ *and*

$$A(s) = \prod_{j=1}^{J} A_j(s) = \Sigma a(m)\,m^{-s}.$$

Then

$$\Sigma a(md)\,m^{-s} = \sum_{d_1\ldots d_J=d}\prod_{j=1}^{J}\sum_{\substack{m=1\\ (m,\prod_{i<j} d_i)=1}}^{\infty} a_j(md_j)\,m^{-s}.$$

This is lemma 3 of Conrey, Ghosh, and Gonek [**1**].

† $\tau_k(\;)$ denotes the k-fold divisor function often denoted $d_k(\;)$.

3. *Main terms*

We first express I as a complex integral. Let $1-s = a-it$, so that $a+it = s-2\delta$ where

$$\delta = \tfrac{1}{2}-a = R\mathscr{L}^{-1}. \tag{9}$$

Then

$$I = \int_T^{T+U} \psi G(a+it)\,\psi G(a-it)\,dt$$
$$= \frac{1}{i}\int_{1-a+iT}^{1-a+i(T+U)} \psi G(s-2\delta)\,\psi G(1-s)\,ds. \tag{10}$$

Now by (1) and the functional equation for $\zeta(s)$,

$$G(1-s) = \chi(1-s)\frac{\zeta'(s)}{\chi'/\chi(s)}, \tag{11}$$

so that

$$I = \frac{1}{i}\int_{1-a+iT}^{1-a+i(T+U)} \chi(1-s)\,\psi(s-2\delta)\,\psi(1-s)\,\zeta'(s)\,\frac{G(s-2\delta)}{\chi'/\chi(s)}\,ds. \tag{12}$$

We simplify this expression via the relation

$$\frac{\chi'}{\chi}(s) = -\log\frac{|t|}{2\pi} + O\left(\frac{1}{1+|t|}\right), \tag{13}$$

which holds uniformly for bounded σ. Therefore,

$$\frac{1}{\chi'/\chi(s)} = \frac{-1}{\mathscr{L}} + O(\mathscr{L}^{-12}) \tag{14}$$

for $T \leqslant t \leqslant T+U$ and bounded σ, so that by (1)

$$\frac{G(s-2\delta)}{\chi'/\chi(s)} = \frac{H(s-2\delta)}{-\mathscr{L}} + O(\mathscr{L}^{-12}\,|\zeta(s-2\delta)| + \mathscr{L}^{-13}\,|\zeta'(s-2\delta)|) \tag{15}$$

for t in the same range, where

$$H(s) = H(s,T) = \zeta(s) + \frac{\zeta'(s)}{\mathscr{L}}. \tag{16}$$

Now $\chi(1-s) \ll 1$ for $s = 1-a+it$, $T \leqslant t \leqslant T+U$, and

$$\int_T^{T+U} |\zeta(\sigma+it)|^4\,dt \ll U\mathscr{L}^4, \tag{17}$$

$$\int_T^{T+U} |\zeta'(\sigma+it)|^4\,dt \ll U\mathscr{L}^8, \tag{18}$$

and

$$\int_T^{T+U} |\psi(\sigma+it)|^4\,dt \ll U\mathscr{L}^4, \tag{19}$$

for $\sigma = \tfrac{1}{2}+O(\mathscr{L}^{-1})$. (The first two of these are classical and the third follows from the mean-value theorem for Dirichlet polynomials

$$\int_{T_1}^{T_2} |\sum_n a_n n^{it}|^2\,dt = \sum_n (T_2-T_1+O(n))\,|a_n|^2. \tag{20}$$

Alternatively, they are consequences of Lemmas 2 and 3.) Therefore, by (12), (15), and Hölder's inequality,

$$I = \frac{-1}{i\mathscr{L}}\int_{1-a+iT}^{1-a+i(T+U)} \chi(1-s)\,\psi(s-2\delta)\,\psi(1-s)\,\zeta'(s)\,H(s-2\delta)\,ds + O(U\mathscr{L}^{-8}). \quad (21)$$

Now we want to move the line of integration to the right of 1. Let Γ be the path consisting of the three line segments $\{\sigma+iT\colon 1-a \leqslant \sigma \leqslant c\}$, $\{c+it\colon T \leqslant t \leqslant T+U\}$, and $\{\sigma+i(T+U)\colon c \geqslant \sigma \geqslant 1-a\}$ where

$$c = 1+4R\mathscr{L}^{-1}. \quad (22)$$

By Cauchy's theorem, the integral in (21) is equal to the integral taken over the path Γ. On the horizontal segments of Γ we have the trivial estimates

$$\chi(1-s) \ll T^{\sigma-\frac{1}{2}}, \quad (23)$$

$$\psi(s) \ll y^{1-\sigma}\mathscr{L}, \quad (24)$$

$$\zeta(s), \zeta'(s) \ll T^{\frac{1}{2}(1-\sigma)}\mathscr{L} \quad (25)$$

whence by (4), (7), and (21),

$$I = \frac{-1}{i\mathscr{L}}\int_{c+iT}^{c+i(T+U)} \chi(1-s)\,\psi(s-2\delta)\,\psi(1-s)\,\zeta'(s)\,H(s-2\delta)\,ds + O(U\mathscr{L}^{-6}). \quad (26)$$

Now we apply Lemma 1 with

$$A(s) = \psi(s-2\delta)\,\zeta'(s)\,H(s-2\delta) = \Sigma a(n)\,n^{-s} \quad (27)$$

for $\sigma > 1+2\delta$. (Note that $H(s) = \Sigma h(n)\,n^{-s}$, $h(n) = 1-\mathscr{L}^{-1}\log n$, is absolutely convergent for $\sigma > 1$.) Thus,

$$a(n) \ll \tau_3(n)\log n$$

so that by Lemma 1,

$$I = \frac{-2\pi}{\mathscr{L}}\sum_{k\leqslant y}\frac{b(k)}{k}\Sigma\, b(j)\,j^{2\delta}h(m)\,m^{2\delta}\log n\, e\left(-\frac{jmn}{k}\right) + O(U\mathscr{L}^{-6}), \quad (28)$$

where the inner sum is for all products jmn such that $j \leqslant y$ and

$$\frac{Tk}{2\pi} \leqslant jmn \leqslant \frac{(T+U)\,k}{2\pi}. \quad (29)$$

Thus
$$I = -\frac{2\pi}{\mathscr{L}}\sum_{k\leqslant y}\frac{b(k)}{k}\left(M\left(\frac{T+U}{2\pi}, k\right) - M\left(\frac{T}{2\pi}, k\right)\right) + O(U\mathscr{L}^{-6}), \quad (30)$$

where
$$M(x,k) = \sum_{\substack{jmn\leqslant xk\\ j\leqslant y}} b(j)\,j^{2\delta}h(m)\,m^{2\delta}\log n\, e(-jmn/k). \quad (31)$$

We evaluate $M(x,k)$ by Perron's formula. Thus, we consider the generating function

$$V(s,k) = \sum_{m,n,j}\frac{b(j)}{j^{s-2\delta}}\frac{h(m)}{m^{s-2\delta}}\frac{\log n}{n^s}e(-mnj/k) \quad (32)$$

($\sigma > 1+2\delta$). We will show that $V(s,k)$ is meromorphic with a double pole at $s = 1$, a double pole at $s = 1+2\delta$ and no other poles. The main term for $M(x)$ arises from the residues at these poles. To exhibit Levinson's main term as soon as possible, we calculate the principal parts of $V(s,k)$ and determine the main parts of $M(x,k)$ and I now, and later we treat the error terms which arise from the use of Perron's formula.

We have

$$V(s,k) = \sum_{j\leqslant y} \frac{b(j)}{j^{s-2\delta}} W(s, J/K) \tag{33}$$

where $j/k = J/K$, $(J,K) = 1$, and

$$W(s, J/K) = \sum_{m,n} \frac{h(m)}{m^{s-2\delta}} \frac{\log n}{n^s} e(-mnJ/K). \tag{34}$$

It is clear that a pole of V is also a pole of W. To determine the poles of W we let

$$\zeta(s,a,K) = \sum_{n\equiv a \bmod K} n^{-s} \tag{35}$$

and recall that $\zeta(s,a,K)$ is analytic everywhere except for a simple pole at $s = 1$ with residue K^{-1}. Thus,

$$\zeta(s,a,K) - K^{-s}\zeta(s) \tag{36}$$

is entire. Now

$$\begin{aligned} W(s,J/K) &= \sum_{a,b=1}^{K} e(-abJ/K) \sum_{m\equiv a(K)} h(m)\, m^{2\delta-s} \sum_{n\equiv b(K)} n^{-s}\log n \\ &= -\sum_{a,b=1}^{K} e\left(\frac{-abJ}{K}\right)\left(1+\frac{1}{\mathscr{L}}\frac{d}{ds}\right)\zeta(s-2\delta,a,K)\frac{d}{ds}\zeta(s,b,K), \end{aligned} \tag{37}$$

whence, by (36),

$$\sum_{a,b=1}^{K} e\left(\frac{-abJ}{K}\right)\left(1+\frac{1}{\mathscr{L}}\frac{d}{ds}\right)(\zeta(s-2\delta,a,K)-K^{2\delta-s}\zeta(s-2\delta))\frac{d}{ds}(\zeta(s,b,K)-K^{-s}\zeta(s)) \tag{38}$$

is entire. Since $\zeta(s,K,K) = K^{-s}\zeta(s)$ and

$$\sum_{a=1}^{K} e\left(-\frac{an}{K}\right) = \begin{cases} K & \text{if} \quad K|n \\ 0 & \text{if} \quad K\nmid n, \end{cases}$$

it follows that the expression in (38) is

$$\begin{aligned} &= W(s,J/K) - K\left(1+\frac{1}{\mathscr{L}}\frac{d}{ds}\right)(K^{2\delta-s}\zeta(s-2\delta))\frac{d}{ds}(K^{-s}\zeta(s)) \\ &= W(s,J/K) - Z(s,K), \end{aligned} \tag{39}$$

say. Thus the principal parts of $W(s,J/K)$ are the same as the principal parts of $Z(s,K)$. In particular, we see that W has double poles at $s = 1$ and $s = 1+2\delta$ and no other poles.

Now, by Perron's formula, with $x \ll T$ and c as in (22), $\mathscr{U} = T^{10}$,

$$\begin{aligned} M(x,k) &= \frac{1}{2\pi i}\int_{c-i\mathscr{U}}^{c+i\mathscr{U}} V(s,k)\,(xk)^s\frac{ds}{s} + O_\epsilon\left(x^\epsilon\left(1+\frac{x}{\mathscr{U}}\right)\right) \\ &= \frac{1}{2\pi i}\int_{\mathscr{R}} V(s,k)\,(xk)^s\frac{ds}{s} - \frac{1}{2\pi i}\int_{\Gamma} V(s,k)\,(xk)^s\frac{ds}{s} + O_\epsilon(T^\epsilon), \end{aligned} \tag{40}$$

where $\mathscr{R}$ is the rectangle with vertices $c \pm i\mathscr{U}$, $\frac{1}{2} \pm i\mathscr{U}$ and Γ consists of the three line segments $\{\sigma - i\mathscr{U}: c \geqslant \sigma \geqslant \frac{1}{2}\}$, $\{\frac{1}{2}+it: -\mathscr{U} \leqslant t \leqslant \mathscr{U}\}$, and $\{\sigma + i\mathscr{U}: \frac{1}{2} \leqslant \sigma \leqslant c\}$. The integral over Γ is the error term to be treated later. By (33) and (38),

$$\frac{1}{2\pi i}\int_{\mathscr{R}} V(s,k)\,(xk)^s\frac{ds}{s} = \sum_{j\leqslant y}\frac{b(j)}{j^{-2\delta}}\int_{\mathscr{R}} Z(s,K)\left(\frac{xk}{j}\right)^s\frac{ds}{s}$$
$$= \sum_{j\leqslant y}\frac{b(j)}{j^{-2\delta}}\left(\operatorname*{res}_{s=1}+\operatorname*{res}_{s=1+2\delta}\right)\left(Z(s,K)\left(\frac{xk}{j}\right)^s s^{-1}\right). \qquad (41)$$

We compute these residues. By (39),

$$Z(s,K) = K^{1+2\delta-2s}\left(\left(1-\frac{\log K}{\mathscr{L}}\right)\zeta(s-2\delta)+\frac{1}{\mathscr{L}}\zeta'(s-2\delta)\right)(\zeta'(s)-\zeta(s)\log K). \qquad (42)$$

Thus,

$$\operatorname*{Res}_{s=1} Z(s,K)\,(xK/J)^s s^{-1}$$
$$=\frac{-xK^{2\delta}}{J}\left(\left(\log\frac{x}{JK}+\log K-1\right)\left(\left(1-\frac{\log K}{\mathscr{L}}\right)\zeta(1-2\delta)+\frac{1}{\mathscr{L}}\zeta'(1-2\delta)\right)\right.$$
$$\left.+\left(1-\frac{\log K}{\mathscr{L}}\right)\zeta'(1-2\delta)+\frac{1}{\mathscr{L}}\zeta''(1-2\delta)\right)$$
$$=-\frac{xK^{2\delta}}{J}\left(\left(\log x-1\right)\left(\zeta(1-2\delta)+\frac{1}{\mathscr{L}}\zeta'(1-2\delta)\right)+\zeta'(1-2\delta)+\frac{1}{\mathscr{L}}\zeta''(1-2\delta)\right.$$
$$-(\log J)\left(\zeta(1-2\delta)+\frac{1}{\mathscr{L}}\zeta'(1-2\delta)\right)$$
$$\left.+\frac{(\log K)}{\mathscr{L}}(\zeta(1-2\delta)(1-\log x)-\zeta'(1-2\delta))+(\log K)(\log J)\frac{\zeta(1-2\delta)}{\mathscr{L}}\right), \qquad (43)$$

and

$$\operatorname*{Res}_{s=1+2\delta} Z(s,K)\,(xK/J)^s s^{-1}$$
$$=\frac{(x/J)^{1+2\delta}}{1+2\delta}\left(\left(-\log\frac{x}{JK}+\frac{1}{1+2\delta}+\mathscr{L}-\log K\right)\frac{1}{\mathscr{L}}(\zeta'(1+2\delta)-\zeta(1+2\delta)\log K)\right.$$
$$\left.-\frac{1}{\mathscr{L}}(\zeta''(1+2\delta)-\zeta'(1+2\delta)\log K)\right)$$
$$=\frac{-(x/J)^{1+2\delta}}{(1+2\delta)\mathscr{L}}\left(\left(\log x-\mathscr{L}-\frac{1}{1+2\delta}\right)\zeta'(1+2\delta)+\zeta''(1+2\delta)-(\log J)\zeta'(1+2\delta)\right.$$
$$-(\log K)\left(\zeta'(1+2\delta)-\zeta(1+2\delta)\left(\mathscr{L}+\frac{1}{1+2\delta}-\log x\right)\right)$$
$$\left.+(\log K)(\log J)\zeta(1+2\delta)\right). \qquad (44)$$

Now by (41), (43) and (44),

$$\sum_{k\leqslant y}\frac{b(k)}{k}\frac{1}{2\pi i}\int_{\mathscr{R}} V(s,k)\,(xk)^s\frac{ds}{s}$$
$$=-x\sum_{j,k\leqslant y}\frac{b(j)\,b(k)}{j^{1-2\delta}k^{1-2\delta}}(j,k)^{1-2\delta}(\alpha_0'+\alpha_1'\log J+\alpha_2'\log K+\alpha_3'\log J\log K)$$
$$+\frac{-x^{1+2\delta}}{1+2\delta}\sum_{j,k\leqslant y}\frac{b(j)\,b(k)}{jk}(j,k)^{1+2\delta}(\beta_0'+\beta_1'\log J+\beta_2'\log K+\beta_3'\log J\log K), \qquad (45)$$

where

$$\left.\begin{aligned}
\alpha_0' &= (\log x - 1)\,(\zeta(1-2\delta) + \mathscr{L}^{-1}\zeta'(1-2\delta)) + \zeta'(1-2\delta) + \mathscr{L}^{-1}\zeta''(1-2\delta),\\
\alpha_1' &= -\zeta(1-2\delta) - \mathscr{L}^{-1}\zeta'(1-2\delta),\\
\alpha_2' &= \mathscr{L}^{-1}(\zeta(1-2\delta)\,(1-\log x) - \zeta'(1-2\delta))\\
\alpha_3' &= \mathscr{L}^{-1}\zeta(1-2\delta),\\
\beta_0' &= \mathscr{L}^{-1}(\log x - \mathscr{L} - (1+2\delta)^{-1})\,\zeta'(1+2\delta) + \mathscr{L}^{-1}\zeta''(1+2\delta),\\
\beta_1' &= -\mathscr{L}^{-1}\zeta'(1+2\delta),\\
\beta_2' &= -\mathscr{L}^{-1}(\zeta'(1+2\delta) - \zeta(1+2\delta)\,(\mathscr{L} + (1+2\delta)^{-1} - \log x)),\\
\text{and}\quad \beta_3' &= \mathscr{L}^{-1}\zeta(1+2\delta).
\end{aligned}\right\} \tag{46}$$

We will later show that

$$\sum_{k\leqslant y} \frac{b(k)}{k} \int_\Gamma V(s,k)\,(xk)^s s^{-1}\,ds \ll x^{\frac{1}{2}} y \mathscr{L}^5. \tag{47}$$

If we assume this for now, then we have, by (40) and (45), that

$$\begin{aligned}\sum_{k\leqslant y} \frac{b(k)}{k} M(x,k) = &-x(S_0\alpha_0' + S_1(\alpha_1' + \alpha_2') + S_3'\alpha_3')\\
&+ \frac{-x^{1+2\delta}}{1+2\delta}(K_0\beta_0' + K_1(\beta_1' + \beta_2') + K_3\beta_3') + O(x^{\frac{1}{2}} y \mathscr{L}^5)\end{aligned} \tag{48}$$

where we have used the notation of the Theorem. We consider the difference of this expression when $x = (T+U)/2\pi$ and when $x = T/2\pi$. Note that by the mean-value theorem,

$$\left.\begin{aligned}
&\frac{T+U}{2\pi}\log\frac{T+U}{2\pi} - \frac{T}{2\pi}\log\frac{T}{2\pi} = \frac{U}{2\pi}(\mathscr{L} + 1 + O(\mathscr{L}^{-10})),\\
&\left(\frac{T+U}{2\pi}\right)^{1+2\delta} - \left(\frac{T}{2\pi}\right)^{1+2\delta} = \frac{U}{2\pi}(1+2\delta)\left(\left(\frac{T}{2\pi}\right)^{2\delta} + O(\mathscr{L}^{-11})\right),\\
&\left(\frac{T+U}{2\pi}\right)^{1+2\delta}\log\left(\frac{T+U}{2\pi}\right) - \left(\frac{T}{2\pi}\right)^{1+2\delta}\log\frac{T}{2\pi}\\
&\qquad = \frac{U}{2\pi}\left(\frac{T}{2\pi}\right)^{2\delta}(1+2\delta)\left(\mathscr{L} + \frac{1}{1+2\delta} + O(\mathscr{L}^{-10})\right).
\end{aligned}\right\} \tag{49}$$

Thus, by (30), (40), and (45)–(49), and since $\zeta(s) = (s-1)^{-1} + C + O(|s-1|)$ for s near 1 and some C, we have the Theorem as soon as we validate the claim made in (47).

4. *Large sieve estimates*

We first illustrate the proof of (47) by an argument which leaves out complicating details. We have, by (32),

$$V(s,k) = \Sigma a(l)\,e(-l/k)\,l^{-s} \tag{51}$$

where

$$a(l) = \sum_{\substack{jmn=l\\ j\leqslant y}} b(j)\,j^{-2\delta}h(m)\,m^{-2\delta}\log n. \tag{52}$$

If $(l,k) = 1$, then

$$e(-l/k) = \frac{1}{\phi(k)}\sum_{\chi \bmod k} \tau(\bar{\chi})\,\chi(-l). \tag{53}$$

Thus, aside from the l which have a prime factor in common with k, $V(s,k)$ is essentially

$$V_1(s,k) = \frac{1}{\phi(k)} \sum_{\chi \bmod k} \tau(\bar{\chi})\,\chi(-1) \sum_l a(l)\,\chi(l)\,l^{-s}$$
$$= \frac{1}{\phi(k)} \sum_{\chi \bmod k} \tau(\bar{\chi})\,\chi(-1)\,B(s,\chi)\left(1+\frac{1}{\mathscr{L}}\frac{d}{ds}\right) L(s-2\delta,\chi)\left(\frac{-d}{ds}\right) L(s,\chi), \qquad (54)$$

where
$$B(s,\chi) = \Sigma b(j)\,\chi(j)\,j^{-s}. \qquad (55)$$

Now by Hölder's inequality and since

$$|\tau(\bar{\chi})| \leqslant k^{\frac{1}{2}}, \qquad (56)$$

we have, with $\mathscr{U} = T^{10}$,

$$\sum_{k \leqslant y} \frac{b(k)}{k} \int_{\frac{1}{2}-i\mathscr{U}}^{\frac{1}{2}+i\mathscr{U}} V_1(s,k) \frac{(xk)^s}{s}\,ds$$
$$\ll x^{\frac{1}{2}} \left(\sum_{k \leqslant y} \frac{1}{\phi(k)} \sum_{\chi} \int_{-\mathscr{U}}^{\mathscr{U}} |B(s,\chi)|^2 \frac{dt}{\frac{1}{2}+|t|^{\frac{1}{2}}} \right)^{\frac{1}{2}} \left(\sum_{k \leqslant y} \frac{1}{\phi(k)} \sum_{\chi} \int_{-\mathscr{U}}^{\mathscr{U}} |L'(s,\chi)|^4 \frac{dt}{\frac{1}{2}+|t|} \right)^{\frac{1}{4}}$$
$$\times \left(\sum_{k \leqslant y} \frac{1}{\phi(k)} \sum_{\chi} \int_{-\mathscr{U}}^{\mathscr{U}} (|L(s-2\delta,\chi)|^4 + \mathscr{L}^4 |L'(s-2\delta,\chi)|^4) \frac{dt}{\frac{1}{2}+|t|} \right)^{\frac{1}{4}}. \qquad (57)$$

If the large sieve estimates of Lemmas 2 and 3 were valid for sums taken over all characters instead of primitive characters then the above would be $\ll x^{\frac{1}{2}} y \mathscr{L}^4$ as required. Unfortunately, the details to be filled in (i.e. the inclusion of terms with $(l,k) > 1$ and the reduction to primitive characters) are a bit complicated.

Instead of (53), we use the formula (5·12) of Conrey, Ghosh, and Gonek [**1**],

$$e(-l/k) = \sum_{q|k} \sum_{\psi \bmod q}^{*} \tau(\bar{\psi}) \sum_{\substack{d|l \\ d|k}} \psi\left(\frac{l}{d}\right) \epsilon(q,k,d,\psi) \qquad (58)$$

where
$$|\epsilon(q,k,d,\psi)| \leqslant \phi\left(\frac{k}{(d,k/q)}\right)^{-1}. \qquad (59)$$

Then, by (51),

$$V(s,k) = \sum_{q|k} \sum_{\psi}^{*} \tau(\bar{\psi}) \sum_{d|k} \frac{\epsilon(q,k,d,\psi)}{d^s} \sum_{l=1}^{\infty} \frac{a(ld)\,\psi(l)}{l^s}. \qquad (60)$$

Now, by (52) and Lemma 4, the sum over l is

$$= \sum_{d_1 d_2 d_3 = d} \left(\sum_{m=1}^{\infty} \frac{h(md_1)\,\psi(m)}{m^{s-2\delta}} \right) \left(\sum_{\substack{m=1 \\ (m,d_1)=1}}^{\infty} \frac{\psi(m) \log md_2}{m^s} \right) \left(\sum_{\substack{m \leqslant y/d_3 \\ (m,d_1 d_2)=1}} \frac{b(md_3)\,\psi(m)}{m^{s-2\delta}} \right)$$
$$= \sum_{d_1 d_2 d_3 = d} d_1^s \left(1 + \mathscr{L}^{-1} \frac{d}{ds}\right) (d_1^{-s} L(s-2\delta,\psi))$$
$$\times d_2^s \frac{d}{ds} (d_2^{-s} L(s,\psi) (\Phi(s,\psi,d_1)) B(s-2\delta,\psi,d_1 d_2,d_3), \qquad (61)$$

where
$$\Phi(s,\psi,d) = \prod_{p|d} (1-\psi(p)\,p^{-s}) \ll \tau(d) \qquad (62)$$

for $\sigma \geqslant 0$, and

$$B(s,\psi,d,e) = \sum_{\substack{m \leqslant y/e \\ (m,d)=1}} b(me)\,\psi(m)\,m^{-s}.$$

Thus

$$\Sigma a(ld)\,\psi(l)\,l^{-s} \ll \tau_4(d)\log^2 d(|L(s,\psi)|+|L'(s,\psi)|)$$
$$\times\left(|L(s-2\delta,\psi)|+\left|\frac{L'(s-2\delta,\psi)}{\mathscr{L}}\right|\right)|\mathscr{B}(s-2\delta,\psi)|, \quad (64)$$

where
$$\mathscr{B}(s,\psi)=\max_{\beta}\left|\sum_{m\leqslant y}\frac{\beta(m)\,\psi(m)}{m^s}\right| \quad (65)$$

and the max is to be taken over uniformly bounded complex numbers $\beta(m)$.

We now bound the integral in (47) taken over the horizontal segments of Γ. We have

$$L(s,\psi),L'(s,\psi)\ll(q\mathscr{U})^{\frac{1}{4}} \quad (66)$$

for $\sigma\geqslant\frac{1}{2}-O((\log\mathscr{U})^{-1})$, $t=\mathscr{U}$; also

$$\mathscr{B}(s,\psi)\ll y^{1-\sigma}\log y. \quad (67)$$

Thus, by (60) and (64),

$$\sum_k\frac{b(k)}{k}V(s,k)\ll\sum_k\frac{1}{k}\sum_{q\mid k}q^{\frac{1}{2}}\phi(q)\sum_{d\mid k}\tau_4(d)\,(q\mathscr{U})^{\frac{1}{2}}y^{\frac{1}{2}}\log^3 y$$
$$\ll_\epsilon y^{\frac{5}{4}+\epsilon}\mathscr{U}^{\frac{1}{2}}, \quad (68)$$

whence
$$\sum_{k\leqslant y}\frac{b(k)}{k}\int_{\frac{1}{2}\pm i\mathscr{U}}^{c\pm i\mathscr{U}}V(s,k)\frac{(xk)^s\,ds}{s}\ll y^2\mathscr{U}^{-\frac{1}{2}}\ll 1. \quad (69)$$

Now we consider the integral on the $\frac{1}{2}$-line. Note that for $\sigma=\frac{1}{2}$, by (59) and (60),

$$\Sigma b(k)\,k^{-1}\,V(s,k)\,(xk)^s\ll x^{\frac{1}{2}}\sum_{k\leqslant y}k^{-\frac{1}{2}}\sum_{q\mid k}q^{\frac{1}{2}}\sum_{\psi}{}^{*}\sum_{d\mid k}\frac{d^{-\frac{1}{2}}}{\phi(k/(d,k/q))}\left|\sum_{l=1}^{\infty}a(ld)\,\psi(l)\,l^{-s}\right|$$
$$\ll x^{\frac{1}{2}}\sum_{k\leqslant y}k^{-\frac{1}{2}}\sum_{q\leqslant y/k}\sum_{\psi}{}^{*}\sum_{d\mid kq}\frac{d^{-\frac{1}{2}}}{\phi(kq/(d,k))}\left|\sum_{l=1}^{\infty}a(ld)\psi(l)\,l^{-s}\right|. \quad (70)$$

Now,
$$\sum_{d\mid kq}\frac{d^{-\frac{1}{2}}\tau_4(d)\log^2 d}{\phi(kq/(d,k))}=\sum_{g\mid k}\frac{g^{-\frac{1}{2}}}{\phi(kq/g)}\sum_{d\mid q}d^{-\frac{1}{2}}\tau_4(dg)\log^2 dg$$
$$\ll\frac{k^{-\frac{1}{2}}}{\phi(q)}\sum_{g\mid k}\frac{g^{\frac{1}{2}}\tau_4(g)}{\phi(g)}(1+\log^2 g)\sum_{d\mid q}\frac{\tau_4(d)\,(1+\log^2 d)}{d^{\frac{1}{2}}}$$
$$=\frac{k^{-\frac{1}{2}}}{\phi(q)}f_1(k)f_2(q), \quad (71)$$

say. Thus, for $\sigma=\frac{1}{2}$, by (64), (70), and (71).

$$\Sigma b(k)\,k^{-1}\,V(s,k)\,(xk)^s\ll x^{\frac{1}{2}}\sum_{k\leqslant y}\frac{f_1(k)}{k}\mathscr{A}(s,y/k), \quad (72)$$

where

$$\mathscr{A}(s,Q)=\sum_{q\leqslant Q}\frac{f_2(q)}{\phi(q)}\sum_{\psi}{}^{*}(|L(s,\psi)|+|L'(s,\psi)|)\,(|L(s2-\delta,\psi)|+|L'(s-2\delta,\psi)|)\,|\mathscr{B}(s-\delta,\psi)|. \quad (73)$$

By Hölder's inequality, for $s = \frac{1}{2}+it$,

$$\begin{aligned}\int_{-\mathscr{U}}^{\mathscr{U}} \mathscr{A}(s,Q)\frac{ds}{s} \ll &\left(\sum_{q\leqslant Q}\frac{f_2(q)^2}{\phi(q)}\sum_{\psi}{}^{*}\int_{-\mathscr{U}}^{\mathscr{U}}(|L(s,\psi)|^4+|L'(s,\psi)|^4\frac{dt}{\frac{1}{2}+|t|}\right)^{\frac{1}{4}}\\ &\left(\sum_{q\leqslant Q}\frac{f_2(q)^2}{\phi(q)}\sum_{\psi}{}^{*}\int_{-\mathscr{U}}^{\mathscr{U}}\left(|L(s-2\delta)|^4+\frac{|L'(s-2\delta,\psi)|^4}{\mathscr{L}^4}\right)\frac{dt}{\frac{1}{2}+|t|}\right)^{\frac{1}{4}}\\ &\left(\sum_{q\leqslant Q}\frac{1}{\phi(q)}\sum_{\psi}{}^{*}\int_{-\mathscr{U}}^{\mathscr{U}}|\mathscr{B}(s-2\delta,\psi)|^2\frac{dt}{\frac{1}{2}+|t|}\right)^{\frac{1}{2}}\\ = &\,\mathscr{I}_1(Q)^{\frac{1}{4}}\mathscr{I}_2(Q)^{\frac{1}{4}}\mathscr{I}_3(Q)^{\frac{1}{2}},\end{aligned} \tag{74}$$

say. By Lemma 3,

$$\mathscr{I}_1(Q) \ll \sum_{q\leqslant Q} f_2(q)^2(\log Q\mathscr{U})^9 \tag{75}$$

and

$$\mathscr{I}_2(Q) \ll \sum_{q\leqslant Q} f_2(q)^2(\log Q\mathscr{U})^5, \tag{76}$$

while by Lemma 2 and (65)

$$\mathscr{I}_3(Q) \ll \sum_{n\leqslant y}(n+Q\log\mathscr{U})\,n^{-1+4\delta} \ll y+Q\log^2\mathscr{U}. \tag{77}$$

Now, by (71),

$$\begin{aligned}\sum_{q\leqslant Q} f_2(q)^2 &\ll \sum_{q\leqslant Q}\Big(\sum_{d|q} d^{-\frac{1}{3}}\Big)^2 = \sum_{d\leqslant Q} d^{-\frac{1}{3}}\sum_{q\leqslant Q/d}\sum_{e|dq} e^{-\frac{1}{3}}\\ &\ll \sum_{d\leqslant Q} d^{-\frac{1}{3}}\Big(\sum_{e|d} e^{-\frac{1}{3}}\Big)\sum_{q\leqslant Q/d}\Big(\sum_{e|q} e^{-\frac{1}{3}}\Big)\\ &\ll \sum_{d\leqslant Q} d^{-\frac{1}{3}}\tau(d)\sum_{e\leqslant Q/d} e^{-\frac{1}{3}}\sum_{q\leqslant Q/de} 1\\ &\ll Q\sum_{d\leqslant Q} d^{-\frac{4}{3}}\tau(d)\sum_{e\leqslant Q} e^{-\frac{4}{3}} \ll Q,\end{aligned} \tag{78}$$

the second line following by the submultiplicativity of $\sigma_{-\frac{1}{3}}$, which is trivial to verify. Hence, by (75)–(78),

$$\begin{aligned}\int_{-\mathscr{U}}^{\mathscr{U}} \mathscr{A}(s,Q)\frac{ds}{s} &\ll Q^{\frac{1}{2}}(\log Q\mathscr{U})^{\frac{7}{2}}(y+Q\log^2\mathscr{U})^{\frac{1}{2}}\\ &\ll Q\mathscr{L}^5+Q^{\frac{1}{2}}y^{\frac{1}{2}}\mathscr{L}^4.\end{aligned} \tag{79}$$

Thus, by (72) and (79), for $s = \frac{1}{2}+it$,

$$\sum_{k\leqslant y}\frac{b(k)}{k}\int_{-\mathscr{U}}^{\mathscr{U}}|V(s,k)\,(xk)^s s^{-1}|\,dt \ll x^{\frac{1}{2}}\sum_{k\leqslant y}\frac{f_1(k)}{k}\left(\frac{y}{k}+\frac{y}{k^{\frac{1}{2}}}\right)\mathscr{L}^5,$$

and since $f_1(k) \ll k^{\frac{1}{4}}$ trivially by (71), this is $\ll x^{\frac{1}{2}}y\mathscr{L}^5$, which proves (47).

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