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THE SIXTH POWER MOMENT OF DIRICHLET L-FUNCTIONS

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Abstract. We prove a formula, with power savings, for the sixth moment of Dirichlet L- functions averaged over all primitive characters $\chi \pmod q$ with $q \le Q$, and over the critical line. Our formula agrees precisely with predictions motivated by random matrix theory. In particular, the constant 42 appears as a factor in the leading order term, exactly as is predicted for the sixth moment of the Riemann zeta-function.

1 Introduction

The study of moments of the Riemann zeta-function and related L-functions has a long history tracing back to the classical work of Hardy and Littlewood. Hardy and Littlewood established an asymptotic formula for the second moment $\int_0^T |\zeta(\frac{1}{2}+it)|^2\,dt$, and Ingham established an asymptotic formula for the fourth moment $\int_0^T |\zeta(\frac{1}{2}+it)|^4\,dt$. Finding asymptotic formulae for the sixth or higher moments of the zeta-function remains an outstanding open problem. The story for moments of L-functions in families is similar, and in many cases a few small moments have been evaluated asymptotically. In general it is difficult even to conjecture an asymptotic formula, and it is only in recent years that a satisfactory picture of the structure of moments has emerged.

The breakthrough came from the work of Keating and Snaith [KS00a, KS00b] who modeled moments of L-values by values of the characteristic polynomials of large random matrices drawn from appropriate classical compact groups. The choice of the group was suggested by the work of Katz and Sarnak [KS99] on the symmetry types for the distribution of zeros of L-functions in families. In this way Keating and Snaith identified the leading order asymptotics for the $2k$th moment of $\zeta(\frac{1}{2}+it)$, and their conjecture agreed with the results of Hardy and Littlewood, and Ingham for $k=1$ and 2, and conjectures derived (using heuristics for shifted divisor problems) by Conrey and Ghosh (for $k=3$, [CG98]) and Conrey and Gonek (for $k=4$, [CG01]). While the Keating–Snaith conjecture gives only the leading order term for

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the moments of L-functions, subsequent work by Conrey et al. [CFKRS05] has led to more precise conjecture for integer moments where the entire asymptotic expansion is identified. An alternative approach, based on multiple Dirichlet series, and leading to the same conjectures was proposed by Diaconu et al. [DGH03]. Towards these conjectures, in many situations we have a lower bound for moments of the conjectured order of magnitude (see [RS05,RS06]), and assuming the truth of the Generalized Riemann Hypothesis we have a corresponding upper bound of almost the right order of magnitude [Sou09]. Further, there is extensive numerical data supporting the conjectures of [CFKRS05], and here it is important to know the full asymptotic expansion because in the range of computations asymptotically lower order terms still contribute significantly.

For the sixth moment of $\zeta(\frac{1}{2}+it)$, the Keating–Snaith conjecture (in this case, the same conjecture was made earlier by Conrey and Ghosh) predicts that

$$\int_0^T |\zeta(1/2+it)|^6\,dt \sim 42\prod_p\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)T\frac{\log^9 T}{9!}. \tag{1}$$

The more precise version due to Conrey, Farmer, Keating, Rubinstein and Snaith predicts that for any $\epsilon > 0$,

$$\int_0^T |\zeta(1/2+it)|^6\,dt = \int_0^T P_3\left(\log\frac{t}{2\pi}\right)\,dt + O(T^{1/2+\epsilon}) \tag{2}$$

where P_3 is a polynomial of degree 9 whose exact coefficients are specified as complicated infinite products and series over primes, and which is given approximately by

$$P_3(x) \approx 0.000005708\,x^9 + 0.0004050\,x^8 + 0.01107\,x^7 + 0.1484\,x^6 \\ +1.0459\,x^5 + 3.9843\,x^4 + 8.6073\,x^3 + 10.2743\,x^2 + 6.5939\,x + 0.9165.$$

As remarked earlier, the precise conjecture (2) is better suited for numerical testing than (1). For example,

$$\int_0^{2{,}350{,}000} |\zeta(1/2+it)|^6\,dt \approx 3{,}317{,}496{,}016{,}044.9 \approx 3.3\times 10^{12}$$

and this compares well with

$$\int_0^{2{,}350{,}000} P_3\left(\log\frac{t}{2\pi}\right)\,dt \approx 3{,}317{,}437{,}762{,}612.4,$$

whereas

$$42\prod_p\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)\times 2{,}350{,}000\times\frac{(\log 2{,}350{,}000)^9}{9!}=4.22\times 10^{11}$$

is nowhere near the prediction.

The proof of formula (1) appears beyond the reach of current technology. Our goal in this paper is to establish an analog of (2) for Dirichlet L-functions suitably averaged. Our formula agrees exactly with the conjecture of [CFKRS05] and so provides, we hope, a new glimpse into the mechanics of moments. We first give a corollary of our work, postponing the more precise technical result to the next section.

Let $\chi \pmod q$ be an even, primitive Dirichlet character and let

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}$$

be its associated L-function. This L-function satisfies the functional equation

$$\Lambda\left(\frac{1}{2}+s,\chi\right):=\left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)L\left(\frac{1}{2}+s,\chi\right)=\epsilon_\chi\Lambda\left(\frac{1}{2}-s,\overline{\chi}\right)$$

where ϵ_χ is a complex number of absolute value 1. Throughout the paper we shall use $\sum^\flat$ to indicate that a sum is over primitive, even Dirichlet characters, and $\phi^\flat(q)$ will denote the number of primitive even Dirichlet characters $\pmod q$. The restriction to even characters is merely a matter of convenience, and we could equally well consider odd characters making appropriate changes to our argument (for example, the Γ-factor in the functional equation will be different). In analogy with (1), we have the following conjecture from [CFKRS05] [note that when $q\equiv 2 \pmod 4$ there are no primitive Dirichlet characters $\pmod q$]:

Conjecture 1. *Put*

$$a_3=\prod_p\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right). \tag{3}$$

Then, as $q\to\infty$ with $q\not\equiv 2 \pmod 4$,

$$\frac{1}{\phi^\flat(q)}\sum_{\chi \bmod q}^\flat\left|L\left(\frac{1}{2},\chi\right)\right|^6\sim 42a_3\prod_{p|q}\frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)}\frac{(\log q)^9}{9!}.$$

Towards this Conjecture, we shall establish:

Corollary 1. *For large Q we have*

$$\sum_{q\le Q}\ \sideset{}{^\flat}\sum_{\chi \bmod q}\int_{-\infty}^{\infty}\left|\Lambda\left(\frac{1}{2}+iy,\chi\right)\right|^6 dy$$

$$\sim 42a_3\sum_{q\le Q}\prod_{p|q}\frac{(1-\frac{1}{p})^5}{(1+\frac{4}{p}+\frac{1}{p^2})}\phi^\flat(q)\frac{(\log q)^9}{9!}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1/2+iy}{2}\right)\right|^6 dy.$$

The number of primitive characters (mod q) is given by $\phi^*(q)=\sum_{dr=q}\mu(d)\phi(r)$, and the number of even primitive characters is $\phi^\flat(q)=\phi^*(q)/2+O(1)$. Thus one can express the main term in Corollary 1 as

$$\sim 42\widetilde{a_3}\frac{Q^2}{2}\frac{\log^9 Q}{9!}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1/2+iy}{2}\right)\right|^6 dy$$

where

$$\widetilde{a_3}=\prod_p\left(1-\frac{1}{p}\right)^5\left(1+\frac{5}{p}-\frac{5}{p^2}+\frac{14}{p^3}-\frac{15}{p^4}+\frac{5}{p^5}+\frac{4}{p^6}-\frac{4}{p^7}+\frac{1}{p^8}\right).$$

However, the form in which we have written Corollary 1 is more suggestive as it reveals that we have established an average form of Conjecture 1, by introducing an average over the moduli q and also an average over points $1/2+iy$. The average over q significantly increases the size of our family of L-functions—from a family of about q L-functions of conductor q we move to a family of about Q^2 L-functions with conductor up to Q. The average over points $1/2+iy$ is more benign—the completed L-function $\Lambda(1/2+iy)$ decays exponentially as $|y|$ increases, and so this average should be thought of as involving only points within a constant distance from the real axis. Nevertheless the average over y is necessary for our argument to work, and it would be nice to develop a corresponding result just at the point $1/2$. In particular, because of the additional averaging over y, we do not conclude anything new about non-vanishing results at the central point.

Recall the large sieve inequality

$$\sum_{q\le Q}\frac{q}{\phi(q)}\ \sideset{}{^*}\sum_{\chi \bmod q}\left|\sum_{n=1}^{N}a_n\chi(n)\right|^2\le (Q^2+N)\sum_{n=1}^{N}|a_n|^2$$

where $\sum^\star$ indicates that the sum is restricted to primitive characters. As an application of the large sieve, Huxley [Hux70] proved that

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q}\left|L\left(\frac{1}{2},\chi\right)\right|^6\ll Q^2\log^9 Q$$

and

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^8 \ll Q^2 \log^{16} Q.$$

Our work may be seen as a refinement of Huxley's sixth moment estimate, and in this regard the current paper is a companion to [CISa] and where we develop similar "asymptotic large sieves" in other contexts. A challenging problem is to obtain a similar asymptotic formula in place of Huxley's estimate for the eighth moment.

2 The Sixth Moment With Shifts

In this section we recall the conjecture from [CFKRS05] for the $2k$th moment of Dirichlet L-functions, and then state our main theorem from which Corollary 1 follows. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$ be two vectors of k complex numbers each, and we suppose that $|\mathrm{Re}(\alpha_j)|, |\mathrm{Re}(\beta_j)| \le 1/4$ for all j. We shall also write $\beta_j = \alpha_{k+j}$, and think of the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as defining a $2k$-tuple $(\alpha_1, \dots, \alpha_{2k})$. Given $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we define

$$\Lambda(s, \chi; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_{j=1}^{k} \Lambda(s + \alpha_j, \chi) \Lambda(s - \beta_j, \overline{\chi}),$$

and set

$$\Lambda(\chi; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \Lambda\left(\frac{1}{2}, \chi; \boldsymbol{\alpha}, \boldsymbol{\beta}\right).$$

Note that any permutation of the coordinates of $\boldsymbol{\alpha}$, and any permutation of the coordinates of β, leaves $\Lambda(s, \chi; \boldsymbol{\alpha}, \boldsymbol{\beta})$ unaltered. Moreover the functional equation gives $\Lambda(s, \chi; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \Lambda(1 - s, \chi; \boldsymbol{\beta}, \boldsymbol{\alpha})$. When $s = 1/2$, the function $\Lambda(\chi; \boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies further symmetry properties. To see this, note that the permutation group S_{2k} acts naturally on the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Writing this pair as a $2k$-tuple, for $\pi \in S_{2k}$ we define $\pi(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha_{\pi(1)}, \dots, \alpha_{\pi(2k)})$ and then take the first k coordinates to be $\pi(\boldsymbol{\alpha})$ and the second k coordinates to be $\pi(\boldsymbol{\beta})$. Now note that the functional equation

$$\Lambda(\chi; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \Lambda(\chi; \pi(\boldsymbol{\alpha}), \pi(\boldsymbol{\beta})),$$

holds for all permutations $\pi \in S_{2k}$. Thus $\Lambda(\chi; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is invariant under S_{2k} while $\Lambda(s, \chi; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is invariant under the subgroup $S_k \times S_k$.

We wish to state the conjecture from [CFKRS05] on the average value of $\Lambda(\chi; \boldsymbol{\alpha}, \boldsymbol{\beta})$. Naturally, the conjectured answer must share the S_{2k}-symmetry described above.

Define

$$\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2} \sum_{j=1}^{k} (\alpha_j - \beta_j), \tag{4}$$

and put

$$G(s;\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{j=1}^{k} \Gamma\left(\frac{s}{2}+\frac{\alpha_j}{2}\right)\Gamma\left(\frac{s}{2}-\frac{\beta_j}{2}\right), \tag{5}$$

so that

$$\Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(\frac{q}{\pi}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} G\left(\frac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\right)\prod_{j=1}^{k} L\left(\frac{1}{2}+\alpha_j,\chi\right)L\left(\frac{1}{2}-\beta_j,\overline{\chi}\right).$$

We define a generalized sum-of-divisors function by

$$\sigma(n;\boldsymbol{\alpha}) = \sum_{n=n_1\cdots n_k} n_1^{-\alpha_1}\cdots n_k^{-\alpha_k}, \tag{6}$$

so that we may write, when the real part of s is sufficiently large

$$\prod_{j=1}^{k} L(s+\alpha_j,\chi)L(s-\beta_j,\overline{\chi}) = \sum_{m,n=1}^{\infty} \frac{\sigma(m;\boldsymbol{\alpha})}{m^s}\frac{\sigma(n;-\boldsymbol{\beta})}{n^s}\chi(m)\overline{\chi(n)}.$$

If we average the above over even primitive characters χ, a candidate for the answer would be the diagonal terms $m=n$ with $(m,q)=(n,q)=1$: namely,

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\sigma(n;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{n^{2s}} = \prod_{p\nmid q}\mathcal{B}_p(s;\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where the Euler factor $\mathcal{B}_p$ is given by

$$\mathcal{B}_p(s;\boldsymbol{\alpha},\boldsymbol{\beta}) := \sum_{r=0}^{\infty} \frac{\sigma(p^r;\boldsymbol{\alpha})\sigma(p^r;-\boldsymbol{\beta})}{p^{2rs}} = \int_0^1 \prod_{j=1}^{k}\left(1-\frac{e(\theta)}{p^{s+\alpha_j}}\right)^{-1}\prod_{\ell=1}^{k}\left(1-\frac{e(-\theta)}{p^{s-\beta_\ell}}\right)^{-1} d\theta. \tag{7}$$

The behavior of this Euler product can be understood by comparing it with an appropriate product of zeta functions. For a prime number p let $\zeta_p(x) = (1-p^{-x})^{-1}$, and define

$$\mathcal{Z}_p(s;\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{j=1}^{k}\prod_{\ell=1}^{k}\zeta_p(2s+\alpha_j-\beta_\ell) \quad \text{and} \quad \mathcal{Z}(s;\boldsymbol{\alpha},\boldsymbol{\beta}) := \prod_{j=1}^{k}\prod_{\ell=1}^{k}\zeta(2s+\alpha_j-\beta_\ell). \tag{8}$$

Further, let

$$\mathcal{A}(s;\boldsymbol{\alpha},\boldsymbol{\beta}) := \prod_p \mathcal{B}_p(s;\boldsymbol{\alpha},\boldsymbol{\beta})\mathcal{Z}_p(s;\boldsymbol{\alpha},\boldsymbol{\beta})^{-1}. \tag{9}$$

With a little calculation, we may see that the conditions $|\mathrm{Re}\ \alpha_j|, |\mathrm{Re}\ \beta_j| \leq 1/4$ ensure that the Euler product for $\mathcal{A}$ converges absolutely. Let $\mathcal{B}_q = \prod_{p|q} \mathcal{B}_p$, and define

$$\mathcal{Q}(q;\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(\frac{q}{\pi}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} G(\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}) \frac{\mathcal{A}\mathcal{Z}}{\mathcal{B}_q}\left(\frac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\right). \tag{10}$$

This is our candidate, based on the diagonal contribution alone, for the average of $\Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta})$.

Note that $\mathcal{Q}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ is symmetric under $S_k \times S_k$, but not under the full group S_{2k}; so it cannot be the full answer for the average of $\Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta})$. We symmetrize this by summing over all $\binom{2k}{k}$ cosets of $S_{2k}/(S_k \times S_k)$. Thus we define

$$\widetilde{\mathcal{Q}}(q;\boldsymbol{\alpha},\boldsymbol{\beta}) := \sum_{\pi \in S_{2k}/(S_k \times S_k)} \mathcal{Q}(q;\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})), \tag{11}$$

and the conjecture of Conrey et al. [CFKRS05] is that this object is in fact close to the average value of $\Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta})$.

CONJECTURE 2. *Assuming that the "shifts" $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ satisfy $|\mathrm{Re}\ \alpha_j|, |\mathrm{Re}\ \beta_j| \leq 1/4$, and $\mathrm{Im}\ \alpha_j, \mathrm{Im}\ \beta_j \ll q^{1-\epsilon}$, we conjecture that*

$$\sideset{}{^\flat}\sum_{\chi \bmod q} \Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \phi^\flat(q)\widetilde{\mathcal{Q}}(q;\boldsymbol{\alpha},\boldsymbol{\beta})(1 + O(q^{-1/2+\epsilon})) \tag{12}$$

where $\sum^\flat$ denotes a sum over even primitive characters.

Even though $\mathcal{Q}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ has singularities (when $\alpha_j = \beta_\ell$) the symmetrized $\widetilde{\mathcal{Q}}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ is in fact analytic in α_j, β_ℓ. Thus one can let all the shifts tend to zero in Conjecture 2, and obtain a conjecture for the $2k$th moment of $L(\frac{1}{2},\chi)$. We refer to [CFKRS05] for the details of this calculation, and note that when $k = 3$ this is what leads to Conjecture 1 stated earlier.

Given $\boldsymbol{\alpha}$ we define $\boldsymbol{\alpha} + s$ to be the translated k-tuple $(\alpha_1 + s, \dots, \alpha_k + s)$, and similarly $\boldsymbol{\beta} + s = (\beta_1 + s, \dots, \beta_k + s)$. Now we are ready to state our main theorem.

Theorem 1. *Let Q be large, and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be 3-tuples with α_j, $\beta_j \ll 1/\log Q$. Let Ψ be a smooth function compactly supported in $[1,2]$. Then*

$$\sum_q \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \sideset{}{^\flat}\sum_{\chi} \Lambda(\chi;\boldsymbol{\alpha}+iy,\boldsymbol{\beta}+iy)\, dy$$
$$= \sum_q \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \phi^\flat(q)\widetilde{\mathcal{Q}}(q;\boldsymbol{\alpha}+iy,\boldsymbol{\beta}+iy)\, dy + O(Q^{19/10+\epsilon}).$$

We could equally well prove a theorem for odd primitive characters. The answer would be similar with just the Gamma-factors changed slightly to reflect the difference in the functional equation for odd primitive Dirichlet L-functions. When the shifts are all 0, this difference disappears in the leading order main term.

In what follows, we focus on establishing Theorem 1. From this, Corollary 1 follows by letting the shifts tend to zero, and by making Ψ approximate the characteristic function of $[1,2]$ and replacing Q by $Q/2, Q/4, \ldots$ Since the calculation of letting the shifts tend to zero is entirely analogous to the derivation of Conjecture 1 from Conjecture 2 in [CFKRS05], we omit the details.

3 The "Approximate" Functional Equation

We formulate a general approximate functional equation for shifted products of L-functions, which we shall later specialize to the case when $k=3$. Let

$$H(s;\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{j=1}^{k}\prod_{\ell=1}^{k}\left(s^2-\left(\frac{\alpha_j-\beta_\ell}{2}\right)^2\right)^3, \tag{13}$$

and put for any $\xi>0$

$$W(\xi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{2\pi i}\int_{(1)} G\left(\frac{1}{2}+s;\boldsymbol{\alpha},\boldsymbol{\beta}\right) H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\xi^{-s}\frac{ds}{s}. \tag{14}$$

Put

$$\Lambda_0(\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(\frac{q}{\pi}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\sum_{m,n=1}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})}{\sqrt{m}}\frac{\sigma(n;-\boldsymbol{\beta})}{\sqrt{n}}\chi(m)\overline{\chi}(n)W\left(\frac{mn\pi^k}{q^k};\boldsymbol{\alpha},\boldsymbol{\beta}\right). \tag{15}$$

Proposition 1. With notation as above, we have

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \Lambda_0(\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) + \Lambda_0(\chi;\boldsymbol{\beta},\boldsymbol{\alpha}).$$

Proof. We begin with

$$\frac{1}{2\pi i}\int_{(1)}\Lambda\left(\frac{1}{2}+s,\chi;\boldsymbol{\alpha},\boldsymbol{\beta}\right)H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\frac{ds}{s}.$$

We move the line of integration to $\mathrm{Re}(s)=-1$, encountering a pole at $s=0$ which leaves the residue $\Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta})H(0;\boldsymbol{\alpha},\boldsymbol{\beta})$. For the remaining integral on the -1-line, we use the functional equation $\Lambda(\frac{1}{2}+s,\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \Lambda(\frac{1}{2}-s,\chi;\boldsymbol{\beta},\boldsymbol{\alpha})$, together with $H(s;\boldsymbol{\alpha},\boldsymbol{\beta}) = H(-s;\boldsymbol{\beta},\boldsymbol{\alpha})$ to conclude that this integral equals (replacing $-s$ by w)

$$-\frac{1}{2\pi i}\int_{(1)}\Lambda\left(\frac{1}{2}+w,\chi;\boldsymbol{\beta},\boldsymbol{\alpha}\right)H\left(w;\boldsymbol{\beta},\boldsymbol{\alpha}\right)\frac{dw}{w}.$$

We conclude that

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{2\pi i}\int\limits_{(1)} \left(\Lambda\left(\frac{1}{2}+s,\chi;\boldsymbol{\alpha},\boldsymbol{\beta}\right)H(s;\boldsymbol{\alpha},\boldsymbol{\beta}) + \Lambda\left(\frac{1}{2}+s,\chi;\boldsymbol{\beta},\boldsymbol{\alpha}\right)H(s;\boldsymbol{\beta},\boldsymbol{\alpha})\right)\frac{ds}{s}.$$

Expanding the L-functions into their Dirichlet series we obtain the proposition. □

For any positive real numbers ξ, η and μ let us define

$$V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\xi,\eta;\mu) = \left(\frac{\mu}{\pi}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\int\limits_{-\infty}^{\infty}\left(\frac{\eta}{\xi}\right)^{it} W\left(\frac{\xi\eta\pi^k}{\mu^k};\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\right)dt. \tag{16}$$

Further, set

$$\Lambda_1(\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{m,n=1}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})}{\sqrt{m}}\frac{\sigma(n;-\boldsymbol{\beta})}{\sqrt{n}}\chi(m)\overline{\chi}(n)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m,n;q). \tag{17}$$

Proposition 2. With notation as above, we have

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\int\limits_{-\infty}^{\infty}\Lambda(\chi;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it)\,dt = \Lambda_1(\chi;\boldsymbol{\alpha},\boldsymbol{\beta}) + \Lambda_1(\chi;\boldsymbol{\beta},\boldsymbol{\alpha}).$$

Proof. This follows readily from Proposition 1 and the observation $\sigma(n;\boldsymbol{\alpha}+it) = \sigma(n;\boldsymbol{\alpha})n^{-it}$. □

Our proof of Theorem 1, starts from the approximate functional equation given in Proposition 2 with $k=3$. We shall analyze the terms $\Lambda_1(\chi;\boldsymbol{\alpha},\boldsymbol{\beta})$ and $\Lambda_1(\chi;\boldsymbol{\beta},\boldsymbol{\alpha})$ when averaged over all characters χ with conductor of size Q. Notice that while the original function $\Lambda(\chi;\boldsymbol{\alpha},\boldsymbol{\beta})$ has S_{2k} symmetry, the terms arising from Proposition 2 have only $S_k \times S_k$ symmetry. Thus at the outset, the symmetry of the final answer has been lost, and this is one reason that the analysis of the main terms in our argument is complicated and we have to work hard to recover the symmetry in the end. It would be interesting to develop an approach which maintains the symmetry throughout the argument, but we don't know how to do this.

In our later work, it will be useful to have an understanding of the weights W and V defined above.

Lemma 1. *The weight $W(\xi;\boldsymbol{\alpha},\boldsymbol{\beta})$ is a smooth function of $\xi \in (0,\infty)$. For large values of ξ, and any non-negative integer ν, we have $W^{(\nu)}(\xi;\boldsymbol{\alpha},\boldsymbol{\beta}) \ll_\nu \exp(-c_0\xi^{\frac{1}{k}})$ for some absolute constant $c_0 > 0$. Further, we have*

$$V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\xi,\eta;\mu) \ll \left(\frac{\mu}{\pi}\right)^{\mathrm{Re}\,\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\exp\left(-c_1\left(\frac{\max(\xi,\eta)^2}{\mu^k}\right)^{\frac{1}{k}}\right).$$

Proof. For any non-negative integer ν and any $c > 0$ we have

$$\frac{d^\nu}{d\xi^\nu} W(\xi; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2\pi i} \int_{(c)} G\left(\frac{1}{2} + s; \boldsymbol{\alpha}, \boldsymbol{\beta}\right) H(s; \boldsymbol{\alpha}, \boldsymbol{\beta}) \left(\frac{d^\nu}{d\xi^\nu} \xi^{-s}\right) \frac{ds}{s}.$$

It follows that W is smooth in ξ. Moreover, for ξ large we choose $c = \xi^{1/k}$, and then a calculation using Stirling's formula reveals that the above is $\ll_\nu \exp(-c_0 \xi^{1/k})$ as stated.

To prove the estimate for $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, assume without loss of generality that $\eta \geq \xi$. In the definition (16) we let $z = it$ denote a complex variable, and insert the definition of W. Thus, using $H(s; \boldsymbol{\alpha} + z, \boldsymbol{\beta}_z) = H(s; \boldsymbol{\alpha}, \boldsymbol{\beta})$, we find that

$$V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\xi, \eta; \mu) = \left(\frac{\mu}{\pi}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} \frac{1}{i} \int_{-i\infty}^{i\infty} \left(\frac{\eta}{\xi}\right)^z \frac{1}{2\pi i} \int_{(c)} G\left(\frac{1}{2} + s; \boldsymbol{\alpha} + z, \boldsymbol{\beta} + z\right) H(s; \boldsymbol{\alpha}, \boldsymbol{\beta}) \times \left(\frac{\xi \eta \pi^k}{\mu^k}\right)^{-s} \frac{ds}{s}.$$

Now we may take the line of integration for z to be any line $\mathrm{Re}(z) = d$, and the line of integration for s to be any $c > 0$, and so long as $|d| \leq c$ no poles of the Γ-functions present in G will be encountered. In the situation when $\eta \geq \xi$, we choose $c = (\eta^2/\mu^k)^{1/k}$ and $d = -c$, and then a calculation with Stirling's formula gives our desired bound. □

To close this section, we remark on why the integral over y is needed in our Theorem. Specializing to the case $k = 3$, in the approximate functional equation of Proposition 1 the variables m and n are restricted so that their product is about size q^3, and it is possible for one variable to be very large (about size q^3) while the other remains small. Such a range is not amenable to the argument we develop. In contrast, the integrated approximate functional equation of Proposition 2 leads (thanks to Lemma 1) to terms where m and n are both at most $q^{\frac{3}{2}+\epsilon}$.

4 First Steps Towards Theorem 1

Henceforth we assume that $k = 3$ and that the shifts $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy $\mathrm{Re}\, \alpha_j, \mathrm{Re}\, \beta_j \ll 1/\log Q$. Our aim is to evaluate asymptotically

$$\mathcal{I} = \mathcal{I}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) := H(0; \boldsymbol{\alpha}, \boldsymbol{\beta}) \sum_q \sideset{}{^\flat}\sum_{\chi \bmod q} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Lambda(\chi; \boldsymbol{\alpha} + it, \boldsymbol{\beta} + it)\, dt, \quad (18)$$

where Ψ is a fixed smooth function, compactly supported in $[1, 2]$. Using Proposition 2, we may write

$$\mathcal{I}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \Delta(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) + \Delta(\Psi, Q; \boldsymbol{\beta}, \boldsymbol{\alpha}) \quad (19)$$

where

$$\Delta(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_q \sideset{}{^\flat}\sum_{\chi \bmod q} \Psi\left(\frac{q}{Q}\right) \Lambda_1(\chi; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{20}$$

LEMMA 2. *Let $\phi^*(q)$ denote the number of primitive characters $\pmod q$, and let $\phi^\flat(q)$ denote the number of even primitive characters $\pmod q$. If m and n are integers with $(mn, q) = 1$ then*

$$\sideset{}{^\flat}\sum_{\chi \pmod q} \chi(m)\overline{\chi}(n) = \frac{1}{2} \sum_{\substack{q=dr \\ r|(m\pm n)}} \mu(d)\phi(r),$$

where we sum over both choices of sign. In particular, $\phi^*(q) = \sum_{dr=q} \mu(d)\phi(r)$, and $\phi^\flat(q) = \phi^*(q)/2 + O(1)$.

Proof. From the orthogonality relations for characters and Möbius inversion we easily see that

$$\sideset{}{^*}\sum_{\chi \pmod q} \chi(m)\overline{\chi(n)} = \sum_{\substack{q=dr \\ r|m-n}} \mu(d)\phi(r),$$

where the sum is over all primitive characters $\pmod q$. Since an even character can be detected by $\frac{1}{2}(1+\chi(-1))$ we obtain the formula for the sum over even primitive characters. □

From the definitions (16), (17) and (20), and Lemma 2 we obtain that

$$\Delta(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{\sigma(m; \boldsymbol{\alpha})}{\sqrt{m}} \frac{\sigma(n; -\boldsymbol{\beta})}{\sqrt{n}} \sum_{\substack{d,r \\ (dr,mn)=1 \\ r|m\pm n}} \phi(r)\mu(d)\Psi\left(\frac{dr}{Q}\right) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m, n; dr). \tag{21}$$

There is a diagonal contribution to (21) from the terms $m = n$, which we call $\mathcal{D}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$. For the terms $m \neq n$, we introduce a parameter D and divide the sum in (21) into two ranges depending on whether $d > D$ or $d \leq D$. Call the first set of terms $\mathcal{S}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ (for small values of r), and the second $\mathcal{G}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ (for greater values of r). Thus, we have the decomposition

$$\Delta(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{D}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) + \mathcal{S}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) + \mathcal{G}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{22}$$

Of these three terms, the diagonal contribution is the easiest to evaluate, and we treat it first.

Lemma 3. *With notations as above,*

$$\mathcal{D}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = H(0; \boldsymbol{\alpha}, \boldsymbol{\beta}) \sum_q \Psi\left(\frac{q}{Q}\right) \phi^\flat(q) \int_{-\infty}^{\infty} \mathcal{Q}(q; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it)\, dt + O(Q^{\frac{5}{4}+\epsilon}).$$

Proof. If $(n,q)=1$ then

$$\frac{1}{2} \sum_{\substack{dr=q \\ r \mid n \pm n}} \mu(d)\phi(r) = \phi^\flat(q),$$

and so we see that

$$\mathcal{D}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_q \phi^\flat(q) \Psi\left(\frac{q}{Q}\right) \sum_{(n,q)=1} \frac{\sigma(n;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{n} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n,n;q). \quad (23)$$

Recalling the definition of V from (16) we see that the sum over n above equals

$$\int_{-\infty}^{\infty} \frac{1}{2\pi i} \int_{(1)} G\left(\frac{1}{2}+s; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it\right) H(s; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it) \left(\frac{q}{\pi}\right)^{3s+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} \times \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\sigma(n;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{n^{1+2s}} \frac{ds}{s}\, dt.$$

Since

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\sigma(n;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{n^{1+2s}} = \frac{\mathcal{AZ}}{\mathcal{B}_q}\left(\frac{1}{2}+s; \boldsymbol{\alpha}, \boldsymbol{\beta}\right),$$

our expression above equals

$$\int_{-\infty}^{\infty} \frac{1}{2\pi i} \int_{(1)} G\left(\frac{1}{2}+s; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it\right) H(s; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it) \times \left(\frac{q}{\pi}\right)^{3s+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} \frac{\mathcal{AZ}}{\mathcal{B}_q}\left(\frac{1}{2}+s; \boldsymbol{\alpha}, \boldsymbol{\beta}\right) \frac{ds}{s}\, dt. \quad (24)$$

In the region $\mathrm{Re}(s) \geq -\frac{1}{4}+\epsilon$, the integrand has a simple pole in s at $s=0$. While $\mathcal{Z}(\frac{1}{2}+s; \boldsymbol{\alpha}, \boldsymbol{\beta})$ has poles at $s = -(\alpha_j - \beta_\ell)/2$ (with $1 \leq j, \ell \leq 3$), these poles are cancelled by the zeros of $H(s; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it) = H(s; \boldsymbol{\alpha}, \boldsymbol{\beta})$ at these points. Thus, moving the line of integration in (24) to $\mathrm{Re}(s) = -\frac{1}{4}+\epsilon$, we obtain

$$\int_{-\infty}^{\infty} G\left(\frac{1}{2}; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it\right) H(0; \boldsymbol{\alpha}, \boldsymbol{\beta}) \frac{\mathcal{AZ}}{\mathcal{B}_q}\left(\frac{1}{2}; \boldsymbol{\alpha}, \boldsymbol{\beta}\right) dt + O(q^{-\frac{3}{4}+\epsilon}).$$

Using this in (23), we obtain the lemma. □

Thus, $\mathcal{D}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ accounts for one of the twenty terms in our conjectured asymptotic formula for the shifted sixth moment, namely the term corresponding to the identity permutation in $S_6/S_3 \times S_3$. Similarly $\mathcal{D}(\Psi, Q; \boldsymbol{\beta}, \boldsymbol{\alpha})$ accounts for another of these twenty terms, namely the one corresponding to the involution $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \to (\boldsymbol{\beta}, \boldsymbol{\alpha})$. The remaining eighteen expressions arise from off-diagonal terms, and we now embark on analyzing their contribution.

5 Evaluating $\mathcal{S}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$

Recall that

$$\mathcal{S}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\sigma(m; \boldsymbol{\alpha})}{\sqrt{m}} \frac{\sigma(n; -\boldsymbol{\beta})}{\sqrt{n}} \sum_{\substack{d > D, r \\ (dr, mn)=1 \\ r | m \pm n}} \mu(d) \phi(r) \Psi\left(\frac{dr}{Q}\right) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m, n; dr).$$

We express the condition $r|(m \pm n)$ above using the characters $\psi \pmod r$. Thus the above equals

$$\sum_{d > D} \sum_{r} \mu(d) \Psi\left(\frac{dr}{Q}\right) \sum_{\substack{\psi \pmod r \\ \psi(-1)=1}} \sum_{\substack{m,n=1 \\ (mn,dr)=1 \\ m \neq n}}^{\infty} \psi(m)\overline{\psi}(n) \frac{\sigma(m; \boldsymbol{\alpha})}{\sqrt{m}} \frac{\sigma(n; -\boldsymbol{\beta})}{\sqrt{n}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m, n; dr). \tag{25}$$

We now isolate the contribution of the principal character $\psi = \psi_0$ above which gives rise to a main term, and we show that the non-principal characters give an acceptable error term.

Proposition 3. We have

$$\mathcal{S}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{MS}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) + O(Q^{2+\epsilon}/D),$$

where

$$\mathcal{MS}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = -\sum_{q} \left(\sum_{\substack{dr=q \\ d \leq D}} \mu(d) \right) \Psi\left(\frac{q}{Q}\right) \sum_{\substack{m,n=1 \\ (mn,q)=1 \\ m \neq n}} \frac{\sigma(m; \boldsymbol{\alpha})}{\sqrt{m}} \frac{\sigma(n; -\boldsymbol{\beta})}{\sqrt{n}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m, n; q).$$

Proof. The principal character $\psi = \psi_0$ contributes to (25) the amount

$$\sum_{q} \left(\sum_{\substack{dr=q \\ d > D}} \mu(d) \right) \Psi\left(\frac{q}{Q}\right) \sum_{\substack{m,n=1 \\ (mn,q)=1 \\ m \neq n}} \frac{\sigma(m; \boldsymbol{\alpha})}{\sqrt{m}} \frac{\sigma(n; -\boldsymbol{\beta})}{\sqrt{n}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m, n; q),$$

and since $\sum_{dr=q}\mu(d)=0$ for $q>1$ this equals the term $\mathcal{MS}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ identified in our proposition.

Now we consider the contribution of the non-principal characters to (25). To estimate this, we begin by reincorporating the terms $m=n$ to (25). The error introduced in doing this is

$$\begin{aligned}&\ll \sum_{\substack{q=dr\\ d>D}} r\Psi\left(\frac{dr}{Q}\right)\sum_n \frac{|\sigma(n;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})|}{n}|V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n,n;q)|\\ &\ll Q^\epsilon \sum_{d>D}\sum_r \Psi\left(\frac{dr}{q}\right) \ll Q^\epsilon \sum_{d>D}\frac{Q^2}{d^2} \ll \frac{Q^{2+\epsilon}}{D},\end{aligned}$$

which is acceptable for our proposition.

We now wish to estimate

$$\sum_{d>D}\sum_r \mu(d)\Psi\left(\frac{dr}{Q}\right)\sum_{\substack{\psi \pmod r\\ \psi(-1)=1\\ \psi\neq\psi_0}}\sum_{\substack{m,n=1\\ (mn,dr)=1}}^{\infty}\psi(m)\overline{\psi(n)}\frac{\sigma(m;\boldsymbol{\alpha})}{\sqrt{m}}\frac{\sigma(n;-\boldsymbol{\beta})}{\sqrt{n}}V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m,n;dr). \tag{26}$$

From the definition of $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ we see that the sum over m and n above is

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{2\pi i}\int_{(1)} G\left(\frac12+s;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\right)H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\left(\frac{dr}{\pi}\right)^{3s+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\\ &\quad\times\sum_{\substack{m,n=1\\ (mn,dr)=1}}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{(mn)^{\frac12+s}}\psi(m)\overline{\psi(n)}\frac{ds}{s}\,dt.\end{aligned}\tag{27}$$

Now the sum over m and n above gives

$$\prod_{j=1}^{3}\prod_{\ell=1}^{3}\frac{L(\frac12+\alpha_j+s,\psi)}{L_{dr}(\frac12+\alpha_j+s,\psi)}\frac{L(\frac12-\beta_\ell+s,\overline{\psi})}{L_{dr}(\frac12-\beta_\ell+s,\overline{\psi})},$$

where $L_{dr}(s,\psi)=\prod_{p|dr}(1-\psi(p)p^{-s})^{-1}$. We use this above, and since ψ is non-principal we may move the line of integration to $\mathrm{Re}(s)=\epsilon>0$ without encountering any poles. Since $G(\frac12+s;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it)$ decreases exponentially in $|t|+|\mathrm{Im}(s)|$, and since the finite Euler products in L_{dr} are $\ll Q^\epsilon$, we find that the quantity in (27) is

$$\ll Q^\epsilon\int_{(\epsilon)}\exp(-|\mathrm{Im}(s)|)\left(\sum_{j=1}^{3}\left|L\left(\frac12+\alpha_j+s,\psi\right)\right|^6+\sum_{\ell=1}^{3}\left|L\left(\frac12-\beta_\ell+s,\overline{\psi}\right)\right|^6\right)|ds|.$$

Using this in (26) we obtain that the quantity there is

$$\ll Q^{\epsilon} \sum_{r \le 2Q/D} \frac{Q}{r} \sum_{\substack{\psi \pmod r \\ \psi \ne \psi_0}} \int_{(\epsilon)} \exp(-|\mathrm{Im}(s)|)$$
$$\times \left(\sum_{j=1}^{3} \left| L\left(\frac{1}{2} + \alpha_j + s, \psi\right)\right|^6 + \sum_{\ell=1}^{3} \left| L\left(\frac{1}{2} - \beta_\ell + s, \overline{\psi}\right)\right|^6 \right) |ds|.$$

Using the large sieve we see that the above is $\ll Q^{2+\epsilon}/D$, completing our proof. □

6 Evaluating $\mathcal{G}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$: Switching to the Complementary Divisor

We now begin our treatment of $\mathcal{G}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ which is the most difficult term in our analysis. Recall that

$$\mathcal{G}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2} \sum_{\substack{m,n=1 \\ m \ne n}}^{\infty} \frac{\sigma(m; \boldsymbol{\alpha})\sigma(n; -\boldsymbol{\beta})}{\sqrt{mn}} \sum_{\substack{d,r \\ (dr,mn)=1 \\ r | m \pm n \\ d \le D}} \phi(r)\mu(d)\Psi\left(\frac{dr}{Q}\right) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m, n; dr). \tag{28}$$

To proceed further, we write $g = (m, n)$ (the gcd of m and n), and put $m = gM$ and $n = gN$ so that $(M, N) = 1$. For a given m, n, the inner sum over d and r above may be written as

$$\sum_{\substack{d \le D \\ (d, gMN)=1}} \mu(d) \sum_{\substack{r | (M \pm N) \\ (r, gMN)=1}} \phi(r)\Psi\left(\frac{dr}{Q}\right) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(gM, gN; dr). \tag{29}$$

We may express the condition that $r|(M \pm N)$ by writing $|M \pm N| = rh$, and our goal now is to eliminate r from the sum above, recasting it in terms of the complementary divisor h. The reason for doing this is that while r is large in this term, the complementary divisor h is small.

First we rewrite the arithmetic function $\phi(r)$ as $\sum_{a\ell=r} \mu(a)\ell$. Then note that the condition $(r, gMN) = 1$ is equivalent to $(r, g) = 1$ (since $(M, N) = 1$ and $r|(M \pm N)$) and this is equivalent to $(a, g) = 1$ and $(\ell, g) = 1$. Thus our expression (29) may be recast as

$$\sum_{\substack{d \le D \\ (d, gMN)=1}} \mu(d) \sum_{(a,g)=1} \mu(a) \sum_{\substack{\ell \\ |M \pm N| = a\ell h \\ (\ell, g)=1}} \ell \Psi\left(\frac{da\ell}{Q}\right) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(gM, gN; da\ell).$$

Now we express the condition that $(\ell, g) = 1$ using $\sum_{b|(\ell,g)} \mu(b)$. Thus the above equals

$$\sum_{\substack{d \le D \\ (d,gMN)=1}} \mu(d) \sum_{(a,g)=1} \mu(a) \sum_{b|g} \mu(b) \sum_{\substack{k \\ |M \pm N| = abkh}} bk\Psi\left(\frac{dabk}{Q}\right) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(gM, gN; dabk).$$

At this juncture, we may replace the sum over k by a sum over the complementary variable h with the condition that $M \pm N \equiv 0 \pmod{abh}$, and then we may write $|M \pm N|/(abh)$ in place of k. Thus we may eliminate the variable k, and recast the above as

$$\sum_{\substack{d \le D \\ (d,gMN)=1}} \sum_{(a,g)=1} \sum_{b|g} \sum_{\substack{h>0 \\ M \equiv \mp N \pmod{abh}}} \mu(d)\mu(a)\mu(b) \\ \times \left(\frac{|M \pm N|}{ah}\right) \Psi\left(\frac{|M \pm N| d}{Qh}\right) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(gM, gM; \frac{d|M \pm N|}{h}\right). \tag{30}$$

To simplify this expression we define, for non-negative real numbers u, x and y, and for each choice of sign,

$$\mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x, y; u) = u|x \pm y| \Psi(u|x \pm y|) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x, y; u|x \pm y|). \tag{31}$$

Recall from (16) (and with $k = 3$ there) that

$$V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\xi, \eta; \mu) = \left(\frac{\mu}{\pi}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} \int_{-\infty}^{\infty} \left(\frac{\eta}{\xi}\right)^{it} W\left(\frac{\xi\eta\pi^3}{\mu^3}; \boldsymbol{\alpha} + it, \boldsymbol{\beta} + it\right) dt,$$

and so, for $c > 0$,

$$c^{-2\delta(\boldsymbol{\alpha},\boldsymbol{\beta})/3} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c\xi, c\eta; \mu c^{2/3}) = V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\xi, \eta; \mu).$$

With this notation the quantity in (30) becomes

$$Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} \sum_{\substack{d \le D \\ (d,gMN)=1}} \sum_{(a,g)=1} \sum_{b|g} \sum_{\substack{h>0 \\ M \equiv \mp N \pmod{abh}}} \frac{\mu(a)\mu(b)\mu(d)}{ad} \mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{gM}{Q^{\frac{3}{2}}}, \frac{gN}{Q^{\frac{3}{2}}}; \frac{Q^{\frac{1}{2}} d}{gh}\right). \tag{32}$$

We now express the condition $M \equiv \mp N \pmod{abh}$ using characters $\psi \pmod{abh}$; note that since $(M, N) = 1$ we have $(MN, abh) = 1$. We isolate the contribution of the principal character which gives rise to main terms, and the remaining characters will contribute an acceptable error term.

Putting the above remarks together, we have shown that

$$\mathcal{G}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{MG}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) + \mathcal{EG}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}), \tag{33}$$

where the main term is

$$\mathcal{MG}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}}{2} \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{\sqrt{mn}}$$

$$\times \sum_{\substack{d\leq D\\ (d,gMN)=1}} \sum_{(a,g)=1} \sum_{b|g} \sum_{\substack{h>0\\ (abh,MN)=1}} \frac{\mu(a)\mu(b)\mu(d)}{ad\phi(abh)} \mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{gM}{Q^{\frac{3}{2}}}, \frac{gN}{Q^{\frac{3}{2}}}; \frac{Q^{\frac{1}{2}}d}{gh}\right), \tag{34}$$

and the error term is

$$\mathcal{EG}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}}{2} \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{\sqrt{mn}}$$

$$\times \sum_{\substack{d\leq D\\ (d,gMN)=1}} \sum_{(a,g)=1} \sum_{b|g} \sum_{\substack{h>0\\ (abh,MN)=1}} \frac{\mu(a)\mu(b)\mu(d)}{ad\phi(abh)}$$

$$\times \sum_{\substack{\psi \pmod{abh}\\ \psi\neq\psi_0}} \psi(M)\overline{\psi}(\mp N) \mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{gM}{Q^{\frac{3}{2}}}, \frac{gN}{Q^{\frac{3}{2}}}; \frac{Q^{\frac{1}{2}}d}{gh}\right). \tag{35}$$

7 Mellin Transforms of the Weight Function

In this section we consider three types of Mellin transforms of the weight functions $\mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x,y;u)$ which we shall find useful. The three types arise by taking Mellin transforms in (a) the variable u, (b) the variables x and y, and (c) all three variables x, y, and u. We now consider each of these in order, and establish some of their key properties.

Let us begin by noting that $\mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x,y;u)$ is smooth in u, x and y for either choice of sign. When the sign is $+$ this is clear from the definition in (31), but one may worry about the $|x-y|$ term when the sign is $-$. This does not cause trouble because Ψ is smooth and supported away from zero.

First we define

$$\widetilde{\mathcal{W}}^{\pm}_{1}(x,y;z) = \int_0^\infty \mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x,y;u) u^z \frac{du}{u}. \tag{36}$$

These transforms will be used in Section 9 in simplifying the contribution of $\mathcal{M}S(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta}) + \mathcal{M}G(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$.

Lemma 4. *Given positive real numbers x and y, the functions $\widetilde{\mathcal{W}}^{\pm}_{1}(x,y;z)$ are analytic for all $z\in\mathbb{C}$. We have the Mellin inversion formula*

$$\mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x,y;u) = \frac{1}{2\pi i}\int_{(c)} \widetilde{\mathcal{W}}^{\pm}_{1}(x,y;z) u^{-z}\, dz, \tag{37}$$

where the integral is taken over the line $Re(z) = c$*, for any real number* c*. The Mellin transforms* $\widetilde{\mathcal{W}}_1^{\pm}(x, y; z)$ *satisfy for any non-negative integer* ν

$$|\widetilde{\mathcal{W}}_1^{\pm}(x, y; z)| \ll_\nu |x \pm y|^{-Re\ z} \prod_{j=1}^{\nu} |z + j|^{-1} \exp\left(-c_1 \max(x, y)^{\frac{1}{3}}\right) \tag{38}$$

for some absolute constant c_1*.*

Proof. From the definition we have

$$\widetilde{\mathcal{W}}_1^{\pm}(x, y; z) = \int_0^\infty u|x \pm y|\Psi(u|x \pm y|)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x, y; u|x \pm y|)u^z \frac{du}{u}.$$

When $x \pm y \neq 0$ (which happens always if the choice of sign is $+$) we may make a change of variables $w = u|x \pm y|$, and then the above becomes

$$|x \pm y|^{-z} \int_0^\infty \Psi(w)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x, y; w)w^z\, dw. \tag{39}$$

Since Ψ is compactly supported away from zero, the above expression gives an analytic function of z for all $z \in \mathbb{C}$. Note that when $x - y = 0$ the transform $\widetilde{\mathcal{W}}_1^{-}(x, x; z)$ is identically zero.

The Mellin inversion formula (37) is standard. Finally, the estimate (38) follows upon integrating the expression in (39) by parts ν times, using Lemma 1 to bound the derivatives of $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. □

Next we define

$$\widetilde{\mathcal{W}}_2^{\pm}(s_1, s_2; u) = \int_0^\infty \int_0^\infty \mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\pm}(x, y; u)x^{s_1}y^{s_2} \frac{dx}{x}\frac{dy}{y}. \tag{40}$$

These transforms will be used in Section 8 to estimate the error terms in (35).

Lemma 5. *Given a positive real number* u*, the functions* $\widetilde{\mathcal{W}}_2^{\pm}(s_1, s_2; u)$ *are analytic in the region* $Re(s_1)$*,* $Re(s_2) > 0$*. We have the Mellin inversion formula*

$$\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\pm}(x, y; u) = \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} \widetilde{\mathcal{W}}_2^{\pm}(s_1, s_2; u)x^{-s_1}y^{-s_2}\, ds_1\, ds_2,$$

where c_1 *and* c_2 *are positive. The Mellin transforms* $\widetilde{\mathcal{W}_2}^{\pm}(s_1, s_2; u)$ *satisfy, for any* $k \geq 1$

$$|\widetilde{\mathcal{W}_2}^{\pm}(s_1, s_2; u)| \ll \frac{(1 + u)^{k-1}}{\max(|s_1|, |s_2|)^k} \exp(-c_1 u^{-\frac{1}{3}}).$$

Proof. For a fixed positive number u, note that $\mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x,y;u)$ is zero unless $1 \leq u|x \pm y| \leq 2$, and decreases rapidly as $\max(x,y)$ gets large. Therefore we see that the Mellin transform $\widetilde{\mathcal{W}}^{\pm}_2$ is analytic in the region $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2) > 0$. Finally, the estimate on the Mellin transform follows upon integrating by parts k times. $\square$

Finally, we define

$$\widetilde{\mathcal{W}}^{\pm}_3(s_1,s_2;z) = \int_0^\infty\int_0^\infty\int_0^\infty \mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x,y;u)u^z x^{s_1} y^{s_2}\frac{du}{u}\frac{dx}{x}\frac{dy}{y}. \tag{41}$$

Further we set

$$\widetilde{\mathcal{W}}_3(s_1,s_2;z) = \widetilde{\mathcal{W}}^{+}_3(s_1,s_2;z) + \widetilde{\mathcal{W}}^{-}_3(s_1,s_2;z). \tag{42}$$

These transforms will play a crucial role in Section 10 where we complete the evaluation of the main terms.

Lemma 6. *For brevity, set below $\omega = (s_1+s_2-z)/2$ and $\xi = (s_1-s_2+z)/2$. In the region $\mathrm{Re}(s_1)$, $\mathrm{Re}(s_2) > 0$, and $|\mathrm{Re}(s_1-s_2)| < \mathrm{Re}(z) < 1$ we have*

$$\widetilde{\mathcal{W}}^{\pm}_3(s_1,s_2;z) = \widetilde{\Psi}(1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})+3\omega+z)\frac{H(\omega;\boldsymbol{\alpha},\boldsymbol{\beta})}{2\omega\pi^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})+3\omega}} \times \int_{-\infty}^{\infty} \mathcal{H}^{\pm}(\xi-it,z)G\left(\frac{1}{2}+\omega;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\right)dt, \tag{43}$$

where

$$\mathcal{H}^{+}(u,v) = \frac{\Gamma(u)\Gamma(v-u)}{\Gamma(u)}, \quad \text{and} \quad \mathcal{H}^{-}(u,v) = \frac{\Gamma(u)\Gamma(1-v)}{\Gamma(1+u-v)} + \frac{\Gamma(v-u)\Gamma(1-v)}{\Gamma(1-u)}. \tag{44}$$

Furthermore, we have in the same region of variables

$$\widetilde{\mathcal{W}}_3(s_1,s_2;z) = \widetilde{\Psi}(1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})+3\omega+z)\frac{H(\omega;\boldsymbol{\alpha},\boldsymbol{\beta})}{2\omega\pi^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})+3\omega}} \times \int_{-\infty}^{\infty} \mathcal{H}(\xi-it,z)G\left(\frac{1}{2}+\omega;\boldsymbol{\alpha}\ +it,\boldsymbol{\beta}+it\right)dt, \tag{45}$$

where

$$\mathcal{H}(u,v) = \mathcal{H}^{+}(u,v)+\mathcal{H}^{-}(u,v) = \pi^{\frac{1}{2}}\frac{\Gamma(\frac{u}{2})\Gamma(\frac{1-v}{2})\Gamma(\frac{v-u}{2})}{\Gamma(\frac{1-u}{2})\Gamma(\frac{v}{2})\Gamma(\frac{1-v+u}{2})}. \tag{46}$$

We have the Mellin inversion formula

$$\mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x,y;u) = \frac{1}{(2\pi i)^3}\int_z\int_{s_1}\int_{s_2}\widetilde{\mathcal{W}}^{\pm}_3(s_1,s_2;z)u^{-z}x^{-s_1}y^{-s_2}\,ds_2\,ds_1\,dz \tag{47}$$

where all of the paths are taken to be the vertical lines with increasing imaginary parts and real parts satisfying the constraints given above, and the integrals over s_1 and s_2 are to be interpreted as being over $|Im(s_1)| \le T_1$ and $|Im(s_2)| \le T_2$ and letting T_1, T_2 tend to infinity. Finally, the Mellin transform $\widetilde{\mathcal{W}}_3(s_1, s_2; z)$ satisfies the bound

$$|\widetilde{\mathcal{W}_3}(s_1, s_2; z)| \ll (1+|z|)^{-A}(1+|\omega|)^{-A}(1+|\xi|)^{Re(z)-1}. \tag{48}$$

Proof. From the definitions we have

$$\widetilde{\mathcal{W}}_3^{\pm}(s_1, s_2; z) = \int_0^\infty\int_0^\infty\int_0^\infty u|x \pm y|\Psi(u|x\pm y|)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x, y; u|x\pm y|)u^z x^{s_1} y^{s_2}\frac{du}{u}\frac{dx}{x}\frac{dy}{y}.$$

The inner integral over u is the Mellin transform $\widetilde{\mathcal{W}}_1^{\pm}$ and so, as in (39), the above equals

$$\int_0^\infty\int_0^\infty\int_0^\infty |x\pm y|^{-z}\Psi(w)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x, y; w)w^z x^{s_1} y^{s_2}\, dw\frac{dx}{x}\frac{dy}{y}.$$

When the sign above is $-$, the integrand is singular on the diagonal $x = y$ and for x and y near 0. Note that the integral is well defined provided $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$ are positive, and so long as $\mathrm{Re}(z) < \mathrm{Re}\ (s_1 + s_2)$.

Keeping to this region of parameters s_1, s_2, and z, let us now consider the integrals over x and y, for a fixed value of w. Making the substitution $x = \lambda y$ we obtain that $\widetilde{\mathcal{W}}_3^{\pm}$ equals

$$\int_0^\infty \Psi(w)w^z\int_0^\infty |\lambda\pm 1|^{-z}\lambda^{s_1}\int_0^\infty V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\lambda y, y; w)y^{s_1+s_2-z}\frac{dy}{y}\frac{d\lambda}{\lambda}\, dw.$$

Employing now the notation $\omega = (s_1 + s_2 - z)/2$ and $\xi = (s_1 - s_2 + z)/2$, by Equations (14) and (16) we have

$$V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\lambda y, y; w) = \left(\frac{w}{\pi}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\int_{-\infty}^{\infty}\lambda^{-it}\frac{1}{2\pi i}\int_{(1)} G\left(\frac{1}{2}+s; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it\right)H(s; \boldsymbol{\alpha}, \boldsymbol{\beta})$$
$$\times\left(\frac{\lambda y^2\pi^3}{w^3}\right)^{-s}\frac{ds}{s}\, dt$$

so that, by Mellin inversion,

$$\int_0^\infty V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\lambda y, y; w)y^{s_1+s_2-z}\frac{dy}{y} = \left(\frac{w}{\pi}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})+3\omega}\lambda^{-\omega}\frac{H(\omega; \boldsymbol{\alpha}, \boldsymbol{\beta})}{2\omega}$$
$$\times\int_{-\infty}^{\infty}\lambda^{-it}G\left(\frac{1}{2}+\omega; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it\right)\, dt.$$

Our work thus far gives, with $\widetilde{\Psi}(s) = \int_0^\infty \Psi(w)w^s dw/w$,

$$\widetilde{\mathcal{W}}_3^{\pm}(s_1, s_2; z) = \frac{H(\omega; \boldsymbol{\alpha}, \boldsymbol{\beta})}{2\omega\pi^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})+3\omega}} \widetilde{\Psi}(1 + z + \delta(\boldsymbol{\alpha}, \boldsymbol{\beta}) + 3\omega) \times \int_0^\infty |\lambda \pm 1|^{-z} \lambda^{s_1-\omega} \int_{-\infty}^\infty \lambda^{-it} G\left(\frac{1}{2} + \omega; \boldsymbol{\alpha} + it, \boldsymbol{\beta} + it\right) dt \frac{d\lambda}{\lambda}. \quad (49)$$

We now evaluate the integral over λ above using the familiar beta-integral. Precisely, for any complex numbers u and v with $0 < \mathrm{Re}(u) < \mathrm{Re}(v) < 1$ we have

$$\int_0^\infty \lambda^u |1-\lambda|^{-v} \frac{d\lambda}{\lambda} = \int_0^1 \lambda^{u-1}(1-\lambda)^{-v} \, d\lambda + \int_0^1 \lambda^{v-u-1}(1-\lambda)^{-v} \, d\lambda = \frac{\Gamma(u)\Gamma(1-v)}{\Gamma(1+u-v)} + \frac{\Gamma(v-u)\Gamma(1-v)}{\Gamma(1-u)} = \mathcal{H}^-(u,v) \quad (50)$$

and

$$\int_0^\infty \lambda^u |1+\lambda|^{-v} \frac{d\lambda}{\lambda} = \int_1^\infty (\lambda-1)^{u-1}\lambda^{-v} \, d\lambda = \int_0^1 \lambda^{v-u-1}(1-\lambda)^{u-1} \, d\lambda = \frac{\Gamma(v-u)\Gamma(u)}{\Gamma(v)} = \mathcal{H}^+(u,v). \quad (51)$$

These remarks establish the result given in (43). The expression given in (46) follows upon summing over both choices of sign and employing the following remarkable identity: (see Lemma 8.2 of [You11])

$$\frac{\Gamma(u)\Gamma(1-v)}{\Gamma(1+u-v)} + \frac{\Gamma(v-u)\Gamma(1-v)}{\Gamma(1-u)} + \frac{\Gamma(v-u)\Gamma(u)}{\Gamma(v)} = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1-v}{2})\Gamma(\frac{u}{2})\Gamma(\frac{v-u}{2})}{\Gamma(\frac{v}{2})\Gamma(\frac{1-u}{2})\Gamma(\frac{1-v+u}{2})}. \quad (52)$$

Finally, using Stirling's formula in (45) we may obtain the estimate for $\widetilde{\mathcal{W}}_3$ given in (48). □

8 Estimating the Error Term $\mathcal{EG}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$

First we dispense with the contribution to (35) from values of a larger than $2Q$. Since $M \neq N$, if $M \equiv \mp N \pmod{abh}$ and $a > 2Q$ then

$$\frac{Q^{\frac{1}{2}}d}{gh} \frac{|gM \pm gN|}{Q^{\frac{3}{2}}} \geq \frac{dab}{Q} \geq 2,$$

so that $\mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(gM/Q^{\frac{3}{2}}, gN/Q^{\frac{3}{2}}; Q^{\frac{1}{2}}d/(gh)) = 0$ for such terms. Therefore the contribution to (35) from terms with $a > 2Q$ is

$$\ll Q^{1+\epsilon} \sum_{m,n=1}^{\infty} \frac{1}{(mn)^{\frac{1}{2}-\epsilon}} \sum_{d \le D} \sum_{a>2Q} \sum_{b|g} \sum_{h>0} \frac{1}{ad\phi(abh)} \left| \mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \left(\frac{m}{Q^{\frac{3}{2}}}, \frac{n}{Q^{\frac{3}{2}}}; \frac{Q^{\frac{1}{2}}d}{gh} \right) \right|.$$

Since $\mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(gM/Q^{\frac{3}{2}}, gN/Q^{\frac{3}{2}}; Q^{\frac{1}{2}}d/(gh))$ is small unless m and n are below $Q^{\frac{3}{2}+\epsilon}$ and $|m \pm n|d/(Qgh) \in [1, 2]$, we find that the above is

$$\ll Q^{1+\epsilon} \sum_{a>2Q} \frac{1}{a^{2-\epsilon}} \sum_{m,n=1}^{\infty} \frac{1}{(mn)^{\frac{1}{2}-\epsilon}} \exp\left(-c \left(\frac{\max(m,n)}{Q^{\frac{3}{2}}} \right)^{\frac{1}{3}} \right) \ll Q^{\frac{3}{2}+\epsilon}.$$

We may therefore restrict attention to the terms with $a \le 2Q$. Here we use the two variable Mellin transform $\widetilde{\mathcal{W}}_2(s_1, s_2; u)$ together with Mellin inversion. Thus the remainder term in (35) may be written as

$$\frac{Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}}{2} \sum_{\substack{a \le 2Q \\ b,h>0}} \sum_{\substack{\psi \,(\mathrm{mod}\ abh) \\ \psi \ne \psi_0}} \sum_{\substack{g \\ b|g,(a,g)=1}} \sum_{\substack{d \le D \\ (d,g)=1}} \frac{\mu(a)\mu(b)\mu(d)}{adg\phi(abh)} \frac{1}{(2\pi i)^2} \int_{(\frac{1}{2}+\epsilon)} \int_{(\frac{1}{2}+\epsilon)} \widetilde{\mathcal{W}}^{\pm}_2$$
$$\times \left(s_1, s_2; \frac{Q^{\frac{1}{2}}d}{gh} \right) \left(\frac{Q^{\frac{3}{2}}}{g} \right)^{s_1+s_2} \sum_{\substack{M,N=1 \\ M \ne N, (M,N)=1}}^{\infty} \frac{\sigma(gM;\boldsymbol{\alpha})\sigma(gN;-\boldsymbol{\beta})}{M^{\frac{1}{2}+s_1} N^{\frac{1}{2}+s_2}} \psi(M)\overline{\psi}(\mp N)\, ds_1\, ds_2. \tag{53}$$

The inner sum over M and N in (53) may be written as

$$\psi(\mp 1) \left(\prod_{j=1}^{3} L\left(\frac{1}{2}+s_1+\alpha_j, \psi\right) L\left(\frac{1}{2}+s_2-\beta_j, \overline{\psi}\right) \mathcal{F}(g,\psi;s_1,s_2) - \sigma(g;\boldsymbol{\alpha})\sigma(g;-\boldsymbol{\beta}) \right),$$

for a suitable function $\mathcal{F}$ which is analytic when $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$ are $> \epsilon$ and is bounded there by $|g|^{\epsilon}$. Since ψ is not principal, the L-functions above are non-trivial and have no poles. Thus we may move the lines of integration to $\mathrm{Re}(s_1) = \mathrm{Re}(s_2) = \epsilon$. In this way, we find that the quantity in (53) is bounded by

$$\ll Q^{1+\epsilon} \sum_{\substack{a \le 2Q \\ b,h>0}} \sum_{\substack{\psi \,(\mathrm{mod}\ abh) \\ \psi \ne \psi_0}} \sum_{\substack{g \\ b|g,(a,g)=1}} \sum_{\substack{d \le D \\ (d,g)=1}} \frac{1}{adg\phi(abh)} \int_{(\epsilon)} \int_{(\epsilon)} \left| \widetilde{\mathcal{W}}^{\pm}_2 \left(s_1, s_2; \frac{Q^{\frac{1}{2}}d}{gh} \right) \right|$$
$$\times \left(1 + \prod_{j=1}^{3} \left| L\left(\frac{1}{2}+s_1+\alpha_j, \psi\right) L\left(\frac{1}{2}+s_2-\beta_j, \overline{\psi}\right) \right| \right) ds_1\, ds_2. \tag{54}$$

Using the bound in Lemma 5 we obtain that for any $k \geq 1$

$$\begin{aligned}
&\sum_{\substack{g \\ b|g}} \frac{1}{g} \sum_{d \leq D} \frac{1}{d} \left| \widetilde{\mathcal{W}}_2^{\pm}\left(s_1, s_2; \frac{Q^{\frac{1}{2}} d}{gh}\right) \right| \\
&\quad \ll \frac{(1+Q^{\frac{1}{2}} D/(bh))^{k-1}}{\max(|s_1|,|s_2|)^k} (\log D) \sum_{b|g} \frac{1}{g} \exp\left(-c\left(\frac{gh}{Q^{\frac{1}{2}} D}\right)^{\frac{1}{3}}\right) \\
&\quad \ll \frac{(1+Q^{\frac{1}{2}} D/(bh))^{k-1}}{\max(|s_1|,|s_2|)^k} \frac{Q^{\epsilon}}{b} \exp\left(-c\left(\frac{bh}{Q^{\frac{1}{2}} D}\right)^{\frac{1}{3}}\right).
\end{aligned}$$

We use this in (54), and divide the remaining variables a, b, h, s_1 and s_2 into dyadic blocks – say $A \leq a < 2A$, $B \leq b < 2B$, $H \leq h < 2H$, $S_1 \leq |s_1| < 2S_1$ and $S_2 \leq |s_2| < 2S_2$. Consider now the contribution of such a dyadic block to (54). For any fixed $k \geq 1$ this is, writing $\ell = abh$

$$\begin{aligned}
&\ll \frac{Q^{1+\epsilon}(1+Q^{\frac{1}{2}} D/(BH))^{k-1}}{A^{2-\epsilon} B^{2-\epsilon} H^{1-\epsilon} \max(S_1,S_2)^k} \exp\left(-c\left(\frac{BH}{Q^{\frac{1}{2}} D}\right)^{\frac{1}{3}}\right) \\
&\quad \times \sum_{ABH \leq \ell < 8ABH} \sum_{\substack{\psi \pmod{\ell} \\ \psi \neq \psi_0}} \int_{\substack{(\epsilon) \\ S_1 \leq |s_1| < 2S_1}} \int_{\substack{(\epsilon) \\ S_2 \leq |s_2| < 2S_2}} \left(1 + \sum_{j=1}^{3} \left(\left|L\left(\frac{1}{2}+s_1+\alpha_j, \psi\right)\right|^6\right.\right. \\
&\quad \left.\left. + \left|L\left(\frac{1}{2}+s_2-\beta_j, \overline{\psi}\right)\right|^6\right)\right) ds_1\, ds_2.
\end{aligned}$$

Using the large sieve we see that the sums and integrals above contribute

$$\ll \left(A^2 B^2 H^2 S_1 S_2 + (ABH \max(S_1,S_2))^{\frac{3}{2}} \min(S_1,S_2)\right)^{1+\epsilon}.$$

We take $k = 1$ if $\max(S_1,S_2) \leq 1 + Q^{\frac{1}{2}} D/(BH)$ and $k = 4$ otherwise. Summing over all dyadic blocks (and keeping in mind that $A \leq Q$) we obtain (with a little calculation) a net estimate of

$$\ll Q^{\frac{3}{2}+\epsilon} D + Q^{\frac{7}{4}+\epsilon} D^{\frac{3}{2}},$$

with the worst case scenario being when A, B and H are small and $\max(S_1,S_2)$ being of size $Q^{\frac{1}{2}} D$.

Thus we have established that

$$\mathcal{E}G(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll Q^{\frac{7}{4}+\epsilon} D^{\frac{3}{2}}. \tag{55}$$

9 The Main Terms: $\mathcal{MS}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta}) + \mathcal{MG}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$

We start by simplifying the expression for $\mathcal{MG}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ which is given in (34). Put

$$F(h, g; MN) = \sum_{(a,gMN)=1} \frac{\mu(a)}{a} \sum_{\substack{b|g \\ (b,MN)=1}} \frac{\mu(b)}{\phi(abh)} = \sum_{(\ell,MN)=1} \frac{\mu(\ell)(\ell, g)}{\ell\phi(\ell h)}, \qquad (56)$$

where the last identity follows upon grouping $\ell = ab$. Then $\mathcal{MG}(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ equals

$$\frac{Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}}{2} \sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} \frac{\sigma(m; \boldsymbol{\alpha})\sigma(n; -\boldsymbol{\beta})}{\sqrt{mn}} \sum_{\substack{d\leq D \\ (d,gMN)=1}} \sum_{\substack{h>0 \\ (h,MN)=1}} \frac{\mu(d)}{d} F(h, g; MN) \mathcal{W}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \times \left(\frac{gM}{Q^{\frac{3}{2}}}, \frac{gN}{Q^{\frac{3}{2}}}; \frac{Q^{\frac{1}{2}}d}{gh} \right). \qquad (57)$$

Now we use the Mellin transform $\widetilde{\mathcal{W}}^{\pm}_1(x, y; z)$ given in (36) together with the Mellin inversion formula and the estimates of Lemma 4. Using (37) with $c = -\epsilon < 0$, we find that the sum over h in (57) is

$$\sum_{\substack{h=1 \\ (h,MN)=1}}^{\infty} F(h, g; MN) \frac{1}{2\pi i} \int_{(-\epsilon)} \widetilde{\mathcal{W}}^{\pm}_1 \left(\frac{gM}{Q^{\frac{3}{2}}}, \frac{gN}{Q^{\frac{3}{2}}}; z \right) \left(\frac{Q^{\frac{1}{2}}d}{gh} \right)^{-z} dz. \qquad (58)$$

We interchange the sum and the integral, and this is legitimate as the sum over h converges absolutely when the real part of z is negative.

Lemma 7. *If Re$(s) > 0$ we have*

$$\sum_{\substack{h=1 \\ (h,MN)=1}}^{\infty} \frac{F(h, g; MN)}{h^s} = \zeta(s + 1)\mathcal{K}(s; g, MN)$$

where, with $\phi(\ell, s) = \prod_{p|\ell}(1 - 1/p^s)$,

$$\mathcal{K}(s; g, MN) = \phi(MN, s + 1) \prod_{p\nmid gmN} \left(1 - \frac{1}{p(p-1)} + \frac{1}{p^{1+s}(p-1)} \right) \times \prod_{\substack{p|g \\ p\nmid MN}} \left(1 - \frac{1}{p^{1+s}} - \frac{1}{p-1} \left(1 - \frac{1}{p^s} \right) \right).$$

Proof. This is a straightforward verification. □

Using Lemma 7 in (58) we obtain that the quantity there equals

$$\frac{1}{2\pi i}\int_{(-\epsilon)} \widetilde{\mathcal{W}}_1^{\pm}\left(\frac{gM}{Q^{\frac{3}{2}}},\frac{gN}{Q^{\frac{3}{2}}};z\right)\zeta(1-z)\mathcal{K}(-z;g,MN)\left(\frac{Q^{\frac{1}{2}}d}{g}\right)^{-z}\,dz. \tag{59}$$

We now move the line of integration to the right, to the line $\mathrm{Re}(z)=\epsilon>0$. We encounter a pole at $z=0$, whose residue (taking into account that the contour is oriented clockwise, and that the residue of $\zeta(1-z)$ at $z=0$ is -1) equals

$$\begin{aligned}
&\widetilde{\mathcal{W}}_1^{\pm}\left(\frac{gM}{Q^{\frac{3}{2}}},\frac{gN}{Q^{\frac{3}{2}}};0\right)\mathcal{K}(0;g,MN)\\
&\quad=\frac{\phi(gMN)}{gMN}\int_0^\infty \mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\pm}\left(\frac{gM}{Q^{\frac{3}{2}}},\frac{gN}{Q^{\frac{3}{2}}};u\right)\frac{du}{u}\\
&\quad=\frac{\phi(gMN)}{gMN}\int_0^\infty u|x\pm y|\Psi(u|x\pm y|)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x,y;u|x\pm y|)\frac{du}{u}\\
&\quad=\frac{\phi(gMN)}{gMN}\int_0^\infty \Psi(u)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{gM}{Q^{\frac{3}{2}}},\frac{gN}{Q^{\frac{3}{2}}};u\right)du,
\end{aligned}\tag{60}$$

where in the middle line we wrote $x=gM/Q^{\frac{3}{2}}$ and $y=gN/Q^{\frac{3}{2}}$ for brevity. Thus the quantity in (59) equals the residue term above, together with the remaining integral on the line $\mathrm{Re}(z)=\epsilon$.

Consider first the contribution of the residue term (60) to $\mathcal{MG}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$—we shall show that this contribution cancels the contribution of $\mathcal{MS}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$. Keeping in mind that the sum in (57) is over both choices of sign $\pm$, and since $\phi(gMN)/(gMN)=\phi(mn)/(mn)$, we see that the residue term (60) contributes

$$Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{\sqrt{mn}}\sum_{\substack{d\le D\\ (d,mn)=1}}\frac{\mu(d)}{d}\frac{\phi(mn)}{mn}\int_0^\infty \Psi(u)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{m}{Q^{\frac{3}{2}}},\frac{n}{Q^{\frac{3}{2}}};u\right)du. \tag{61}$$

Now consider $\mathcal{MS}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ which we recall from Proposition 3 equals

$$-\sum_{\substack{m,n=1\\ m\neq n}}\frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{\sqrt{mn}}\sum_{\substack{d\le D\\ (d,mn)=1}}\mu(d)\sum_{\substack{r\\ (r,mn)=1}}\Psi\left(\frac{dr}{Q}\right)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m,n;dr).$$

Since

$$\sum_{\substack{r\le x\\ (r,mn)=1}}1=\frac{\phi(mn)}{mn}x+O((mn)^{\epsilon}),$$

and $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is small unless $mn \le Q^{3+\epsilon}$ we find that the above is

$$-Q\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{\sqrt{mn}}\sum_{\substack{d\le D\\ (d,mn)=1}}\frac{\mu(d)}{d}\frac{\phi(mn)}{mn}\int_0^\infty \Psi(u)V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m,n;uQ)du+O(DQ^{\frac{3}{2}+\epsilon}).$$

Since $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m,n;uQ)=Q^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(mQ^{-\frac{3}{2}},nQ^{-\frac{3}{2}};u)$ the main term above exactly cancels the quantity in (61)!

Summarizing our discussions above we have established that $\mathcal{MS}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})+\mathcal{MG}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ equals, up to an error $O(DQ^{\frac{3}{2}+\epsilon})$,

$$\frac{Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}}{2}\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{\sqrt{mn}}$$
$$\times\frac{1}{2\pi i}\int_{(\epsilon)}\widetilde{\mathcal{W}}_1^{\pm}\left(\frac{gM}{Q^{\frac{3}{2}}},\frac{gN}{Q^{\frac{3}{2}}};z\right)\zeta(1-z)\mathcal{K}(-z;g,MN)\left(\frac{Q^{\frac{1}{2}}}{g}\right)^{-z}\sum_{\substack{d\le D\\ (d,gMN)=1}}\frac{\mu(d)}{d^{1+z}}\,dz. \tag{62}$$

We move the line of integration above to $\mathrm{Re}(z)=1-\epsilon$, and then extend the sum over d to include all values of d. Using Lemma 4 the error in extending our sum over d is

$$\ll Q^{1+\epsilon}\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{1}{(mn)^{\frac{1}{2}-\epsilon}}\int_{(1-\epsilon)}\left|\frac{m\pm n}{Q^{\frac{3}{2}}}\right|^{-1+\epsilon}|z|^{-10}\exp(-c_1(\max(m,n)/Q^{\frac{3}{2}})^{\frac{1}{3}})$$
$$\times\left(\frac{Q^{\frac{1}{2}}}{g}\right)^{-1+\epsilon}D^{-1+\epsilon}|dz|$$
$$\ll Q^{2+\epsilon}D^{-1}.$$

Moving now the line of integration back to $\mathrm{Re}(z)=\epsilon$, we conclude that with an error $O(Q^{2+\epsilon}D^{-1}+DQ^{\frac{3}{2}+\epsilon})$ the quantity $\mathcal{MS}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})+\mathcal{MG}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ equals

$$\frac{Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}}{2}\sum_{m,n=1}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{\sqrt{mn}}$$
$$\times\frac{1}{2\pi i}\int_{(\epsilon)}\widetilde{\mathcal{W}}_1^{\pm}\left(\frac{gM}{Q^{\frac{3}{2}}},\frac{gN}{Q^{\frac{3}{2}}};z\right)\frac{\zeta(1-z)\mathcal{K}(-z;g,MN)}{\zeta(1+z)\phi(gMN,1+z)}\left(\frac{Q^{\frac{1}{2}}}{g}\right)^{-z}dz. \tag{63}$$

Note that above we reintroduced the terms $m=n$ with an acceptable error of $O(Q^{1+\epsilon})$.

10 Identifying the Non-Diagonal Main Terms

We now consider, thinking of z as fixed, the sum over m and n in (63) above. We use the three variable Mellin transform $\widetilde{\mathcal{W}}_3^{\pm}$ discussed in Lemma 6. Since we are summing over both choices of sign, the quantity in (63) may be written as

$$\frac{Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}}{2}\frac{1}{(2\pi i)^3}\int_{(\epsilon)}\int_{(\frac{1}{2}+\epsilon)}\int_{(\frac{1}{2}+\epsilon)}\widetilde{\mathcal{W}}_3(s_1,s_2;z)\frac{\zeta(1-z)}{\zeta(1+z)}Q^{\frac{3}{2}(s_1+s_2)-\frac{z}{2}}$$
$$\times\sum_{m,n=1}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{m^{\frac{1}{2}+s_1}n^{\frac{1}{2}+s_2}}\frac{g^z\mathcal{K}(-z;g,MN)}{\phi(gMN,1+z)}\,ds_2\,ds_1\,dz. \tag{64}$$

Thus we are led to consider

$$\mathcal{F}(s_1,s_2;z)=\sum_{m,n=1}^{\infty}\frac{\sigma(m;\boldsymbol{\alpha})\sigma(n;-\boldsymbol{\beta})}{m^{\frac{1}{2}+s_1}n^{\frac{1}{2}+s_2}}\frac{g^z}{\phi(gMN,1+z)}\mathcal{K}(-z;g,MN). \tag{65}$$

The sum over m and n above has an obvious multiplicative structure, and we can therefore write

$$\mathcal{F}(s_1,s_2;z)=\prod_p\mathcal{F}_p(s_1,s_2;z),$$

where, recalling the definition of $\mathcal{K}$ from Lemma 7,

$$\mathcal{F}_p(s_1,s_2;z)=1+\frac{(p^z-1)}{p(p-1)}+\sum_{\substack{a,b\geq 0\\ \max(a,b)\geq 1}}\frac{\sigma(p^a;\boldsymbol{\alpha})\sigma(p^b;-\boldsymbol{\beta})}{p^{a(\frac{1}{2}+s_1)}p^{b(\frac{1}{2}+s_2)}}p^{z\min(a,b)}\frac{1-1/p^{1-z}}{1-1/p^{1+z}}$$
$$+\frac{1}{(p-1)}\frac{p^z-1}{1-1/p^{1+z}}\sum_{k=1}^{\infty}\frac{\sigma(p^k;\boldsymbol{\alpha})\sigma(p^k;-\boldsymbol{\beta})}{p^{k(1+s_1+s_2-z)}}.$$

The behavior of $\mathcal{F}_p$ is dominated by the contributions from $(a,b)=(1,0)$, $(0,1)$, $(1,1)$ and $k=1$ terms above. Thus we write

$$\mathcal{F}(s_1,s_2;z)=\zeta(2-z)\prod_{j=1}^{3}\frac{\zeta(\frac{1}{2}+s_1+\alpha_j)}{\zeta(\frac{1}{2}+s_1+\alpha_j+1-z)}\prod_{\ell=1}^{3}\frac{\zeta(\frac{1}{2}+s_2-\beta_\ell)}{\zeta(\frac{1}{2}+s_2-\beta_\ell+1-z)}$$
$$\times\prod_{j,\ell=1}^{3}\zeta(1+s_1+s_2-z+\alpha_j-\beta_\ell)\mathcal{G}(s_1,s_2;z).$$

Here $\mathcal{G}(s_1,s_2;z)=\prod_p\mathcal{G}_p(s_1,s_2;z)$ is absolutely convergent in a wider region of s_1, s_2 and z.

For a fixed value of z with $\mathrm{Re}(z)=\epsilon$, $\mathcal{F}(s_1,s_2;z)$ has nine poles at $s_1=\frac{1}{2}-\alpha_j$ and $s_2=\frac{1}{2}+\beta_\ell$. Keeping z fixed, we move the lines of integration in s_1 and s_2 to $\mathrm{Re}(s_1)=2\epsilon$ and $\mathrm{Re}(s_2)=2\epsilon$, and pick up the residues at these poles, and the integrals on the remaining lines is acceptably small. To move the lines of integration carefully, we first truncate the integrals in s_1 and s_2 at a height $T = Q^2$ and then move the line of integration over s_2 first and then the line over s_1. We use the estimate (48) and find that the accumulated error may be bounded by $O(Q^{\frac{7}{4}+\epsilon})$.

Now we work out the contribution of the residues. As noted above, there are nine such terms and for simplicity we shall work out the contribution from $s_1 = \frac{1}{2} - \alpha_1$ and $s_2 = \frac{1}{2} + \beta_1$, the other cases being similar. Our goal is to show that this term contributes

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) \sum_q \Psi\left(\frac{q}{Q}\right) \phi^\flat(q) \int_{-\infty}^{\infty} \mathcal{Q}(q;\pi(\boldsymbol{\alpha})+iy,\pi(\boldsymbol{\beta})+iy)dy + O(Q), \tag{66}$$

where $\pi \in S_6/(S_3 \times S_3)$ is the permutation with $\pi(\boldsymbol{\alpha}) = (\beta_1, \alpha_2, \alpha_3)$ and $\pi(\boldsymbol{\beta}) = (\alpha_1, \beta_2, \beta_3)$.

The residue of $\mathcal{F}(s_1, s_2; z)$ equals

$$\prod_{j=2}^{3} \frac{\zeta(1+\alpha_j-\alpha_1)\zeta(1+\beta_1-\beta_j)}{\zeta(2+\alpha_j-\alpha_1-z)\zeta(2-\beta_j+\beta_1-z)} \prod_{\substack{j,\ell=1 \\ (j,\ell)\neq(1,1)}}^{3} \zeta(2+\beta_1-\alpha_1-z+\alpha_j-\beta_\ell)$$
$$\times \mathcal{G}\left(\frac{1}{2}-\alpha_1, \frac{1}{2}+\beta_1; z\right). \tag{67}$$

We use this in (64), and move the line of integration in z to $\mathrm{Re}(z) = \frac{3}{2}-\epsilon$. In doing so, we encounter a simple pole at $z = 1-\alpha_1+\beta_1$ (from the $\widetilde{\mathcal{W}}_3(\frac{1}{2}-\alpha_1, \frac{1}{2}+\beta_1; z)$ term) and the residue here is the dominant contribution. Note that there are potential poles at $z = 1-\alpha_1+\beta_1+\alpha_j-\beta_\ell$ (for $(j,\ell) \neq (1,1)$) but these are offset by the corresponding zeros of $H((1-\alpha_1+\beta_1-z)/2;\boldsymbol{\alpha},\boldsymbol{\beta})$ at these points. Taking the residue at $z = 1-\alpha_1+\beta_1$, the expression (67) simplifies to

$$\frac{\mathcal{Z}(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}{\zeta(1+\beta_1-\alpha_1)} \mathcal{G}\left(\frac{1}{2}-\alpha_1, \frac{1}{2}+\beta_1; 1+\beta_1-\alpha_1\right). \tag{68}$$

Moreover, the residue of

$$\frac{Q^{1+\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}}{2} \widetilde{\mathcal{W}}_3\left(\frac{1}{2}-\alpha_1, \frac{1}{2}+\beta_1; z\right) \frac{\zeta(1-z)}{\zeta(1+z)} Q^{\frac{3}{2}(1+\beta_1-\alpha_1)-\frac{z}{2}}$$

at $z = 1-\alpha_1+\beta_1$ gives, using (45),

$$\frac{Q^{2+\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}\zeta(\alpha_1-\beta_1)}{2\pi^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\zeta(2-\alpha_1+\beta_1)} \widetilde{\Psi}(2+\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})))H(0;\boldsymbol{\alpha},\boldsymbol{\beta})$$
$$\times \int_{-\infty}^{\infty} G\left(\frac{1}{2};\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\right) \mathcal{H}\left(\frac{1}{2}-\alpha_1-it, 1-\alpha_1+\beta_1\right) dt.$$

Using now (46) and the functional equation connecting $\zeta(\alpha_1-\beta_1)$ and $\zeta(1-\alpha_1+\beta_1)$, the above simplifies to give

$$\frac{Q^{2+\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}\zeta(1-\alpha_1+\beta_1)}{2\pi^{\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}\zeta(2-\alpha_1+\beta_1)}\widetilde{\Psi}(2+\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})))H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) \times \int_{-\infty}^{\infty} G\left(\frac{1}{2};\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it\right)dt. \tag{69}$$

Combining (68) and (69) we obtain that the contribution of this term to (68) is

$$\frac{Q^{2+\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}}{2\pi^{\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}}\mathcal{Z}\left(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\right)\widetilde{\Psi}(2+\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})))H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) \times\frac{\mathcal{G}(\frac{1}{2}-\alpha_1,\frac{1}{2}+\beta_1;1+\beta_1-\alpha_1)}{\zeta(2-\alpha_1+\beta_1)}\int_{-\infty}^{\infty} G\left(\frac{1}{2};\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it\right)dt. \tag{70}$$

It remains now to match up the quantity in (70) above with our desired object in (66). Let us begin by first simplifying the expression in (66). The main term there equals

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\sum_q \Psi\left(\frac{q}{Q}\right)\phi^\flat(q)\left(\frac{q}{\pi}\right)^{\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}\frac{\mathcal{AZ}}{\mathcal{B}_q}\left(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\right) \times\int_{-\infty}^{\infty} G\left(\frac{1}{2};\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it\right)dt.$$

Using that $\phi^\flat(q)=\frac{1}{2}\phi^*(q)+O(1)$, and the function ϕ^* is multiplicative with $\phi^*(p)=p-2$ and $\phi^*(p^k)=p^{k-2}(p-1)^2$ for $k\geq 2$, we may evaluate using a standard contour shift argument the sum over q above. Thus the above becomes

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\left(\int_{-\infty}^{\infty} G\left(\frac{1}{2};\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it\right)dt\right)\frac{Q^{2+\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}}{2\pi^{\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}}\widetilde{\Psi}(2+\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))) \times\mathcal{AZ}\left(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\right)\prod_p\left(1-\frac{1}{p}\right)\left(1+\frac{1}{\mathcal{B}_p(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}\left(\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right)\right). \tag{71}$$

Comparing (70) and (71) we note that many of the terms match up, and what remains is to match up the Euler products on both sides. For this it suffices to check that the Euler factors at each prime match up. This entails checking whether

$$\frac{\mathcal{G}_p(\frac{1}{2}-\alpha_1,\frac{1}{2}+\beta_1;1+\beta_1-\alpha_1)}{\zeta_p(2-\alpha_1+\beta_1)}$$

equals

$$\mathcal{A}_p\left(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\right)\left(1-\frac{1}{p}\right)\left(1+\frac{1}{\mathcal{B}_p(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}\left(\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right)\right)?$$

A little calculation reduces this to checking whether

$$\mathcal{F}_p\left(\frac{1}{2}-\alpha_1,\frac{1}{2}+\beta_1;1+\beta_1-\alpha_1\right)\left(1-\frac{1}{p}\right)\frac{\zeta_p(1-\alpha_1+\beta_1)}{\zeta_p(2-\alpha_1+\beta_1)}$$

equals

$$\mathcal{B}_p(\tfrac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))+\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}?$$

With a little calculation, this may be verified upon using the following Parseval identities

$$\begin{aligned}\sum_{a,b\geq 0}&\frac{\sigma(p^a;\boldsymbol{\alpha})\sigma(p^b;-\boldsymbol{\beta})}{p^{a(1-\alpha_1)}p^{b(1+\beta_1)}}p^{(1-\alpha_1+\beta_1)\min(a,b)}\\ &=\int_0^1\left(\sum_{a=0}^{\infty}\frac{\sigma(p^a;\boldsymbol{\alpha})e(a\theta)}{p^{a(1-\alpha_1-\beta_1)/2}}\right)\left(\sum_{b=0}^{\infty}\frac{\sigma(p^b;-\boldsymbol{\beta})e(-b\theta)}{p^{b(1+\alpha_1+\beta_1)/2}}\right)\\ &\quad\times\left(1+\sum_{k=1}^{\infty}\frac{e(k\theta)}{p^{k(1-\alpha_1+\beta_1)/2}}+\sum_{\ell=1}^{\infty}\frac{e(-\ell\theta)}{p^{\ell(1-\alpha_1+\beta_1)/2}}\right)d\theta,\end{aligned}$$

and

$$\sum_{k=0}^{\infty}\frac{\sigma(p^k;\boldsymbol{\alpha})\sigma(p^k;-\boldsymbol{\beta})}{p^k}=\int_0^1\prod_{j=1}^{3}\left(1-\frac{e(\theta)}{p^{\frac{1}{2}+\alpha_j}}\right)^{-1}\left(1-\frac{e(-\theta)}{p^{\frac{1}{2}-\beta_j}}\right)^{-1}d\theta.$$

11 Completion of the Proof of Theorem 1

Recall from Section 4 that our goal is to evaluate $\mathcal{I}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})=\Delta(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})+\Delta(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ [see (18), (19), (20)]. We then decomposed $\Delta(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})=\mathcal{D}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})+\mathcal{S}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})+\mathcal{G}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ [see (22)]. In Lemma 3 we evaluated $\mathcal{D}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ obtaining one of the twenty terms in our desired asymptotic formula, together with an error term of $O(Q^{\frac{5}{4}+\epsilon})$.

The work in Section 5 extracts a main term $\mathcal{MS}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ out of $\mathcal{S}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ with an error term of $O(Q^{2+\epsilon}/D)$. Correspondingly in Section 6 we extract a main term $\mathcal{MG}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ out of $\mathcal{G}(\Psi,Q;\boldsymbol{\alpha},\boldsymbol{\beta})$ with an error that is estimated in Section 8 as $O(Q^{\frac{7}{4}+\epsilon}D^{\frac{3}{2}})$. The two main terms $\mathcal{MS}$ and $\mathcal{MG}$ are combined in Section 9 to obtain a single term given in (63) with an error $O(Q^{2+\epsilon}D+Q^{\frac{3}{2}+\epsilon}D)$. The choice $D=Q^{\frac{1}{10}}$ minimizes our errors, and the total error is $O(Q^{\frac{19}{10}+\epsilon})$.

The main term of (63) is evaluated in Section 10 by an involved residue calculation. Nine terms in our desired asymptotic formula arise here, corresponding to the nine transpositions $\pi = (\alpha_j \beta_\ell)$. Thus $\Delta(\Psi, Q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ leads to ten terms in our desired asymptotic formula, and the remaining ten come from the $\Delta(\Psi, Q; \boldsymbol{\beta}, \boldsymbol{\alpha})$ contribution.

Putting everything together we conclude that

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) \sum_q \sideset{}{^\flat}\sum_{\chi \pmod q} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Lambda(\chi; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it)\, dt$$

equals

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) \sum_q \Psi\left(\frac{q}{Q}\right) \phi^\flat(q) \int_{-\infty}^{\infty} \widetilde{\mathcal{Q}}(q; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it)\, dt + O(Q^{\frac{19}{10}+\epsilon}).$$

If the α_j and β_ℓ are bounded away from each other, and from zero in such a way that $H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) \gg Q^{-\epsilon}$ then we may divide both sides by $H(0;\boldsymbol{\alpha},\boldsymbol{\beta})$ and obtain the Theorem in this case. The general case follows from this by noting that the two expressions above must both be analytic in the variables α_j and β_ℓ.

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