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On the orthogonal symmetry of L-functions of a family of Hecke Grössencharacters

by

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1. Introduction. L-functions are fundamental objects in number theory that carry a lot of arithmetic information. Probably the most famous example is the Birch and Swinnerton-Dyer conjecture that equates the rank of an elliptic curve with the order of vanishing of its L-function at the central point. It is generally believed that the vanishing of an L-function at its central point indicates some arithmetic-geometric structure. There are many theorems concerning the first-order vanishing of elliptic curve L-functions and random matrix theory has been used to model the frequency of second-order vanishing [CKRS02]. In addition, the Langlands philosophy predicts that for any L-function arising from an automorphic representation there is a new L-function associated with the rth symmetric power representation. Combining these ideas, Barry Mazur asked the following question: Given the L-function of an elliptic curve $E/\mathbb{Q}$, is it true that the central value of the L-function of its nth symmetric power vanishes, if ever, for at most finitely many values of n? He admitted that it would likely be too difficult to answer this question, but further asked if random matrix theory could provide a model for this question.

We investigated this interesting question and quickly agreed that it was much too difficult to answer. At this stage, we cannot even decide whether, generically, the collection of $\{L(\mathrm{sym}^n E, s)\}$ constitutes a family in the sense of [KS99] or [CFK$^+$05]. But we did address his question in the interesting special case that E is an elliptic curve with complex multiplication. In this situation, $L(\mathrm{sym}^n E, s)$ is no longer a primitive L-function, i.e. it factors into other L-functions. Basically $L(\mathrm{sym}^n E, s)$ is a product of L-functions associated with cusp forms of increasing weight; see the discussion at equation (1.1). If one of these L-functions happens to vanish at its central point,

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then, since that L-function will show up in the factorizations of infinitely many of the $L(\mathrm{sym}^n E, s)$, we have a somewhat trivial answer to Mazur's question. A better question in this case is whether infinitely often the new primitive part of each of $L(\mathrm{sym}^n E, s)$ can vanish at its central point. This is a question that has received attention from an automorphic and p-adic perspective, notably in works of Gross–Zagier [GZ80], Villegas–Zagier [RVZ93] and Greenberg [Gre83]. In particular it is known in some instances that when the functional equation has even type, there is never any vanishing. This is far better than what one could hope for by analytic methods, where the best one could possibly achieve would be non-vanishing in 100% of the cases. Nevertheless, in view of the success by algebraic methods, it is interesting to compare what happens with a classical approach.

With this classical method we have had some partial success. In particular, the primitive parts alluded to above do seem to form an orthogonal family, and we can model this family using random matrix theory. We can also take some theoretical steps and in particular can prove an asymptotic formula, with power savings, for the first moment of the L-functions in this family. This improves an asymptotic formula with no error term proven by Greenberg [Gre83] and Villegas–Zagier [RVZ93]. We can also give an upper bound that is probably too large by only one logarithm for the second moment of the L-functions in this family. We conclude, by Cauchy's inequality, that at least $N/\log^2 N$ of the first N L-functions in this family do not vanish at their central point. Moreover, if we assume that the Riemann Hypothesis holds for this family, then we can compute the one-level density for this family with a restricted class of test functions, from which it follows that at least $1/4$ of the L-functions in this family do not vanish at their central point.

The family of L-functions we consider are associated with a sequence of Hecke Grössencharacters. To make things concrete, we deal with one specific case, the Grössencharacters associated with the field $\mathbb{Q}(\sqrt{-7})$, following the paper by Gross and Zagier [GZ80]. Many families of L-functions have been studied with a view to determining whether they show the unitary, orthogonal or symplectic symmetry type of random matrix theory, as proposed by Katz and Sarnak [KS99]. We investigate the moments at the central point and the one-level density of the zeros of the Hecke L-functions and find that these agree with the hypothesis that the symmetry type of this family is *orthogonal*.

1.1. Background to Hecke L-functions. The book of Iwaniec and Kowalski [IK04, Section 3.8] is a good reference for the material in this section, as is [GZ80], from which much of this material is taken. See also [RVZ93]. It should be noted that we use the analytic normalization which places the central point at 1/2. The integers of $\mathbb{Q}(\sqrt{-7})$ are all numbers of the form $a + b\eta$ where a and b are integers and

$$\eta = \frac{1+\sqrt{-7}}{2}.$$

The norm of $a + b\eta$ is

$$N(a+b\eta) = (a+b\eta)(a+b\overline{\eta}) = a^2 + ab + 2b^2.$$

The field $\mathbb{Q}(\sqrt{-7})$ has class number 1 so that the ideals are generated by the integers $a + b\eta$. The only units are $+1$ and -1, so that each ideal has two generators.

The Dedekind zeta-function of the field $K = \mathbb{Q}(\sqrt{-7})$ is

$$\zeta_K(s) = \frac{1}{2} \sum_{(a,b)\neq(0,0)} \frac{1}{N(a+b\eta)^s} = \zeta(s)L(s,\chi_{-7})$$

where $\chi_{-7}(n) = \left(\frac{n}{7}\right)$ is the Legendre symbol.

Note that

$$\frac{1}{2} \sum_{(a,b)\neq(0,0)} q^{a^2+ab+2b^2} = 1+\sum_{n=1}^{\infty} a_n q^n = 1+q+2q^2+3q^4+q^7+4q^8+2q^{11}+\cdots$$

so that $\zeta_K(s)$ is given by

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)\left(1+\frac{1}{2^s}-\frac{1}{3^s}+\frac{1}{4^s}-\frac{1}{5^s}-\frac{1}{6^s}+\frac{1}{8^s}+\cdots\right).$$

The Hecke character we are interested in is defined by

$$\chi((a+b\eta)) = \epsilon_{a,b}(a+b\eta)$$

provided that $a + b\eta$ is relatively prime to $\sqrt{-7}$ (otherwise the value of χ is 0). Here the choice of $\epsilon = \pm 1$ is determined by

$$(a+b\eta)^3 \equiv \epsilon_{a,b} \bmod \sqrt{-7}.$$

This amounts to whether $a^3 - 2a^2b - ab^2 + b^3$ is congruent to ± 1 modulo 7. Thus,

$$\epsilon_{a,b} = \left(\frac{a^3 - 2a^2b - ab^2 + b^3}{7}\right).$$

The Hecke L-function is

$$L(s,\chi) = \frac{1}{2} \sum_{(a,b)\neq(0,0)} \frac{\chi((a+b\eta))}{(a^2+ab+2b^2)^{s+1/2}},$$

which can be more simply written as

$$L(s,\chi) = \frac{1}{2} \sum_{(a,b)\neq(0,0)} \frac{(a+b\eta)\left(\frac{a^3-2a^2b-ab^2+b^3}{7}\right)}{(a^2+ab+2b^2)^{s+1/2}}.$$

This is the L-function of a cusp form of level 49 and weight 2 and is the L-function of the elliptic curve $y^2 + xy = x^3 - x^2 - 2x - 1$, a rank 0 CM elliptic curve of conductor 49. The L-function $L_E(s) = L(s,\chi)$ satisfies the functional equation

$$\left(\frac{7}{2\pi}\right)^s \Gamma(s+1/2)L(s,\chi) = \Phi(s) = \Phi(1-s).$$

We are interested in the primitive parts of the L-functions of the symmetric powers of $L(s,\chi)$. This amounts to looking at a sequence of Hecke Grössencharacters, denoted by χ^{2n-1}, $n = 1, 2, \ldots$. The series for $L(s,\chi^{2n-1})$ is

$$L(s,\chi^{2n-1}) = \frac{1}{2}\sum_{(a,b)\neq(0,0)} \frac{(a+b\eta)^{2n-1}\left(\frac{a^3-2a^2b-ab^2+b^3}{7}\right)}{(a^2+ab+2b^2)^{s+n-1/2}}.$$

(Note that $L(s,\chi^{2n})$ is identically zero.) The Euler product for $L(s,\chi^{2n-1})$ is

$$\begin{aligned}(1.1)\qquad & L(s,\chi^{2n-1}) \\ &= \prod_{p=a^2+ab+2b^2}\left(1 - \frac{\epsilon_{a,b}(a+b\eta)^{2n-1}}{p^{s+n+1/2}}\right)^{-1}\left(1 - \frac{\epsilon_{a,b}(a+b\overline{\eta})^{2n-1}}{p^{s+n+1/2}}\right)^{-1}.\end{aligned}$$

In general, if

$$L(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1}\left(1 - \frac{\overline{\alpha_p}}{p^s}\right)^{-1}$$

with $|\alpha_p| = 1$, then the symmetric kth power is (up to some bad factors)

$$\begin{aligned}&L(s,\mathrm{sym}^k) \\ &= \prod_p \left(1 - \frac{\alpha_p^k}{p^s}\right)^{-1}\left(1 - \frac{\alpha_p^{k-2}}{p^s}\right)^{-1}\ldots\left(1 - \frac{\overline{\alpha_p^{k-2}}}{p^s}\right)^{-1}\left(1 - \frac{\overline{\alpha_p^k}}{p^s}\right)^{-1}.\end{aligned}$$

Thus we see in our situation for the symmetric powers of the L-function of a CM elliptic curve that

$$L(s,\chi,\mathrm{sym}^{2n-1}) = L(s,\chi^{2n-1})L(s,\chi^{2n-3})L(s,\chi^{2n-5})\cdots L(s,\chi).$$

See [RS08, Section 4] for more explicit details.

It is convenient to define the function $\chi^{(2n-1)}$ at positive rational integers m by

$$\chi^{(2n-1)}(m) = \frac{1}{2} \sum_{a^2+ab+2b^2=m} \chi^{2n-1}((a+b\eta)).$$

Then

$$L(s,\chi^{2n-1}) = \sum_{m=1}^{\infty} \frac{\chi^{(2n-1)}(m)}{m^{s+n-1/2}}.$$

The functional equation for $L(s,\chi^{2n-1})$ is

$$\left(\frac{7}{2\pi}\right)^{s} \Gamma(s+n-1/2)L(s,\chi^{2n-1}) = \Phi_{2n-1}(s) = (-1)^{n-1}\Phi_{2n-1}(1-s)$$

and in asymmetric form

$$L(s,\chi^{2n-1}) = (-1)^{n-1} X_{2n-1}(s) L(1-s,\chi^{2n-1})$$

where

$$X_{2n-1}(s) = \left(\frac{7}{2\pi}\right)^{1-2s} \frac{\Gamma(1-s+n-1/2)}{\Gamma(s+n-1/2)}.$$

Here the center of the critical strip is at $s=1/2$.

Using Hecke's standard method, if $L(s) = \sum_{m=1}^{\infty} a_m m^{-s}$ is entire then the functional equation

$$Q^s\Gamma(s+a)L(s) = Q^{1-s}\Gamma(1-s+a)L(1-s)$$

is equivalent (via Mellin transforms) to

$$f(y) := \sum_{m=1}^{\infty} m^a a_m e^{-my/Q} = y^{-2a-1} f(1/y).$$

Therefore,

$$\begin{aligned} Q^s\Gamma(s+a)L(s) &= Q^{-a} \int_0^{\infty} f(y) y^{s+a}\,\frac{dy}{y} \\ &= Q^{-a} \int_1^{\infty} (f(y)y^{s+a} + f(1/y)y^{-s-a})\,\frac{dy}{y} \end{aligned}$$

whence

$$\begin{aligned} L(1/2) &= \frac{2Q^{-a-1/2}}{\Gamma(1/2+a)} \int_1^{\infty} f(y) y^{a+1/2}\,\frac{dy}{y} \\ &= \frac{2}{\Gamma(1/2+a)} \sum_{m=1}^{\infty} \frac{a_m}{m^{1/2}} \int_{m/Q}^{\infty} e^{-y} y^{a+1/2}\,\frac{dy}{y}. \end{aligned}$$

We apply this formula with $a=n-1/2$ and $Q=7/2\pi$ and use the formula for the incomplete gamma function:

$$\Gamma(b,z)=\int_z^\infty y^b e^{-y}\,\frac{dy}{y}.$$

In this way, if n is odd, we obtain

$$L(1/2,\chi^{2n-1})=\frac{2}{(n-1)!}\sum_{m=1}^{\infty}\frac{\chi^{(2n-1)}(m)}{m^n}\Gamma\Big(n,\frac{2\pi m}{7}\Big).$$

2. Moments of the L-function at the central point. One way to test the symmetry type of a family of L-functions is to compute average values of the L-functions evaluated at the central point. In families displaying orthogonal symmetry the average of the kth power of the central value of the L-function grows like the $k(k-1)/2$ power of the asymptotic variable, where as for unitary symmetry the growth is like the k^2 power and symplectic symmetry shows $k(k+1)/2$ power growth. In the case of our family of Hecke characters, the asymptotic parameter is $\log N$, so to agree with the predictions of orthogonal symmetry we expect the first moment to be asymptotically constant and the second moment to grow like $\log N$.

2.1. The first moment. We wish to compute

$$\mathcal{M}_r(N):=\frac{1}{N}\sum_{n=1}^{N}L(1/2,\chi^{4n-3})^r$$

asymptotically when $r=1$ and to give an upper bound when $r=2$.

We note that Greenberg (see [Gre83, p. 258]) states that such an asymptotic formula for this first moment (with no explicit error term) would follow from a formula of his, provided it were known that $L(1/2,\chi^{4n-3})\ge 0$. Villegas and Zagier [RVZ93] prove this non-negativity. It is instructive to give a direct treatment from first principles. In addition we have an explicit error term.

Theorem 2.1. *As $N\to\infty$, we have*

$$\mathcal{M}_1(N)=\frac{1}{N}\sum_{n=1}^{N}L(1/2,\chi^{4n-3})=\frac{2\pi}{\sqrt{7}}+O\Big(\frac{\log N}{\sqrt{N}}\Big).$$

Proof. We have

$$\mathcal{M}_1(N)=\frac{2}{N}\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\sum_{n=1}^{N}\frac{\Gamma(2n-1,2\pi m/7)}{\Gamma(2n-1)}\,\frac{\chi^{(4n-3)}(m)}{m^{2n-3/2}}.$$

Now

$$\chi^{(4n-3)}(m)=\frac{1}{2}\sum_{a^2+ab+2b^2=m}\Big(\frac{a^3-2a^2b-ab^2+b^3}{7}\Big)(a+b\eta)^{4n-3}$$

so that

$$\mathcal{M}_1(N) = \frac{1}{N}\sum_{(a,b)\neq(0,0)} \frac{\left(\frac{a^3-2a^2b-ab^2+b^3}{7}\right)}{\sqrt{a^2+ab+2b^2}} \sum_{n=1}^{N} \frac{\Gamma(2n-1, 2\pi(a^2+ab+2b^2)/7)}{\Gamma(2n-1)} \delta_{a,b}^{4n-3}$$

where

$$\delta_{a,b} = \frac{a+b\eta}{\sqrt{a^2+ab+2b^2}} = e(\theta_{a,b}),$$

say, where $e(x) = \exp(2\pi i x)$. Now suppose that $f(x)$ is a positive, smooth, increasing function on $[0,\infty)$. Then for any real number θ,

$$\sum_{n=1}^{N} f(n)e(n\theta) = \int_{1^-}^{N} f(u)\,d\Sigma(u) = f(N)\Sigma(N) - \int_{1}^{N} f'(u)\Sigma(u)\,du$$

where

$$\Sigma(u) = \sum_{n\le u} e(n\theta) = \frac{e^{2\pi i(1+[u])\theta} - e^{2\pi i\theta}}{1-e^{2\pi i\theta}} = \frac{e^{2\pi i(1/2+[u])\theta} - e^{\pi i\theta}}{e^{-\pi i\theta} - e^{\pi i\theta}}.$$

It follows that

$$|\Sigma(u)| \le \frac{1}{|\sin \pi\theta|}.$$

Thus,

$$\sum_{n=1}^{N} f(n)e(n\theta) \ll \frac{|f(N)|}{|\sin \pi\theta|}.$$

Now

$$f(n) = \frac{\Gamma(2n-1, 2\pi(a^2+ab+2b^2)/7)}{\Gamma(2n-1)}$$

does have the properties described above, and

$$\frac{\Gamma(2n-1,x)}{\Gamma(2n-1)} \ll e^{-x/n}.$$

Thus,

$$\mathcal{M}_1(N) = \frac{1}{N}\sum_{\substack{(a,b)\neq(0,0)\\ 4\theta_{a,b}\in\mathbb{Z}}} \frac{\left(\frac{a^3-2a^2b-ab^2+b^3}{7}\right)}{\sqrt{a^2+ab+2b^2}} \sum_{n=1}^{N} \frac{\Gamma(2n-1, 2\pi(a^2+ab+2b^2)/7)}{\Gamma(2n-1)} \delta_{a,b}^{4n-3}$$
$$+ O\left(\frac{1}{N}\sum_{4\theta_{a,b}\notin\mathbb{Z}} \frac{e^{-(a^2+ab+2b^2)/N}}{\sqrt{a^2+ab+2b^2}}\,\frac{1}{|\sin 4\pi\theta_{a,b}|}\right)$$

Now

$$\delta_{a,b} = \cos 2\pi\theta_{a,b} + i \sin 2\pi\theta_{a,b} = \frac{a+b\eta}{\sqrt{a^2+ab+2b^2}} = e(\theta_{a,b})$$

so that

$$\sin 4\pi\theta_{a,b} = 2\sin 2\pi\theta_{a,b}\cos 2\pi\theta_{a,b} = \frac{(a+b/2)b\sqrt{7}}{a^2+ab+2b^2}.$$

If $4\theta_{a,b} \notin \mathbb{Z}$ then

$$\begin{aligned} \frac{1}{|\sin 4\pi\theta_{a,b}|} &\ll \frac{a^2+ab+2b^2}{|a+b/2|\,|b|} \\ &\ll \sqrt{a^2+ab+2b^2}\max\{1/|a+b/2|, 1/|b|\} \end{aligned} \tag{2.1}$$

since $a^2+ab+2b^2 = (a+b/2)^2 + 7b^2/4$. Thus, the O-term above is

$$\begin{aligned} &\ll \frac{1}{N}\sum_{\substack{b\neq 0\\ 2a\neq -b}} e^{-(a^2+ab+2b^2)/N}\max\left\{\frac{1}{|a+b/2|}, \frac{1}{|b|}\right\} \\ &\ll \frac{1}{N}\sum_{\substack{b\neq 0\\ 2a\neq -b}} e^{-(a^2+ab+2b^2)/N}\left(\frac{1}{|a+b/2|} + \frac{1}{|b|}\right) \ll N^{-1/2}\log N. \end{aligned}$$

If $4\theta_{a,b} \in \mathbb{Z}$ then either $b = 0$, or $a = -b/2$ and $\delta_{a,b}^4 = 1$. In the situation that $a = -b/2$ we see that $\delta_{a,b} = a\sqrt{-7}$ is not coprime to $\sqrt{-7}$. So these terms do not contribute anything. Thus, we have

$$\begin{aligned} &\mathcal{M}_1(N) \\ &= \frac{1}{N}\sum_{\substack{b=0\\ a\neq 0}} \frac{\left(\frac{a^3-2a^2b-ab^2+b^3}{7}\right)}{\sqrt{a^2+ab+2b^2}}\delta_{a,b}^{-3}\sum_{n=1}^{N}\Gamma\left(2n-1, \frac{2\pi(a^2+ab+2b^2)}{7}\right)/\Gamma(2n-1) \\ &\quad + O(N^{-1/2}\log N). \end{aligned}$$

Lemma 2.2 (Tricomi [Tri50])). *Suppose that $b > 0$ and $n > 0$. Let*

$$\gamma(n,b) = \int_0^b e^{-x}x^n\,\frac{dx}{x}.$$

Then, as $n \to \infty$,

$$\frac{\gamma(n+1, n-y\sqrt{2n})}{\Gamma(n+1)} = \frac{1}{2}\operatorname{erfc}(y) - \frac{\sqrt{2}}{3\sqrt{\pi n}}(1+y^2)e^{-y^2} + O\left(\frac{1}{n}\right).$$

Here y is any real number and

$$\operatorname{erfc}(y) = \frac{2}{\sqrt{\pi}}\int_y^\infty e^{-t^2}\,dt.$$

For $y > 0$,

$$\operatorname{erfc}(y) = \frac{e^{-y^2}}{\sqrt{\pi} y}\left(1 - \frac{1}{2y^2} + \frac{3}{4y^4} + \cdots\right)$$

while

$$\operatorname{erfc}(-y) = 1 - \operatorname{erfc}(y).$$

We have, recalling $\left(\frac{-1}{7}\right) = -1$ and $\delta_{a,0} = a/|a|$,

$$\mathcal{M}_1(N) = \frac{2}{N}\sum_{a=1}^{\infty}\frac{\left(\frac{a}{7}\right)}{a}\sum_{n=1}^{N}\frac{\Gamma(2n-1, 2\pi a^2/7)}{\Gamma(2n-1)} + O(N^{-1/2}\log N).$$

We split the sum into four pieces: $(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4)/N$ where

$$\Sigma_1 = 2\sum_{n=1}^{N}\sum_{a^2 \le C_1 n - C_2\sqrt{n}}\frac{\left(\frac{a}{7}\right)}{a}\,\frac{\Gamma(2n-1, 2\pi a^2/7)}{\Gamma(2n-1)},$$

$$\Sigma_2 = 2\sum_{n=1}^{N}\sum_{C_1 n - C_2\sqrt{n} < a^2 \le C_1 n + C_2\sqrt{n}}\frac{\left(\frac{a}{7}\right)}{a}\,\frac{\Gamma(2n-1, 2\pi a^2/7)}{\Gamma(2n-1)},$$

$$\Sigma_3 = 2\sum_{n=1}^{N}\sum_{C_1 n + C_2\sqrt{n} < a^2 \le 2n}\frac{\left(\frac{a}{7}\right)}{a}\,\frac{\Gamma(2n-1, 2\pi a^2/7)}{\Gamma(2n-1)},$$

$$\Sigma_4 = 2\sum_{n=1}^{N}\sum_{a^2 > 2n}\frac{\left(\frac{a}{7}\right)}{a}\,\frac{\Gamma(2n-1, 2\pi a^2/7)}{\Gamma(2n-1)};$$

here $C_1 = 7/\pi$ and C_2 is a large constant. C_1 is chosen so that the two arguments in the incomplete gamma function are approximately equal; see Lemma 2.2 which describes this transition range.

We now have

$$\Sigma_1 = 2\sum_{n=1}^{N}\sum_{a^2 \le C_1 n - C_2\sqrt{n}}\frac{\left(\frac{a}{7}\right)}{a} - \Sigma_{1,a}$$

where

$$\Sigma_{1,a} = 2\sum_{n=1}^{N}\sum_{a^2 \le C_1 n - C_2\sqrt{n}}\frac{\left(\frac{a}{7}\right)}{a}\,\frac{\gamma(2n-1, 2\pi a^2/7)}{\Gamma(2n-1)}.$$

Now

$$\sum_{a^2 \le C_1 n - C_2\sqrt{n}}\frac{\left(\frac{a}{7}\right)}{a} = \sum_{a=1}^{\infty}\frac{\left(\frac{a}{7}\right)}{a} + O\left(\frac{1}{\sqrt{n}}\right)$$

so that

$$\Sigma_1 = 2L(1, \chi_{-7})N + O(\sqrt{N}) - \Sigma_{1,a}.$$

Using the lemma, we have

$$\Sigma_{1,a} \ll \sum_{n=1}^{N} \sum_{a^2 \le C_1 n - C_2\sqrt{n}} a^{-1}(\operatorname{erfc}(y) + y^2 e^{-y^2} n^{-1/2} + O(1/n))$$

where $y = (2n - 2 - 2\pi a^2/7)/\sqrt{4n-4}$. We illustrate how to estimate this sum with a simplified version that omits the constants:

$$\begin{aligned}
\sum_{n=1}^{N} \sum_{a^2 \le n-\sqrt{n}} \frac{1}{a} \operatorname{erfc}\Big(\frac{n-a^2}{\sqrt{n}}\Big) &\ll \sum_{n=1}^{N} \sum_{a^2 \le n-\sqrt{n}} a^{-1} \frac{e^{-(n-a^2)^2/n}}{(n-a^2)/\sqrt{n}} \\
&\ll \sum_{n=1}^{N} \int_{1}^{\sqrt{n-\sqrt{n}}} u^{-1} \frac{e^{-(n-u^2)^2/n}}{(n-u^2)/\sqrt{n}}\, du \\
&= \sum_{n=1}^{N} \int_{\sqrt{n}}^{n-1} \frac{e^{-v^2/n}}{v/\sqrt{n}} \frac{dv}{n-v} \\
&= \sum_{n=1}^{N} \sqrt{n} \int_{1}^{\sqrt{n}-1/\sqrt{n}} \frac{e^{-t^2}}{t} \frac{dt}{n - t\sqrt{n}}.
\end{aligned}$$

Now split the integral into $1 \le t \le \sqrt{n}/2$ and $\sqrt{n}/2 \le t \le \sqrt{n} - 1/\sqrt{n}$ to see that this sample sum is $\ll \sum_{n=1}^{N} 1/\sqrt{n} \ll \sqrt{N}$. We can treat the part with $y^2 e^{-y^2}/\sqrt{n}$ in a similar way and the $1/n$ part trivially. In this way we have

$$\Sigma_{1,a} \ll \sqrt{N}.$$

The inner sum over a of Σ_2 has a bounded number of terms, each of which is $\ll 1/a \ll 1/\sqrt{n}$. Thus $\Sigma_2 \ll \sqrt{N}$. We can treat Σ_3 exactly as we did $\Sigma_{1,a}$. Finally, the estimation of Σ_4 is like the estimation of

$$\begin{aligned}
&\sum_{n=1}^{N} \sum_{a^2 > 2n} a^{-1} \int_{a^2}^{\infty} e^{-t} t^{n-1}\, dt/\Gamma(n) \\
&\qquad \ll \sum_{n=1}^{N} \int_{2n}^{\infty} e^{-t} t^{n-1} \sum_{a \le \sqrt{t}} \frac{1}{a}\, dt/\Gamma(n) \\
&\qquad \ll \sum_{n=1}^{N} e^{-2n} (2n)^{n-1} (\log 2n)/\Gamma(n) \ll \sum_{n=1}^{N} e^{-n} \ll 1.
\end{aligned}$$

Thus, we conclude that

$$\mathcal{M}_1(N) = 2L(1, \chi_{-7}) + O\Big(\frac{\log N}{\sqrt{N}}\Big) = \frac{2\pi}{\sqrt{7}} + O\Big(\frac{\log N}{\sqrt{N}}\Big). \quad \blacksquare$$

2.2. The second moment. We can give an upper bound for the second moment $\mathcal{M}_2(N)$.

THEOREM 2.3. *We have*

$$\mathcal{M}_2(N) \ll \log^2 N.$$

Note that this result implies the convexity bound

$$L(1/2, \chi^{4n-3}) \ll n^{1/2} \log n,$$

addressing Greenberg's question on the size of these L-functions (see [Gre83, p. 258]).

Proof of Theorem 2.3. The proof follows exactly along the lines above. The only thing extra that we need is a bound for summing the inverse of a quadratic form.

LEMMA 2.4. *Let Q be a non-degenerate quadratic form in four variables with integer coefficients. Then*

$$\sum_{\substack{a,b,A,B \le X \\ Q(a,b,A,B) \neq 0}} \frac{1}{|Q(a,b,A,B)|} \ll X^2 \log^2 X.$$

For example,

$$\sum_{\substack{a,b,A,B \le X \\ a \neq b,\, A \neq B}} \frac{1}{|(a-b)(A-B)|} \ll X^2 \log^2 X.$$

We leave the proof as an interesting exercise for the reader.

Here is a sketch of the proof of the theorem. Take the square of the formula for the central value and average over n:

$$\mathcal{M}_2(N) = \frac{1}{N} \sum_{\substack{(a,b) \neq (0,0) \\ (A,B) \neq (0,0)}} \frac{\left(\frac{a^3-2a^2b-ab^2+b^3}{7}\right)}{\sqrt{a^2+ab+2b^2}} \frac{\left(\frac{A^3-2A^2B-AB^2+B^3}{7}\right)}{\sqrt{A^2+AB+2B^2}}$$
$$\times \sum_{n=1}^{N} \frac{\Gamma(2n-1, 2\pi(a^2+ab+2b^2)/7)}{\Gamma(2n-1)}$$
$$\times \frac{\Gamma(2n-1, 2\pi(A^2+AB+2B^2)/7)}{\Gamma(2n-1)} (\delta_{a,b}\delta_{A,B})^{4n-3}.$$

The inner sum is a geometric series with a smooth weight, as in the proof of Theorem 2.1. Now, with $\eta = (1+\sqrt{-7})/2$,

$$\delta_{a,b}\delta_{A,B} = \frac{a+b\eta}{|a+b\eta|} \frac{A+B\eta}{|A+B\eta|} = \frac{(aA-2bB)+(Ab+aB+bB)\eta}{\sqrt{a^2+ab+2b^2}\sqrt{A^2+AB+2B^2}}$$
$$= \delta_{aA-2bB,Ab+aB+bB} = e(\theta_{aA-2bB,Ab+aB+bB}).$$

Just as above, we have

$$\frac{1}{|\sin 4\pi\theta_{aA-2bB,Ab+aB+bB}|} \ll \sqrt{a^2+ab+2b^2}\,\sqrt{A^2+AB+2B^2} \times \max\{1/|2aA-4bB+Ab+aB+bB|, 1/|Ab+aB+bB|\}.$$

Thus, by Lemma 2.4, the terms with $4\pi\theta_{aA-2bB,Ab+aB+bB} \notin \mathbb{Z}$ contribute an amount which is $\ll N\log^2 N$. If $4\pi\theta_{aA-2bB,Ab+aB+bB} \in \mathbb{Z}$, then it must be the case that either $Ab+aB+bB = 0$ or else $2aA-4bB+Ab+aB+bB = 0$. As before, in the second case the coefficient of this term is 0. Thus, $Ab+aB+bB = 0$. If $(a,b) = 1 = (A,B)$, then we have $B(a+b) = -Ab$ and $b(A+B) = -aB$ so that $b \mid B$ and $B \mid b$. If $B = 0$, then $b = 0$ and vice versa. If $B = -b$ then $A = a+b$, and if $B = b \neq 0$ then $A = -a-b$. In any of these events we have

$$\sum_{\substack{a,b,A,B\le X\\ Ab+aB+bB=0}} \frac{1}{\sqrt{a^2+ab+2b^2}\,\sqrt{A^2+AB+2B^2}} \ll \log^2 X.$$

We conclude that, in this diagonal case, the sum over n is N and the sum over a, A, b, B is $\ll \log^2 N$. Thus, we have shown that $\mathcal{M}_2(N) \ll \log^2 N$ as desired. ∎

Corollary 2.5. *For at least $N/\log^2 N$ values of $n \le N$ we have*

$$L(2n-1, \chi^{4n-3}) \neq 0.$$

This follows from a standard use of Cauchy's inequality:

$$\Big|\sum_{n=1}^{N} L(1/2, \chi^{4n-3})\Big|^2 \le \Big(\sum_{\substack{n\le N\\ L(1/2,\chi^{4n-3})\neq 0}} 1\Big)\Big(\sum_{n=1}^{N} |L(1/2, \chi^{4n-3})|^2\Big)$$

whence

$$\sum_{\substack{n\le N\\ L(1/2,\chi^{4n-3})\neq 0}} 1 \ge \frac{(N\mathcal{M}_1(N))^2}{N\mathcal{M}_2(N)} \gg \frac{N^2}{N\log^2 N} \gg \frac{N}{\log^2 N}.$$

3. Moment conjectures. In this section we use the moment conjectures described in [CFK+05] to calculate the first and second moments of our family of L-functions. The first moment agrees with the previous section and both of the first two moments support the hypothesis that the family has orthogonal symmetry.

3.1. The first moment. We want to use the moment conjecture recipe to find the first moment at the central point of the family of L-functions $L(s, \chi^{4n-3})$ as we vary n. We want to calculate

$$\frac{1}{N}\sum_{n=1}^{N} L(1/2+\alpha,\chi^{4n-3}).$$

The functional equation for $L(s,\chi^{4n-3})$ looks like

$$\left(\frac{7}{2\pi}\right)^s \Gamma(s+2n-3/2)L(s,\chi^{4n-3}) = \Phi_{4n-3}(s) = \Phi_{4n-3}(1-s),$$

defining

$$X_{4n-3}(s) := \left(\frac{7}{2\pi}\right)^{1-2s}\frac{\Gamma(1-s+2n-3/2)}{\Gamma(s+2n-3/2)},$$

and so the main two terms in the approximate functional equation give us

$$\frac{1}{N}\sum_{n=1}^{N} L(1/2+\alpha,\chi^{(4n-3)}) = \frac{1}{N}\sum_{n=1}^{N}\left(\sum_m \frac{\chi^{(4n-3)}(m)}{m^{2n-1+\alpha}} + X_{4n-3}(1/2+\alpha)\sum_m \frac{\chi^{(4n-3)}(m)}{m^{2n-1-\alpha}}\right).$$

The recipe instructs us to perform the average over n over the characters and the X factor from the functional equation. The quantity we need to understand is

$$\delta(m) = \left\langle \frac{\chi^{(4n-3)}(m)}{m^{2n-3/2}}\right\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{\chi^{(4n-3)}(m)}{m^{2n-3/2}}. \tag{3.1}$$

We claim that δ is multiplicative. To prove this claim it suffices to show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{(a+b\eta)(c+d\eta)}{|a+b\eta|\,|c+d\eta|}\right)^{4n} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{a+b\eta}{|a+b\eta|}\right)^{4n}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{c+d\eta}{|c+d\eta|}\right)^{4n}$$

whenever $(N(a+b\eta),N(c+d\eta))=1$. The only time that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{A+B\eta}{|A+B\eta|}\right)^{4n}$$

is not zero is when

$$\left(\frac{A+B\eta}{|A+B\eta|}\right)^{4} = 1$$

or, equivalently, when $(A+B\eta)^4$ is real and positive, i.e. $A+B\eta$ is either real or purely imaginary, which translates to either $B=0$ or $A=-B/2$. Thus, under the assumption that $(N(a+b\eta),N(c+d\eta))=1$, it suffices to prove the assertion that $(a+b\eta)(c+d\eta)$ is either real or purely imaginary if and only if each of $a+b\eta$ and $c+d\eta$ is either real or purely imaginary.

One direction is clear. In the other direction, we consider two cases. In the first case suppose that $(a+b\eta)(c+d\eta) = A + B\eta$ with $B = 0$. Then

$$a'd + c'b = 0 \tag{3.2}$$

where $a' = 2a + b$ and $c' = 2c + d$ (note that a', b, c', d are all integers). Write $a' = gA'$ and $b = gB$ with $g = (a', b)$. Then equation (3.2) becomes $A'd + c'B = 0$. Since $(A', B) = 1$ it must be the case that $B \mid d$ and $A' \mid c'$, say $d = tB$ and $c' = -tA'$. Substituting back we have

$$a = g(A' - B)/2, \qquad b = gB, \qquad c = -t(A' + B)/2, \qquad d = Bt.$$

Then

$$\begin{aligned} N(a+b\eta) &= a^2 + ab + 2b^2 = (A'^2 + 7B^2)g^2/4, \\ N(c+d\eta) &= c^2 + cd + 2d^2 = (A'^2 + 7B^2)t^2/4. \end{aligned}$$

If $B \neq 0$ then clearly $(N(a+b\eta), N(c+d\eta)) > 1$. Therefore $B = 0$, which implies that $b = d = 0$. Similar sorts of arguments work in the case that $(a+b\eta)(c+d\eta) = A + B\eta$ with $A = -B/2$.

Thus, δ is multiplicative. Now we need to evaluate it at prime power arguments. It is not hard to show that δ vanishes at non-square arguments and that

$$\delta(p^2) = \begin{cases} 0 & \text{if } p = 7, \\ +1 & \text{if } \left(\frac{p}{7}\right) = 1, \\ -1 & \text{if } \left(\frac{p}{7}\right) = -1, \end{cases}$$

and that

$$\delta(p^{2k}) = \delta(p^2)^k.$$

That is,

$$\delta(m) = \begin{cases} 0 & \text{if } m \neq \Box, \\ \left(\frac{\sqrt{m}}{7}\right) & \text{if } m = \Box. \end{cases}$$

So, by the recipe we have

$$\begin{aligned} &\frac{1}{N}\sum_{n=1}^{N} L(1/2+\alpha, \chi^{4n-3}) \\ &\qquad\approx \frac{1}{N}\sum_{n=1}^{N}\left(\sum_m \frac{\delta(m)}{m^{1/2+\alpha}} + \langle X_{4n-3}(1/2+\alpha)\rangle \sum_m \frac{\delta(m)}{m^{1/2-\alpha}}\right) \\ &\qquad= \frac{1}{N}\sum_{n=1}^{N}\left(\sum_m \frac{\delta(m^2)}{m^{1+2\alpha}} + \langle X_{4n-3}(1/2+\alpha)\rangle \sum_m \frac{\delta(m^2)}{m^{1-2\alpha}}\right) \\ &\qquad= \frac{1}{N}\sum_{n=1}^{N}\left(\sum_m \frac{\left(\frac{m}{7}\right)}{m^{1+2\alpha}} + \langle X_{4n-3}(1/2+\alpha)\rangle \sum_m \frac{\left(\frac{m}{7}\right)}{m^{1-2\alpha}}\right). \end{aligned}$$

So, the moment conjecture in this case would be

$$\frac{1}{N}\sum_{n=1}^{N} L(1/2+\alpha,\chi^{4n-3}) = L(1+2\alpha,\chi_{-7}) + \left(\frac{7}{2\pi}\right)^{-2\alpha}\frac{1}{N}\sum_{n=1}^{N}\frac{\Gamma(2n-1-\alpha)}{\Gamma(2n-1+\alpha)}L(1-2\alpha,\chi_{-7}) + O(N^{-1/2+\epsilon}).$$

When $\alpha=0$,

$$\frac{1}{N}\sum_{n=1}^{N} L(1/2,\chi^{4n-3}) = 2L(1,\chi_{-7}) + O(N^{-1/2+\epsilon}).$$

Thus we have

Conjecture 3.1. *By the moment conjecture recipe from* [CFK+05],

$$\frac{1}{N}\sum_{n=1}^{N} L(1/2,\chi^{4n-3}) = \frac{2\pi}{\sqrt{7}} + O(N^{-1/2+\epsilon}).$$

3.2. The second moment. Now we calculate the second moment using the moment conjecture:

$$\begin{aligned}\sum_{n=1}^{N} & L(1/2+\alpha,\chi^{4n-3})L(1/2+\beta,\chi^{4n-3})\\ &\approx \sum_{n=1}^{N}\Big(\sum_{\ell}\frac{\chi^{(4n-3)}(\ell)}{\ell^{2n-1+\alpha}} + X_{4n-3}(1/2+\alpha)\sum_{\ell}\frac{\chi^{(4n-3)}(\ell)}{\ell^{2n-1-\alpha}}\Big)\\ &\qquad\times\left(\sum_{m}\frac{\chi^{(4n-3)}(m)}{m^{2n-1+\beta}} + X_{4n-3}(1/2+\beta)\sum_{m}\frac{\chi^{(4n-3)}(m)}{m^{2n-1-\beta}}\right)\\ &\approx \sum_{n=1}^{N}\Big(\sum_{\ell,m}\frac{\delta(\ell,m)}{\ell^{1/2+\alpha}m^{1/2+\beta}} + \langle X_{4n-3}(1/2+\alpha)\rangle\sum_{\ell,m}\frac{\delta(\ell,m)}{\ell^{1/2-\alpha}m^{1/2+\beta}}\\ &\qquad + \langle X_{4n-3}(1/2+\beta)\rangle\sum_{\ell,m}\frac{\delta(\ell,m)}{\ell^{1/2+\alpha}m^{1/2-\beta}}\\ &\qquad + \langle X_{4n-3}(1/2+\alpha)X_{4n-3}(1/2+\beta)\rangle\sum_{\ell,m}\frac{\delta(\ell,m)}{\ell^{1/2-\alpha}m^{1/2-\beta}}\Big).\end{aligned}$$

Here we define

$$\delta(\ell,m) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{\chi^{(4n-3)}(\ell)}{\ell^{2n-3/2}}\,\frac{\chi^{(4n-3)}(m)}{m^{2n-3/2}}.$$

We find that δ is multiplicative. That is, if $(\ell_1\ell_2, m_1m_2) = 1$, then

$$\delta(\ell_1 m_1, \ell_2 m_2) = \delta(\ell_1, \ell_2)\delta(m_1, m_2). \tag{3.3}$$

We leave the proof to the reader; basically it is an elaboration of the proof for the one-variable δ given immediately after (3.1). The behavior of δ is summarized as (for $a < b$)

$$\delta(p^a, p^b) = \begin{cases} 0 & \text{if } a+b \text{ is odd, } p = 1,2,4 \bmod 7, \\ a+1 & \text{if } a+b \text{ is even, } p = 1,2,4 \bmod 7, \\ 0 & \text{if } a \text{ or } b \text{ is odd, } p = 3,5,6 \bmod 7, \\ (-1)^{(a+b)/2} & \text{if } a \text{ and } b \text{ are even, } p = 3,5,6 \bmod 7. \end{cases}$$

The practical use of being multiplicative is that we can write the sum over δ as an Euler product:

$$\sum_{\ell,m} \frac{\delta(\ell,m)}{\ell^{1/2+\alpha} m^{1/2+\beta}} = \prod_p \sum_{a,b} \frac{\delta(p^a,p^b)}{p^{(1/2+\alpha)a+(1/2+\beta)b}}. \tag{3.4}$$

For primes $p = 1,2,4 \bmod 7$,

$$\begin{aligned} &\sum_{a,b=0}^{\infty} \delta(p^a,p^b) p^{-(\alpha+1/2)a} p^{-(\beta+1/2)b} \\ &\qquad = \sum_{a=0}^{\infty} \sum_{\substack{b=0 \\ a+b\,\text{even}}}^{a} (b+1) p^{-(\alpha+1/2)a} p^{-(\beta+1/2)b} \\ &\qquad\quad + \sum_{b=0}^{\infty} \sum_{\substack{a=0 \\ a+b\,\text{even}}}^{b} (a+1) p^{-(\alpha+1/2)a} p^{-(\beta+1/2)b} \\ &\qquad\quad - \sum_{a=0}^{\infty} (a+1) p^{-(\alpha+1/2)a} p^{-(\beta+1/2)a} \\ &\qquad = \frac{\left(1-\frac{1}{p^{1+2\alpha}}\right)^{-1}\left(1-\frac{1}{p^{1+\alpha+\beta}}\right)^{-1}\left(1-\frac{1}{p^{1+2\beta}}\right)^{-1}}{\left(1+\frac{1}{p^{1+\alpha+\beta}}\right)^{-1}}. \end{aligned} \tag{3.5}$$

Note that if all the primes contributed an expression of this form, the Euler product would yield

$$\frac{\zeta(1+2\alpha)\zeta(1+\alpha+\beta)^2\zeta(1+2\beta)}{\zeta(2+2\alpha+2\beta)},$$

which has a fourth order pole as α and β approach zero.

For primes $p = 3, 5, 6 \bmod 7$,

$$(3.6)\qquad \sum_{a,b=0}^{\infty} \delta(p^a, p^b) p^{-(\alpha+1/2)a} p^{-(\beta+1/2)b}$$
$$= \sum_{a,b=0}^{\infty} (-1)^{a+b} p^{-2a(\alpha+1/2)} p^{-2b(\beta+1/2)}$$
$$= \left(1 + \frac{1}{p^{1+2\alpha}}\right)^{-1}\left(1 + \frac{1}{p^{1+2\beta}}\right)^{-1} = \frac{\left(1 - \frac{1}{p^{2+4\alpha}}\right)^{-1}\left(1 - \frac{1}{p^{2+4\beta}}\right)^{-1}}{\left(1 - \frac{1}{p^{1+2\alpha}}\right)^{-1}\left(1 - \frac{1}{p^{1+2\beta}}\right)^{-1}}.$$

If all the primes had this contribution the resulting product would be

$$\frac{\zeta(2+4\alpha)\zeta(2+4\beta)}{\zeta(1+2\alpha)\zeta(1+2\beta)},$$

which has a second order zero. Thus, half the primes lead to a fourth order pole and the other half to a second order zero; taken together over all the primes we have a pole of order $\frac{4}{2} - \frac{2}{2} = 1$, which is what we expect from an orthogonal family second moment.

Combining (3.5) and (3.6) we find that (3.4) is equal to

$$\frac{L(1+2\alpha, \chi_{-7})L(1+2\beta, \chi_{-7})\zeta_7(1+\alpha+\beta)L(1+\alpha+\beta, \chi_{-7})}{\zeta_7(2+2\alpha+2\beta)},$$

where

$$\zeta_7(s) = \prod_{p\neq 7}\left(1 - \frac{1}{p^s}\right)^{-1}.$$

Thus the second moment would be

$$(3.7)\qquad \frac{1}{N}\sum_{n=1}^{N} L(1/2+\alpha, \chi^{4n-3})L(1/2+\beta, \chi^{4n-3})$$
$$= \frac{1}{N}\sum_{n=1}^{N}\Bigg(\zeta(1+\alpha+\beta)F(\alpha,\beta)$$
$$+ \left(\frac{7}{2\pi}\right)^{-2\alpha}\frac{\Gamma(2n-1-\alpha)}{\Gamma(2n-1+\alpha)}\zeta(1-\alpha+\beta)F(-\alpha,\beta)$$
$$+ \left(\frac{7}{2\pi}\right)^{-2\beta}\frac{\Gamma(2n-1-\beta)}{\Gamma(2n-1+\beta)}\zeta(1+\alpha-\beta)F(\alpha,-\beta)$$
$$+ \left(\frac{7}{2\pi}\right)^{-2\alpha-2\beta}\frac{\Gamma(2n-1-\alpha)}{\Gamma(2n-1+\alpha)}\frac{\Gamma(2n-1-\beta)}{\Gamma(2n-1+\beta)}\zeta(1-\alpha-\beta)F(-\alpha,-\beta)\Bigg)$$
$$+ O(N^{-1/2+\epsilon}),$$

where

$$\text{(3.8)}\quad F(\alpha,\beta) = \frac{L(1+2\alpha,\chi_{-7})L(1+2\beta,\chi_{-7})L(1+\alpha+\beta,\chi_{-7})\big(1-\frac{1}{7^{1+\alpha+\beta}}\big)}{\zeta(2+2\alpha+2\beta)\big(1-\frac{1}{7^{2+2\alpha+2\beta}}\big)}.$$

Now we want to let $\alpha,\beta\to 0$. We group the first and fourth terms in the numerator of (3.8) and in the second and third term send $\alpha\to-\alpha$ (allowable because we are going to take the limit $\alpha\to 0$) and then factor out the exponential and gamma factors

$$\left(\frac{7}{2\pi}\right)^{2\alpha}\frac{\Gamma(2n-1+\alpha)}{\Gamma(2n-1-\alpha)}$$

to leave exactly the first and fourth terms again. The expression (3.8) tends to 1 as $\alpha,\beta\to 0$. Thus we expect that the second moment evaluated at the central point is

$$\begin{aligned}
\mathcal{M}_2(N) &:= \lim_{\alpha,\beta\to 0}\frac{1}{N}\sum_{n=1}^{N} L(1/2+\alpha,\chi^{4n-3})L(1/2+\beta,\chi^{4n-3})\\
&= \lim_{\alpha,\beta\to 0}\frac{1}{N}\sum_{n=1}^{N}\left(1+\left(\frac{7}{2\pi}\right)^{2\alpha}\frac{\Gamma(2n-1+\alpha)}{\Gamma(2n-1-\alpha)}\right)\bigg(\zeta(1+\alpha+\beta)F(\alpha,\beta)\\
&\quad+\left(\frac{7}{2\pi}\right)^{-2\alpha-2\beta}\frac{\Gamma(2n-1-\alpha)}{\Gamma(2n-1+\alpha)}\,\frac{\Gamma(2n-1-\beta)}{\Gamma(2n-1+\beta)}\zeta(1-\alpha-\beta)F(-\alpha,-\beta)\bigg)\\
&\quad+O(N^{-1/2+\epsilon})\\
&= 4\left(f_0\gamma+f_1-f_0\log\frac{2\pi}{7}+f_0\frac{1}{N+1}\sum_{n=1}^{N}\frac{\Gamma'(2n-1)}{\Gamma(2n-1)}\right)\\
&\quad+O(N^{-1/2+\epsilon}),
\end{aligned}$$

where we expand $F(a,b)$ around $a=0$, $b=0$ as

$$F(a,b)=f_0+f_1a+f_1b+\cdots$$

with

$$f_0=\frac{L(1,\chi_{-7})^3}{\zeta(2)}\,\frac{7}{8}=\left(\frac{\pi}{\sqrt{7}}\right)^3\frac{6}{\pi^2}\,\frac{7}{8}=\frac{3\pi}{4\sqrt{7}}$$

and

$$\frac{\partial}{\partial\alpha}F(\alpha,\beta)\bigg|_{\alpha,\beta=0}=f_1=f_0\left(3\frac{L'}{L}(1,\chi_{-7})-2\frac{\zeta'}{\zeta}(2)+\frac{\log 7}{8}\right).$$

So, our conjecture is that

$$\mathcal{M}_2(N) = \frac{3\pi}{\sqrt{7}}\biggl(\gamma + 3\frac{L'}{L}(1,\chi_{-7}) - 2\frac{\zeta'}{\zeta}(2) + \frac{\log 7}{8} - \log\frac{2\pi}{7} + \frac{1}{N}\sum_{n=1}^{N}\frac{\Gamma'(2n-1)}{\Gamma(2n-1)}\biggr) + O(N^{-1/2+\epsilon}). \tag{3.9}$$

It can be shown that this reduces to

Conjecture 3.2.

$$\mathcal{M}_2(N) = \frac{3\pi}{\sqrt{7}}(\log N + C) + O(N^{-1/2+\epsilon})$$

where

$$C = 4\gamma - 3\log\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)} - 2\frac{\zeta'}{\zeta}(2) + \frac{\log 7}{8} + \log 7\pi^2 + 3\log 2 - 1.$$

With $N = 469$, computing L-values and evaluating $\mathcal{M}_2(N)$ numerically gives 28.37. The main term of equation (3.9) gives 28.35.

4. One-level density. In this section we assume the Riemann Hypothesis for the family of $L(s,\chi^{4n-3})$ and calculate the one-level density for this family, valid for test functions whose Fourier transforms are supported in $[-\alpha,\alpha]$ where $\alpha < 1$. As a consequence of our calculation we can show

Theorem 4.1. *If the Riemann Hypothesis for the family $\{L(s,\chi^{4n-3})\}$ is true, then $L(1/2,\chi^{4n-3}) \neq 0$ for at least $(1/4-\epsilon)N$ values of $n \le N$.*

Let $\phi(t) = F(1/2+it)$ be an even test function whose Fourier transform has compact support. Then F is entire and decays quickly with $|t|$. We derive an explicit formula. Defining

$$-\frac{L'(s,\chi^{4n-3})}{L(s,\chi^{4n-3})} = \sum_{k=1}^{\infty}\frac{\Lambda_{4n-3}(k)}{k^s}, \tag{4.1}$$

we write

$$\begin{aligned}\frac{1}{2\pi i}\int_{(2)}\frac{L'}{L}(s,\chi^{4n-3})F(s)\,ds &= -\sum_{k=1}^{\infty}\Lambda_{4n-3}(k)\frac{1}{2\pi i}\int_{(2)}F(s)k^{-s}\,ds \\ &= -\sum_{k=1}^{\infty}\Lambda_{4n-3}(k)\frac{1}{2\pi\sqrt{k}}\int_{-\infty}^{\infty}\phi(t)e^{-it\log k}\,dt \\ &= -\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{\Lambda_{4n-3}(k)}{\sqrt{k}}\hat{\phi}\Bigl(\frac{\log k}{2\pi}\Bigr).\end{aligned} \tag{4.2}$$

Here we use the definition

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(u)e(-ut)\,du.$$

On the left side of (4.2) we move the contour to the vertical line with real part 1/4 (note we have assumed RH and so have encircled all the zeros). Thus we get, with $\rho_n = 1/2 + \gamma_n$ a generic (1) zero,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(2)} \frac{L'}{L}(s,\chi^{4n-3})F(s)\,ds \\
&\quad= \sum_{\rho_n} F(\rho_n) + \frac{1}{2\pi i}\int_{(1/4)} \frac{L'}{L}(s,\chi^{4n-3})F(s)\,ds \\
&\quad= \sum_{\gamma_n} \phi(\gamma_n) + \frac{1}{2\pi i}\int_{(1/4)} \left(\frac{X'_{4n-3}}{X_{4n-3}}(s) - \frac{L'}{L}(1-s,\chi^{4n-3})\right)F(s)\,ds \\
&\quad= \sum_{\gamma_n} \phi(\gamma_n) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X'_{4n-3}}{X_{4n-3}}(1/2+it)\phi(t)\,dt \\
&\qquad - \frac{1}{2\pi i}\int_{(3/4)} \frac{L'}{L}(s,\chi^{4n-3})F(1-s)\,ds.
\end{aligned}$$

In the last integral the contour can be moved to the vertical line with real part 2:

$$\begin{aligned}
-\frac{1}{2\pi i}\int_{(2)} \frac{L'}{L}(s,\chi^{4n-3})F(1-s)\,ds &= \sum_{k=1}^{\infty} \Lambda_{4n-3}(k)\,\frac{1}{2\pi i}\int_{(2)} F(1-s)k^{-s}\,ds \\
&= \sum_{k=1}^{\infty} \Lambda_{4n-3}(k)\,\frac{1}{2\pi i}\int_{(1/2)} F(s)k^{s-1}\,ds \\
&= \sum_{k=1}^{\infty} \Lambda_{4n-3}(k)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi(t)}{\sqrt{k}}k^{it}\,dt \\
&= \frac{1}{2\pi}\sum_{k=1}^{\infty} \frac{\Lambda_{4n-3}(k)}{\sqrt{k}}\hat{\phi}\left(\frac{\log k}{2\pi}\right).
\end{aligned}$$

So,

$$\sum_{\gamma} \phi(\gamma) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X'}{X}(1/2+it)\phi(t)\,dt - \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\Lambda_{4n-3}(k)}{\sqrt{k}}\hat{\phi}\left(\frac{\log k}{2\pi}\right).$$

(1) The subscript n on ρ_n merely means that it is a zero of $L(s,\chi^{4n-3})$ and should not be mistaken for the nth zero.

We really want to work with scaled zeros in the explicit formula, so now we define a set of zeros $\widetilde{\gamma}_n$ that have average consecutive spacing of 1. We need to know how many zeros $L(s,\chi^{4n-3})$ has in the interval $0<t<T$, where T is large but bounded. We calculate the change in argument of
$$\xi(s,\chi^{4n-3})=\left(\frac{7}{2\pi}\right)^s\Gamma(s+2n-3/2)L(s,\chi^{4n-3})$$
around a contour enclosing these zeros.

We have, with the contour C defined as the rectangle with corners 2, $2+iT$, $-1+iT$ and -1,
$$\begin{aligned}
&\#\{\gamma_n\le T: L(1/2+i\gamma,\chi^{4n-3})=0\}\\
&\quad=\frac{1}{2\pi}\Delta_C\arg(\xi(s,\chi^{4n-3}))\\
&\quad=\frac{1}{\pi}\Delta\arg\left(\left(\frac{7}{2\pi}\right)^s\Gamma(s+2n-3/2)\right)\Bigg|_{s=1/2}^{s=1/2+iT}+O\left(\frac{\log 2n}{\log\log 3n}\right)\\
&\quad=\frac{T}{\pi}\log 2n+O\left(\frac{\log 2n}{\log\log 3n}\right).
\end{aligned}$$
In the third line we consider just half the contour C (from $s=1/2$, through $s=2$ and $s=2+iT$, to $s=1/2+iT$) because the functional equation gives $\xi(\sigma+it)=\xi(1-\sigma-it)=\overline{\xi(1-\sigma+it)}$. The change in argument of $L(s,\chi^{4n-3})$ is contained in the error term and follows by standard methods on assuming the Riemann Hypothesis for this L-function. For the final line we use Stirling's formula and so we see that the zeros, γ_n, of $L(s,\chi^{4n-3})$ need to be scaled as
$$\widetilde{\gamma}_n=\frac{\log 2n}{\pi}\gamma_n$$
in order to have approximate unit mean spacing.

We now define a new test function (2)
$$\phi(x)=f\left(\frac{x\log N}{\pi}\right).$$
We note that
$$\begin{aligned}
\hat{\phi}(x):={}&\int_{-\infty}^{\infty}\phi(u)e(-ux)\,du=\int_{-\infty}^{\infty}f\left(\frac{u\log N}{\pi}\right)e(-ux)\,du\\
&=\frac{\pi}{\log N}\int_{-\infty}^{\infty}f(u)e\left(-\frac{u\pi x}{\log N}\right)du=\hat{f}\left(\frac{\pi x}{\log N}\right)\frac{\pi}{\log N}.
\end{aligned}$$

With the support of $\hat{f}$ restricted, $\operatorname{supp}\hat{f}\subset[-\alpha,\alpha]$, $\alpha<1$, we have a scaled explicit formula:

(2) For convenience we scale up by $\log N$ instead of the asymptotically equal $\log 2n$.

$$
\begin{aligned}
(4.3)\qquad & \frac{1}{N}\sum_{n=1}^{N}\sum_{\gamma_n} f\left(\frac{\gamma_n \log N}{\pi}\right) \\
&= \sum_{n=1}^{N}\left(-\frac{1}{2\pi N}\int_{-\infty}^{\infty}\frac{X'_{4n-3}}{X_{4n-3}}(1/2+it) f\left(\frac{t\log N}{\pi}\right)dt \right.\\
&\qquad \left. -\frac{1}{N\log N}\sum_{k=1}^{\infty}\frac{\Lambda_{4n-3}(k)}{\sqrt{k}}\hat{f}\left(\frac{\log k}{2\log N}\right)\right).
\end{aligned}
$$

To evaluate the right side of this formula we start with

$$
\begin{aligned}
&\sum_{n=1}^{N}\frac{X'_{4n-3}}{X_{4n-3}}(1/2+it) \\
&\quad = \sum_{n=1}^{N}\left(-\frac{\Gamma'}{\Gamma}(2n-1-it)-\frac{\Gamma'}{\Gamma}(2n-1+it)-2\log\frac{7}{2\pi}\right) \\
&\quad = -2\sum_{n=1}^{N}(\log 2n+O(1)) = -2N\log N+O(N)
\end{aligned}
$$

for bounded t. So we have

$$
\begin{aligned}
\left(\frac{\log N}{\pi}+O(1)\right)\int_{-\infty}^{\infty} f\left(\frac{t\log N}{\pi}\right)dt &= \left(1+O\left(\frac{1}{\log N}\right)\right)\int_{-\infty}^{\infty} f(t)\,dt \\
&= \left(1+O\left(\frac{1}{\log N}\right)\right)\hat{f}(0) \\
&\to \hat{f}(0) \quad \text{as } N\to\infty.
\end{aligned}
$$

Next we address the sum in (4.3) containing $\Lambda_{4n-3}(k)$. With $\eta = (1+\sqrt{-7})/2$, we recall

$$
L(s,\chi^{4n-3}) = \sum_{k=1}^{\infty}\frac{\chi^{(4n-3)}(k)}{k^{s+2n-3/2}} = \frac{1}{2}\sum_{k=1}^{\infty}\sum_{a^2+ab+2b^2=k}\epsilon_{a,b}\frac{(a+b\eta)^{4n-3}}{k^{s+2n-3/2}}.
$$

The Euler product is

$$
\begin{aligned}
L(s,\chi^{4n-3}) &= \prod_{p\neq 7}\left(1-\frac{\chi^{(4n-3)}(p)}{p^{s+2n-3/2}}+\frac{p^{4n-3}}{p^{2s}}\right)^{-1} \\
&= \prod_{p\neq 7}\left(1-\frac{\alpha_n(p)}{p^s}\right)^{-1}\left(1-\frac{\overline{\alpha_n(p)}}{p^s}\right)^{-1},
\end{aligned}
$$

where the α_p have the properties

$$\begin{aligned}
|\alpha_n(p)| &= 1,\\
\frac{\chi^{(4n-3)}(p)}{p^{2n-3/2}} &= \alpha_n(p)+\overline{\alpha_n(p)} = \frac{1}{2}\,\frac{\sum_{a^2+ab+2b^2=p}\epsilon_{a,b}(a+b\eta)^{4n-3}}{p^{2n-3/2}},\\
\frac{\chi^{(4n-3)}(p^2)}{p^{4n-3}} &= \alpha_n(p)^2+1+\overline{\alpha_n(p)}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{L'}{L}(s,\chi^{4n-3}) &= \frac{d}{ds}\bigg(-\sum_{p\neq 7}\log\Big(1-\frac{\alpha_n(p)}{p^s}\Big)+\log\Big(1-\frac{\overline{\alpha_n(p)}}{p^s}\Big)\bigg)\\
&= \frac{d}{ds}\sum_{p\neq 7}\sum_{r=1}^{\infty}\bigg(\frac{\alpha_n(p)^r}{rp^{rs}}+\frac{\overline{\alpha_n(p)}^r}{rp^{rs}}\bigg)\\
&= -\sum_{p\neq 7}\log p\sum_{r=1}^{\infty}\frac{(\alpha_n(p)^r+\overline{\alpha_n(p)}^r)}{p^{rs}},
\end{aligned}$$

and so by (4.1) we have $\Lambda_{4n-3}(k)=0$ if k is not a prime power and

$$\Lambda_{4n-3}(p^r)=\log p\,(\alpha_n(p)^r+\overline{\alpha_n(p)}^r).$$

For all $k=p^r$, with $r\geq 3$, the sum over the coefficients $\Lambda_{4n-3}(k)$ in (4.3) is, with the sum over p running over primes,

$$\frac{1}{N\log N}\sum_{n=1}^{N}\sum_{p}\sum_{r\geq 3}\frac{\log p\,(\alpha_n(p)^r+\overline{\alpha_n(p)}^r)}{p^{r/2}}\hat{f}\Big(\frac{r\log p}{2\log N}\Big) \ll \frac{1}{\log N}\sum_p\frac{\log p}{p^{3/2}}\ll\frac{1}{\log N}.$$

Here the first "$\ll$" follows because $\hat{f}$ and $\alpha_n(p)$ are bounded and once removed leave a geometric series in r and no dependence on n.

When $k=p$ is a prime, we have the sum

$$\frac{-1}{2N\log N}\sum_{n=1}^{N}\sum_{p\leq N^{2\alpha}}\hat{f}\Big(\frac{\log p}{2\log N}\Big)\frac{\log p}{\sqrt{p}}\sum_{a^2+ab+2b^2=p}\xi_{a,b}e(4n\theta_{a,b}), \tag{4.4}$$

where

$$\xi_{a,b}=\epsilon_{a,b}e(-3\theta_{a,b})\quad\text{and}\quad e(\theta_{a,b})=\frac{a+b\eta}{\sqrt{a^2+ab+2b^2}}.$$

In this case $4\theta_{a,b}\notin\mathbb{Z}$ because $e(\theta_{a,b})\neq 1,-1,i,-i$: if $b=0$ then $e(\theta_{a,b})=\pm1$ but then $a^2+0+0\neq p$ for any prime p and if $a+b/2=0$ then $e(\theta_{a,b})$ is purely imaginary, but $a^2-2a^2+2(2a)^2=7a^2\neq p$ for any prime p. Consequently, as in the proof of the first moment (see (2.1)), we have

$$\Big|\sum_{n=1}^{N} e(4n\theta)\Big| = \left|\frac{e(4(N+1)\theta)-e(4\theta)}{e(4\theta)-1}\right| \le \frac{1}{|\sin 4\pi\theta|}. \tag{4.5}$$

We then have the estimate

$$\frac{1}{\sin 4\pi\theta_{a,b}} \ll \sqrt{a^2+ab+2b^2}\cdot \max\bigg\{\frac{1}{|a+b/2|}, \frac{1}{|b|}\bigg\}. \tag{4.6}$$

So, we approximate (4.4) with

$$\frac{-1}{2N\log N}\sum_{a,b\le N^\alpha} \max\bigg\{\frac{1}{|a+b/2|}, \frac{1}{|b|}\bigg\} \ll \frac{1}{N\log N} N^\alpha \log N.$$

In the line above we have discarded the bounded quantities $\hat{f}\big(\frac{\log p}{2\log N}\big)$ and $\xi_{a,b}$, replaced the sum over n of $e(4n\theta_{a,b})$ using (4.5) and (4.6), and removed the requirement that $a^2+2ab+2b^2$ be a prime; we allow it to be any integer but drop the $\log p$ from the equation. The final line comes from the sum over a and b. The $1/|b|$ sum can be evaluated as $\sum_{a=1}^{N^\alpha} 1$ times $\sum_{b=1}^{N^\alpha} 1/b$, and the $1/|a+b/2|$ sum is $\sum_{n=1}^{3N^\alpha} f(n)/(n/2)$, where $f(n)$, the number of ways to obtain $a+b/2=n$, is certainly less than N^α, giving a result of $N^\alpha \log N$.

The final case is $k=p^2$ for all primes p. Recall that

$$\Lambda_{4n-3}(p^2) = \log p(\alpha_n(p)^2 + \overline{\alpha_n(p)}^2) = \log p\left(\frac{\chi^{(4n-3)}(p^2)}{p^{4n-3}} - 1\right)$$

So, the relevant term from the explicit formula is

$$\frac{-1}{N\log N}\sum_{n=1}^{N}\sum_{p\le N^\alpha} \hat{f}\left(\frac{\log p}{\log N}\right)\frac{\log p}{p}\left(\frac{1}{2}\sum_{a^2+ab+2b^2=p^2} \xi_{a,b} e(4n\theta_{a,b}) - 1\right). \tag{4.7}$$

If we consider first just the term with the -1 in the final bracket we get, using the Prime Number Theorem, and remembering that $\hat{f}$ is even and has compact support in $[-\alpha,\alpha]$,

$$\begin{aligned}\frac{1}{\log N}\sum_{p\le N^\alpha} \hat{f}\left(\frac{\log p}{\log N}\right)\frac{\log p}{p} &\sim \frac{1}{\log N}\int_1^{N^\alpha} \hat{f}\left(\frac{\log u}{\log N}\right)\frac{du}{u} \\ &= \int_0^\alpha \hat{f}(\beta)\,d\beta = \frac{1}{2}\int_{-1}^{1} \hat{f}(\beta)\,d\beta.\end{aligned}$$

The remaining part of (4.7) is

$$\frac{-1}{2N\log N}\sum_{n=1}^{N}\sum_{p\le N^\alpha}\hat f\left(\frac{\log p}{\log N}\right)\frac{\log p}{p}\sum_{a^2+ab+2b^2=p^2}\xi_{a,b}e(4n\theta_{a,b}) \tag{4.8}$$

$$= \frac{-1}{2N\log N}\sum_{p\le N^\alpha}\hat f\left(\frac{\log p}{\log N}\right)\frac{\log p}{p}$$

$$\times\sum_{a^2+ab+2b^2=p^2}\xi_{a,b}\begin{cases}\frac{e(4(N+1)\theta_{a,b})-e(4\theta_{a,b})}{e(4\theta_{a,b})-1} & \text{if } 4\theta_{a,b}\notin\mathbb{Z},\\ N & \text{if } 4\theta_{a,b}\in\mathbb{Z}.\end{cases}$$

For $4\theta_{a,b}\notin\mathbb{Z}$, the contribution is

$$\ll \frac{1}{N\log N}\sum_{a,b\le N^\alpha}\frac{\max\{\frac{1}{|a+b/2|},\frac{1}{|b|}\}}{\sqrt{a^2+ab+2b^2}} \le \frac{1}{N\log N}\sum_{a,b\le N^\alpha}\frac{\frac{1}{|a+b/2|}+\frac{1}{|b|}}{\sqrt{a^2+ab+2b^2}}$$

$$\ll \frac{1}{N\log N}\sum_{a,b\le N^\alpha}\frac{1}{|a+b/2|^2}+\frac{1}{|b|^2},$$

because $a^2+ab+2b^2=(a+b/2)^2+7b^2/4$. So the $4\theta_{a,b}\notin\mathbb{Z}$ contribution is

$$\ll \frac{1}{N\log N}N^\alpha.$$

When $4\theta_{a,b}\in\mathbb{Z}$, then $b=0$, arguing as after (4.6). The contribution to (4.8) is

$$-\frac{1}{2\log N}\sum_{p\le N^\alpha}\hat f\left(\frac{\log p}{\log N}\right)\frac{\log p}{p}\sum_{a=\pm p}\xi_{a,0}. \tag{4.9}$$

We see that

$$\xi_{\pm p,0}=\epsilon_{\pm p,0}\,e(-3\theta_{\pm p,0})=\left(\frac{\pm p^3}{7}\right)\times\pm 1=\pm\left(\frac{\pm p}{7}\right)=\left(\frac{p}{7}\right),$$

because $\left(\frac{-1}{7}\right)=-1$. Thus (4.9) becomes

$$-\frac{1}{\log N}\sum_{p\le N^\alpha}\int_{-\infty}^{\infty}f(t)e\left(\frac{-t\log p}{\log N}\right)dt\left(\frac{p}{7}\right)\frac{\log p}{p}$$

$$\ll \frac{1}{\log N}\int_{-\infty}^{\infty}f(t)\sum_p\frac{\left(\frac{p}{7}\right)\log p}{p^{1+\frac{2\pi it}{\log N}}}\,dt$$

$$\ll \frac{1}{\log N}\int_{-\infty}^{\infty}f(t)\frac{L'}{L}\left(1+\frac{2\pi it}{\log N},\chi_{-7}\right)dt \ll \frac{1}{\log N},$$

where the final approximation follows because L'/L grows slowly on the 1-line and $f(t)$ decays fast enough to keep the integral bounded.

Thus, for large N, (4.3) becomes

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\sum_{\gamma_n} f\left(\frac{\gamma_n \log N}{\pi}\right) &= \hat{f}(0) + \frac{1}{2}\int_{-1}^{1} \hat{f}(\beta)\,d\beta + o(1)\\ &= \int_{-\infty}^{\infty} \hat{f}(\beta)\left(\delta(\beta) + \frac{1}{2}I_{[-1,1]}(\beta)\right)d\beta + o(1)\\ &= \int_{-\infty}^{\infty} f(y)\left(1 + \frac{\sin 2\pi y}{2\pi y}\right)dy + o(1),\end{aligned}$$

where $I_{[-1,1]}$ is the characteristic function of the interval $[-1,1]$. This is consistent with even orthogonal symmetry for this family of L-functions: recall that for the group $\mathrm{SO}(2N)$ the limiting form of the one-level density is

$$1 + \frac{\sin 2\pi y}{2\pi y}$$

and its Fourier transform is

$$\delta(u) + \frac{1}{2}I_{[-1,1]}(u).$$

To derive the theorem we argue as in [ILS00]. Suppose that our test function f is non-negative, that $f(0) = 1$ and that the Fourier transform $\hat{f}$ is supported in $[-1,1]$. We write the above equation as

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{\gamma_n} f\left(\frac{\gamma_n \log N}{\pi}\right) = v + o(1).$$

Let

$$p_m(N) = \frac{1}{N}\,\#\{n \le N : \text{the order of the zero of } L(s,\chi^{4n-3}) \text{ at } s = 1/2 \text{ is } m\}.$$

Then

$$\sum_{m=1}^{\infty} m p_m(N) \le \frac{1}{N}\sum_{n=1}^{N}\sum_{\gamma_n} f\left(\frac{\gamma_n \log N}{\pi}\right) = v + o(1).$$

Clearly,

$$\sum_{m=0}^{\infty} p_m(N) = 1.$$

Also, since all of the $L(s,\chi^{4n-3})$ have even functional equations it follows that $p_m(N) = 0$ if m is odd. Therefore,

$$\sum_{m=1}^{\infty} mp_m(N) = 2p_2(N) + 4p_4(N) + 6p_6(N) + \cdots$$
$$\geq 2p_2(N) + 2p_4(N) + 2p_6(N) + \cdots = 2 - 2p_0(N).$$

Thus,

$$\lim_{N\to\infty} p_0(N) \geq \frac{2-v}{2}.$$

With the choice

$$f(y) = \left(\frac{\sin \pi y}{\pi y}\right)^2$$

we have

$$\hat{f}(x) = \max\{0, 1 - |x|\}$$

so that $v = 3/2$. It follows that $p_0(N) \geq 1/4 - \epsilon$. In words, at least one-fourth of the L-functions in our family do not vanish at their central points.

5. One-level density from the ratios conjecture. The previous section demonstrated a rigorous calculation of the one-level density for the family of L-functions. The limitation of this technique is always the restricted support of the Fourier transform of the test function $f(y)$. In the following we remove the restriction on the support, but the result is conditional on a ratios conjecture of the type introduced by Conrey, Farmer and Zirnbauer [CFZ08]. We are interested in a conjecture for the quantity

$$\sum_{n=1}^{N} \frac{L(1/2+\alpha, \chi^{4n-3})}{L(1/2+\gamma, \chi^{4n-3})}.$$

We use the notation

$$L(s, \chi^{4n-3}) = \prod_{p\neq 7}\left(1 - \frac{\alpha_n(p)}{p^s}\right)^{-1}\left(1 - \frac{\overline{\alpha_n}(p)}{p^s}\right)^{-1} = \sum_{m=1}^{\infty} \frac{a_n(m)}{m^s},$$
$$\frac{1}{L(s, \chi^{4n-3})} = \prod_{p\neq 7}\left(1 - \frac{\alpha_n(p)}{p^s}\right)\left(1 - \frac{\overline{\alpha_n}(p)}{p^s}\right) = \sum_{m=1}^{\infty} \frac{\mu_n(m)}{m^s},$$

so

$$\mu_n(p^\ell) = \begin{cases} 1 & \text{if } \ell = 0, \\ -a_n(p) & \text{if } \ell = 1, \\ 1 & \text{if } \ell = 2, \\ 0 & \text{if } \ell \geq 3. \end{cases}$$

Note that $a_n(m) = \chi^{(4n-3)}(m)/m^{2n-3/2}$ in our previous notation. We are interested in averages of the form

$$\delta_\mu(p^m,p^\ell):=\langle a_n(p^m)\mu_n(p^\ell)\rangle=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_n(p^m)\mu_n(p^\ell).$$

It can be shown that

$$\delta_\mu(p^m,p^\ell)=\begin{cases}0 & \text{if } m \text{ is odd, } \ell=0,2,\\ 1 & \text{if } m \text{ is even, } \ell=0,2,\ p\equiv 1,2,4 \bmod 7,\\ (-1)^{m/2} & \text{if } m \text{ is even, } \ell=0,2,\ p\equiv 3,5,6 \bmod 7,\\ -2 & \text{if } m \text{ is odd, } \ell=1,\ p\equiv 1,2,4 \bmod 7,\\ 0 & \text{if } m \text{ is even, } \ell=1,\ p\equiv 1,2,4 \bmod 7,\\ 0 & \text{if } \ell=1,\ p\equiv 3,5,6 \bmod 7,\\ 0 & \text{if } \ell\ge 3,\\ 0 & \text{if } p=7.\end{cases}$$

Following the ratios conjecture recipe, we take the contribution to the L-function in the numerator from the first term in the approximate functional equation:

$$L(s,\chi^{4n-3})=\sum_{m<x}\frac{a_n(m)}{m^s}+X_{4n-3}(s)\sum_{m<y}\frac{a_n(m)}{m^{1-s}}.$$

Thus we are interested in the sum

$$\text{(5.1)}\qquad \sum_{n=1}^{N}\Bigg[\sum_m\frac{a_n(m)}{m^{1/2+\alpha}}\sum_\ell\frac{\mu_n(\ell)}{\ell^{1/2+\gamma}}\Bigg]$$

$$=\sum_{m,\ell}\frac{1}{m^{1/2+\alpha}}\,\frac{1}{\ell^{1/2+\gamma}}\sum_{n=1}^{N}a_n(m)\mu_n(\ell)$$

$$=\prod_p\sum_{m,\ell}\frac{\delta_\mu(p^m,p^\ell)}{p^{(1/2+\alpha)m}p^{(1/2+\gamma)\ell}}$$

$$=\prod_{p\equiv 1,2,4 \bmod 7}\left(-\frac{2}{p^{1+\alpha+\gamma}}\sum_{m=0}^{\infty}\frac{1}{p^{(1+2\alpha)m}}+\left(1+\frac{1}{p^{1+2\gamma}}\right)\sum_{m=0}^{\infty}\frac{1}{p^{(1+2\alpha)m}}\right)$$

$$\times\prod_{p\equiv 3,5,6 \bmod 7}\left(\left(1+\frac{1}{p^{1+2\gamma}}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{p^{(1+2\alpha)m}}\right).$$

Now we want to extract from (5.1) all the factors that do not converge like $1/p^2$. Thus, the limit as $N\to\infty$ of the expression in (5.1) divided by N equals

$$\frac{\zeta(1+2\gamma)L(1+2\alpha,\chi_{-7})}{\zeta(1+\alpha+\gamma)L(1+\alpha+\gamma,\chi_{-7})}A(\alpha,\gamma),$$

with the convergent Euler product $A(\alpha,\gamma)$ given by

$$(5.2)\quad A(\alpha,\gamma)=\prod_p \frac{1-\frac{1}{p^{1+2\gamma}}}{1-\frac{1}{p^{1+\alpha+\gamma}}} \times \prod_{p\equiv 1,2,4 \bmod 7} \frac{1-\frac{1}{p^{1+2\alpha}}}{1-\frac{1}{p^{1+\alpha+\gamma}}}\biggl(-\frac{2}{p^{1+\alpha+\gamma}}\sum_{m=0}^{\infty}\frac{1}{p^{(1+2\alpha)m}} + \Bigl(1+\frac{1}{p^{1+2\gamma}}\Bigr)\sum_{m=0}^{\infty}\frac{1}{p^{(1+2\alpha)m}}\biggr) \times \prod_{p\equiv 3,5,6 \bmod 7} \frac{1+\frac{1}{p^{1+2\alpha}}}{1+\frac{1}{p^{1+\alpha+\gamma}}}\biggl(\Bigl(1+\frac{1}{p^{1+2\gamma}}\Bigr)\sum_{m=0}^{\infty}\frac{(-1)^m}{p^{(1+2\alpha)m}}\biggr).$$

Using the second sum in the approximate functional equation merely exchanges $-\alpha$ for α and introduces a factor $X_{4n-3}(1/2+\alpha)$. Thus we arrive at the conjecture, making the usual assumptions about the error term and applying the usual restrictions on the parameters α and γ:

CONJECTURE 5.1. *Let $-1/4 < \Re\alpha < 1/4$, $1/\log N \ll \Re\gamma < 1/4$ and $\Im\alpha, \Im\gamma \ll_\epsilon N^{1-\epsilon}$. Then following the ratios conjecture recipe* [CFZ08] *we have*

$$\sum_{n=1}^{N}\frac{L(1/2+\alpha,\chi^{4n-3})}{L(1/2+\gamma,\chi^{4n-3})} = \sum_{n=1}^{N}\biggl(\frac{\zeta(1+2\gamma)L(1+2\alpha,\chi_{-7})}{\zeta(1+\alpha+\gamma)L(1+\alpha+\gamma,\chi_{-7})}A(\alpha,\gamma) + \Bigl(\frac{7}{2\pi}\Bigr)^{-2\alpha}\frac{\Gamma(2n-1-\alpha)}{\Gamma(2n-1+\alpha)}\,\frac{\zeta(1+2\gamma)L(1-2\alpha,\chi_{-7})}{\zeta(1-\alpha+\gamma)L(1-\alpha+\gamma,\chi_{-7})}A(-\alpha,\gamma)\biggr) + O(N^{1/2+\epsilon}).$$

with $A(\alpha,\gamma)$ defined at (5.2).

We note that in arriving at this conjecture we have used the ratios conjecture method exactly as was done for ratios of quadratic twists of the L-function associated with Ramanujan's tau-function, $L_\Delta(s,\chi_d)$, Conjecture 2.9 of [CS07]. That conjecture and Conjecture 5.1 have exactly the same structure and in a straightforward analogy with the lines leading up to Theorem 2.10 in [CS07] and Theorem 3.2 in that paper we see without need to include further workings that the form of the one-level density that follows from the ratios Conjecture 5.1 has the following form:

THEOREM 5.2. *Assuming Conjecture* 5.1 *and assuming for simplicity that $f(z)$ is even, holomorphic in the strip $|\Im z| < 2$, real on the real line,* *and that* $f(x) \ll 1/(1+x^2)$ *as* $x \to \infty$,

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}\sum_{\gamma_n} f(\gamma_n)\\
&\quad= \frac{1}{2\pi N}\int_{-\infty}^{\infty} f(t)\sum_{n=1}^{N}\bigg(2\log\frac{7}{2\pi}+\frac{\Gamma'}{\Gamma}(2n-1+it)+\frac{\Gamma'}{\Gamma}(2n-1-it)\\
&\qquad+2\bigg(-\frac{\zeta'(1+2it)}{\zeta(1+2it)}+\frac{L'(1+2it,\chi_{-7})}{L(1+2it,\chi_{-7})}+A'(it,it)\\
&\qquad-\left(\frac{7}{2\pi}\right)^{-2it}\frac{\Gamma(2n-1-it)}{\Gamma(2n-1+it)}\,\frac{\zeta(1+2it)L(1-2it,\chi_{-7})}{L(1,\chi_{-7})}A(-it,it)\bigg)\bigg)\,dt\\
&\qquad+O(N^{-1/2+\epsilon}),
\end{aligned}$$

where $A'(r,r)$ *is defined as*

$$\frac{d}{d\alpha}A(\alpha,\gamma)\bigg|_{\alpha=\gamma=r}.$$

If the zeros are scaled by $(1/\pi)\log N$ so as to have, asymptotically, unit density, then we see as in [CS07] that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{\gamma_n} f\left(\frac{\gamma_n \log N}{\pi}\right)=\int_{-\infty}^{\infty} f(y)\left(1+\frac{\sin 2\pi y}{2\pi y}\right)dy.$$

Here there is no restriction on the support of the Fourier transform of the test function, but the result relies on the correct form of the main terms in Conjecture 5.1.

6. Discretization. Our family of L-functions seems to have orthogonal symmetry type, which means that it is potentially possible to have lots of small values or zeros of these L-functions at the critical point. For modeling this family with random matrix theory we need to know how the family is "discretized". In practice, the central values of the L-functions of an orthogonal family can be given by a nice conjectural formula which expresses the value in terms of the arithmetic, or geometry, of the family. For example, if d is a fundamental discriminant, and $\chi_d(n)$ the associated quadratic Dirichlet character, then the central value of the dth quadratic twist of an elliptic curve L-function is

$$L_E(1/2,\chi_d)=\kappa\frac{(c(|d|))^2}{\sqrt{|d|}}$$

where $c(|d|)$ is an integer. Thus, we know that if the L-function has a value smaller than $\kappa/\sqrt{|d|}$ then it must be 0. This is what we mean by the discretization. In this situation, it appears that the $c(|d|)$ may take any integral values subject to some mild conditions arising from consideration of Tamagawa factors that lead to some powers of 2 that must divide $c(|d|)$, depending on the primes that divide d.

Our situation here is different. If we try to discretize in a similar way, we need a formula for the central values and we need to know just how small our values $L(1/2, \chi^{4n-3})$ can be without being 0. In this case, Rodriguez-Villegas and Zagier [RVZ93] have proven a formula, conjectured by Gross and Zagier [GZ80], for the central value of the $L(s, \chi^{2n-1})$, namely

$$L(1/2, \chi^{2n-1}) = 2\frac{(2\pi/\sqrt{7})^n \Omega^{2n-1} A(n)}{(n-1)!}$$

where

$$\Omega = \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{4\pi^2} = 0.81408739831\ldots.$$

By the functional equation, $A(n) = 0$ whenever n is even. For odd n, Gross and Zagier [GZ80] conjectured that $A(n)$ is a square and gave the following table (in the later notation of Rodriguez-Villegas and Zagier):

n	$A(n)$	$L(1/2, \chi^{2n-1})$
1	$1/4$	0.9666
3	1	4.7890
5	1	0.9885
7	3^2	0.7346
9	7^2	0.1769
11	$(3^2 \cdot 5 \cdot 7)^2$	9.8609
13	$(3 \cdot 7 \cdot 29)^2$	0.6916
15	$(3 \cdot 7 \cdot 103)^2$	0.1187
17	$(3 \cdot 5 \cdot 7 \cdot 607)^2$	1.0642
19	$(3^3 \cdot 7 \cdot 4793)^2$	1.7403
21	$(3^2 \cdot 5 \cdot 7 \cdot 29 \cdot 2399)^2$	6.6396
23	$(3^3 \cdot 5 \cdot 7^2 \cdot 10091)^2$	0.3302
25	$(3^2 \cdot 7^2 \cdot 29 \cdot 61717)^2$	0.2072
27	$(3^2 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 53^2 \cdot 79)^2$	1.2823
29	$(3^4 \cdot 5^2 \cdot 7^2 \cdot 113 \cdot 127033)^2$	8.4268
31	$(3^5 \cdot 5 \cdot 7^2 \cdot 71 \cdot 1690651)^2$	0.6039
33	$(3^4 \cdot 5 \cdot 7^2 \cdot 1291 \cdot 1747169)^2$	0.0591

Rodriguez-Villegas and Zagier [RVZ93] proved that $A(n) = B(n)^2$ where $B(1) = 1/2$ and $B(n)$ is an integer for $n > 1$. In fact they prove a remarkable recursion formula:

Define sequences of polynomials $a_k(x)$, $b_k(x)$ by the recursions

$$a_{k+1}(x) = \sqrt{(1+x)(1-27x)}\left(x\frac{d}{dx} - \frac{2k+1}{3}\right)a_k(x) - \frac{k^2}{9}(1-5x)a_{k-1}(x)$$

and

$$\begin{aligned}21b_{k+1}(x) = \bigg((32kx - 56k + 42) - (x-7)(64x-7)\frac{d}{dx}\bigg)b_k(x)\\ - 2k(2k-1)(11x+7)b_{k-1}(x)\end{aligned}$$

with initial conditions $a_0(x) = 1$, $a_1(x) = -\frac{1}{3}\sqrt{(1-x)(1+27x)}$, $b_0(x) = 1/2$, and $b_1(x) = 1$. Then, with A and B defined as above,

$$A(2n+1) = \frac{a_{2n}(-1)}{4} \quad \text{and} \quad B(2n+1) = b_n(0).$$

Equation (6) of [RVZ93] states that for odd n,

$$B(n) \equiv -n \bmod 4,$$

a result that in one fell swoop proves the non-vanishing of $L(1/2, \chi^{2n-1})$ for all odd n.

It would be interesting to use these recursion formulae to try to understand a discretization of the values of this family of L-functions, from which one might profitably apply a random matrix model to infer more detailed statistical behavior of these values. The integers $B(n)$ that appear in the formula of Villegas–Zagier are growing quickly, presumably to counteract, by virtue of the expected Lindelöf Hypothesis, the $C^n(n-1)!$ growth in the denominator. The question of just how small these L-values can be is an interesting one.

7. Conclusion. With a mind toward modeling the symmetric powers of the L-function of a general elliptic curve, and to consider whether their central values vanish, we have taken a few steps toward the much simpler problem of modeling the family of L-functions associated with the symmetric powers of the L-function of an elliptic curve with complex multiplication. We have used generally applicable methods from analytic number theory even though for our particular family there are powerful algebraic methods available. We have given some evidence that the family has orthogonal symmetry type. Some unresolved questions are to:

(1) Try to determine if the symmetric power L-functions in the non-CM case form a family. This seemed very nebulous to us, but see the letter [Sar07] of Sarnak to Mazur for some interesting calculations relevant to this issue.

(2) Try to determine a discretization for the central values of an L-function associated with a weight k newform. This seems to be a whole new

direction that has not been considered. In particular, for the Gross–Zagier family we consider, explaining the genesis of the large values of the integers $A(n)$ that appear as factors in the central values is a challenge.

(3) Try to obtain an asymptotic formula for the second moment of our family. Our upper bound fell just short of achieving an asymptotic formula.

(4) Prove a bound of the form $L(1/2, \chi^{4n-3}) \ll n^{1/2-\lambda}$ for some $\lambda > 0$ (i.e. a subconvexity bound) for this family.

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