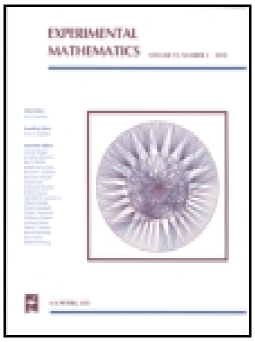


Experimental Mathematics

ISSN: 1058-6458 (Print) 1944-950X (Online) Journal homepage: http://www.tandfonline.com/loi/uexm20

Smooth Neighbors

J. B. Conrey , M. A. Holmstrom & T. L. McLaughlin

To cite this article: J. B. Conrey , M. A. Holmstrom & T. L. McLaughlin (2013) Smooth Neighbors, Experimental Mathematics, 22:2, 195-202, DOI: 10.1080/10586458.2013.768483

To link to this article: http://dx.doi.org/10.1080/10586458.2013.768483

Published online: 25 Apr 2013.

Submit your article to this journal

Article views: 55

View related articles

Full Terms & Conditions of access and use can be found at
http://www.tandfonline.com/action/journalInformation?journalCode=uexm20

Download by: [171.67.34.69] **Date:** 23 May 2016, At: 23:25

Experimental Mathematics, 22(2):195–202, 2013
Copyright © Taylor & Francis Group, LLC
ISSN: 1058-6458 print / 1944-950X online
DOI: 10.1080/10586458.2013.768483

Taylor & Francis
Taylor & Francis Group

Smooth Neighbors

J. B. Conrey, M. A. Holmstrom, and T. L. McLaughlin

Downloaded by [171.67.34.69] at 23:25 23 May 2016

CONTENTS

2000 AMS Subject Classification: Primary 11Y50; Secondary 11N25
Keywords: smooth neighbors, ABC-conjecture, Stormer's problem

We give a new algorithm that quickly finds z-smooth neighbors, where a number is z-smooth if none of its prime factors exceeds z, and if b is a solution of $p \mid b(b+1) \implies p \le z$, then the pair $(b, b+1)$ are called z-smooth neighbors.

1. INTRODUCTION

We say that a number is z-smooth if none of its prime factors exceeds z. In this paper, we search for solutions b of

$$p \mid b(b+1) \implies p \le z. \tag{1–1}$$

If b is a solution of (1–1), then we refer to the pair $(b, b+1)$ as z-smooth *neighbors*. It has been known since 1898 [Størmer 98] that for every z, there are only finitely many z-smooth neighbors. In 1964, D. H. Lehmer found all 869 of the 41-smooth neighbors [Lehmer 64]. To do this, he improved on Størmer's method, which relied on solving a finite number of Pell's equations. In fact, Lehmer showed that if $b(b+1)$ is z-smooth, then $4b(b+1)$ is the "y part" of the nth power of the fundamental solution $x_0 + \sqrt{d}y_0$ of the Pell's equation

$$x^2 - dy^2 = 1,$$

where d is square-free and z-smooth, and $n < (d+1)/2$. Lehmer solved all of these Pell's equations to see which led to 41-smooth neighbors.

In [Luca and Najman 11], the authors used a modified version of Lehmer's method to find 100-smooth neighbors. In a calculation that took 15 days on a quad-core 2.66-GHz processor, they found 13 325 neighbors and claimed that this was all of the possible 100-smooth neighbors. The calculation was especially difficult because solutions to Pell's equations for square-free 100-smooth integers can have as many as 10^{10^6} digits. Consequently, they had to use a special method to represent the solutions. In an erratum [Luca and Najman 13], they found 49 more solutions that they had missed previously.

We have found a fast, amazingly simple algorithm that finds almost all z-smooth neighbors much more quickly.

Downloaded by [171.67.34.69] at 23:25 23 May 2016

In fact, when we ran our method to find 100-smooth neighbors, it completed in 20 minutes (on a similar machine to that used by Luca and Najman) and found 13 333 100-smooth neighbors. We were missing 37 solutions that Luca and Najman found.

Subsequently, we searched for all 200-smooth neighbors. This computation took about two weeks and produced a list of 346 192 solutions.[1] This list included all but one of the solutions from the (completed) Luca–Najman list, namely 9 591 468 737 351 909 375. We determined that this 100-smooth number missing from our list would be found using our method when searching for 227-smooth neighbors.

2. THE ALGORITHM

Suppose we have a set S of positive integers. For any two elements $b < B$ of S, form the ratio

$$\frac{\beta}{\beta'} = \frac{b}{b+1} \times \frac{B+1}{B},$$

where $\gcd(\beta, \beta') = 1$. Sometimes, it will be the case that $\beta' = \beta + 1$, for example,

$$\frac{15}{16} = \frac{3}{4} \times \frac{5}{4}.$$

We are particularly interested in when this happens, i.e., we are interested in the solutions β of

$$\frac{\beta}{\beta+1} = \frac{b}{b+1} \times \frac{B+1}{B}, \qquad (2\text{–}1)$$

where b and B are in S. Given S, we form a new set, S', which is the union of S and all of the solutions β to (2–1). We can repeat this process to form $(S')' = S''$, and so on. Ultimately, by Størmer's theorem [Størmer 98], we will arrive at a set $S^{(n)}$ (meaning n iterations of priming) for which $(S^{(n)})' = S^{(n)}$, i.e., there are no new solutions to be added. We let $\delta(S)$ denote this set $S^{(n)}$, which can no longer be enlarged by this process. As an example, suppose that

$$S = \{1, 2, 3, 4, 5\}.$$

Then it is easy to check that

$$S' = \{1, 2, 3, 4, 5, 8, 9, 15, 24\}$$

and

$$S'' = \{1, 2, 3, 4, 5, 8, 9, 15, 24, 80\}.$$

[1] These numbers may be downloaded from http://www.aimath.org/conrey/SmoothNeighbors/.

After that, we have $S''' = S''$, so

$$\delta(\{1, 2, 3, 4, 5\}) = \{1, 2, 3, 4, 5, 8, 9, 15, 24, 80\}.$$

Recall that Lehmer gave a complete list of 41-smooth solutions to (1–1)). In particular, we can see from his list that the above set $\{1, 2, 3, 4, 5, 8, 9, 15, 24, 80\}$ is the complete list of 5-smooth solutions to (1–1). In other words, we found all of the $z = 5$ solutions to (1–1) by starting with the set $\{1, 2, 3, 4, 5\}$ and then repeatedly adding in solutions to (2–1). This good fortune is not always the case. For example,

$$\begin{aligned}\delta(\{1,2,3,4,5,6,7\}) &= \big\{1, 2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 20, 24, 27, 35, 48, 49, 63,\\ &\quad 80, 125, 224, 2400\big\},\end{aligned}$$

whereas from Lehmer's table, we see that the complete set of 7-smooth solutions to (1–1) includes all of these numbers along with 4374. However, it is the case that

$$4374 \in \delta(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}).$$

Actually,

$$\begin{aligned}\delta(\{1, 2, 5, 6, 10\}) = \{&1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 20,\\ &21, 24, 27, 32, 35, 44, 48, 49, 54, 55, 63, 80, 98, 99, 120, 125,\\ &175, 224, 242, 384, 440, 539, 2400, 3024, 4374, 9800\}\end{aligned}$$

contains all of the solutions to (1–1) with $z = 7$.

Looking at Lehmer's tables, we find for all primes p up to 41, with the exception of $p = 7$ and $p = 41$, that

$$\delta_p := \delta(\{1, 2, \ldots, p\})$$

coincides exactly with the complete set of p-smooth solutions of (1–1). In the case $p = 41$, we found that δ_{41} has 890 of Lehmer's solutions; it is missing the largest solution, 63 927 525 375. Note that

$$\frac{63\,927\,525\,375}{63\,927\,525\,376} = \frac{3^3 \times 5^3 \times 7^7 \times 23}{2^{13} \times 11^4 \times 13 \times 41}.$$

The least n for which

$$63\,927\,525\,375 \in \delta(\{1, 2, \ldots, n\})$$

is $n = 52$.

We calculated δ_{199} in a week on a PC using Mathematica. It has 346 192 elements. A histogram of the logarithms of these numbers is shown in Figure 1. The numbers seem to be normally distributed, but we do not have a conjecture for their mean or variance.

The prime before 199 is 197. The number of new b in δ_{199}, i.e., those not in δ_{197}, is 43 215. For all but 300 of these b, it is the case that $199 \mid b(b+1)$. The other 300 have a smaller largest prime factor. Table 1 shows the numbers of these b sorted by the largest prime factor of $b(b+1)$.

Figure 2 shows a plot of n versus the number of new b's we get for the nth prime. It appears to grow like $n^5/4800$. If this is correct, the running time to calculate δ_{p_n} should be around n^{10}.

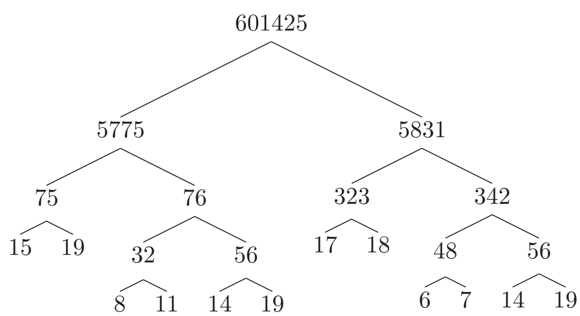
3. TREES

In this section, we illustrate the way in which 601 425 enters into $\delta_{20} = \delta(\{1, 2, \ldots, 20\})$.

The factorization

$$\begin{aligned}\frac{601425}{601426} &= \left(\frac{5775}{5776}\right)\left(\frac{5831}{5832}\right)^{-1} \\ &= \left(\left(\frac{75}{76}\right)\left(\frac{76}{77}\right)^{-1}\right)\left(\left(\frac{323}{324}\right)\left(\frac{342}{343}\right)^{-1}\right)^{-1} \\ &\cdots \\ &= \left(\frac{6}{7}\right)\left(\frac{7}{8}\right)^{-1}\left(\frac{8}{9}\right)^{-1}\left(\frac{11}{12}\right)\left(\frac{15}{16}\right)\left(\frac{17}{18}\right)^{-1} \\ &\quad \times \left(\frac{18}{19}\right)\left(\frac{19}{20}\right)^{-1}.\end{aligned}$$

is illustrated by the following tree:



4. SOLVING THE DIOPHANTINE EQUATION

In this section we analyze the solutions of

$$\frac{b}{b+1} \times \frac{B+1}{B} = \frac{\beta}{\beta+1}. \tag{4–1}$$

Proposition 4.1. *Given β, there is a one-to-one correspondence between pairs (u, v) with $u < v$, $u \mid \beta$, $v \mid (\beta+1)$,*

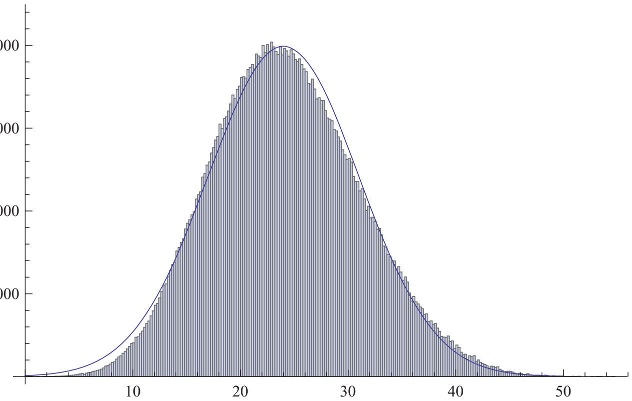


FIGURE 1. A histogram of the set $\{\log b : p \mid b(b+1) \implies p \leq 199\}$ together with the probability density function of the normal distribution (color figure available online).

and pairs (b, B) with $b < B$ and

$$\frac{b}{b+1} \times \frac{B+1}{B} = \frac{\beta}{\beta+1}.$$

Proof. Equation (4–1) can be written as

$$b(B+1)(\beta+1) = (b+1)B\beta.$$

Let $\gcd(b, B) = g$ and put

$$b = gu, \quad B = gv,$$

where clearly $\gcd(u, v) = 1$. Then we have

$$u(B+1)(\beta+1) = (b+1)v\beta.$$

Now, $\gcd(u, (b+1)v) = 1$, so it must be the case that $u \mid \beta$, say $\beta = hu$. Then we have

$$(B+1)(\beta+1) = (b+1)vh.$$

Then $v \mid (\beta+1)$, say $\beta + 1 = vx$, and $h \mid (B+1)$, say $B + 1 = hy$. Thus, in terms of the variables g, h, u, v, x, y, we have the three equations

$$\begin{aligned} hy - gv &= 1, \\ vx - hu &= 1, \\ xy - gu &= 1, \end{aligned}$$

along with $b = gu$, $B = gv$, and $\beta = hu$. If we solve for x and y in the first two equations and substitute the results

Prime	127	131	137	139	149	151	157	163	167	173	179	181	191	193	197
Number	1	1	2	2	5	5	10	8	13	16	25	36	43	51	82

TABLE 1. The number of elements b in δ_{199} but not in δ_{197} for which the largest prime factor of $b\ (b+1)$ is p.

Downloaded by [171.67.34.69] at 23:25 23 May 2016

Downloaded by [171.67.34.69] at 23:25 23 May 2016

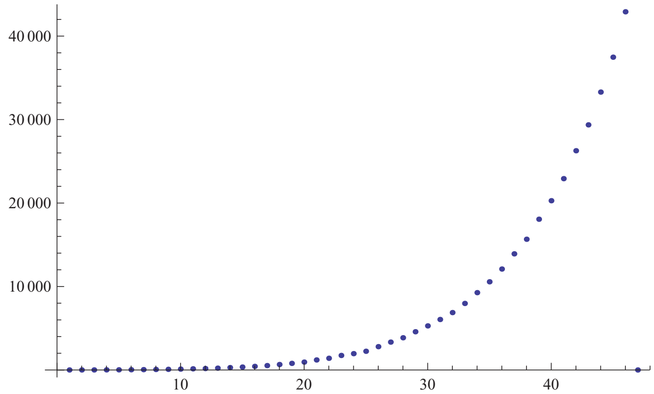


FIGURE 2. A plot of $(n, |\delta_{p_n}| - |\delta_{p_{n-1}}|)$, where p_n denotes the nth prime (color figure available online).

into the third, we have

$$uh + gv = hv - 1. \qquad (4\text{–}2)$$

Notice that if h and v are positive coprime integers, then there exist unique positive integers u and g that satisfy (4–2). Then x and y can be determined as well as b, B, and β. Thus, a coprime pair (h, v) leads to a triple (b, B, β), and conversely.

From another perspective, one may ask how to find a pair (b, B) from a given β. Take any divisor u of β and any divisor v of $\beta + 1$. Then $h = \beta/u$ and $x = (\beta + 1)/v$, from which we see that

$$y = \frac{v - u}{vx - hu}, \quad g = \frac{h - x}{vx - hu}.$$

Thus, the pair (u, v) leads to the pair (b, B) given by

$$b = \frac{uh - ux}{vx - hu} = \beta - \frac{u}{v}(\beta + 1),$$
$$B = \frac{hv - xv}{vx - hu} = \frac{v}{u}\beta - (\beta + 1) = \frac{vb}{u},$$

where we assume that $u < v$. And we have already seen that given $b < B$, we can define

$$u = \frac{b}{(b, B)}, \quad v = \frac{B}{(b, B)}.$$

Thus, given β, there is a one-to-one correspondence between pairs (u, v) with $u < v$, $u \mid \beta$, $v \mid (\beta + 1)$, and pairs (b, B) with $b < B$ and

$$\frac{b}{b+1} \times \frac{B+1}{B} = \frac{\beta}{\beta + 1},$$

which is what we were trying to prove. □

5. THE LARGEST LUCA–NAJMAN SOLUTION

The 79-smooth neighbor pair that starts with

$$\beta = 9\,591\,468\,737\,351\,909\,375$$

was found by Luca and Najman but is not on our list (which we call δ_{199}) of 199-smooth neighbors. We can prove that it will first appear in our method when we search for 227-smooth neighbors.

Given a number β, there is a one-to-one correspondence between pairs (b, B) with

$$\frac{b\,(B+1)}{(b+1)\,B} = \frac{\beta}{\beta + 1}$$

and pairs (u, v) with $u \mid \beta$ and $v \mid (\beta + 1)$. In one direction, this correspondence is given by

$$b = \beta - \frac{u}{v}(\beta + 1), \quad B = \frac{v}{u}\beta - (\beta + 1) = \frac{vb}{u}.$$

There are 1440 divisors of β and 5632 divisors of $\beta + 1$. This means that there is a total of 8 110 080 pairs (u, v) to consider. For each of these pairs, we computed the pair (b, B) and then computed the largest prime factor of $b(b + 1)B(B + 1)$. The least of all of these largest prime factors was $p = 227$, which appeared for several pairs (b, B), in particular for the pair

$$b = 285\,406\,166\,331\,883\,519,$$
$$B = 294\,159\,243\,066\,390\,624.$$

Therefore, β cannot appear as a solution in our method unless we go up to $p = 227$.

Further analysis led us to find an explicit tree for β all of whose bottom nodes are either in δ_{199} or else appear within the first two iterations (which are easy to compute) arising in the computation of δ_{227}; thus we can show that $\beta \in \delta_{227}$ without computing all of δ_{227}. The tree is shown as Figure 3.

Finally, $o, t, y \in \delta_{227}$. This last fact is determined by computing the first two iterations in the process for determining δ_{227}, which takes only a few seconds.

6. COMPUTATIONAL TIME

We have computed 346 192 solutions to

$$p \mid b(b + 1) \implies p \le 199.$$

Note that 199 is the 46th prime. Would it be possible to use the Pell's equation method to find the complete list of 199-smooth neighbors? No. To use the Pell's equation method [Lehmer 64] to find all of the solutions, one would

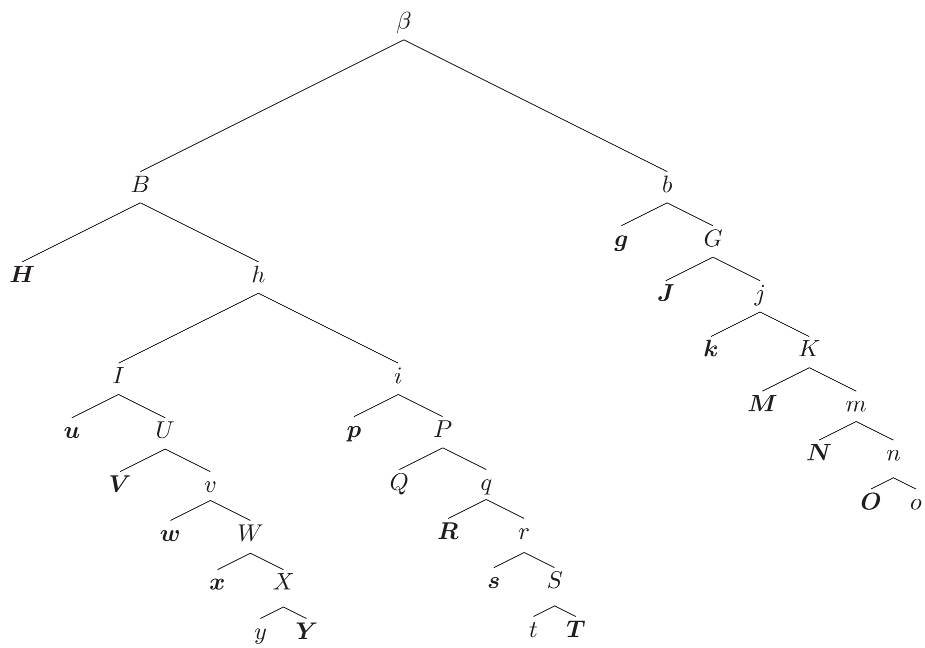


FIGURE 3. A tree showing that $\beta = 9\,591\,468\,737\,351\,909\,375$ is in δ_{227}. A bold letter in the tree indicates a number that is in δ_{199}. Here $G = 2\,247\,272\,709\,023\,744$, $g = 2\,229\,716\,045\,541\,599$, $H = 510\,640\,590\,102\,749\,183$, $h = 186\,642\,247\,267\,999\,999$, $I = 911\,608\,699\,868\,750$, $i = 907\,177\,810\,312\,319$, $J = 1\,672\,934\,505\,788$, $j = 1\,671\,690\,051\,584$, $K = 341\,681\,535$, $k = 341\,611\,712$, $M = 301\,040$, $m = 300\,775$, $N = 3444$, $n = 3405$, $O = 524$, $o = 454$, $P = 199\,846\,415\,040$, $p = 199\,802\,399\,641$, $Q = 16\,504\,943$, $q = 16\,503\,580$, $R = 581\,624$, $r = 561\,824$, $S = 54\,480$, $s = 49\,664$, $T = 7783$, $t = 6810$, $U = 122\,557\,101\,693\,480$, $u = 108\,033\,083\,250\,000$, $V = 10\,639\,238\,337$, $v = 10\,638\,314\,820$, $W = 1\,451\,438$, $w = 1\,451\,240$, $X = 99\,198$, $x = 92\,852$, $Y = 4508$, $y = 4312$.

Downloaded by [171.67.34.69] at 23:25 23 May 2016

first have to find the fundamental solutions of

$$2^{46} - 1 = 70\,368\,744\,177\,663$$

different Pell's equations and then check up to the 100th solution of each. The difficulty in finding the continued fraction expansion of $\sqrt{d}$ grows dramatically with d. We expect that for the prime 199, we would find d such that the period of $\sqrt{d}$ is as large as 10^{40}. Even using the subexponential algorithm employed in [Luca and Najman 11], it would still be impossible to go this far.

Would it be possible by this method to find all of the solutions we found? No. For the numbers b on our list δ_{199}, the continued fraction expansions of $\sqrt{b(b+1)}$ have unusually small periods; in fact, the largest such period is only 38. Suppose one performed a calculation whereby one checked all 2^{46} square-free d's that are 200-smooth to see which lead to short continued-fraction expansions as described above; even this calculation we estimate would take, at a conservative minimum, at least 3000 years on the computer we used.

7. LARGEST SOLUTIONS

Table 2 shows a list of (q, b), where q is a prime number up to 197 and b is the largest element of δ_{199} for which $q \mid (b(b+1)$ and $p \mid b(b+1) \implies p \le q$.

Remark 7.1. Note that the Luca–Najman number $9\,591\,468\,737\,351\,909\,375$ would correspond to $q = 79$ above, where we have $1\,383\,713\,998\,733\,898$.

8. SMOOTHNESS AND THE ABC CONJECTURE

The ABC equation is

$$A + B = C. \tag{8–1}$$

The 1985 ABC conjecture of Masser and Oesterlé asserts that for every $\epsilon > 0$, there is a $\kappa(\epsilon) > 0$ such that for all solutions to (8–1), the inequality

$$C < \kappa(\epsilon)\,\mathrm{rad}(ABC)^{1+\epsilon}$$

holds, where $\mathrm{rad}(n) := \prod_{p|n} p$, called the *radical* of n, is the product of the prime divisors of n.

$(2,1)$, $(3,8)$, $(5,80)$, $(7,4374)$, $(11,9800)$, $(13,123200)$, $(17,336140)$, $(19,11859210)$, $(23,5142500)$, $(29,177182720)$,
$(31,1611308699)$, $(37,3463199999)$, $(41,63927525375)$, $(43,421138799639)$, $(47,1109496723125)$, $(53,1453579866024)$,
$(59,20628591204480)$, $(61,31887350832896)$, $(67,12820120234375)$, $(71,119089041053696)$, $(73,2286831727304144)$,
$(79,1383713998733898)$, $(83,17451620110781856)$, $(89,166055401586083680)$, $(97,49956990469100000)$,
$(101,4108258965739505499)$, $(103,19316158377073923834000)$, $(107,386539843111191224)$, $(109,90550606380841216610)$,
$(113,205142063213188103639)$, $(127,20978372743774437375)$, $(131,1043073004436787852800)$,
$(137,6524436000407205500)$, $(139,15229574576965658738 4)$, $(149,6025407960052311234299)$,
$(151,1801131756071318295624)$, $(157,27776569503477226248 7)$, $(163,1149394259345749379424)$,
$(167,2201197005772848768608)$, $(173,4574658033790609920000)$, $(179,9021820053747825025975)$,
$(181,1398996021795812890312 4)$, $(191,7512199659128762773503 9)$, $(193,4441710634686533148581 75)$,
$(197,2545031605607422002864 0)$, $(199,5898644396087169912015 60)$

TABLE 2. Pairs (b,p) with the largest $b \in \delta_{199}$ such that p is the largest prime factor of $b(b+1)$.

We introduce a new measure of the size of a z-smooth solution of (8–1) with a quantity we call the *smoothness index* given by

$$s(A,B,C) := \frac{\log C}{\log z},$$

where we take z to be the largest prime factor of ABC. For example,

$$2 + 25 = 27$$

has $C = 27$, and the largest prime factor of $2 \times 25 \times 27 = 2 \times 3^3 \times 5^2$ is 5, so that $s(2,25,27) = \log 27/\log 5 = 2.04782$.

Remark 8.1. It is shown in [Hildebrand 85] that $s(A,B,C)$ is unbounded.

Remark 8.2. The strong form of the Lagarias–Soundararajan xyz conjecture [Lagarias and Soundararajan 11] implies that there are triples with $s(A,B,C)$ about as large as $z^{2/3}/\log z$.

Thus, theoretically, we know that $s(A,B,C)$ can be arbitrarily large; for us, the challenge is actually finding (A,B,C) with $s(A,B,C)$ large. In other words it is a computational challenge rather than a theoretical challenge to find large values of $s(A,B,C)$. Our algorithm finds large values of

$$s(1,B,B+1).$$

The largest value we found is for

$$B = 19\,316\,158\,377\,073\,923\,834\,000;$$

we have

$$s(1,B,B+1) = 11.0719.$$

The value of B here is approximately 1.9×10^{22}, which is beyond the range where systematic study of the ABC conjecture has been conducted. We know of no larger values of s.

9. SOME DATA ABOUT SMOOTHNESS

We call a triple (A,B,C) *maximally smooth* if $s(A_1,B_1,C_1) < s(A,B,C)$ for all $C_1 < C$ (or for $C_1 = C$ and $A_1 < A$). Here are the first thirteen maximally smooth triples, together with their smoothness indices:

$1+3=4$	1.262
$3+5=8$	1.292
$1+8=9$	2.000
$2+25=27$	2.048
$5+27=32$	2.153
$1+80=81$	2.730
$3+125=128$	3.015
$32+343=375$	3.046
$49+576=625$	3.308
$5+1024=1029$	3.565
$1+2400=2401$	4.000
$1+4374=4375$	4.308
$7168+78125=85293$	4.427

The triples on this list seem very appealing. In particular,

$$\begin{aligned} 3 + 5^3 &= 2^7, \\ 1 + 5 \times 2^4 &= 3^4, \\ 2^5 + 7^3 &= 3 \times 5^3, \\ 5 + 2^{10} &= 3 \times 7^3, \\ 1 + 3 \times 2^7 &= 7^4, \\ 7 \times 2^{10} + 5^7 &= 13 \times 3^8, \end{aligned}$$

are all of the form

$$ax^\ell + by^m = cz^n$$

Downloaded by [171.67.34.69] at 23:25 23 May 2016

Downloaded by [171.67.34.69] at 23:25 23 May 2016

p	h=2	h=3	h=4	h=5	h=6	h=7
3	8					
5	80	8				
7	4374	48				
11	9800	54				
13	123200	350	63	24		
17	336140	440	63	48		
19	11859210	2430	168	48		
23	11859210	2430	322	48		
29	177182720	13310	322	54		
31	1611308699	13454	1518	152		
37	3463199999	17575	1518	152		
41	63927525375	212380	1680	1517	285	
43	421138799639	212380	10878	1517	340	
47	1109496723125	212380	17575	1517	340	184
53	1453579866024	1205644	17575	1517	340	184
59	20628591204480	1205644	17575	1517	528	527
61	31887350832896	1205644	17575	1767	528	527
67	31887350832896	2018978	17575	5828	1271	527
71	119089041053696	3939648	17575	5828	1271	527
73	2286831727304144	3939648	70224	5828	1271	527
79	2286831727304144	15473808	70224	5828	3476	527
83	17451620110781856	15473808	97524	5828	4897	4896
89	166055401586083680	407498958	97524	5828	4897	4896
97	166055401586083680	407498958	97524	7565	7564	4896
...	...	...	...	...	...	...
199	589864439608716991201560	768026327418	61011223	1448540	44250	18904

TABLE 3. The largest n we found for which $n(n+1)\dots(n+h-1)$ is p-smooth.

with a, b, c, x, y, z each equal to 1 or a prime, and $\ell, m, n > 2$.

A triple (A, B, C) is called an *ABC triple* if $C \geq \mathrm{rad}(ABC)$. The first 22 763 667 triples are available for download from the abc@home web-site.[2] These include all of the triples for $C < 10^{18}$. The largest C for a triple on this list is for the triple

$$131\,854\,322\,526\,743\,011 + 9\,091\,517\,323\,167\,918\,864 = 9\,223\,371\,645\,694\,661\,875,$$

or in factored form,

$$13^4 \times 16651^3 + 2^4 \times 3^8 \times 53^4 \times 3313^2 = 5^4 \times 7^7 \times 37 \times 59^2 \times 373^2.$$

This triple has smoothness index 4.493. Here we have $C = 9.2 \times 10^{18}$.

Among these more than 22 million ABC triples, the triple with largest smoothness index is

$$176\,202\,799\,695\,875 + 3\,178\,472\,661\,789\,594\,624 = 3\,178\,648\,864\,589\,290\,499,$$

with smoothness index 11.0653. In factored form, this is

$$\left(5^3 \times 23^3 \times 42^5\right) + \left(2^{10} \times 3^7 \times 7^6 \times 13^3 \times 17^2 \times 19\right) = 11^4 \times 31 \times 37^4 \times 43^3 \times 47.$$

10. LEHMER'S TABLE

For each prime $z \leq 41$ and each $h \leq 6$, Lehmer found the largest n such that

$$p \mid n(n+1)\cdots(n+h-1) \implies p \leq z$$

[Lehmer 65]. For example, for $z = 37$ and $h = 3$, he obtained $n = 17575$, which means that

$$17\,575 \times 17\,576 \times 17\,577 = 2^3 \times 3^4 \times 5^2 \times 7 \times 13^3 \times 19 \times 31 \times 37$$

has no prime factor larger than 37 and that 17 575 is the largest number with that property. Our algorithm produces many more such data points, but in each case, we can say only that our number is a lower bound for the actual number, since we don't know for any $z > 41$ whether our list of b is complete.

[2] Available at abcathome.com.

Downloaded by [171.67.34.69] at 23:25 23 May 2016

Let $\ell(n)$ denote the largest prime factor of n. We found that

$$\ell(134848 \times 134849 \times 134850) = 43,$$
$$\ell(192510 \times 192511 \times 192512) = 47,$$
$$\ell(1205644 \times 1205645 \times 1205646) = 53.$$

Table 3 shows largest solutions found so far by our algorithm.

We note some minor errors in Lehmer's tables: His $h = 5$ entry for $p = 41$ should be 1517. Also, his Table IIA is missing the entries 109 35, 12 901 781, and 26 578 125 in the 29 column, and 4807, 12 495, 16 337, 89 375 from the 31 column.

11. FUTURE DIRECTIONS

The unanswered question in all of this is, why does it work? Can we really get all of the b for which $b(b+1)$ has only small prime factors by repeatedly applying δ to a small initial set?

A reasonable thing to do would be to run this for a year on a supercomputer and generate millions of smooth neighbors, perhaps some with as many as 30 digits.

It is true that computations get longer as you raise the smoothness barrier. But the length of computation time is due mainly to the vast number of solutions, not the complexity in finding them.

The same technique we have elucidated above works surprisingly well to find solutions to the more general problem

$$p \mid b\,(b+k) \implies p \le z$$

when k is an odd integer. Fix a difference k. Start with an initial set S and let

$$S' = \left\{\beta : \frac{b}{b+k} \times \frac{B+k}{B} = \frac{\beta}{\beta+k},\ \text{with } b, B \in S\right\}.$$

Then $S'' = (S')'$, and so on. Iterate this procedure until it stabilizes and call the result $\delta_k(S)$. It would be interesting to do extensive computations with a range of differences k.

For even integers, the process has to be modified somewhat. For example, to generate solutions of

$$p \mid b\,(b+2) \implies p \le z,$$

one should start with the set $\delta_p = \delta_1(\{1, 2, \ldots, p\})$ for some p and then look for solutions of

$$\frac{b}{b+1} \times \frac{B+1}{B} = \frac{\beta}{\beta+2}$$

with $b, B \in \delta_p$.

ACKNOWLEDGMENTS

The authors would like to thank the referee for many helpful comments.

REFERENCES

[Hildebrand 85] Adolf Hildebrand. "On a Conjecture of Balog." *Proc. Amer. Math. Soc.* 95 (1985), 517–523.

[Lagarias and Soundararajan 11] Jeffrey C. Lagarias and Kannan Soundararajan. "Smooth Solutions to the ABC Equation: The xyz Conjecture." *J. Théor. Nombres Bordeaux* 23 (2011), 209–234.

[Lehmer 64] D. H. Lehmer. "On a Problem of Størmer." *Illinois J. Math.* 8 (1964) 57–79.

[Lehmer 65] D. H. Lehmer. "The Prime Factors of Consecutive Integers." *American Mathematical Monthly* 72 (1965), 19–20.

[Luca and Najman 11] Florian Luca and Filip Najman. "On the Largest Prime Factor of $x^2 - 1$." *Math. Comp.* 80 (2011), 429–435.

[Luca and Najman 13] Florian Luca and Filip Najman. "Errata to 'On the Largest Prime Factor of $x^2 - 1$.'" To appear in *Math. Comp.*, 2013.

[Størmer 98] C. Størmer. "Sur une équation indéterminée." *C. R. Acad. Sci. Paris* 127 (1898), 752–754.

J. B. Conrey, American Institute of Mathematics, 360 Portage Avenue, Palo Alto, CA 94306 (conrey@aimath.org)

M. A. Holmstrom, American Institute of Mathematics, 360 Portage Avenue, Palo Alto, CA 94306 (rubiks3.14@gmail.com)

T. L. McLaughlin, American Institute of Mathematics, 360 Portage Avenue, Palo Alto, CA 94306 (taram0327@yahoo.com)