ON THE DISTRIBUTION OF GAPS BETWEEN ZEROS OF THE ZETA-FUNCTION

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LET $\zeta(s)=\sum n^{-s}$ be the Riemann zeta-function. Assuming the Riemann Hypothesis we may write the non-real zeros of $\zeta(s)$ as $\frac{1}{2}\pm i\gamma$ where $\gamma>0$. Let $0<\gamma\leqslant\gamma'$ be consecutive ordinates of zeros and let

$$\delta(\gamma)=(\gamma'-\gamma).$$

On average, $\delta(\gamma)$ is $2\pi/\log\gamma$. We wish to investigate the distribution of the (normalized) numbers

$$f(\gamma)=\delta(\gamma)(2\pi)^{-1}\log\gamma.$$

To this end, we may define upper and lower distribution functions

$$D^+(\alpha)=\limsup_{T\to\infty} D(\alpha, T)$$

and

$$D^-(\alpha)=\liminf_{T\to\infty} D(\alpha, T),$$

where

$$D(\alpha, T)=\sum_{\substack{0<\gamma\leqslant T\\ f(\gamma)\leqslant\alpha}} 1 \Big/ \sum_{0<\gamma\leqslant T} 1.$$

It is expected that $D^+(\alpha)=D^-(\alpha)$ $(=D(\alpha))$ for all α and that $D(0)=0$, $D(\alpha)<1$ for all α, and $D(\alpha)$ is continuous. In fact, from a "multiple correlation" conjecture as in Montgomery [5] it is possible to determine the behavior of $D(\alpha)$. However, there is little evidence for such a conjecture. As far as what is known about $D^\pm(\alpha)$, Selberg proved that for some $\alpha_1<1$,

$$D^-(\alpha_1)>0$$

and for some $\alpha_2>1$,

$$D^+(\alpha_2)<1.$$

The α_1 and α_2 were never determined explicitly, nor were bounds for D^- and D^+ given.

In this paper we present a new method for obtaining information about $D^\pm(\alpha)$. In fact we show, assuming the Riemann Hypothesis,

THEOREM (ON RH). *If* $\alpha>0.77$ *then* $D^-(\alpha)>0$. *If* $\alpha<1.33$ *then* $D^+(\alpha)<1$.

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Remarks. 1. It is possible by our method to calculate, explicitly, bounds for $D^{\pm}(\alpha)$, although we have not done so here.

2. Our method allows for a certain amount of flexibility in the choice of some coefficients, and we have made no serious attempt to obtain the best result our method would give.

It would be of interest to have sharp bounds for $D^{\pm}(\alpha)$ since proper bounds for these could be applied to give effective lower bounds for the class number $h(-d)$ of a complex quadratic field $K = Q((-d)^{\frac{1}{2}})$. This is because Montgomery and Weinberger [**7**] have shown that for A, $\varepsilon > 0$ there exists an effectively computable $d_0 = d_0(A, \varepsilon)$ such that if $d > d_0$ and $h(-d) \leq A$ then all the zeros $\beta + i\gamma$ of $\zeta_K(s)$ in $0 < \sigma < 1$, $0 \leq t \leq d^{\frac{1}{2}-\varepsilon}$ have $\beta = \frac{1}{2}$ and satisfy

$$\frac{(1-\varepsilon)2\pi}{\log d(\gamma+1)^2} < \delta(\gamma) < \frac{(1+\varepsilon)2\pi}{\log d(\gamma+2)^2}.$$

In particular for γ near $d^{\frac{1}{2}-\varepsilon}$ we have

$$\frac{\delta(\gamma)\log\gamma}{2\pi}$$

is about $\frac{1}{4}$. Thus, to obtain effective bounds for $h(-d)$, we need only prove (assuming *RH*) that there are gaps between zeta zeros which are not near $\frac{1}{4}$ times their average. This could be accomplished with enough information about $D^{\pm}(\alpha)$ with α near 1 (but not equal to 1).

§2

In what follows, let L denote $1/2\pi \log T$. Let $F(s, y)$ be a Dirichlet polynomial of length y. Let α and η be arbitrary positive numbers and define for $k \geq 0$

$$h_k(F, \eta, \alpha) = \lim_{T\to\infty} \frac{\int_{-\alpha/2L}^{\alpha/2L} \sum_{T<\gamma\leq 2T} |F(\gamma+t; T^{\eta})|^{2k}\, dt}{\int_T^{2T} |F(t; T^{\eta})|^{2k}\, dt} \tag{1}$$

$$= \lim_{T\to\infty} h_k(F, \eta, \alpha, T)$$

say, whenever the limit exists. The function $h_k(F, \eta, \alpha)$ has been evaluated in [**1**] for $k = 1$, $\eta < 1$. In [4], $F(s, y)$ was taken to be $\zeta(s)$ and $k = 1$, while in [**2**], it is the product of the zeta-function with a Dirichlet polynomial. We shall see later that for suitable functions F_1 and F_2, and suitable numbers μ and λ, one can show

$$h_1(F_1, \tfrac{1}{2}, \mu) > 1 > h_1(F_2, \tfrac{1}{2}, \lambda).$$

For the proof of our theorem we shall also need the case $k=2$.

We order the zeros of $\zeta(s)$ in ascending order $\gamma_1 \leqslant \gamma_2 \leqslant \cdots$, where it will be understood that if a zero is multiple, with multiplicity m, then it will appear precisely m times consecutively in the sequence above. For $T<\gamma_i \leqslant 2T$, we define

$$\delta^+(\gamma_i)=(\gamma_{i+1}-\gamma_i)L,$$

$$\delta^-(\gamma_i)=(\gamma_i-\gamma_{i-1})L, \; (i \geqslant 2).$$

Next, let

$$\delta_0(\gamma_i)=\min\,(\delta^+(\gamma_i), \delta^-(\gamma_i)),$$

$$\delta_1(\gamma_i)=\max\,(\delta^+(\gamma_i), \delta^-(\gamma_i)).$$

The results we prove will show that a positive proportion of the zeros satisfy

$$\delta_0(\gamma)<0.77 \quad \text{and} \quad \delta_1(\gamma')>1.33,$$

from which the theorem follows.

Let $F(t, y)$ be a function satisfying

$$|F(t, y)| \ll T^{1-\varepsilon}$$

for $\varepsilon>0$. Also for convenience, put $y=T^{\eta}$.

For the small gaps, we have

$$\begin{aligned}
&\int_T^{2T} |F(t, y)|^2\, dt \\
&= \sum_{T<\gamma\leqslant 2T} \int_{\gamma-(1/2L)\delta^-(\gamma)}^{\gamma+(1/2L)\delta^+(\gamma)} |F(t, y)|^2\, dt + O(T^{1-\varepsilon}) \\
&= \sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)<\mu}} \int_{-(1/2L)\delta^-(\gamma)}^{(1/2L)\delta^+(\gamma)} |F(t+\gamma, y)|^2\, dt \\
&\quad + \sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)\geqslant\mu}} \int_{-(1/2L)\delta^-(\gamma)}^{(1/2L)\delta^+(\gamma)} |F(t+\gamma, y)|^2\, dt + O(T^{1-\varepsilon}) \\
&\geqslant \sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)<\mu}} \int_{-(1/2L)\delta^-(\gamma)}^{(1/2L)\delta^+(\gamma)} |F(t+\gamma, y)|^2\, dt \\
&\quad + \int_{-(1/2L)\mu}^{(1/2L)\mu} \sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)\geqslant\mu}} |F(t+\gamma, y)|^2\, dt + O(T^{1-\varepsilon})
\end{aligned} \tag{2}$$

$$\geqslant h_1(F, \eta, \mu, T) \int_T^{2T} |F(t, y)|^2 \, dt$$

$$- \sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)<\mu}} \int_{\delta_0(\gamma)/2L}^{\mu/2L} \{|F(t+\gamma, y)|^2 + |F(-t+\gamma, y)|^2\} \, dt + O(T^{1-\epsilon}).$$

Thus, we conclude that

$$\sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)<\mu}} \int_{(1/2L)\delta_0(\gamma)}^{\mu/2L} \{|F(t+\gamma, y)|^2 + |F(-t+\gamma, y)|^2\} \, dt$$
$$\geqslant (h_1(F, \eta, \mu, T) - 1) \int_T^{2T} |F(t, y)|^2 \, dt + O(T^{1-\epsilon}).$$

Applying Cauchy's inequality to the left-hand-side above shows it to be bounded by

$$\left(\frac{\mu}{L}\right)^{\frac{1}{2}} \Big(\sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)<\mu}} 1\Big)^{\frac{1}{2}}$$
$$\times \Big(\sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)<\mu}} \int_{\delta_0(\gamma)/2L}^{\mu/2L} \{|F(t+\gamma, y)|^4 + |F(-t+\gamma, y)|^4\} \, dt\Big)^{\frac{1}{2}}$$
$$\ll (\mu/L)^{\frac{1}{2}} \Big(\sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)<\mu}} 1\Big)^{\frac{1}{2}} \Big(h_2(F, \eta, \mu, T) \int_T^{2T} |F(t, y)|^4 \, dt\Big)^{\frac{1}{2}}.$$

Thus, one concludes that if $h_1(F, \eta, \mu) > 1$ for suitable choices of the function F and the constants η and μ, then

$$\sum_{\substack{T<\gamma\leqslant 2T \\ \delta_0(\gamma)<\mu}} 1 \gg \frac{\{h_1(F, \eta, \mu, T) - 1 - o(1)\}^2 \Big(\int_T^{2T} |F(t, y)|^2 \, dt\Big)^2}{h_2(F, \eta, \mu, T) \int_T^{2T} |F(t, y)|^4 \, dt} L \qquad (3)$$

as $T \to \infty$. Here we have assumed that

$$\int_T^{2T} |F(t, y)|^2 \, dt \ll T,$$

as will indeed be the case. By Cauchy's inequality one sees that if $h_1(F, \eta, \mu) > 0$, and if

$$\lim_{T\to\infty} \frac{\left(\int_T^{2T} |F(t, y)|^2 \, dt\right)^2}{T \int_T^{2T} |F(t, y)|^4 \, dt}$$

exists and is non-zero for a suitable* function F, then $h_2(F, \eta, \mu) \neq 0$. This is, in part, the motivation behind the choice of our function F.

One considers the large gaps in a similar way. Starting from (2), it is easily seen that

$$\int_T^{2T} |F(t, y)|^2 \, dt \leq \sum_{\substack{T<\gamma\leq 2T \\ \delta_1(\gamma)\leq\lambda}} \int_{-\lambda/2L}^{\lambda/2L} |F(\gamma+t, y)|^2 \, dt$$
$$+ \sum_{\substack{T<\gamma\leq 2T \\ \delta_1(\gamma)>\lambda}} \int_{-\delta^-(\gamma)/2L}^{\delta^+(\gamma)/2L} |F(\gamma+t, y)|^2 \, dt + O(T^{1-\epsilon}),$$
$$\leq h_1(F, \eta, \lambda, T) \int_T^{2T} |F(t, y)|^2 \, dt$$
$$+ \sum_{\substack{T<\gamma\leq 2T \\ \delta_1(\gamma)>\lambda}} \int_{-\delta^-(\gamma)/2L}^{\delta^+(\gamma)/2L} |F(\gamma+t, y)|^2 \, dt + O(T^{1-\epsilon}),$$

so that

$$\sum_{\substack{T<\gamma\leq 2T \\ \delta_1(\gamma)>\lambda}} \int_{-\delta^-(\gamma)/2L}^{\delta^+(\gamma)/2L} |F(\gamma+t, y)|^2 \, dt$$
$$\geq (1 - h_1(F, \eta, \lambda, T)) \int_T^{2T} |F(t, y)|^2 \, dt - O(T^{1-\epsilon}).$$

Applying Cauchy's inequality to the left-hand-side above shows it to be bounded by

$$\left(\sum_{\substack{T<\gamma\leq 2T \\ \delta_1(\gamma)>\lambda}} 1\right)^{\frac{1}{4}} \left(\sum_{T<\gamma\leq 2T} \delta_1(\gamma)^2\right)^{\frac{1}{4}} \left(\int_T^{2T} |F(t, y)|^4 \, dt\right)^{\frac{1}{2}}.$$

* Such a function will be called "normal".

Thus, if F, η and λ are chosen such that $h_1(F, \eta, \lambda) < 1$, we have, as $T \to \infty$

$$\sum_{\substack{T<\gamma\leq 2T \\ \delta_1(\gamma)>\lambda}} 1 \gg \frac{\{1-h_1(F, \eta, \lambda, T)-o(1)\}^4}{\left(\sum_{T<\gamma\leq 2T} \delta_1(\gamma)^2\right)} \frac{\left(\int_T^{2T} |F(t, y)|^2\, dt\right)^4}{\left(\int_T^{2T} |F(t, y)|^4\, dt\right)^2}. \tag{4}$$

The rest of the paper is devoted to the evaluation of the expressions in (3) and (4), with a suitable choice of the function F.

§3

We will take F to be a Dirichlet polynomial,

$$F(t, y) = F(t) = \sum_{n\leq y} f(n) n^{it}$$

where $y = T^{\eta}$, $\eta = \frac{1}{2} - \varepsilon$ and the coefficients $f(n)$ will be specified later. By the mean-value theorem for Dirichlet polynomials,

$$\int_T^{2T} |F(t)|^2\, dt = (T + O(y)) \sum_{n\leq y} |f(n)|^2. \tag{5}$$

Also, assuming *RH*, we can show that

$$\sum_{T<\gamma\leq 2T} |F(t+\gamma)|^2 = TL \sum_{n\leq y} |f(n)|^2 - \frac{T}{\pi} \operatorname{Re} \sum_{nk\leq y} \frac{\Lambda(k)\overline{f(n)} f(nk) k^{it}}{k^{\frac{1}{2}}} + O\left(yL \sum_{n\leq y} |f(n)|^2\right) \tag{6}$$

for $|t| \ll L^{-1}$. Then

$$\int_{-\alpha/2L}^{\alpha/2L} \sum_{T<\gamma\leq 2T} |F(\gamma+t)|^2\, dt = \alpha T \sum_{n\leq y} |f(n)|^2 - T \sum_{nk\leq y} \frac{\Lambda(k)\overline{f(n)} f(nk) g(k)}{k^{\frac{1}{2}}} + O\left(y \sum_{n\leq y} |f(n)|^2\right), \tag{7}$$

where

$$g(n) = \frac{2 \sin(\alpha(\log n)/(2L))}{\pi \log n},$$

so that

$$h_1(F, \eta, \alpha, T) = \alpha - \left(\sum_{nk\leq y} \frac{\Lambda(k)\overline{f(n)} f(nk) g(k)}{k^{\frac{1}{2}}}\right) \Big/ \left(\sum_{n\leq y} |f(n)|^2\right) + O(y/T). \tag{8}$$

(For details see Conrey, Ghosh, and Gonek [**1**], or Montgomery and Odlyzko [**6**].)

Recalling the remark after (3), we choose the coefficients $f(n)$ so that F is "normal". The simplest choice is to have $f(n)=0$ if n is not 1 or a prime. Then, the optimal choice of coefficients (by Cauchy's inequality) is $f(1)=1$, $f(p)=\pm Cg(p)p^{-\frac{1}{2}}\log p$ for a constant $C>0$ to be specified later. The ambiguous sign is taken to be + when dealing with small gaps (3) and − for large gaps (4). Then

$$\sum_{nk\leq y}\frac{\Lambda(k)\overline{f(n)}f(nk)g(k)}{k^{\frac{1}{2}}}=\pm C\sum_{p\leq y}\frac{(g(p)\log p)^2}{p}$$
$$=\pm C\frac{4}{\pi^2}\sum_{p\leq y}\frac{\sin^2(\alpha(\log p)/(2L))}{p}. \qquad (9)$$

By the prime number theorem and Stieltjes integration this is seen to be (after a change of variables)

$$=\pm C\frac{4}{\pi^2}\int_0^\beta \frac{\sin^2 v}{v}\,dv+o(1)$$

where $\beta=\beta(y)=(\alpha\log y)/(2L)$. Similarly,

$$\sum_{n\leq y}|f(n)|^2=1+C^2\frac{4}{\pi^2}\int_0^\beta \frac{\sin^2 v}{v}+o(1). \qquad (10)$$

The optimal choice for C is clearly

$$C=\left(\frac{4}{\pi^2}\int_0^\beta \frac{\sin^2 v}{v}\,dv\right)^{-\frac{1}{2}}, \qquad (11)$$

whence the left side of (9) is $=\pm C^{-1}+o(1)$ and $\sum_{n\leq y}|f(n)|^2=2+o(1)$. Thus,

$$h_1(F_\pm,\eta,\alpha,T)=\alpha\pm\frac{1}{\pi}\left(\int_0^\beta \frac{\sin^2 v}{v}\,dv\right)^{\frac{1}{2}}+o(1) \qquad (12)$$

since $y=o(T)$. We remark that with $\eta=\frac{1}{2}$ (i.e. $y=T^{\frac{1}{2}}$), so that $\beta=\pi\alpha$, it can be easily calculated that

$$h_1(F_+,\tfrac{1}{2},0.77,T)>1+o(1) \qquad (13)$$

and

$$h_1(F_-,\tfrac{1}{2},1.33,T)<1+o(1). \qquad (14)$$

Next we estimate h_2 and the mean fourth power of F. Although it is not difficult to give asymptotic formulae for these, it is even easier to give upper bounds which suffice to prove our theorem. Since we do not compute explicit bounds for $D^{\pm}(\alpha)$ we choose the latter course. By the mean-value theorem for Dirichlet polynomials,

$$\int_T^{2T} |F(t)|^4\,dt = \int_T^{2T} |F(t)^2|^2\,dt$$

$$= (T+O(y^2)) \sum_{N\leq y^2} \Big|\sum_{mn=N} f(m)f(n)\Big|^2. \tag{15}$$

The sum on N is

$$= \sum_{\substack{m_1n_1\leq y^2\\ m_2n_2\leq y^2\\ m_1n_1=m_2n_2}} f(m_1)\overline{f(m_2)}f(n_1)\overline{f(n_2)}$$

$$\leq \sum_{\substack{mn\leq y^2\\ m_1n_1\leq y^2\\ m_1n_1=mn}} |f(m)f(n)|^2$$

by Cauchy's inequality and this is

$$\leq 2\sum_{mn\leq y^2} |f(m)f(n)|^2$$

$$= 2\Big(\sum_{n\leq y} |f(n)|^2\Big)^2$$

since $f(m)=0$ if $m>y$ or if m has more than one prime factor. Therefore,

$$\int_T^{2T} |F(t)|^4\,dt \leq 2(T+O(y^2))\Big(\sum_{n\leq y}|f(n)|^2\Big)^2$$

$$\leq \frac{(2+o(1))}{T}\Big(\int_T^{2T} |F(t)|^2\,dt\Big)^2. \tag{16}$$

Now we give an upper bound for h_2 by applying Cauchy's inequality to the estimate (8). If α is bounded then $g(k)\ll L^{-1}$, so that

$$h_1(F,\eta,\alpha,T) \leq |\alpha| + \frac{\Big(\sum_{nk\leq y} \frac{\Lambda(k)\,|g(k)|^2}{k}|f(n)|^2\Big)^{\frac12}\Big(\sum_{nk\leq y}|f(nk)|^2(k)\Big)^{\frac12}}{\sum_{n\leq y}|f(n)|^2} + O(y/T)$$

$$\ll 1+\frac{\left(\sum_{k\leq y}\frac{\Lambda(k)}{k}L^{-2}\sum_{n\leq y}|f(n)|^2\right)^{\frac{1}{2}}\left(\sum_{m\leq y}|f(m)|^2\log m\right)^{\frac{1}{2}}}{\sum_{n\leq y}|f(n)|^2}+O(y/T)$$

$$\ll 1$$

provided only that $y=T^{\eta}=o(T)$. Since this estimate holds for an arbitrary F, we conclude that for our particular F with $\eta=\frac{1}{2}-\varepsilon$,

$$h_2(F,\eta,\alpha,T)=h_1(F^2,2\eta,\alpha,T)\ll 1. \tag{17}$$

Finally, we remark that

$$\sum_{T<\gamma\leq 2T}\delta_1(\gamma)^2\ll TL \tag{18}$$

follows directly from a result that can be found in Fujii [3].

Our theorem is now a consequence of (3), (4), (16), (17), and (18).

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