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# An Optimal Choice of Dirichlet Polynomials for the Nyman–Beurling Criterion

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*In memory of Professor A. A. Karatsuba*
*on the 75th anniversary of his birth*

## 1. INTRODUCTION

The Nyman–Beurling–Báez-Duarte approach to the Riemann hypothesis asserts that the Riemann hypothesis is true if and only if

$$\lim_{N\to\infty} d_N^2 = 0,$$

where

$$d_N^2 = \inf_{A_N} \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|1-\zeta A_N\left(\frac{1}{2}+it\right)\right|^2 \frac{dt}{1/4+t^2},$$

and the infimum is over all Dirichlet polynomials $A_N(s)=\sum_{n=1}^N a_n/n^s$ of length $N$ (see [1] for a nice account of this).

An open question is to determine what the rate of convergence of $d_n$ to zero is, assuming the Riemann hypothesis. Balazard and de Roton showed that, if the Riemann hypothesis is true, then

$$d_N^2 \ll \frac{(\log\log N)^{5/2+\varepsilon}}{\sqrt{\log N}}$$

for all $\varepsilon>0$. On the other hand Báez-Duarte, Balazard, Landreau and Saias [2, 3] showed (unconditionally) that $d_N^2$ can not decay faster than a constant times $1/\log N$. More precisely, they showed that

$$\liminf_{N\to\infty} d_N^2 \log N \ge \sum_{\mathrm{Re}(\rho)=1/2} \frac{1}{|\rho|^2},$$

where the sum is restricted to distinct zeros of the Riemann zeta function on the critical line. The constant was later improved by Burnol [4] who showed

$$\liminf_{N\to\infty} d_N^2 \log N \ge \sum_{\mathrm{Re}(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2},$$

where $m(\rho)$ denotes the multiplicity of $\rho$. This lower bound is believed to be optimal and one expects that

$$d_N^2 \sim \frac{1}{\log N}\sum_{\mathrm{Re}(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2}. \tag{1}$$

Notice that under the Riemann hypothesis, one has

$$\sum_{\mathrm{Re}(\rho)=1/2} \frac{m(\rho)}{|\rho|^2} = 2+\gamma-\log 4\pi$$

and in particular, if all the non-trivial zeros of $\zeta(s)$ are simple, then (1) can be rewritten as

$$d_N^2 \sim \frac{2+\gamma-\log 4\pi}{\log N}.$$

It is the purpose of this note to prove (1) under the Riemann Hypothesis and assuming a mild condition on the growth of the mean value of $1/|\zeta'(\rho)|^2$ over the non-trivial zeros $|\rho| \le T$ of $\zeta(s)$. This will be achieved by using the Dirichlet polynomial

$$V_N(s) := \sum_{n=1}^{N} \left(1 - \frac{\log n}{\log N}\right) \frac{\mu(n)}{n^s}.$$

**Theorem 1.** *If the Riemann hypothesis is true and if*

$$\sum_{|\,\mathrm{Im}(\rho)|\le T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{3/2-\delta} \tag{2}$$

*for some* $\delta > 0$, *then*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - \zeta V_N\left(\frac{1}{2}+it\right)\right|^2 \frac{dt}{1/4+t^2} \sim \frac{2+\gamma-\log 4\pi}{\log N}.$$

The condition (2) implicitly assumes that the zeros of the Riemann zeta function are all simple. Moreover, this upper bound is "mild" in the sense that a conjecture, due to Gonek and recovered by a different heuristic method of Hughes, Keating, and O'Connell [5], predicts that

$$\sum_{|\rho|\le T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{6}{\pi^3} T.$$

We remark that Theorem 1 is in contrast to what one might have expected after viewing the graphs of Landreau and Richards [6] which at first sight suggest that $V_N$ is not optimal.

This behaviour of the Riemann zeta function resembles that of polynomials. In fact, Grenander and Rosenblatt [7] (see also [4, Theorem 2.1]) showed that for a polynomial $P(z)$ one has that the zeros of $P$ are all located outside or on the unit circle if and only if $\lim_{N\to\infty} \delta_N = 0$, where

$$\delta_N^2 = \frac{1}{2\pi} \inf_{Q_N} \int_0^{2\pi} |1 - P(z)Q_N(z)|^2 \, d\theta,$$

where $z = e^{i\theta}$ and the infimum is over polynomials $Q_N$ of degree at most $N$. Moreover, if this happens, then

$$\lim_{N\to\infty} N\delta_N^2 = \sum_{|\rho|=1} m(\rho)^2,$$

where the sum is restricted to the distinct zeros $\rho$ of $P(z)$ lying on the unit circle and $m(\rho)$ is again the multiplicity of $\rho$.

This analogy seems to apply also to the choices of optimal polynomials.

**Theorem 2.** *Let* $P(z)$ *be a polynomial whose zeros are all simple and lie outside or on the unit circle. Let*

$$W_N(z) := \sum_{n=0}^{N} \left(1 - \frac{n}{N}\right) a_n z^n, \tag{3}$$

*where*

$$\frac{1}{P(z)} = \sum_{n\geq 0} a_n z^n$$

*is the Taylor expansion at $z=0$ of the inverse of $P(z)$ (i.e., it is the formal power series inverse of $P(z)$). Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |1-P(z)W_N(z)|^2\, d\theta \sim \frac{1}{N}\sum_{|\rho|=1} m(\rho)^2,$$

*where $z=e^{i\theta}$.*

We remark that the proofs of Theorems 1 and 2 are very similar, the main difference being that the Riemann zeta function has infinitely many zeros. This generates some issues concerning the convergence of certain sums of $1/\zeta'(\rho)$, which force us to assume condition (2).

## 2. POLYNOMIALS

**Lemma 1.** *Let $P(s)$ be a polynomial with $P(0)\neq 0$. We have*

$$W_N(s) = \frac{1}{P(s)}\left(1+\frac{s}{N}\frac{P'}{P}(s)\right) - \frac{s}{N}Y_N(s),$$

*where $W_N(s)$ is defined in (3),*

$$Y_N(s) := \sum_{\rho} \operatorname*{Res}_{z=\rho} \frac{s^N}{P(z)(z-s)^2 z^N},$$

*and the sum is over distinct zeros $\rho$ of $P(z)$.*

**Proof.** Since $P(0)\neq 0$, we can take an $\varepsilon>0$ such that all the zeros of $P(z)$ lie outside of the circle $|z|=\varepsilon$. Now, observe that we can assume $0<|s|<\varepsilon$, since the result will then extend to all $\mathbb{C}$ by analytic continuation. Denoting by $\mathcal{C}_y$ the circle of radius $y>0$ (oriented in the positive direction), by the residue theorem we have that

$$a_n = \frac{1}{2\pi i}\int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)}\frac{dz}{z^{n+1}},$$

therefore

$$W_N(s) = \frac{1}{2\pi i}\int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)}\sum_{n=0}^{N}\left(1-\frac{n}{N}\right)\left(\frac{s}{z}\right)^n \frac{dz}{z}.$$

Now,

$$\sum_{n=0}^{N}\left(1-\frac{n}{N}\right)z^n = -\frac{1}{N}\frac{z-z^{N+1}}{(1-z)^2} + \frac{1}{1-z}$$

and thus

$$W_N(s) = \frac{1}{2\pi i}\int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)}\left(-\frac{1}{N}\frac{sz^N - s^{N+1}}{(z-s)^2 z^N} + \frac{1}{z-s}\right)dz.$$

Now, by the residue theorem

$$\frac{1}{2\pi i}\int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)}\left(-\frac{1}{N}\frac{s}{(z-s)^2} + \frac{1}{z-s}\right)dz = \frac{1}{P(s)}\left(1+\frac{s}{N}\frac{P'(s)}{P(s)}\right),$$

whereas, moving the line of integration to $\mathcal{C}_y$ and letting $y$ tend to infinity, one has that

$$\frac{1}{2\pi i N}\int_{\mathcal{C}_\varepsilon}\frac{1}{P(z)}\frac{s^{N+1}}{(z-s)^2 z^N}\,dz = -\frac{s}{N}Y_N(s)$$

and the Lemma follows.

**Proof of Theorem 2.** Let $\delta > 1$ be such that $P(s)$ does not have any zero on $1 < |s| \le \delta$. We have

$$\frac{1}{2\pi}\int_0^{2\pi}|1-P(z)W_N(z)|^2\,d\theta = \frac{1}{2\pi i}\int_{\mathcal{C}_1}(1-P(s)W_N(s))\Big(1-\overline{P}\Big(\frac{1}{s}\Big)\overline{W}_N\Big(\frac{1}{s}\Big)\Big)\frac{ds}{s}$$
$$= \frac{1}{2\pi i}\int_{\mathcal{C}_\delta}(1-P(s)W_N(s))\Big(1-\overline{P}\Big(\frac{1}{s}\Big)\overline{W}_N\Big(\frac{1}{s}\Big)\Big)\frac{ds}{s}.$$

Therefore, by Lemma 1, this is

$$\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\Big(\frac{P'}{P}(s)-P(s)Y_N(s)\Big)\Big(\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)-\overline{P}\Big(\frac{1}{s}\Big)\overline{Y}_N\Big(\frac{1}{s}\Big)\Big)\frac{ds}{s}.$$

Now, for $|s| = \delta$ one has

$$Y_N(s)\overline{Y}_N\Big(\frac{1}{s}\Big) = O(1),$$

therefore

$$\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\Big(\frac{P'}{P}(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)+P(s)Y_N(s)\overline{P}\Big(\frac{1}{s}\Big)\overline{Y}_N\Big(\frac{1}{s}\Big)\Big)\frac{ds}{s} = O\Big(\frac{1}{N^2}\Big).$$

Moreover for $s \in \mathcal{C}_\delta$ one has that $\overline{Y}_N(1/s) = O(\delta^{-N})$, thus

$$-\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\Big(\frac{P'}{P}(s)\overline{P}\Big(\frac{1}{s}\Big)\overline{Y}_N\Big(\frac{1}{s}\Big)\Big)\frac{ds}{s} = O\Big(\frac{\delta^{-N}}{N^2}\Big).$$

Finally, by the residue theorem,

$$-\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)\frac{ds}{s}$$
$$= -\frac{1}{N^2}\sum_{|\rho|=1}\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)\frac{1}{s} - \frac{1}{2\pi i N^2}\int_{\mathcal{C}_{1/\delta}}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)\frac{ds}{s}$$
$$= -\frac{1}{N^2}\sum_{|\rho|=1}\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big) + O\Big(\frac{\delta^{-N}}{N^2}\Big).$$

The Theorem then follows by observing that

$$\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)\frac{1}{s} = -N + O(1).$$

## 3. THE RIEMANN ZETA-FUNCTION

We start with the following lemma, which is the analogue of Lemma 1. We remark that this lemma is unconditional.

**Lemma 2.** *If* $0 < \mathrm{Re}(s) < 1$, *then*

$$V_N(s) = \frac{1}{\zeta(s)}\left(1 - \frac{1}{\log N}\frac{\zeta'}{\zeta}(s)\right) + \frac{1}{\log N}\sum_{\rho} R_N(\rho, s) + \frac{1}{\log N}F_s\left(\frac{1}{N}\right),$$

*where the sum is over distinct non-trivial zeros* $\rho$ *of* $\zeta(s)$ *with*

$$R_N(\rho, s) = \operatorname*{Res}_{z=\rho} \frac{N^{z-s}}{\zeta(z)(z-s)^2},$$

*and where*

$$F_s(z) = \pi z^s \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n+1} z^{2n}}{(2n)!\,\zeta(2n+1)(2n+s)^2}$$

*is an entire function of* $z$.

**Proof.** We have

$$V_N(s) = \frac{1}{\log N}\frac{1}{2\pi i}\int_{(2)} \frac{N^w}{\zeta(s+w)}\frac{dw}{w^2},$$

where we use the notation $\int_{(c)}$ to mean an integration up the vertical line from $c - i\infty$ to $c + i\infty$. Now we move the path of integration to $\mathrm{Re}(w) = -\mathrm{Re}(s) - 2M - 1$ for a large integer $M$. The residue at $w = \rho - s$ is $R_N(\rho, s)/\log N$. The residue at $s + w = -2n$ is

$$\frac{N^{-2n-s}}{\zeta'(-2n)(2n+s)^2 \log N}$$

and the integral on the new path is $\ll N^{-2M-1}$. Letting $M \to \infty$ and using

$$\zeta'(-2n) = \frac{(-1)^n \pi (2n)!\,\zeta(2n+1)}{(2\pi)^{2n+1}}$$

we obtain the result.

**Lemma 3.** *Let* $\varepsilon > 0$. *Assume the Riemann hypothesis and that all the zeros of* $\zeta(s)$ *are simple. Then, if condition* (2) *holds, for* $\mathrm{Re}(s) = 1/2 \pm \varepsilon$ *one has*

$$\sum_{\rho} R_N(\rho, s) \ll N^{\mp\varepsilon}|s|^{3/4-\delta/2+\varepsilon}. \qquad (4)$$

**Proof.** Firstly observe that, by the Cauchy–Schwartz inequality, (2) implies

$$\sum_{|\rho|\le T} \frac{1}{|\zeta'(\rho)|} \ll \sqrt{N(T)\sum_{|\rho|\le T}\frac{1}{|\zeta'(\rho)|^2}} \ll T^{5/4-\delta/2}\sqrt{\log T},$$

since

$$N(T) := \frac{1}{2}\sum_{|\rho|\le T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

Therefore, by partial summation, we have that the series

$$\sum_{\rho} \frac{1}{|\zeta'(\rho)||\rho|^{\alpha}}$$

is convergent for any $\alpha > 5/4 - \delta/2$. Now, for a simple zero $\rho$, we have

$$R_N(\rho, s) = \sum_{\rho} \frac{N^{\rho-s}}{\zeta'(\rho)(\rho-s)^2}.$$

Therefore

$$\begin{aligned} N^{\pm\varepsilon}\sum_{\rho} R_N(\rho,s) &\ll \sum_{|\rho-s|<|\rho|/2}\frac{1}{|\zeta'(\rho)||\rho-s|^2}+\sum_{|\rho-s|\geq|\rho|/2}\frac{1}{|\zeta'(\rho)||\rho-s|^2}\\ &\ll \sum_{|\rho-s|<|\rho|/2}\frac{1}{|\zeta'(\rho)||\rho-s|^2}+\sum_{|\rho-s|\geq|\rho|/2}\frac{1}{|\zeta'(\rho)||\rho|^2}\\ &\ll \sum_{|\rho-s|<|\rho|/2}\frac{1}{|\zeta'(\rho)||\rho-s|^2}+1. \end{aligned} \tag{5}$$

Now, by the Cauchy–Schwartz inequality,

$$\sum_{|\rho-s|<|\rho|/2}\frac{1}{|\zeta'(\rho)||\rho-s|^2}\ll\sqrt{\biggl(\sum_{|\rho|<2|s|}\frac{1}{|\zeta'(\rho)|^2}\biggr)\biggl(\sum_{|\rho|<2|s|}\frac{1}{|\rho-s|^4}\biggr)}\ll|s|^{3/4-\delta/2+\varepsilon},$$

since, by partial summation,

$$\sum_{|\rho|<2|s|}\frac{1}{|\rho-s|^4}\ll\log(|s|+2).$$

This completes the proof of the lemma.

**Proof of Theorem 1.** We have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\biggl|1-\zeta V_N\biggl(\frac12+it\biggr)\biggr|^2\frac{dt}{1/4+t^2}&=\frac{1}{2\pi i}\int_{(1/2)}(1-\zeta V_N(s))(1-\zeta V_N(1-s))\frac{ds}{s(1-s)}\\ &=\frac{1}{2\pi i}\int_{(1/2-\varepsilon)}(1-\zeta V_N(s))(1-\zeta V_N(1-s))\frac{ds}{s(1-s)}.\end{aligned}$$

By Lemma 2, this is

$$\begin{aligned}\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{(1/2-\varepsilon)}\biggl(\frac{\zeta'}{\zeta^2}(s)-\sum_{\rho}R_N(\rho,s)-F_s\biggl(\frac1N\biggr)\biggr)\\ \times\biggl(\frac{\zeta'}{\zeta^2}(1-s)-\sum_{\rho}R_N(\rho,1-s)-F_{1-s}\biggl(\frac1N\biggr)\biggr)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds.\end{aligned} \tag{6}$$

Now, we have

$$\begin{aligned}\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{(1/2-\varepsilon)}\sum_{\rho_1,\rho_2}R_N(\rho_1,s)R_N(\rho_2,1-s)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds\\ \ll\frac{1}{\log^2 N}\int_{(1/2-\varepsilon)}\sum_{|\rho-s|<|\rho|/2}\frac{1}{|\zeta'(\rho)||\rho-s|^2}\frac{|ds|}{|s|^{5/4+\delta/2-5\varepsilon}}+O\biggl(\frac{1}{\log^2 N}\biggr),\end{aligned}$$

where we used (4), (5) and the bound $\zeta(1/2\pm\varepsilon\pm it)\ll|t|^{2\varepsilon}$ (which is a consequence of the Lindelöf hypothesis). Reversing the order of summation and integration, we have that this is bounded by

$$\begin{aligned}\frac{1}{\log^2 N}\sum_{\rho}\frac{1}{|\zeta'(\rho)|}\int_{1/2-\varepsilon+i(\mathrm{Im}(\rho)-|\rho|/2)}^{1/2-\varepsilon+i(\mathrm{Im}(\rho)+|\rho|/2)}\frac{|ds|}{|\rho-s|^2|s|^{5/4+\delta/2-5\varepsilon}}+O\biggl(\frac{1}{\log^2 N}\biggr)\\ \ll\frac{1}{\log^2 N}\sum_{\rho}\frac{1}{|\zeta'(\rho)||\rho|^{5/4+\delta/2-5\varepsilon}}\ll\frac{1}{\log^2 N},\end{aligned}$$

if $\varepsilon<\delta/10$.

Now, by Lemma 3 and the trivial estimate $F_s(z) = O(N^{-5/2})$, all the other terms in (6) are trivially $O(1/\log^2 N)$ apart from

$$-\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{(1/2-\varepsilon)}\frac{\zeta'}{\zeta}(1-s)\sum_{\rho}R_N(\rho,s)\frac{\zeta(s)}{s(1-s)}\,ds. \tag{7}$$

The integrand has a double pole at every zero $\rho$ of residue

$$\begin{aligned}\operatorname*{Res}_{s=\rho}\bigg(\frac{\zeta'}{\zeta}(1-s)\sum_{\rho}R_N(\rho,s)\frac{\zeta(s)}{s(1-s)}\bigg) &= \bigg(\log N - \frac{1}{2}\frac{\zeta''(\rho)}{\zeta'(\rho)} + \frac{\chi'}{\chi}(\rho) + \frac{1-2\rho}{|\rho|^2}\bigg)\frac{1}{|\rho|^2}\\ &= \frac{\log N}{|\rho|^2} + O\bigg(\frac{1}{|\rho|^{2-\varepsilon}|\zeta'(\rho)|} + \frac{1}{|\rho|^2}\bigg),\end{aligned}$$

where we used the bound $\zeta''(1/2+it) \ll |t|^{\varepsilon}$, which follows from the Lindelöf hypothesis and Cauchy's estimate for the derivatives of a holomorphic function. It follows that moving the line of integration in (7) to $\mathrm{Re}(s) = 1/2+\varepsilon$ we get that the integral is equal to

$$\frac{1}{\log N}\sum_{\rho}\frac{1}{|\rho|^2} + O\bigg(\frac{1}{\log^2 N}\bigg),$$

and Theorem 1 then follows.

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