SMALL POPULATIONS OF ZEROS OF L-FUNCTIONS

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§ 1. INTRODUCTION

Given an L-function of arithmetic nature

$$L(s) = \sum_{1}^{\infty} \lambda(n) n^{-s} \tag{1.1}$$

chances are that all its zeros $\rho = \beta + i\gamma$ in the critical strip $s = \sigma + it, 0 < \sigma < 1$ are actually on the critical line $\sigma = \frac{1}{2}$ (the symmetry line of the functional equation). This is a Generalized Riemann Hypothesis. So far not a single L-function is known which satisfies the GRH. Thus it is appropriate to ask how many zeros can possibly be off the line. Specifically we consider the zeros in the rectangles

$$s = \sigma + it \text{ with } \alpha \leqslant \sigma < 1, \ |t| \leqslant T$$

where $\frac{1}{2} < \alpha < 1, T \geqslant 2$, and we wish to show that the number of such zeros (counted with multiplicity), say $N(\alpha, T)$, is relatively small. For many L-functions we can show that

$$N(\alpha, T) \ll T^{1-\eta} \tag{1.2}$$

with some positive constant $\eta = \eta(\alpha)$. Since the number of all zeros in the rectangle $0 < \sigma < 1, |t| \leqslant T$ is

$$N(T) \asymp T \log T$$

the bound (1.2) tells us that almost all zeros are located near the critical line.

Although we believe in the validity of the GRH, so (1.2) is just a bound for the cardinality of the empty set, there is great interest for applications to make (1.2) as strong as possible.

DENSITY CONJECTURE. For $\frac{1}{2} < \alpha < 1$ and $T \geqslant 2$ we have

$$N(\alpha, T) \leqslant T^{2(1-\alpha)+\varepsilon} \tag{1.4}$$

where $\varepsilon = \varepsilon(T) \to 0$ as $T \to \infty$.

For many applications the bound (1.4) would be as helpful as the GRH, however the best known results are a bit short. For example, in the case of the Riemann zeta function

$$\zeta(s) = \sum_n n^{-s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1.5}$$

we know that

$$N(\alpha, T) \leqslant T^{c(\alpha)(1-\alpha)+\varepsilon} \tag{1.6}$$

with $c(\alpha) = 3/(2-\alpha)$ due to A.E. Ingham [I] and $c(\alpha) = 3/(3\alpha - 1)$ due to M.N. Huxley [H], where $\varepsilon = \varepsilon(T) \to 0$ as $T \to \infty$.

The classical Lindelöf Hypothesis, which asserts that

$$\zeta(\frac{1}{2} + it) \ll (|t| + 1)^{\varepsilon} \tag{1.7}$$

with any $\varepsilon > 0$, the implied constant depending on ε, does imply (1.4) for the Riemann zeta function. By the way, the LH is equivalent with the assertion that (see [B])

$$N(\alpha, T + 1) = N(\alpha, T) + o(\log T) \tag{1.8}$$

for any $\frac{1}{2} < \alpha < 1$.

In these notes we are going to investigate various estimates for sums over the zeros off the critical line which clearly support the validity of the DC in considerable generality. In particular we shall prove unconditionally that the "solitary" zeros satisfy the DC.

REMARKS. Professor R. Balasubramanian has kindly pointed out that he and K. Ramachandra (see [BR],[R]) have also established "Density Conjecture" for isolated zeros in some sense. We also learned from the MathSciNet review of [BR] by D.R. Heath-Brown that the reviewer has shown (unpublished, 1975) the density hypothesis for the "ε-isolated" zeros. However, this "ε-isolated" concept is more restrictive than our "small populations". After presentation of our results in Chennai, January 2012, by the second author J. Kaczorowski revealed that he also has established similar results (unpublished).

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$$\sigma > \beta + \frac{1}{\log T}, \quad |t-\gamma| < \log T. \tag{2.9}$$

Let $N^*(\alpha, T)$ denote the number of extremal zeros $\rho_r = \beta_r + i\gamma_r$ with $\beta_r \geqslant \alpha, |\gamma_r| \leqslant T$. Then we have

$$N(\alpha, T - \log^2 T) \ll N^*(\alpha, T)(\log T)^2, \tag{2.10}$$

because for every zero $\rho = \beta + i\gamma$ with $\beta \geq \alpha, |\gamma| \leqslant T - \log^2 T$ there is an extremal zero $\rho_r = \beta_r + i\gamma_r$ with $\beta_r \geqslant \alpha, |\gamma_r - \gamma| \leqslant \log T$.

In view of (2.10) we can restrict the counting of all zeros in rectangles to these which are extremal without missing a chance for the Density Conjecture. Actually we are going to take into account all the zeros with certain weights which are heavier when being closer to an extremal zero. Specifically we consider

$$W(it) = \sum_{\frac{1}{2}<\beta<1} T^{\rho}(\rho - it)^{-k} \tag{2.11}$$

where $k \geqslant 2, T \geqslant 2$ and t is a real number. Note that $W(it)$ depends on k and T (which property we have not displayed for notational simplicity). We shall play with the variables k and T in Section 6. The series (2.11) converges rapidly; only the zeros $\rho = \beta + i\gamma$ of height γ close to t makes an essential contribution. Eventually we shall choose $t = \gamma_r$, the height of the extremal zero. In this case the extremal zero $\rho_r = \beta_r + i\gamma_r$ contributes to $W(i\gamma_r)$ exactly

$$V_r = T^{\beta_r} \beta_r^{-k}. \tag{2.12}$$

We shall get a lower bound for $|W(i\gamma_r)|$ at special points which is not much smaller than (2.12). This goal is easy for the solitary zeros $\rho_r = \beta_r + i\gamma_r$ (see Section 5) and quite hard when there is a large population of zeros $\rho = \beta + i\gamma$ near ρ_r. We shall treat small populations of zeros by the Power Sums Method of P. Turán [T] in Section 6. Our results are similar and conditional like these of Turán (see Chapter 37 of [T]). However there are some differences in the arguments. For instance we are able to select special subsets of zeros (extremal, dominant, solitary,...), drop the others by positivity features getting estimates of the Density Conjecture for the number of these distinguished zeros.

Complete density theorems are derived in modern time by means of "Large Values of Dirichlet Polynomials" which play a role of individual zero detectors. However, these methods work only for primitive L-functions of degree one and two whereas the arguments described in these notes (and in the original paper of Turán) have no such limitations, the obtained estimates are strong but partial.

§ 2. Assumptions

We require a little information about the L-functions (1.1). Actually we could forget that $\rho = \beta + i\gamma$ are the zeros of $L(s)$, because we only need to know that they are simple poles of a meromorphic function $Q(s)$ which has a Dirichlet series expansion

$$Q(s) = \sum_{1}^{\infty} q(n) n^{-s}, \quad \text{if } Re\, s > 1 \tag{2.1}$$

with coefficients satisfying

$$|q(n)| \leqslant c \log n \tag{2.2}$$

where c is a positive constant. In particular, $q(1) = 0$. Of course, we have in mind

$$Q(s) = -L'(s)/L(s) \tag{2.3}$$

in which case the Dirichlet series expansion (2.1) follows by the Euler product for $L(s)$ and the coefficients $q(n)$ are supported on prime powers. The estimate (2.2) can be somewhat weaker; as stated it is a kind of Ramanujan's conjecture.

We assume that $L(s)$ has meromorphic continuation to the half-plane $Re\, s > 0$ with a pole at $s = 1$ and zeros at $s = \rho = \beta + i\gamma$, so $Q(s)$ has simple poles at $s = 1$ and $s = \rho$. The number of these poles (counted with multiplicities) in unit squares satisfy

$$\#\{\rho = \beta + i\gamma;\ 0 < \beta < 1,\ |\beta - t| < \frac{1}{2}\} \leqslant c \log(|t| + 3) \tag{2.4}$$

for every t, where c is a positive constant. We claim that

$$Q(s) = \frac{m}{s-1} - \sum_{|\rho - s| < 1} \frac{1}{s - \rho} + O(\log(|s| + 3)), \quad \text{if } Re\, s > \frac{1}{2}. \tag{2.5}$$

Let $T \geqslant 2$.

Using (2.4) one can draw a continuous curve $\mathscr{C}$ (composed of horizontal and vertical segments) in the strip

$$\frac{1}{2} < Re\, s < \frac{1}{2} + \frac{1}{\log T} \tag{2.6}$$

such that every point $s \in \mathscr{C}$ is distant from every pole ρ of $Q(s)$ by

$$|s - \rho| \gg 1/\log(|s| + T). \tag{2.7}$$

Hence using (2.5) and (2.4) we get

$$Q(s) \ll \log^2(|s| + T), \quad \text{if } s \in \mathscr{C}. \tag{2.8}$$

A zero $\rho = \beta + i\gamma$ of $L(s)$ (i.e. a pole of $Q(s)$) with $\frac{1}{2} < \beta < 1$ and $|\gamma| \leqslant T$ is called "extremal" if there are no zeros in the rectangle $s = \sigma + it$ with

$$\sigma > \beta + \frac{1}{\log T}, \quad |t - \gamma| < \log T. \tag{2.9}$$

Let $N^*(\alpha, T)$ denote the number of extremal zeros $\rho_r = \beta_r + i\gamma_r$ with $\beta_r \geqslant \alpha, |\gamma_r| \leqslant T$. Then we have

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§ 3. Mean Value of $|W(it_r)|^2$

Let $\mathscr{T}$ be a sequence of real numbers t_r which is not very dense anywhere, specifically we asume that

$$\#\{r; |t_r - t| \leqslant 1\} \leqslant c \log(|t| + 3) \tag{3.1}$$

for every t, where c is a positive constant.

Proposition 3.1. *For $k \geqslant 2$ and $T \geqslant 2$ we have*

$$\sum_{|t_r| \leqslant T} | \sum_{\frac{1}{2} < \beta < 1} T^{\rho} (\rho - it_r)^{-k} |^2 \ll 4^k T^2 (\log T)^7 \tag{3.2}$$

where the implied constant depends only on c in (3.1), (2.4) and (2.2).

We begin the proof of (3.2) by introducing very short sums of the coefficients of $Q(s)$;

$$\Psi(x) = \sum_1^{\infty} q(n)\, g(\frac{n}{x}), \quad x > 0, \tag{3.3}$$

where $g(u)$ is a smooth bump function supported on $1 \leqslant u \leqslant 1 + \frac{1}{T}$. We assume that $0 \leqslant g(u) \leqslant 2, g'(u) \ll T, g''(u) \ll T^2$ and

$$g(u) \geqslant 1 \;\text{ if }\; 1 + \frac{1}{3T} < u < 1 + \frac{2}{3T}.$$

Note that if $0 < x < T$ the series (3.3) has at most one non-zero term.

The Mellin transform

$$h(s) = \int_0^{\infty} g(u) u^{s-1} du \tag{3.4}$$

satisfies the following estimates

$$h(s) \ll \frac{1}{T} \left(1 + \frac{|s|}{T}\right)^{-2},$$

$$|h(s)| \asymp \frac{1}{T}, \qquad \text{if} \quad |t| \leqslant T,$$

$$|h(s') - h(s)| \ll |s' - s|\, T^{-2}.$$

Our goal is to evaluate the integral

$$I(\nu) = \int_0^{\infty} \Psi(x)\, G(x) x^{\nu - 1} dx \tag{3.5}$$

on the line $Re\, \nu = 0$, where $G(x)$ is a nice continuous function. There are many interesting choices of $G(x)$ which would lead to equivalent results. We take $G(x)$ depending on a parameter k;

$$G(x) = \frac{1}{\Gamma(k)} (\log \frac{T}{x})^{k-1}, \quad \text{if} \quad 0 < x < T \tag{3.6}$$

and $G(x) = 0$ if $x \geqslant T$. Here $k \geqslant 2$ is any real number, not necessarily an integer. We have

$$G(x) = \frac{1}{2\pi i} \int_{(\sigma)} \left(\frac{T}{x}\right)^s \frac{ds}{s}, \quad \sigma > 0. \tag{3.7}$$

Hence by Mellin's inversion

$$\int_0^\infty G(x)\, x^{s-1} dx = T^s s^{-k}. \tag{3.8}$$

Note that G^2 is of type (3.6) so (3.8) becomes (by re-scaling)

$$\int_0^\infty G^2(x) x^{s-1}\, dx = \frac{\Gamma(2k-1)}{\Gamma(k)^2} T^s\, s^{1-2k}.$$

Introducing (3.3) and (3.7) to (3.5) we get

$$I(\nu) = \frac{1}{2\pi i} \int_{(2)} T^{s+\nu} Q(s) h(s) (s+\nu)^{-k} ds.$$

Next we move the line of integration to the curve $\mathscr{C}$ described in (2.6) and (2.7) getting

$$I(\nu) = mh(1) T^{1+\nu} (1+\nu)^{-k} - \sum_\rho h(\rho) T^{\rho+\nu} (\rho+\nu)^{-k} + I_{\mathscr{C}}(\nu),$$

where ρ runs over the poles of $Q(s)$ (the zeros of $L(s)$ counted with multiplicities) to the right of $\mathscr{C}$ and $I_{\mathscr{C}}(\nu)$ is the contour integral along the curve $\mathscr{C}$. By (2.8) we estimate this integral as follows

$$I_{\mathscr{C}}(\nu) \ll \frac{2^k}{\sqrt{T}} \int_{\mathscr{C}} \left(1 + \frac{|s|}{T}\right)^{-2} \frac{\log^2(|s|+T)}{|s+\nu|^2} |ds| \ll 2^k T^{-\frac{1}{2}} \log^2 T.$$

We can extend the sum over ρ to all poles with $Re\rho > \frac{1}{2}$ up to the same error term as the above estimate for the contour integral. Indeed, the residues of poles ρ' lying between the critical line and the curve $\mathscr{C}$ which were not previously captured contribute

$$\sum_{\rho'} h(\rho') T^{\rho'+\nu} (\rho'+\nu)^{-k} \ll \frac{2^k}{\sqrt{T}} \sum_{\rho'} (1 + \frac{|\rho'|}{T})^{-2} |\rho'+\nu|^{-2} \ll 2^k T^{-\frac{1}{2}} \log T.$$

Hence

$$I(\nu) = mh(1)T^{1+\nu}(1+\nu)^{-k} - \sum_{\frac{1}{2}<\beta<1} h(\rho)T^{\rho+\nu}(\rho+\nu)^{-k} + O(2^k T^{-\frac{1}{2}} \log^2 T).$$

Here we can replace $h(\rho)$ by $h(-\nu)$ up to the correcting terms

$$(h(\rho) - h(-\nu))T^{\rho+\nu}(\rho+\nu)^{-k} \ll |\rho+\nu|^{1-k}T^{\beta-2}.$$

These correcting terms contribute to $I(\nu)$ much less than the existing error term (use (2.4)). Therefore, we conclude the following formula

$$I(\nu) = mh(1)T^{1+\nu}(1+\nu)^{-k} - h(-\nu)T^{\nu}W(-\nu) + O(2^k T^{-\frac{1}{2}} \log^2 T). \tag{3.9}$$

for any ν with $Re\,\nu = 0$. Hence

$$|W(it)| \ll T|I(it)| + T(1+|t|)^{-k} + 2^k T^{\frac{1}{2}} \log^2 T \tag{3.10}$$

for any real t with $|t| \leqslant T$. We use this estimate at the points $t = t_r$ of the set $\mathscr{T}$. By our assumption (3.1) we get

$$\sum_{|t_r|\leqslant T} |W(it_r)|^2 \ll T^2 \sum_{|t_r|\leqslant T} |I(it_r)|^2 + 4^k T^2 (\log T)^6. \tag{3.11}$$

On the right side we estimate the sum over the set $\mathscr{T}$ using the orthogonality of x^{it_r}. First we smooth out the summation by the Sobolev-Gallagher inequality (c.f.[G])

$$\sum_{|t_r|\leqslant T} |I(it_r)|^2 = \sum_{|t_r|\leqslant T} \Big| \int_1^T \Psi(x)G(x)x^{it_r-1}dx\Big|^2 \ll (\mathscr{I}(T) + \mathscr{I}_1(T)) \log T$$

where

$$\mathscr{I}(T) = \int_{-T-\frac{1}{2}}^{T+\frac{1}{2}} \Big| \int_1^T \Psi(x)G(x)x^{it-1}dx\Big|^2 dt$$

$$\mathscr{I}_1(T) = \int_{-T-\frac{1}{2}}^{T+\frac{1}{2}} \Big| \int_1^T \Psi(x)G(x)(\log x)x^{it-1}dx\Big|^2 dt.$$

By the Fourier transform

$$\int_{-\infty}^{\infty} (1+\frac{t^2}{T^2})^{-1}(\frac{x}{y})^{it} \mathrm{dt} = \pi T \exp(-T|\log\frac{x}{y}|)$$

we get

$$\mathscr{I}(T) \leqslant 3\int_{-\infty}^{\infty} (1+\frac{t^2}{T^2})^{-1}|\ldots|^2 dt = 6\pi T \iint_{1<x<y<T} \Psi(x)\Psi(y)G(x)G(y)\Big(\frac{x}{y}\Big)^T \frac{dxdy}{xy}.$$

Here we have

$$\Psi(x)\Psi(y) \leqslant \sum_{\substack{x<m<x(1+1/T)\\ y<n<y(1+1/T)}}\sum q(m)q(n),$$

$2 \leqslant m \leqslant n \leqslant T+1$, $G(x) \leqslant G(\frac{mT}{T+1})$, $G(y) \leqslant G(\frac{nT}{T+1})$, $(\frac{x}{y})^T \leqslant e(\frac{m}{n})^T$, because $G(x)$ is decreasing, getting

$$\begin{aligned}
\mathscr{J}(T) &\leqslant 4\pi eT \sum_{m\leqslant n}\sum q(m)q(n)g(\frac{mT}{T+1})\,G(\frac{nT}{T+1})(\frac{m}{n})^T \log^2(1+\frac{1}{T})\\
&\leqslant 4\pi ecT\ \log(T+1)\ \log^2(1+\frac{1}{T}) \sum_m q(m)\,G^2(\frac{mT}{T+1}) \sum_{n\geqslant m}(\frac{m}{n})^T\\
&\leqslant 4\pi ecT\ \log(T+1)\ \log^2(1+\frac{1}{T}) \sum_m q(m)\,G^2(\frac{mT}{T+1})(1+\frac{m}{T-1})\\
&\leqslant \frac{8\pi e}{T-1}\Big(cT\ \log(T+1)\log(1+\frac{1}{T})\Big)^2 \sum_m G^2(\frac{mT}{T+1}),
\end{aligned}$$

where c is the constant in (2.2). By (3.8) we derive

$$\sum_m G^2(\frac{mT}{T+1}) \leqslant \int_0^\infty G^2(\frac{xT}{T+1})\,dx = \frac{T+1}{T}\int_0^\infty G^2(x)\,dx = \frac{\Gamma(2k-1)}{\Gamma(k)^2}\,(T+1).$$

From the above estimates we conclude that

$$\mathscr{J}(T) \ll 4^k(\log T)^4.$$

Similarly

$$\mathscr{J}_1(T) \ll 4^k(\log T)^6.$$

Hence

$$\sum_{|t_r|\leqslant T} |I(it_r)|^2 \ll 4^k(\log T)^7. \tag{3.12}$$

Finally, introducing (3.12) to (3.11) we complete the proof of Proposition 3.1.

Corollary 3.2. *Let $1 \leqslant K < k \leqslant 2K$ and $T \geqslant 2$. Then we have*

$$\sum_{|t_r|\leqslant T} \Big|\sum_{\frac12<\beta<1} T^{\rho k/K}(\rho - it_r)^{-k}\Big|^2 \ll 4^k T^{2k/K}\,(\log T)^7. \tag{3.13}$$

Proof. Apply (3.2) for T changed to $T^{k/K}$. □

§ 4. Density of Dominant zeros

We shall use Proposition 3.1 for the series $W(it_r)$ given by (2.11) at the points $t_r = \gamma_r$ which are the heights of certain zeros $\rho_r = \beta_r + i\gamma_r$ with $\frac{1}{2} < \beta_r < 1$ and $|\gamma_r| \leqslant T$. Let $0 < \varepsilon \leqslant \frac{1}{2}$. We say that ρ_r is "dominant" if

$$|W(i\gamma_r)| \geqslant T^{\beta_r - \varepsilon}. \tag{4.1}$$

Let $2 \leqslant k \leqslant \varepsilon \log T$. Then (3.2) yields

$$\sum_{\rho}^{\#} T^{2\beta_r} \ll T^{2+4\varepsilon} \tag{4.2}$$

where the # restricts the summation to the dominant zeros of height up to T. Hence the number of dominant zeros with $\beta_r \geqslant \alpha$, say $N^{\#}(\alpha, T)$, satisfies the estimate of the Density Conjecture

$$N^{\#}(\alpha, T) \ll T^{2(1-\alpha)+4\varepsilon}. \tag{4.3}$$

Our condition (4.1) for a zero ρ_r to be dominant is quite reasonable, and it seems to hold for every zero of any natural L-function. Recall that the single term $\rho = \rho_r$ in the series (2.11) contributes (2.12) which is larger than the required lower bound (4.1). A problem may occur with the surrounding zeros which could cause a considerable cancellation of terms in (2.11) and ruin the bound (4.1). However, this scenario is improbable for judiciously chosen k as it can be ruled out by an appeal to the Lindelöf Hypothesis. In the next sections we shall select some zeros which do satisfy (4.1).

§ 5. Solitary zeros

Let $\rho_r = \beta_r + i\gamma_r$ be an extremal zero, which means $\frac{1}{2} < \beta_r < 1, |\gamma_r| \leqslant T$ and every other zero $\rho = \beta + i\gamma$ satisfies either

$$\beta \leqslant \beta_r + 1/\log T \quad \text{or} \quad |\gamma - \gamma_r| \geqslant \log T.$$

First we are going to remove from the series

$$W(i\gamma_r) = \sum_{\frac{1}{2}<\beta<1} T^{\rho}(\rho - i\gamma_r)^{-k}$$

the zeros ρ which are quite distant from ρ_r by estimating trivially their contribution. Throughout we assume $2 \leqslant k \leqslant \frac{1}{2}\log T$.

Let $0 < a < \frac{1}{2}$. Note that the function $T^{\beta}|\beta + i\tau|^{-k}$ is increasing in $\beta > \frac{1}{2}$. Hence a zero $\rho = \beta + i\gamma$ with $\frac{1}{2} < \beta \leqslant \beta_r - a$ contributes to $W(i\gamma_r)$ at most

$$\begin{aligned} T^{\beta_r - a}(\beta_r - a)^{2-k}|\beta + i(\gamma - \gamma_r)|^{-2} &\leqslant T^{\beta_r - a}(\beta_r - a)^{-k}(\tfrac{1}{4} + (\gamma - \gamma_r)^2)^{-1} \\ &\leqslant T^{\beta_r - a}\beta_r^{-k}(1 - 2a)^{-k}(\tfrac{1}{4} + (\gamma - \gamma_r)^2)^{-1}. \end{aligned}$$

Let $b > 0$. Then a zero $\rho = \beta + i\gamma$ with $\beta \leqslant \beta_r + 1/\log T$ and $|\gamma - \gamma_r| \geqslant b$ contributes to $W(i\gamma_r)$ at most

$$\begin{aligned} eT^{\beta_r}\,|\beta_r + i(\gamma - \gamma_r)|^{-k} &\leqslant eT^{\beta_r}\,\beta_r^{-k}\,(1 + (\gamma - \gamma_r)^2)^{-k/2} \\ &\leqslant eT^{\beta_r}\,\beta_r^{-k}(1 + b^2)^{-k/2}(1 + (\gamma - \gamma_r)^2)^{-1}. \end{aligned}$$

1- k/2

Any zero $\rho = \beta + i\gamma$ with $|\gamma - \gamma_r| \geqslant \log T$ contributes to $W(i\gamma_r)$ at most

$$\begin{aligned} T^{\beta}|\beta + i(\gamma - \gamma_r)|^{-k} &\leqslant T(1 + (\gamma - \gamma_r)^2)^{-k/2} \leqslant T^{(}\log T)^{2-k}(1 + (\gamma - \gamma_r)^2)^{-1} \\ &\leqslant T^{\beta_r}\,\beta_r^{-k}\,T^{\frac{1}{2}}(\log T)^{2-k}(1 + (\gamma - \gamma_r)^2)^{-1}. \end{aligned}$$

Now we are left with the zeros $\rho = \beta + i\gamma$ in the box

$$-a < \beta - \beta_r < 1/\log T \tag{5.1}$$

$$|\gamma - \gamma_r| < b. \tag{5.2}$$

Let $W^b(i\gamma_r)$ denote the partial sum of $W(i\gamma_r)$ over the zeros restricted by (5.1) - (5.2). The above inequalities yield the following estimate

$$|W(i\gamma_r) - W^b(i\gamma_r)| \leqslant 4V_r(T^{-a}(1-2a)^{-k} + (1+b^2)^{1-k/2} + T^{\frac{1}{2}}(\log T)^{2-k}) \sum_{\frac{1}{2}<\beta<1} (1+(\gamma-\gamma_r)^2)^{-1}.$$

The last sum is bounded by $O(\log T)$. Hence we conclude

Proposition 5.1. *Let $2 \leqslant k \leqslant \frac{1}{2}\log T, 0 < a < \frac{1}{2}$ and $b > 0$. Then for the extremal zero $\rho_r = \beta_r + i\gamma_r$ we have*

$$W(i\gamma_r) = W^b(i\gamma_r) + E(i\gamma_r) \tag{5.3}$$

with

$$E(i\gamma_r) \ll V_r(T^{-a}(1-2a)^{-k} + (1+b^2)^{1-k/2} + T^{\frac{1}{2}}(\log T)^{2-k})\log T \tag{5.4}$$

where the implied constant depends on c in (2.4).

Obviously the zero $\rho_r = \beta_r + i\gamma_r$ satisfies (5.1) - (5.2) and it contributes to $W^b(i\gamma_r)$ exactly $V_r = T^{\beta_r}\beta_r^{-k}$.

We say that an extremal zero

$$\rho_r = \beta_r + i\gamma_r \text{ is "solitary"}$$

if there are no other zeros $\rho = \beta + i\gamma$ in the box (5.1) - (5.2).

Of course, this concept depends on the size of the box (one may say the "backyard" of ρ_r) which appears to be quite small; it measures $a + 1/\log T$ and $2b$ in the horizontal and vertical directions, respectively. We shall specify a, b in forthcoming applications of Proposition 5.1. We could build a small house in the backyard of ρ_r which accommodate any zero $\rho = \beta + i\gamma$ with $|\rho - \rho_r| < 1/\log T$ without changing the results.

By the definition for a solitary zero we have $W^b(i\gamma_r) = V_r$ and Proposition 5.1 yields

$$W(i\gamma_r) = V_r\{1 + O(T^{-a}(1-2a)^{-k} + (1+b^2)^{1-k/2} + T^{\frac{1}{2}}(\log T)^{2-k})\log T)\}.$$

Taking $k = \varepsilon \log T$ with $0 < \varepsilon < \frac{1}{4}$ this yields (4.1) by choosing

$$a = a(T) = \frac{2\log\log T}{\log T} \tag{5.5}$$

$$b = b(T) = \left(\frac{e\log\log T}{\varepsilon \log T}\right)^{\frac{1}{2}}. \tag{5.6}$$

In our words the solitary zero ρ_r with respect to $a = a(T)$ and $b = b(T)$ given above is dominant. Therefore we conclude

Corollary 5.2. *The number of solitary zeros with respect to $a(T), b(T)$ given by (5.5) - (5.6) satisfies the bound of the Density Conjecture;*

$$N_{sol}(\alpha, T) \ll T^{2(1-\alpha)+2\varepsilon}. \tag{5.7}$$

§ 6. Small populations of zeros

An extremal zero $\rho_r = \beta_r + i\gamma_r$ does not need to be solitary in order to be dominant. Our arguments work if the number of zeros close to ρ_r is relatively small. To this end we are going to use the following inequality of P. Turán [T].

Proposition 6.1. *For any complex numbers $z_1, \dots, z_N$ there exists an integer k with $K < k \leqslant K + N$ such that*

$$|z_1^k + \dots + z_N^k| \geqslant 2\Big(\frac{N}{8e(K+N)}\Big)^N \max |z_n|^k. \tag{6.1}$$

Let $1 \leqslant N \leqslant K$. For any k with $K < k \leqslant K + N$ we get by Proposition 5.1 (change T to $T^{k/K}$)

$$\sum_\rho T^{\rho k/K} \quad (\rho - i\gamma_r)^{-k} = \sum_\rho{}^\flat\, T^{\rho k/K}(\rho - i\gamma_r)^{-k}$$
$$+O(T^{\beta_r k/K}\, \beta_r^{-k}(T^{-a}(1-2a)^{-k} + (1+b^2)^{1-k/2} + T(\log T)^{2-k}) \log T)$$

where the $\flat$ restricts the summation over the zeros $\rho = \beta + i\gamma$ in the box (5.1) - (5.2). Suppose there are at most N such zeros, so (6.1) yields

$$|\sum_\rho{}^\flat| \geqslant 2\Big(\frac{N}{16eK}\Big)^N\, T^{\beta_r k/K}\, \beta_r^{-k}$$

for some k with $K < k \leqslant K + N$. We want this lower bound to be larger than the above error term. Specifically we require the following conditions

$$T^a(1-2a)^{2K} \geqslant C\big(16e\frac{K}{N}\big)^N \log T \tag{6.2}$$

$$(1+b^2)^{\frac{K}{2}-1} \geqslant C\big(16e\frac{K}{N}\big)^N \log T \tag{6.3}$$

$$(\log T)^{K-3} \geqslant C\big(16e\frac{K}{N}\big)^N\, T \tag{6.4}$$

where C is a sufficiently large constant, three times larger than the implied constant in the error term. Assuming these conditions we get

$$|\sum_\rho T^{\rho k/K}\,(\rho - i\gamma_r)^{-k}| \geqslant \Big(\frac{N}{16eK}\Big)^N\, T^{\beta_r k/K} \beta_r^{-k}$$

for some k with $K < k \leqslant K + N$ (which may depend on γ_r). Hence

$$\sum_{K < k \leqslant K+N} |\sum_\rho T^{\rho k/K}(\rho - i\gamma_r)^{-k}|^2 \geqslant (8e)^{-2K}\, T^{2\beta_r}. \tag{6.5}$$

Combining (6.5) with (3.13) we arrive at

$$\sum_{\rho_r}^{(N)} T^{2\beta_r} \ll (32e)^K T^{2(1+N/K)} (\log T)^7 \tag{6.6}$$

where the sum $\Sigma^{(N)}$ runs over the extremal zeros $\rho_r = \beta_r + i\gamma_r$ with $\frac{1}{2} < \beta_r < 1, |\gamma_r| \leqslant T$ for which there are at most N zeros $\rho = \beta + i\gamma$ in the box (5.1) - (5.2).

Recall that (6.6) holds subject to the conditions (6.2) - (6.4). It remains to secure these conditions. First we take

$$K = \varepsilon \log T \tag{6.7}$$

where ε is a small positive number and we assume that T is sufficiently large in terms of ε. We fix N with

$$1 \leqslant N \leqslant \varepsilon^2 \log T. \tag{6.8}$$

Then (6.2) holds for

$$a = \frac{2N}{\log T} \log\left(\frac{\log T}{N}\right), \tag{6.9}$$

and (6.3) holds for

$$b^2 = \frac{3N}{\varepsilon \log T} \log\left(\frac{\log T}{N}\right). \tag{6.10}$$

Also (6.4) holds because T is large in terms of ε. Hence we obtain

Theorem 6.2. *Let ε be any small positive number and T be any sufficiently large number in terms of ε. Let N, a, b be given by (6.8) - (6.10). Then*

$$\sum_{\rho_r}^{(N)} T^{2\beta_r} \ll T^{2+\text{[illegible]}\varepsilon}. \tag{6.11}$$

We end these notes with the final self-contained statement.

Corollary 6.3. *Let $L(s)$ be an L-function having the properties described in (2.1) - (2.5). Let ε be a small positive number, T be a large number in terms of ε and N, a, b be given by (6.8) - (6.10). Let $\rho_r = \beta_r + i\gamma_r$ denote the zeros of $L(s)$ with $\frac{1}{2} < \beta_r < 1, |\gamma_r| \leqslant T$ such that there are no zeros $\rho = \beta + i\gamma$ of $L(s)$ with*

$$\beta - \beta_r \geqslant 1/\log T, \quad |\gamma| \leqslant \log T$$

and at most N zeros in the box

$$-a < \beta - \beta_r < 1/\log T, \quad |\gamma - \gamma_r| < b.$$

Then the number of these zeros $\rho_r = \beta_r + i\gamma_r$ with $\beta_r \geqslant \alpha$, say $N'(\alpha, T)$, satisfies

$$N'(\alpha, T) \ll T^{2(1-\alpha)+\text{[illegible]}\varepsilon}$$

for any $\frac{1}{2} < \alpha < 1$, the implied constant depending on ε and the constant c in (2.2) and (2.4).

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