RIEMANN'S HYPOTHESIS

BRIAN CONREY

1. Gauss

There are 4 prime numbers less than 10; there are 25 primes less than 100; there are 168 primes less than 1000, and 1229 primes less than 10000. At what rate do the primes thin out? Today we use the notation $\pi(x)$ to denote the number of primes less than or equal to x; so $\pi(1000) = 168$.

Carl Friedrich Gauss in an 1849 letter to his former student Encke provided us with the answer to this question. Gauss described his work from 58 years earlier (when he was 15 or 16) where he came to the conclusion that the likelihood of a number n being prime, without knowing anything about it except its size, is

$$\frac{1}{\log n}.$$

Since $\log 10 = 2.303\dots$ the means that about 1 in 16 seven digit numbers are prime and the 100 digit primes are spaced apart by about 230. Gauss came to his conclusion empirically: he kept statistics on how many primes there are in each sequence of 100 numbers all the way up to 3 million or so! He claimed that he could count the primes in a chiliad (a block of 1000) in 15 minutes! Thus we expect that

$$\pi(N) \approx \frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \cdots + \frac{1}{\log N}.$$

This is within a constant of

$$\mathrm{li}(N) = \int_0^N \frac{du}{\log u}$$

so Gauss' conjecture may be expressed as

$$\pi(N) = \mathrm{li}(N) + E(N)$$

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where $E(N)$ is an error term. What is stunning is how small $E(N)$ is! Here is some data from Gauss' letter:

N	$\pi(N)$	$\mathrm{li}(N)$	$E(N)$
500000	41556	41606.4	+50.4
1000000	78501	78627.5	+126.5
1500000	114112	114263.1	+151.1
2000000	148883	149054.8	+171.8
2500000	183016	183245.0	+229.0
3000000	216745	216970.6	+225.6

Note that there is an asymptotic expansion

$$\mathrm{li}(N) = \frac{N}{\log N} - \frac{N}{\log^2 N} + \dots.$$

2. Riemann

In 1859 G. B. F. Riemann proposed a pathway to understand on a large scale the distribution of the prime numbers. He studied

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

as a function of a complex variable $s = \sigma + it$ and proved that $\zeta(s)$ has a meromorphic continuation to the whole s – plane with its only singularity a simple pole at $s = 1$. Moreover, he proved a functional equation that relates $\zeta(s)$ to $\zeta(1-s)$ in a simple way and reveals that $\zeta(-2) = \zeta(-4) = \dots = 0$. His famous hypothesis is that all the other zeros have real part equal to 1/2. He even computed the first few of these non-real zeros:

$$1/2 + i14.13\dots, /2 + i21.02\dots, 1/2 + i25.01\dots, \dots$$

A good way to be convinced that these are indeed zeros is to use the easily proven formula

$$(1 - 2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} \pm \dots.$$

The alternating series on the right converges for $\Re s > 0$ and so

$$s = 1/2 + i14.134725141734693790457251983562 4\dots$$

can be substituted into a truncation of this series (using your favorite computer algebra system) to see that it is very close to 0. (See www.lmfdb.org to find a list of high precision zeros of $\zeta(s)$ as well as a wealth of information about $\zeta(s)$ and similar functions called L-functions.)

Euler had earlier proved that $\zeta(s)$ can be expressed as an infinite product over primes:

$$\begin{aligned} \zeta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \dots\right)\left(1 + \frac{1}{5^s} + \dots\right) \\ &= \prod_p (1 - p^{-s})^{-1}. \end{aligned}$$

Euler's formula is essentially an analytic encoding of the fundamental theorem of arithmetic that each integer can be expressed uniquely as a product of primes. Euler's formula provided a clue for Riemann to use complex analysis to investigate the primes. We can almost immediately see a consequence of Riemann's Hypothesis about the zeros of $\zeta(s)$ if we invert Euler's formula:

$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}) = 1 - 2^{-s} - 3^{-s} - 5^{-s} + 6^{-s} + \cdots = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

where μ is known as the Möbius mu-function. A simple way to explain the value of $\mu(n)$ is that it is 0 if n is divisible by the square of any prime, while if n is squarefree then it is $+1$ if n has an even number of prime divisors and -1 if n has an odd number of prime divisors. Not surprisingly the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

converges if the real part of s is bigger than the real part of any zero ρ of $\zeta(s)$. This actually translates into an equivalent formulation of the Riemann Hypothesis (RH):

$$\text{RH is true if and only if for any } \epsilon > 0, \ \sum_{n \le x} \mu(n) \le C(\epsilon) x^{1/2+\epsilon}.$$

Thus, we expect that the integers with an even number of prime factors are equally numerous as the integers with an odd number of prime factors and the difference between the two counts is related to the size of the rightmost zero of $\zeta(s)$.

Riemann had an ambitious plan to find an *exact* formula for the number of primes up to x and he did so:

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{1/n})$$

where

$$f(x) = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) - \ln 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}.$$

Here the $\rho = \beta + i\gamma$ are the zeros of $\zeta(s)$. The upshot of all of this is that the error term $E(x)$ from Gauss' conjecture is no more than $x^{\beta_0} \log x$ where β_0 is the supremum of the real parts of the zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. So, if Riemann's hypothesis is true then Gauss' formula is correct with a square-root sized error term! Whereas even one zero not on the 1/2–line will lead to a an error term larger by a power of x. Note that by Riemann's functional equation the zeros are symmetric with respect to the 1/2-line. And $\zeta(s)$ is real for real s so the zeros are symmetric with respect to the real axis. Thus, if $\rho = \beta + i\gamma$ is a zero then so are $\beta - i\gamma$ and if $\beta \ne 1/2$ then $1 - \beta \pm i\gamma$ are zeros, too.

There are no zeros with real part greater than 1; we know this because of Euler's product formula. But if a zero had a real part equal to 1 then the error term in Gauss formula would be as large as the main term! So, it was a huge advance in 1896 when Hadamard and de la Vallée Poussin independently proved that $\zeta(1+it) \neq 0$ and concluded that

$$\pi(N) \sim \mathrm{li}(N)$$

a theorem which is known as the prime number theorem.

3. How many zeros are there

The non-trivial zeros $\rho = \beta + i\gamma$ all satisfy $0 < \beta < 1$ and Riemann's Hypothesis is that $\beta = 1/2$ always. It has been proven that each of the first 10 trillion zeros have real part equal to 1/2! This is very compelling evidence. Riemann gave us an accurate count of the zeros:

$$N(T) := \#\{\rho = \beta + i\gamma | 0 < \gamma \leq T\} = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Thus, on average there are $\frac{\log T}{2\pi}$ zeros with γ between T and $T+1$. So the zeros are becoming denser as we go up the critical strip $0 < sigma < 1$. Hardy was the first person to prove that infinitely many of the zeros are on the 1/2-line! If we let

$$N_0(T) := \#\{\rho = \beta + i\gamma | \beta = 1/2 \text{ and } 0 < \gamma \leq T\}$$

then Hardy and Littlewood proved that

$$N_0(T) \geq CT$$

for some constant $C > 0$. In 1946 Selberg showed that

$$N_0(T) \geq CN(T)$$

for a $C > 0$, i.e. that a positive proportion of zeros are on the critical line $\sigma = 1/2$. Levinson [L], in 1974, by a method different than Selberg's showed that $C = 1/3$ is admissible. Conrey [C] in 1987 proved that $C = 0.4088$ is allowable; the current record is due to Feng: $C = 0.412$.

There are also density results. Roughly speaking these say that almost all of the zeros of $\zeta(s)$ are very near to the critical line $\sigma = 1/2$. A classical theorem (see [T], section 9.19) is that

$$N(\sigma, T) := \#\{\rho = \beta + i\gamma : \beta \geq \sigma \text{ and } \gamma \leq T\} < CT^{(3/2-\sigma)} \log^5 T.$$

In particular, for any $\sigma > 1/2$ the number of zeros to the right of the σ line with imaginary parts at most T is bounded by T to a power smaller than 1 and so is $o(T)$.

4. Approaches to RH

The Riemann zeta-function $\zeta(s)$ is the last elementary function that we do not understand. There are numerous approaches to RH and hundreds of equivalent formulations in nearly every field of mathematics. But, at the moment there is no clear trail; RH stands alone as a singular monument. It is generally regarded as the most important unsolved problem in all of mathematics. It is pretty universally believed to be true. There are near counter examples and many many wrong proofs. These usually involve Dirichlet series that satisfy functional equations similar to that of $\zeta(s)$ but do not have an Euler product. The belief is that one must use these two ingredients appropriately to make any progress. In this sense, one might say that RH is a fundamental statement about a relationship between addition and multiplication that we still do not understand.

4.1. **Almost periodicity.** A tempting strategy is to try to prove that if $\zeta(s)$ has one zero off the line then it has infinitely many off the line. Bombieri has come closest to achieving this. One reason this is tempting is the analogy with almost periodic functions in the sense of Bohr. Dirichlet polynomials $\sum_{n=1}^N a_n n^{-s}$ are almost periodic. They have the property that a zero 'repeats itself' approximately. If you find one zero then moving vertically in the s-plane one finds $> CT$ zeros up to a height T. If the zeta-function were almost periodic in the half plane $\sigma > 1/2$ then one would expect that a zero $\beta + i\gamma$ with $\beta > 1/2$ off the line would lead to CT zeros off the line in some half plane $\sigma \geq \sigma_0 > 1/2$. But this would contradict the density theorem mentioned above. Here is a conjecture that might encapsulate this idea:

Conjecture 1. *Suppose that the Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

converges for $\sigma > 0$ and has a zero in the half-plane $\sigma > 1/2$. Then there is a number $C_F > 0$ such that $F(s)$ has $> C_F T$ zeros in $\sigma > 1/2$, $|t| \leq T$.

This seemingly innocent conjecture implies the Riemann Hypothesis for virtually any primitive L-function (except curiously possibly the Riemann zeta-function itself!). And it seems that the Euler product condition has already been used (in the density result above); i.e. the hard part is already done. Note that the "1/2" in the conjecture needs to be there as the example

$$\sum_{n=1}^{\infty} \frac{\mu(n)/n^{1/2}}{n^s}$$

demonstrates. Assuming RH this series converges for $\sigma > 0$ and its lone zero is at $s = 1/2$. This example is possibly at the boundary of what is possible.

5. A spectral interpretation

Hilbert and Pólya are reputed to have suggested that the zeros of $\zeta(s)$ should be interpreted as eigenvalues of an appropriate operator.

In the 1950s physicists predicted that excited nuclear particles emit energy at levels which are distributed like the eigenvalues of random matrices. This was verified experimentally in the 1970s and 1980s; Oriol Bohigas was the first to put this data together in a way that demonstrated this law.

In 1972 Hugh Montgomery, then a graduate student at Cambridge, delivered a lecture at a symposium on analytic number theory in St. Louis, outlining his work on the spacings between zeros of the Riemann zeta-function. This was the first time anyone had considered such a question. On his flight back to Cambridge he stopped over in Princeton to show his work to Selberg. At afternoon tea at the Institute for Advanced Study, Chowla insisted that Montgomery meet the famous physicist - and former number theorist - Freeman Dyson. When Montgomery explained to Dyson the kernel he had found that seemed to govern the spacings of pairs of zeros, Dyson immediately responded that it was the same kernel that governs pairs of eigenvalues of random matrices.

In 1980, Andrew Odlyzko and Schonhage invented an algorithm which allowed for the very speedy calculation of many values of $\zeta(s)$ at once. This led Odlyzko to do compile extensive statistics about the zeros at enormous heights - up to 10^{20} and higher. His famous graphs showed an incredible match between data for zeros of $\zeta(s)$ and for the proven statistical distributions for random matrices.

These amazing graphs reminded people of the Pólya and Hilbert philosophy and prompted Odlyzko to write to Pólya. Here is the text of Odlyzko's letter, dated Dec. 8, 1981.

> Dear Professor Pólya:
>
> I have heard on several occasions that you and Hilbert had independently conjectured that the zeros of the Riemann zeta function correspond to the eigenvalues of a self-adjoint hermitian operator. Could you provide me with any references? Could you also tell me when this conjecture was made, and what was your reasoning behind this conjecture at that time?
>
> The reason for my questions is that I am planning to write a survey paper on the distribution of zeros of the zeta function. In addition to some theoretical results, I have performed extensive computations of the zeros of the zeta function, comparing their distribution to that of random hermitian matrices, which have been studied very seriously by physicists. If a hermitian operator associated to the zeta function exists, then in some respects we might expect it to behave like a random hermitian operator, which in turn ught to resemble a random hermitian matrix. I have discovered that the distribution of zeros of the zeta function does indeed resemble the distribution of eigenvalues of random hermitian matrices of unitary type.
>
> Any information or comments you might care to provide would be greatly appreciated.
>
> Sincerely yours,
> Andrew Odlyzko

and Pólya's response, dated January 3, 1982.

> Dear Mr. Odlyzko,
>
> Many thanks for your letter of Dec. 8. I can only tell you what happened to me.
>
> I spent two years in Göttingen ending around the beginning of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann Hypothesis should be true?"
>
> This would be the case, I answered, if the non-trivial zeros of the ζ function were so connected with the physical problem that the Riemann Hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.
>
> I never published this remark, but somehow it became known and it is still remembered.
>
> With best regards.
>
> Yours sincerely,
> George, Pólya

Now we have the challenge of not only explaining why all of the zeros are on a straight line, but also why they are distributed on this line the way they are! The connections with Random Matrix theory first discovered by Montgomery and Dyson have received a great deal of support from seminal papers of Katz and Sarnak and Keating and Snaith. The last 15 years have seen an explosion of work around these ideas. In particular, it definitely seems like there should be a spectral interpretation of the zeros à la Hilbert and Pólya.

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