MOMENTS OF ZETA AND CORRELATIONS OF DIVISOR-SUMS: II

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ABSTRACT. This is part II of our examination of the second and fourth moments and shifted moments of the Riemann zeta-function on the critical line using long Dirichlet polynomials and divisor correlations.

1. Introduction

In part I, see [CK], we completed the analysis of the second moment of the Riemann zeta-function using the long Dirichlet polynomial method of Goldston and Gonek [GG] and we initiated the study of the fourth moment by this approach. In particular we calculated the contributions from the off-diagonal terms arising from coefficient correlations of the form $\sum_{n\leq X} d(n)d(n+h)$ and identified the terms that are missed in this approach. In this paper we show how to evaluate these new terms that were missing and in doing so we introduce a new technique that is a discrete analog of the circle method. This analysis gives a concrete introduction to how we will approach higher moments through this circle method approach. In a subsequent paper we will show how to obtain the "full-moment" conjecture for the 2kth moment of $\zeta(s)$ on the critical line, i.e. the full polynomial of degree k^2 which comprises the main term. The idea for this method originates in the work of Bogomolny and Keating; see [BK].

Thus, we will calculate the contribution of what we call the type II sums (after [BK]) which arise in the evaluation of

$$\int_0^\infty \psi\left(\frac{t}{T}\right) \sum_{m < X} \frac{\tau_{\alpha,\beta}(m)}{m^s} \sum_{m < X} \frac{\tau_{\gamma,\delta}(n)}{n^{1-s}} dt$$

where s = 1/2 + it and $\tau_{\alpha,\beta}(n) = \sum_{de=n} d^{-\alpha}e^{-\beta}$. (See [CK] for further notation and introduction.) To describe the type II sums we observe that integrating term-by-term we find

Date: November 6, 2018.

We gratefully acknowledge support under EPSRC Programme Grant EP/K034383/1 LMF: L-Functions and Modular Forms. Research of the first author was also supported by the American Institute of Mathematics and by a grant from the National Science Foundation. JPK is grateful for the following additional support: a grant from the Leverhulme Trust, a Royal Society Wolfson Research Merit Award, a Royal Society Leverhulme Senior Research Fellowship, and a grant from the Air Force Office of Scientific Research, Air Force Material Command, USAF (number FA8655-10-1-3088). He is also pleased to thank the American Institute of Mathematics for hospitality during a visit where this work started.

that the above is

$$T \sum_{m,n \le X} \frac{\tau_{\alpha,\beta}(m)\tau_{\gamma,\delta}(n)}{\sqrt{mn}} \hat{\psi}(\frac{T}{2\pi}\log(m/n)) = T\mathcal{D} + T\mathcal{O} + \mathcal{E}$$

where \mathcal{D} is the diagonal

$$\mathcal{D} = \hat{\psi}(0) \sum_{n \le X} \frac{\tau_{\alpha,\beta}(n) \tau_{\gamma,\delta}(n)}{n};$$

 \mathcal{O} is the off-diagonal

$$\mathcal{O} = \sum_{\substack{m \neq n \\ 0 < |m-n| < m/\tau}} \frac{\tau_{\alpha,\beta}(m)\tau_{\gamma,\delta}(n)}{\sqrt{mn}} \hat{\psi}(\frac{T}{2\pi}\log(m/n));$$

and $\mathcal{E} \ll T^{\epsilon}$ is an error term; here $\tau = T^{1-\epsilon}$ and the Fourier transform is defined by

$$\hat{\psi}(v) = \int_{\mathbb{R}} \psi(u)e(uv) \ du$$

where $e(x) = \exp(2\pi i x)$.

If we evaluate \mathcal{O} here in the traditional manner, eg. as in [GG], we would now solve the shifted convolution problem which consists of evaluating

$$\sum_{n \le x} \tau_{\alpha,\beta}(n) \tau_{\gamma,\delta}(n+h)$$

and summing by parts. This analysis was carried out in I. Here we use a new approach. We first make use of the fact that $\tau_{\alpha,\beta}$ and $\tau_{\gamma,\delta}$ are convolutions to write

$$\mathcal{O} = \sum_{\substack{m_1 m_2, n_1 n_2 \leq X \\ 0 < |m_1 m_2 - n_1 n_2| < m_1 m_2 / \tau}} \frac{m_1^{-\alpha} m_2^{-\beta} n_1^{-\gamma} n_2^{-\delta}}{m_1 m_2} \hat{\psi}(\frac{T}{2\pi} \log((n_1 n_2) / (m_1 m_2)).$$

Now we embark on a discrete analog of the circle method which basically consists of approximating a ratio, say m_1/n_1 by a rational number with a small denominator, say M/N, and then sum all of the terms with m_1/n_1 close to M/N.

To this end we introduce a parameter Q and subdivide the interval [0,1] into Farey intervals associated with the fractions M/N with $1 \le M \le N \le Q$ and (M,N) = 1 from the Farey sequence \mathcal{F}_Q . The Farey interval $\mathcal{M}_{M,N}$ determined by the fraction M/N is defined to be

$$\mathcal{M}_{M,N} = \left[\frac{M}{N} - \frac{M + M''}{N + N''}, \frac{M}{N} + \frac{M + M'}{N + N'} \right)$$

where $\frac{M''}{N''}$, $\frac{M}{N}$, $\frac{M'}{N'}$ are three consecutive terms in the Farey sequence \mathcal{F}_Q . Now given such an M and N we sum over the terms m_1 and n_1 for which $m_1/n_1 \in \mathcal{M}_{M,N}$; for such a pair we define

$$h_1 := m_1 N - n_1 M.$$

The possible range of h_1 may be computed by

$$|h_1| = \left| \frac{m_1}{n_1} - \frac{M}{N} \right| n_1 N \le \left(\frac{M}{N} - \frac{M + M''}{N + N''} \right) n_1 N = \frac{n_1}{N + N''} \approx \frac{n_1}{Q}$$

since adjacent denominators satisfy Q < N + N'' < 2Q. In general, the rapid decay of $\hat{\psi}$ governs the range of h_1 and h_2 defined below.

Also, we note that if Q is not too large then $m_1/n_1 \in \mathcal{M}_{M,N}$ implies that $n_2/m_2 \in \mathcal{M}_{M,N}$ as well. This is because the distance from m_1/n_1 to n_2/m_2 is

$$\left| \frac{m_1}{n_1} - \frac{n_2}{m_2} \right| = \frac{|m_1 m_2 - n_1 n_2|}{n_1 m_2} \le \frac{m_1 m_2}{\tau n_1 m_2} \le \frac{1}{\tau}.$$

On the other hand

$$\left|\frac{M}{N} - \frac{M'}{N'}\right| \gg \frac{1}{Q^2}$$

so if $Q^2 = o(\tau)$ then our assertion follows.

Now we define

$$h_2 := m_2 M - n_2 N$$
.

We have

$$m_1 m_2 M N - n_1 n_2 M N = h_1 m_2 M + h_2 m_1 N - h_1 h_2$$

so that

$$\frac{m_1 m_2 - n_1 n_2}{m_1 m_2} = \frac{h_1}{m_1 N} + \frac{h_2}{m_2 M} - \frac{h_1 h_2}{m_1 m_2 M N}$$

and

$$\log \frac{n_1 n_2}{m_1 m_2} = \frac{h_1}{m_1 N} + \frac{h_2}{m_2 M} + O\left(\frac{h_1 h_2}{m_1 m_2 M N}\right).$$

The error term is negligible so we have now arranged the sum as

$$\sum_{M \leq N \leq Q \atop (M,N)=1} \sum_{h_1,h_2} \sum_{m_1 m_2 \leq X \atop (*_1),(*_2)} \frac{m_1^{-\alpha} m_2^{-\beta} n_1^{-\gamma} n_2^{-\delta}}{m_1 m_2} \hat{\psi} \left(\frac{Th_1}{2\pi m_1 N} + \frac{Th_2}{2\pi m_2 M} \right)$$

where

$$(*_1): m_1N - n_1M = h_1$$
 and $(*_2): m_2M - n_2N = h_2$

Note that for a given m_1, n_1 and h_1 the condition $(*_1)$ implies that $m_1/n_1 \in \mathcal{M}_{M,N}$ so we don't need to write that condition.

2. Smoothing the sums over M and N

We introduce another smooth weight function $\phi(y)$, which is an approximation to the characteristic function $\chi_{(0,1]}(y)$ to help with the summation over M and N. We will then have sums of the form

$$S_Q(\xi, \eta) := \sum_{\substack{1 \le M \le N \\ (M, N) = 1}} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) M^{-1-\xi} N^{-1-\eta}$$

for a finite set of choices of ξ and η which are of the form

$$\epsilon_1 \alpha + \epsilon_2 \beta + \epsilon_3 \gamma + \epsilon_4 \delta$$

where the $\epsilon_i \in \{1, 0, 1\}$. We require that

$$\phi(y) = \frac{1}{2\pi i} \int_{(1)} \tilde{\phi}(s) y^{-s} ds$$

where $\tilde{\phi}(s)$ has the properties that

$$\operatorname{Res}_{s=0} \tilde{\phi}(s) = 1 \quad \text{and} \quad \tilde{\phi}(\xi) = 0$$

for all of the eligible values of ξ that arise, and that $\tilde{\phi}(s)$ is analytic in $\Re s \geq -1/2$ and has rapid decay vertically in this region. In practice $S_Q(\xi, \eta)$ will be combined with $S_Q(\eta, \xi)$ to obtain

$$S_Q(\xi,\eta) + S_Q(\eta,\xi) = \phi \left(\frac{1}{Q}\right)^2 + \sum_{(M,N)=1} \phi \left(\frac{M}{Q}\right) \phi \left(\frac{N}{Q}\right) M^{-1-\xi} N^{-1-\eta}$$

The second term is

$$\begin{split} & \sum_{d} \frac{\mu(d)}{d^{2+\xi+\eta}} \sum_{M} \phi\left(\frac{Md}{Q}\right) M^{-1-\xi} \sum_{N} \phi\left(\frac{Nd}{Q}\right) M^{-1-\eta} \\ & = \sum_{d} \frac{\mu(d)}{d^{2+\xi+\eta}} \left(\frac{1}{2\pi i} \int_{(1)} \tilde{\phi}(w) \zeta(w+1+\xi) \left(\frac{Q}{d}\right)^{w} \ dw\right) \left(\frac{1}{2\pi i} \int_{(1)} \tilde{\phi}(z) \zeta(z+1+\eta) \left(\frac{Q}{d}\right)^{z} \ dz\right). \end{split}$$

The first integral is $= \zeta(1+\xi)+O((Q/d)^{-1/3})$ as can be seen by moving the path of integration to the left to $\Re w = -1/3$ and accounting for the residue at the pole w=0; note that since $\tilde{\phi}(-\xi)=0$, there is no pole at $w=-\xi$. Thus, altogether we have

(1)
$$S_Q(\xi,\eta) + S_Q(\eta,\xi) = \phi \left(\frac{1}{Q}\right)^2 + \frac{\zeta(1+\xi)\zeta(1+\eta)}{\zeta(2+\xi+\eta)} + O(Q^{-1/3}).$$

3. The case of
$$h_2 = 0$$

We remark first of all that the terms with $h_1 = h_2 = 0$ are precisely the diagonal terms. Now we consider what happens if $h_2 = 0$ and $h_1 \neq 0$. We call this a "semi-diagonal" term after [BK].

If $h_2 = 0$ then $m_2 M = n_2 N$. Since (M, N) = 1 it follows that $m_2 = N\ell$ and $n_2 = M\ell$ for some ℓ . Thus we have

$$\sum_{\substack{M \leq N \\ (M,N)=1}} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) \sum_{h_1} \sum_{\substack{m_1,n_1,\ell \\ (*_1) \\ n_1 > |h_1|Q}} \frac{m_1^{-\alpha} (N\ell)^{-\beta} n_1^{-\gamma} (M\ell)^{-\delta}}{m_1 m_2} \hat{\psi}\left(\frac{Th_1}{2\pi m_1 N}\right)$$

where

$$(*_1): m_1N - n_1M = h_1.$$

We replace m_1 by a smooth variable u_1 and n_1 by m_1N/M . We have $u_1\ell N = m_1m_2 \leq X$ and so our sum is

$$\sum_{\substack{M \leq N \\ (M,N)=1}} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) M^{-\delta+\gamma-1} N^{-\beta-\gamma-1} \sum_{h_1} \sum_{\ell} \ell^{-1-\beta-\delta} \int_{u_1 \ell \leq \frac{X}{N}} u_1^{-1-\alpha-\gamma} \hat{\psi}\left(\frac{Th_1}{2\pi u_1 N}\right) du_1$$

We save the term with $h_1 = 0$ for later and we group the terms with h_1 and $-h_1$ together and use $\hat{\psi}(-v) = \hat{\psi}(v)$. We make the substitution $v_1 = \frac{Th_1}{2\pi u_1 N}$ in the integral and switch the integral over v_1 with the sum over h_1 and ℓ . Then (with $h_1 > 0$) we have that

$$\frac{\ell NTh_1}{2\pi v_1 N} = u_1 \ell N \le X$$

implies that

$$\ell h_1 \le \frac{2\pi X v_1}{T}$$

Thus we have

$$\sum_{\substack{M \leq N \\ (M|N)=1}} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) M^{-\delta+\gamma-1} N^{\alpha-\beta-1} \int_0^\infty v_1^{-1+\alpha+\gamma} (2\Re \hat{\psi}(v_1)) \sum_{h_1 \ell \leq \frac{2\pi X v_1}{T}} h_1^{-\alpha-\gamma} \ell^{-1-\beta-\delta} \ dv_1.$$

The sum over h_1 and ℓ is

$$\frac{1}{2\pi i} \int_{(2)} \zeta(s+1+\beta+\delta) \zeta(s+\alpha+\gamma) \left(\frac{2\pi v_1 X}{T}\right)^s \frac{ds}{s}$$

Together with the integral over v_1 this is

$$\int_0^\infty v_1^{-1+\alpha+\gamma} \hat{\psi}(v_1) \frac{2}{2\pi i} \int_{(2)} \zeta(s+1+\beta+\delta) \zeta(s+\alpha+\gamma) \left(\frac{2\pi v_1 X}{T}\right)^s \frac{ds}{s} dv_1.$$

Now, as we've seen before, if $\Re s > 0$ then

$$\int_0^\infty (2\Re \hat{\psi}(v)) v^s \ dv = \chi(1-s) \int_0^\infty \psi(t) t^{-s} \ dt.$$

Thus, the above is

$$\int_0^\infty t^{-1-\alpha-\gamma} \psi(t) \frac{2}{2\pi i} \int_{(2)} \zeta(s+1+\beta+\delta) \zeta(1-s-\alpha-\gamma) \left(\frac{2\pi X}{tT}\right)^s \frac{ds}{s} dt$$

We move the s-path left to $\Re s = -1/2$, thus crossing the poles at s = 0, $s = -\alpha - \gamma$ and $s = -\beta - \delta$. Thus the above is

$$\int_{0}^{\infty} t^{-1-\alpha-\gamma} \psi(t) \left(\zeta(1+\beta+\delta) \zeta(1-\alpha-\gamma) - \frac{\zeta(1-\alpha+\beta-\gamma+\delta) \left(\frac{2\pi X}{tT}\right)^{-\beta-\delta}}{\beta+\delta} + \frac{\zeta(1-\alpha+\beta-\gamma+\delta) \left(\frac{2\pi X}{tT}\right)^{-\alpha-\gamma}}{\alpha+\gamma} \right) dt$$

and altogether we have

$$\sum_{\substack{M \leq N \\ (M,N)=1}} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) M^{-\delta+\gamma-1} N^{\alpha-\beta-1}$$

$$\times \int_{0}^{\infty} t^{-1-\alpha-\gamma} \psi(t) \left(\zeta(1+\beta+\delta)\zeta(1-\alpha-\gamma) - \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\left(\frac{2\pi X}{tT}\right)^{-\beta-\delta}}{\beta+\delta} + \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\left(\frac{2\pi X}{tT}\right)^{-\alpha-\gamma}}{\alpha+\gamma}\right) dt$$

All of the above is predicated on $m_1/n_1 < 1$. The contribution from the terms where $n_1 < m_1$ will be exactly as above but with the quadruple $(\alpha, \beta, \gamma, \delta)$ replaced with $(\gamma, \delta, \alpha, \beta)$. In particular, $\alpha + \gamma$ will be replaced by $\beta + \gamma$ prior to summing over M and N. This will give another term

$$\sum_{\substack{M \leq N \\ (M,N)=1}} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) M^{\alpha-\beta-1} N^{-\delta+\gamma-1}$$

$$\times \int_{0}^{\infty} t^{-1-\beta-\delta} \psi(t) \left(\zeta(1+\beta+\delta)\zeta(1-\alpha-\gamma) - \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\left(\frac{2\pi X}{tT}\right)^{-\beta-\delta}}{\beta+\delta} + \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\left(\frac{2\pi X}{tT}\right)^{-\alpha-\gamma}}{\alpha+\gamma}\right) dt$$

Now we consider what happens when $h_1 = 0$ and $h_2 \neq 0$. These terms will contribute the "complements" to the above two expressions so that we will be in the situation described in (1) and so we can execute the sums over M and N as described there, replacing the sums

over M and N by ratios of zeta functions with small error terms. Thus, we obtain

$$\int_{0}^{\infty} t^{-1-\alpha-\gamma} \psi(t) \left(\frac{\zeta(1+\beta+\delta)\zeta(1-\alpha-\gamma)\zeta(1-\gamma+\delta)\zeta(1-\alpha+\beta)}{\zeta(2-\alpha+\beta-\gamma+\delta)} - \left(\frac{2\pi X}{tT} \right)^{-\beta-\delta} \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\zeta(1-\alpha+\beta)\zeta(1-\gamma+\delta)}{(\beta+\delta)\zeta(2-\alpha+\beta-\gamma+\delta)} + \left(\frac{2\pi X}{tT} \right)^{-\alpha-\gamma} \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\zeta(1-\alpha+\beta)\zeta(1-\gamma+\delta)}{(\alpha+\gamma)\zeta(2-\alpha+\beta-\gamma+\delta)} \right) dt$$

and the complimentary term with $\alpha + \gamma$ replaced by $\beta + \delta$ and vice-versa.

This is identical with one of the one-swap terms identified by descending as previously described.

There are further semi-diagonal terms. If we do the exact same analysis as throughout this entire section but now focusing on the ratio m_1/n_2 instead of m_1/n_1 then the effect will be to switch the roles of γ and δ in the expression above. Then we end up with two more terms and a total of four terms. These terms are identical with the four terms obtained by the "descent" method described in section 8 of [CK].

A question of whether we have over-counted some terms may arise. But the "duplicate" terms for which $m_1/n_1 \in \mathcal{M}_{M,N}$ and simultaneously $m_1/n_2 \in \mathcal{M}_{M',N'}$ with $N \leq Q$ and $N' \leq Q$ contribute an insignificant amount to the total and so may be regarded as part of the error term.

4. The case of $h_1h_2 \neq 0$

Now we consider

$$\sum_{M \leq N \atop (M,N)=1} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) \sum_{h_1h_2 \neq 0} \sum_{\substack{m_1m_2 \leq X \\ (*_1),(*_2)}} \frac{m_1^{-\alpha} m_2^{-\beta} n_1^{-\gamma} n_2^{-\delta}}{m_1 m_2} \hat{\psi}\left(\frac{Th_1}{2\pi m_1 N} + \frac{Th_2}{2\pi m_2 M}\right).$$

In this case we have a bound for h_2 similar to that for h_1 :

$$|h_2| \ll \frac{m_2}{O} \ll \frac{n_2 M}{ON}.$$

In particular, we have

$$|h_1 h_2| \ll \frac{n_1 n_2 M}{Q^2 N} \ll \frac{X}{Q^2}.$$

Now we replace the sums over m_1, m_2, n_1, n_2 subject to $(*_1)$ and $(*_2)$ by their averages. As before, we replace m_1 by u_1 and now we replace m_2 by u_2 . We replace n_1 and n_2 by u_1N/M and u_2M/N respectively. We then have

$$\sum_{\substack{M \leq N \\ (M,N)=1}} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) M^{\gamma-\delta-1} N^{\delta-\gamma-1} \sum_{h_1 h_2 \neq 0} \int_{u_1 u_2 \leq X} u_1^{-\alpha-\gamma-1} u_2^{-\beta-\delta-1} \hat{\psi}\left(\frac{Th_1}{2\pi u_1 N} + \frac{Th_2}{2\pi u_2 M}\right) du_1 du_2.$$

Now there are four cases to consider according to the four sign choices of h_1 and h_2 . We make the substitutions

$$v_1 = \frac{T|h_1|}{2\pi u_1 N}$$
 and $v_2 = \frac{T|h_2|}{2\pi u_2 M}$

and move the sums over h_1 and h_2 to the inside. The condition $u_1u_2 \leq X$ implies that

$$\frac{T^2|h_1h_2|}{4\pi^2v_1v_2MN} = u_1u_2 \le X$$

or

$$|h_1 h_2| \le \frac{4\pi^2 X M N v_1 v_2}{T^2}.$$

We get

$$\begin{split} & \big(\frac{T}{2\pi}\big)^{-\alpha-\beta-\gamma-\delta} \sum_{M \leq N \atop (M,N)=1} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) M^{\gamma+\beta-1} N^{\delta+\alpha-1} \iint_{v_1,v_2} v_1^{\alpha+\gamma-1} v_2^{\beta+\delta-1} \sum_{0 < |h_1h_2| \leq \frac{4\pi^2 X M N v_1 v_2}{T^2}} \\ & \times h_1^{-\alpha-\gamma} h_2^{-\beta-\delta} \left(\hat{\psi}(v_1+v_2) + \hat{\psi}(v_1-v_2) + \hat{\psi}(-v_1+v_2) + \hat{\psi}(-v_1-v_2)\right) \ dv_1 \ dv_2. \end{split}$$

Using

$$\hat{\psi}(v_1 + v_2) = \int_0^\infty \psi(t)e(t(v_1 + v_2)) dt$$

we see that

$$\hat{\psi}(v_1+v_2)+\hat{\psi}(v_1-v_2)+\hat{\psi}(-v_1+v_2)+\hat{\psi}(-v_1-v_2)=\int_0^\infty \psi(t)\big(e(tv_1)+e(-tv_1)\big)\big(e(tv_2)+e(-tv_2)\big)\,dt;$$

Also

$$\sum_{h_1 h_2 \le \frac{4\pi^2 X M N v_1 v_2}{T^2}} h_1^{-\alpha - \gamma} h_2^{-\beta - \delta} = \frac{1}{2\pi i} \int_{(2)} \zeta(s + \alpha + \gamma) \zeta(s + \beta + \delta) \frac{\left(\frac{4\pi^2 X M N v_1 v_2}{T^2}\right)^s}{s} ds$$

and

$$\int_0^\infty v_1^{s+\alpha+\gamma-1}(e(tv_1) + e(-tv_1)) \ dv_1 = t^{-s-\alpha-\gamma}\chi(1-s-\alpha-\gamma),$$

and similarly for the integral over v_2 . Incorporating these, we have simplified things to

$$\left(\frac{T}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} \sum_{\substack{M \leq N \leq Q \\ (M,N)=1}} \phi\left(\frac{M}{Q}\right) \phi\left(\frac{N}{Q}\right) M^{\gamma+\beta-1} N^{\delta+\alpha-1}$$

$$\times \int_{0}^{\infty} \psi(t) t^{-\alpha-\beta-\gamma-\delta} \frac{1}{2\pi i} \int_{(2)} \zeta(1-s-\alpha-\gamma) \zeta(1-s-\beta-\delta) \frac{\left(\frac{4\pi^{2}XMN}{t^{2}T^{2}}\right)^{s}}{s} ds dt.$$

The above expression is unchanged if (α, γ) is interchanged with (β, δ) . So the result of summing terms for which $n_1/m_1 \leq 1$ rather than $m_1/n_1 \leq 1$ allows for summing over M and N as in Section 9; we obtain

$$\left(\frac{T}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} \frac{1}{(2\pi i)^2} \int_{z,w} \tilde{\phi}(z) \tilde{\phi}(w) \frac{\zeta(1-\beta-\gamma-s+z)\zeta(1-\alpha-\delta-s+w)}{\zeta(2-\alpha-\beta-\gamma-\delta-2s+z+w)}$$

$$\times \int_0^\infty \psi(t) t^{-\alpha-\beta-\gamma-\delta} \frac{1}{2\pi i} \int_{(2)} \zeta(1-s-\alpha-\gamma)\zeta(1-s-\beta-\delta) \frac{(\frac{X}{t^2T^2})^s Q^{z+w}}{s} \ ds \ dw \ dz \ dt.$$

Moving the s-path to the right to ∞ and the z and w paths to the left to -1/4, say we obtain

$$\int_{0}^{\infty} \psi(t) \left(\left(\frac{tT}{2\pi} \right)^{-\alpha-\beta-\gamma-\delta} \frac{\zeta(1-\alpha-\gamma)\zeta(1-\beta-\delta)\zeta(1-\beta-\gamma)\zeta(1-\alpha-\delta)}{\zeta(2-\alpha-\beta-\gamma-\delta)} \right.$$

$$+ X^{-\alpha-\gamma} \left(\frac{tT}{2\pi} \right)^{\alpha-\beta+\gamma-\delta} \frac{\zeta(1+\alpha-\beta+\gamma-\delta)\zeta(1+\alpha-\beta)\zeta(1+\gamma-\delta)}{(\alpha+\gamma)\zeta(2+\alpha-\beta+\gamma-\delta)}$$

$$+ X^{-\beta-\delta} \left(\frac{tT}{2\pi} \right)^{-\alpha+\beta-\gamma+\delta} \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\zeta(1-\alpha+\beta)\zeta(1-\gamma+\delta)}{(\beta+\delta)\zeta(2-\alpha+\beta-\gamma+\delta)}$$

$$+ X^{-\alpha-\delta} \left(\frac{tT}{2\pi} \right)^{\alpha-\beta-\gamma+\delta} \frac{\zeta(1-\gamma+\delta)\zeta(1+\alpha-\beta-\gamma+\delta)\zeta(1+\alpha-\beta)}{(\alpha+\delta)\zeta(2+\alpha-\beta-\gamma+\delta)}$$

$$+ X^{-\beta-\gamma} \left(\frac{tT}{2\pi} \right)^{-\alpha+\beta+\gamma-\delta} \frac{\zeta(1-\alpha+\beta)\zeta(1-\alpha+\beta+\gamma-\delta)\zeta(1+\gamma-\delta)}{(\beta+\gamma)\zeta(2-\alpha+\beta+\gamma-\delta)} \right) dt$$

with an error term of $O(Q^{-1/4})$. This expression is exactly what we were hoping for; it is identical to the "two-swap" terms found in the descent approach of section 9 of [CK].

5. Conclusion

We have shown how to reproduce the complete conjecture for the shifted fourth moment of ζ by analyzing the mean square of long Dirichlet polynomials whose coefficients are convolutions of two smooth arithmetic functions. In the next paper we will carry this analysis out for coefficients which are convolutions of an arbitrary number of convolutions and use this to reproduce the full conjecture for the 2kth moment of ζ for an arbitrary k.

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