Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial

By *R. Balasubramanian* at Bombay, *J. B. Conrey* at Stillwater and *D. R. Heath-Brown* at Oxford

§ 1. Statement of results

We prove an asymptotic formula for

$$I=\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left|A\left(\frac{1}{2}+it\right)\right|^2 dt$$

where ζ is Riemann's zeta-function and

$$A(s)=\sum_{m\leqq M} a(m)\,m^{-s}$$

is a Dirichlet polynomial. Our work is motivated by that of Iwaniec [5] who considers upper bounds for I. Our result is

Theorem 1. *Let I be defined as above and suppose that $a(m)\ll_\varepsilon m^\varepsilon$ for any $\varepsilon>0$ and $\log M\ll\log T$. Then*

$$I=T\sum_{h,k\leqq M}\frac{a(h)\,\overline{a(k)}}{h\quad k}(h,k)\left(\log\frac{T(h,k)^2}{2\pi\,hk}+2\gamma-1\right)+\mathscr{E}$$

where $\mathscr{E}=\mathscr{E}_1+\mathscr{E}_2$ with $\mathscr{E}_1\ll_B T(\log T)^{-B}$ for any $B>0$ and, in general,

(A) $$\mathscr{E}_2\ll_\varepsilon M^2T^\varepsilon$$

for any $\varepsilon>0$. If Hooley's Conjecture R^* (see Hooley [4]) is true then for any $\varepsilon>0$,

(B) $$\mathscr{E}_2\ll_\varepsilon M^{\frac{7}{4}}T^\varepsilon.$$

Further, if $M\leqq T^{\frac{3}{5}}$ and $a(m)$ has the special form

$$a(m)=\mu(m)\,F(m)$$

where $F(x)\ll 1$ and $F'(x)\ll\frac{1}{x}$ for $1\leqq x\leqq M$ (abbreviated by $F\in\mathscr{F}_M$) then

(C) $$\mathscr{E}_2\ll_\varepsilon M^{\frac{17}{12}}T^{\frac{1}{4}+\varepsilon}$$

for any $\varepsilon>0$.

Note that we have asymptotic formulae in general if $M \ll T^{\frac{1}{2}-\delta}$ for some $\delta > 0$ or, assuming R^*, if $M \ll T^{\frac{4}{7}-\delta}$ for some $\delta > 0$. If $a(m) = \mu(m)\, F(m)$ where $F \in \mathscr{F}_M$, then we have an asymptotic formula if $M \ll T^{\frac{9}{17}-\delta}$ for some $\delta > 0$.

We can also prove that if $a(m) = 1$ for $1 \leqq m \leqq M$, then Theorem 1 holds with

(D) $$\mathscr{E}_1 + \mathscr{E}_2 \ll_{\varepsilon, B} T(\log T)^{-B} + MT^{\varepsilon}$$

which gives an asymptotic formula if $M \ll T^{1-\delta}$ for some $\delta > 0$. Moreover, we can show that, in this case, the asymptotic formula fails if $M = T^{1+\delta}$ for any $\delta > 0$. We conjecture that for general $a(m) \ll_{\varepsilon} m^{\varepsilon}$ the formula

$$I = T \sum_{h,k \leqq M} \frac{a(h)\,\overline{a(k)}}{hk}(h,k)\left(\log \frac{T(h,k)^2}{2\pi\, hk} + (2\gamma - 1)\right) + o(T)$$

holds provided that $M = T^{1-\delta}$ for some $\delta > 0$.

As an application of Theorem 1 we have

Corollary. *As* $T \to \infty$,

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \left|\sum_{m \leqq M} \mu(m)\left(1 - \frac{\log m}{\log M}\right) m^{-\frac{1}{2} - it}\right|^2 dt \sim T\left(1 + \frac{\log T}{\log M}\right)$$

uniformly for $M \leqq T^{\frac{9}{17}-\varepsilon}$ *for any* $\varepsilon > 0$. *If* $\rho = \beta + i\gamma$ *denotes a zero of* $\zeta(s)$, *then*

$$\sum_{\substack{0 < \gamma < T \\ \beta > \frac{1}{2}}} \left(\beta - \frac{1}{2}\right) \leqq (0.0845 + o(1))\, T$$

as $T \to \infty$.

This gives an explicit estimate for a well-known result of Selberg [6], Section 7.

It is likely that a further application of the method could be made to improve the constant α in the relation

$$N_0(T) \geqq (\alpha + o(1))\, N(T)$$

where, as usual, $N(T)$ is the number of zeros of $\zeta(s)$ with $0 < \sigma < 1$ and $0 < t < T$ and $N_0(T)$ is the number of zeros of $\zeta\left(\frac{1}{2} + it\right)$ with $0 < t < T$. The best known result is $\alpha \geqq 0.3658$ due to Conrey [1] which corresponds to Theorem 1 with the error term in (A). If the appropriate mean-value theorem were proved with the estimate $\frac{17}{9}$ in (C), then one could show $\alpha \geqq 0.38$. Thus this result should be accessible unconditionally. If R^* is assumed (see estimate (B)) then $\alpha \geqq 0.4077$ could be obtained. If our conjecture is assumed (see estimate (D)) then $\alpha \geqq 0.55$ follows.

The estimate (C) depends on the Riemann Hypothesis for curves over finite fields (via estimates for Kloosterman sums) and so the improvement in the value of α would depend on the non-classical Riemann Hypothesis!

Heath-Brown first considered these mean squares in the case where the polynomial is

$$\sum_{m \leqq M} \mu(m)\, m^{-s}$$

and obtained an upper bound of $T^{1+\varepsilon}$ with $M = T^{\frac{9}{17}}$ (unpublished). $\left(\text{The } \text{“}\frac{9}{17}\text{”} \text{ has been incorrectly stated as } \text{“}\frac{8}{15}\text{”} \text{ in various places.}\right)$

§ 2. Some lemmas

To prove our formula we need some lemmas.

Lemma 1 (Estermann [2]). *Let*

$$S\left(x, \frac{h}{k}\right) = \sum_{n=1}^{\infty} d(n)\, e\left(\frac{nh}{k}\right) e^{2\pi i n x}$$

for $\operatorname{Im} x > 0$ *and* $k \geqq 1$ *and let*

$$D\left(s, \frac{h}{k}\right) = \sum_{n=1}^{\infty} d(n)\, e\left(\frac{nh}{k}\right) n^{-s}$$

for $\operatorname{Re} s > 1$. *Let* $z = z(x) = -2\pi i x$. *Then*

$$S\left(x, \frac{h}{k}\right) = \frac{1}{zk^*}(\gamma - \log z - 2\log k^*) + D\left(0, \frac{h^*}{k^*}\right)$$
$$- i \int_{(c)} (2\pi)^{-2s} \frac{\Gamma(s)\, k^{*2s-1}}{\sin \pi s} \left(D\left(s, \frac{\overline{h^*}}{k^*}\right) + (\cos \pi s)\, D\left(s, -\frac{\overline{h^*}}{k^*}\right)\right) z^{s-1}\, ds$$

where $h^* = \frac{h}{(h,k)}$, $k^* = \frac{k}{(h,k)}$, $\overline{h^*} h^* \equiv 1 \pmod{k^*}$, *and* $1 < c < 2$. *Moreover*

$$\left| D\left(0, \frac{h^*}{k^*}\right) \right| \leqq k^* (\log 2k^*)^2.$$

Next we evaluate an integral.

Lemma 2. *Suppose that* $1 < c < 2$ *and, as usual,*

$$\chi(1-s) = 2(2\pi)^{-s}\, \Gamma(s) \cos \frac{\pi s}{2}.$$

Let

$$J(y) = J(s_0, \Delta, y) = \frac{1}{i\Delta\sqrt{\pi}} \int_{(c)} e^{(s-s_0)^2 \Delta^{-2}} \chi(1-s)\, y^{-s}\, ds.$$

Then

$$J(y) = \int_0^{\infty} v^{s_0} \exp\left(\frac{-\Delta^2 \log^2 v}{4}\right) (e^{-2\pi i y v} + e^{2\pi i y v}) \frac{dv}{v}$$

for any $y \neq 0$, s_0, *and* $\Delta > 0$.

Proof. We write $J=(i\Delta\sqrt{\pi})^{-1}(J_1+J_2)$ where

$$J_1=\int_{(c)} e^{(s-s_0)^2\Delta^{-2}}\Gamma(s)\,(2\pi e^{\frac{\pi i}{2}}y)^{-s}\,ds$$

and

$$J_2=\int_{(c)} e^{(s-s_0)^2\Delta^{-2}}\Gamma(s)\,(2\pi e^{-\frac{\pi i}{2}}y)^{-s}\,ds.$$

Now we use the fact from the theory of Mellin transforms (see Titchmarsh [7], for example) that

$$\frac{1}{2\pi i}\int_{(c)} F(s)\,G(s)\,x^{-s}ds=\int_0^\infty f\left(\frac{1}{v}\right)g(xv)\frac{dv}{v}$$

where

$$f(v)=\frac{1}{2\pi i}\int_{(c)} F(s)\,v^{-s}ds$$

and

$$g(v)=\frac{1}{2\pi i}\int_{(s)} G(s)\,v^{-s}ds.$$

We apply this formula to J_1 with

$$F_1(s)=e^{(s-s_0)^2\Delta^{-2}},$$

$$G_1(s)=\Gamma(s)\,(2\pi e^{\frac{\pi i}{2}}e^{-i\varepsilon})^{-s},$$

and $x_1=ye^{i\varepsilon}$ where $\varepsilon>0$ and to J_2 with

$$F_2(s)=e^{(s-s_0)^2\Delta^{-2}},$$

$$G_2(s)=\Gamma(s)\,(2\pi e^{-\frac{\pi i}{2}}e^{i\varepsilon})^{-s},$$

and $x_2=ye^{-i\varepsilon}$ where we assume, for now, that y is real and positive. It is easy to see that if $|\arg x|<\pi$, then

$$\int_{(c)} e^{(s-s_0)^2\Delta^{-2}}x^{-s}ds=\exp\left(-s_0\log x-\frac{\Delta^2\log^2 x}{4}\right)\int_{(c)}\exp\frac{\left(s-s_0-\frac{1}{2}\Delta^2\log x\right)^2}{\Delta^2}\,ds$$
$$=i\Delta\sqrt{\pi}\,x^{-s_0}\exp\left(-\frac{\Delta^2\log^2 x}{4}\right),$$

and it is well-known that if $|\arg x|<\frac{\pi}{2}$, then

$$\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\,x^{-s}\,ds=e^{-x}.$$

Thus,

$$f_1(v)=f_2(v)=\frac{i\Delta\sqrt{\pi}}{2\pi i}v^{-s_0}\exp\left(-\frac{\Delta^2\log^2 v}{4}\right),$$

$$g_1(v)=\exp(-2\pi i v e^{-i\varepsilon}),$$

and

$$g_2(v)=\exp(2\pi i v e^{i\varepsilon}).$$

This proves the lemma for real y. It follows for complex y by analytic continuation.

Let L_δ denote the straight line path from 0 to $e^{i\delta}\infty$.

Lemma 3. *Let* $s_0=\frac{1}{2}+iu$, $0\leqq\delta\leqq\frac{\pi}{2}$, *and* $0<\Delta<u$. *Then*

$$\int_{L_\delta} v^{s_0}\exp\left(-\frac{\Delta^2\log^2 v}{4}\right)\frac{dv}{v(v-1)}-\int_{L_{-\delta}} v^{s_0}\exp\left(-\frac{\Delta^2\log^2 v}{4}\right)\frac{dv}{v(v-1)}=-2\pi i,$$

and if

$$K=\int_{L_\delta} v^{s_0}\exp\left(-\frac{\Delta^2\log^2 v}{4}\right)\frac{\log(-i(v-1))\,dv}{v(v-1)}-\int_{L_{-\delta}} v^{s_0}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)\frac{\log(i(v-1))\,dv}{v(v-1)},$$

$\left(\textit{with } \log(\mp i(v-1))=\pm\frac{i\pi}{2} \textit{ at } v=0\right)$ *then*

$$K=2\pi i\left(\log u+\sum_{n=0}^{N-1}c_n\left(\frac{\Delta}{u}\right)^{2n}\right)+O\left(\left(\frac{\Delta}{u}\right)^{2N}\right),$$

where c_0 *is a certain constant,* N *is any positive integer and* $c_n=c_n(\Delta)\ll_n 1$ *uniformly for* $\Delta\geqq 1$. *Later it will follow that* $c_0=\gamma$.

Proof. The first assertion follows immediately from the residue theorem since the paths of the two integrals join together and the new path they form can be moved arbitrarily far to the right (across the simple pole at $v=1$).

To prove the second assertion we change the paths of integration to the positive real axis except for an indentation around $v=1$ with $\operatorname{Im} v>0$ in the first integral and $\operatorname{Im} v<0$ in the second integral. Then we make the change of variable $v=e^x$ in the two integrals. Thus,

$$K=\int\exp\left(iux-\frac{\Delta^2x^2}{4}\right)\frac{\log(-i(e^x-1))}{2\sinh\frac{x}{2}}\,dx-\int\exp\left(iux-\frac{\Delta^2x^2}{4}\right)\frac{\log(i(e^x-1))}{2\sinh\frac{x}{2}}\,dx$$

where the path in the first integral is from $-\infty$ to $-\varepsilon$ on the real axis, then along a semicircle C_ε^+ in the upper half plane to $+\varepsilon$, and then along the real axis to $+\infty$.

The path in the second integral is the same except that the semicircle C_ε^- is in the lower half-plane. Let C_ε be the union of C_ε^- and the reversal of C_ε^+. Then

$$K=-\int_{C_\varepsilon} \exp\left(iux-\frac{\Delta^2x^2}{4}\right)\frac{\log(e^x-1)}{2\sinh\frac{x}{2}}\,dx-2\pi i\int_\varepsilon^\infty \frac{(\cos ux)\exp\left(-\frac{\Delta^2x^2}{2}\right)}{2\sinh\frac{x}{2}}\,dx-\frac{\pi i}{2}R_\varepsilon$$

where the log in the first integral has its principal value and

$$R_\varepsilon=\int_{C_\varepsilon^+\cup C_\varepsilon^-} \exp\left(iux-\frac{\Delta^2x^2}{4}\right)\left(2\sinh\frac{x}{2}\right)^{-1}dx.$$

Thus,

$$K=-\int_{C_\varepsilon}\frac{\log x}{x}\,dx-R_\varepsilon'-2\pi i\int_\varepsilon^\infty\frac{\cos ux}{x}\,dx-2\pi iP_\varepsilon-\frac{\pi i}{2}R_\varepsilon$$

where

$$R_\varepsilon'=\int_{C_\varepsilon}\left(\frac{\exp\left(iux-\frac{\Delta^2x^2}{4}\right)\log(e^x-1)}{2\sinh\frac{x}{2}}-\frac{\log x}{x}\right)dx$$

and

$$P_\varepsilon=\int_\varepsilon^\infty(\cos ux)\left\{\exp\left(-\frac{\Delta^2x^2}{4}\right)\left(2\sinh\frac{x}{2}\right)^{-1}-\frac{1}{x}\right\}dx.$$

It is easy to calculate that

$$\int_{C_\varepsilon}\frac{\log x}{x}\,dx=2\pi i\log\varepsilon$$

and

$$\int_\varepsilon^\infty\frac{\cos ux}{x}\,dx=\int_{u\varepsilon}^\infty\frac{\cos x}{x}\,dx=-\log u\varepsilon+\int_{u\varepsilon}^1\frac{\cos x-1}{x}\,dx+\int_1^\infty\frac{\cos x}{x}\,dx.$$

The $\log\varepsilon$ terms cancel and we let $\varepsilon\to0^+$. It is easy to see that R_ε and $R_\varepsilon'\to0$ so that

$$K=2\pi i\log u-2\pi i\left(\int_0^1\frac{\cos x-1}{x}\,dx+\int_1^\infty\frac{\cos x}{x}\,dx\right)-2\pi iP$$

where $P=P_0$. The middle term here is a constant which we call $-c_0$.

By a change of variable,

$$P=\int_0^\infty\cos\left(\frac{u}{\Delta}x\right)F(x)\,dx$$

where

$$F(x)=\frac{\exp\left(-\frac{x^2}{4}\right)}{2\Delta\sinh\frac{x}{2\Delta}}-\frac{1}{x}.$$

Now if f is analytic at 0 and all of its derivatives vanish at ∞ then we can integrate by parts to obtain

$$\int_0^\infty (\cos\alpha x) f(x)\,dx = \sum_{n=1}^{N} \frac{(-1)^n f^{(2n-1)}(0)}{\alpha^{2n}} + (-1)^{N+1} \int_0^\infty \frac{(\sin\alpha x) f^{(2N+1)}(x)}{\alpha^{2N+1}}\,dx.$$

It is easy to verify that if $f(x) = F(x)$ then

$$f^{(2n-1)}(0) \ll_n 1$$

and

$$\int_0^\infty |f^{(2n-1)}(x)|\,dx \ll_n 1.$$

This proves the lemma with $c_n = (-1)^{n+1} F^{(2n-1)}(0)$.

Lemma 4. *If $\Delta > 0$, then*

$$H = \int_0^\infty v^{s_0} \exp\left(-\frac{\Delta^2 \log^2 v}{4}\right)\frac{dv}{v} = \frac{2\sqrt{\pi}}{\Delta} \exp\left(\frac{s_0^2}{\Delta^2}\right).$$

Proof. Let $x = \frac{1}{2}\Delta \log v$. Then

$$H = \frac{2}{\Delta} \int_{-\infty}^{\infty} \exp\left(-x^2 + \frac{2xs_0}{\Delta}\right) dx$$
$$= \frac{2}{\Delta} \exp\left(\frac{s_0^2}{\Delta^2}\right) \int_{-\infty}^{\infty} \exp\left(-\left(x - \frac{s_0}{\Delta}\right)^2\right) dx.$$

By Cauchy's theorem the integral here is

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}.$$

This proves the lemma.

Lemma 5. *Let $s_0 = \frac{1}{2} + iu$, $T \leqq |u| \leqq 2T$, $\delta = \frac{1}{T}$, $\exp(5(\log T)^{\frac{1}{2}}) < \Delta < \frac{T}{\log^2 T}$, and $c = \eta$ or $c = 1 + \eta$ where $\eta > 0$ is a small, fixed number. Let*

$$W = \int_{L_\delta} \int_{(c)} (1 + |s|) \left| v^{s_0} \exp\left(-\frac{\Delta^2 \log^2 v}{4}\right) \frac{\Gamma(s)}{\sin \pi s} (\cos \pi s)\, e^{\frac{-\pi i s}{2}} (v-1)^{s-1}\, ds \frac{dv}{v}\right|.$$

Then $W \ll_{\varepsilon,\eta} \Delta^{-c-\frac{5}{2}} T^{\frac{5}{2}+\eta+\varepsilon}$ for any $\varepsilon > 0$. The same estimate holds if the $\cos \pi s$ term is omitted or if L_δ is replaced by $L_{-\delta}$ and $e^{\frac{-\pi i s}{2}}$ by $e^{\frac{\pi i s}{2}}$.

Proof. Let $v = xe^{i\delta}$. Then

$$|v^{s_0}| = x^{\frac{1}{2}} e^{-\delta u} \leqq x^{\frac{1}{2}} e^{\delta |u|} \ll x^{\frac{1}{2}},$$

$$\left| \exp\left(-\frac{\Delta^2 \log^2 v}{4}\right)\right| = \exp\left(-\frac{\Delta^2 \log^2 x}{4}\right) \exp\left(\frac{\Delta^2 \delta^2}{4}\right) \ll \exp\left(\frac{-\Delta^2 \log^2 x}{4}\right),$$

$$|\Gamma(s)| \ll (1+|t|)^{c-\frac{1}{2}} e^{-\frac{\pi}{2}|t|},$$

$$\left|\frac{\cos \pi s}{\sin \pi s}\right| \ll 1,$$

$$|e^{-\frac{\pi i s}{2}}| = e^{\frac{\pi}{2} t},$$

$$\frac{1}{|v|} \ll \frac{1}{x},$$

and

$$|(v-1)^{s-1}| = |v-1|^{c-1} e^{-t \arg(v-1)} = a_c(x, \delta) \exp(-t b(x, \delta))$$

where

$$a_c(x, \delta) = \left((x-1)^2 + 2x(1-\cos\delta)\right)^{\frac{c-1}{2}}$$

and

$$b(x, \delta) = \arctan \frac{x \sin \delta}{x \cos \delta - 1}.$$

We also have the estimates

$$a_c(x, \delta) \ll \begin{cases} \delta^{c-1-\eta} & \text{if } x \leqq 1, \\ x^\eta \delta^{c-1-\eta} & \text{if } x \geqq 1, \end{cases}$$

and, for some positive constant C,

$$b(x, \delta) \begin{cases} > \delta, & x \geqq 0, \\ \gg \dfrac{\delta \Delta}{\log \Delta}, & |x-1| \leqq \dfrac{\log \Delta}{\Delta}, \\ \leqq \pi - C \dfrac{\delta \Delta}{\log \Delta}, & |x-1| \leqq \dfrac{\log \Delta}{\Delta}, \\ \leqq \pi - C x \delta, & 0 \leqq x \leqq \dfrac{1}{4}, \\ \leqq \pi - C \delta, & \dfrac{1}{4} \leqq x. \end{cases}$$

We split the x integral into four pieces. Thus $W=W_1+W_2+W_3+W_4$ where W_1 is the integral with $0\leqq x\leqq\frac{1}{4}$, W_2 is the integral with $\frac{1}{4}\leqq x\leqq 1-\frac{\log\Delta}{\Delta}$, W_3 is the integral with $1-\frac{\log\Delta}{\Delta}\leqq x\leqq 1+\frac{\log\Delta}{\Delta}$, and W_4 is the integral with $1+\frac{\log\Delta}{\Delta}\leqq x\leqq\infty$. It is easy to see that

$$\int_0^\infty (1+t)^{c+\frac{1}{2}} e^{-\varepsilon t}dt \ll \varepsilon^{-c-\frac{3}{2}}$$

as $\varepsilon\to 0^+$. Thus, by our estimate for $b(x,\delta)$,

$$\int_{-\infty}^{\infty} (1+|t|)^{c+\frac{1}{2}} e^{\frac{\pi}{2}(t-|t|)} e^{-tb(x,\delta)}dt \ll \begin{cases} \delta^{-c-\frac{3}{2}}, & x\geqq 1+\frac{\log\Delta}{\Delta}, \\ \left(\frac{\log\Delta}{\Delta\delta}\right)^{c+\frac{3}{2}}, & |x-1|\leqq\frac{\log\Delta}{\Delta}, \\ \delta^{-c-\frac{3}{2}}, & \frac{1}{4}\leqq x\leqq 1-\frac{\log\Delta}{\Delta}, \\ (\delta x)^{-c-\frac{3}{2}}, & 0<x\leqq\frac{1}{4}. \end{cases}$$

Hence

$$W_1 \ll \delta^{-\frac{5}{2}-\eta}\int_0^{\frac{1}{4}} x^{-c-2}\exp\left(-\frac{\Delta^2\log^2 x}{4}\right)dx \ll \delta^{-\frac{5}{2}-\eta}\exp\left(-\frac{\Delta^2\log^2 4}{4}\right),$$

$$W_2 \ll \delta^{-\frac{5}{2}-\eta}\int_{\frac{1}{4}}^{1-\frac{\log\Delta}{\Delta}} \exp\left(-\frac{\Delta^2\log^2 x}{4}\right)dx \ll \delta^{-\frac{5}{2}-\eta}\exp\left(-\frac{\log^2\Delta}{4}\right),$$

$$W_3 \ll \frac{\log\Delta}{\Delta}\left(\frac{\log\Delta}{\Delta\delta}\right)^{c+\frac{3}{2}}\delta^{c-1-\eta} \ll_\varepsilon \Delta^{-c-\frac{5}{2}}T^{\frac{5}{2}+\eta+\varepsilon},$$

and

$$W_4 \ll \delta^{-\frac{5}{2}-\eta}\int_{1+\frac{\log\Delta}{\Delta}}^{\infty} \exp\left(-\frac{\Delta^2\log^2 x}{4}\right)dx \ll \delta^{-\frac{5}{2}-\eta}\exp\left(-\frac{\log^2\Delta}{8}\right).$$

The lemma follows from these estimates and the range of Δ.

A special case of Hooley's Conjecture R^* is that if $v_2-v_1\leqq k$, then for any $\varepsilon>0$,

$$V(v_1,v_2,l,k)=\sum_{\substack{v_1<h\leqq v_2\\(h,k)=1}} e\left(\frac{l\bar{h}}{k}\right) \ll_\varepsilon k^\varepsilon (v_2-v_1)^{\frac{1}{2}}(l,k)^{\frac{1}{2}}$$

uniformly in all the variables, where $h\bar{h}\equiv 1 \bmod k$.

Lemma 6. *Assume Hooley's Conjecture R^*. Suppose that $a(m) \ll_\varepsilon m^\varepsilon$ for any $\varepsilon > 0$ and $1 \leqq m \leqq M$ where $\log M \ll \log T$, and that $s = 1 + \eta + it = c + it$, where $\eta > 0$ is small and fixed. Then, for any $\varepsilon > 0$,*

$$\mathcal{M} = \max_t \left| \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \sum_{h,k \leqq M} \frac{a(h)\overline{a(k)}\,(h,k)^{1-2s}}{(hk)^{1-s}} e\left(\frac{n\overline{h^*}}{k^*}\right) \right| \ll_{\varepsilon,\eta} M^{\frac{7}{4}+2\eta+\varepsilon}$$

where $h^* = \dfrac{h}{(h,k)}$ *and* $k^* = \dfrac{k}{(h,k)}$.

Proof. First of all, we note that

$$\sum_{n=1}^{\infty} d(n)^2 n^{-1-\eta} \ll_\eta 1$$

for any $\eta > 0$. Thus, it suffices to prove that

$$\sum_{h,k \leqq M} a(h)\overline{a(k)}\,(h,k)^{1-2s}(hk)^{s-1} e\left(\frac{n\overline{h^*}}{k^*}\right) \ll_\varepsilon d(n)\, M^{\frac{7}{4}+\varepsilon+2\eta}$$

uniformly in n. We group pairs h and k according to their greatest common divisor g. Thus, it suffices to show that

$$\sum_{h,k \leqq M_1}{}' a(hg)\overline{a(kg)}\,(hk)^{s-1} e\left(\frac{n\overline{h}}{k}\right) \ll_{\varepsilon,\eta} d(n)\, M^{\frac{7}{4}+\varepsilon+2\eta}$$

uniformly for $1 \leqq g \leqq M$ with $M_1 = \dfrac{M}{g}$ where $\sum'$ indicates that the sum is for $(h,k) = 1$. We apply Cauchy's inequality to the sum on h and see that the left side is

$$\begin{aligned} &\ll \Big(\sum_{h \leqq M_1} |a(hg)|^2 h^{2\eta}\Big)^{\frac{1}{2}} (\Sigma_1)^{\frac{1}{2}} \\ &\ll_\varepsilon M^{\frac{1}{2}+\eta+\varepsilon} \Sigma_1^{\frac{1}{2}} \end{aligned}$$

where

$$\Sigma_1 = \sum_{h \leqq M_1} \left| \sum_{k \leqq M_1}{}' \overline{a(kg)}\, k^{s-1} e\left(\frac{n\overline{h}}{k}\right) \right|^2.$$

Now it suffices to show that

$$\Sigma_1 \ll_\varepsilon d(n)\, M^{\frac{5}{2}+2\eta+\varepsilon}.$$

To estimate Σ_1, we interchange the sums on h and k. Thus

$$\Sigma_1 = \sum_{k_1,k_2} \overline{a(k_1 g)}\, a(k_2 g)\, k_1^{s-1} k_2^{\bar{s}-1} \sum_h{}'' e\left(\frac{n\bar{h}}{k_1}\right) e\left(-\frac{n\bar{h}}{k_2}\right)$$

where $\sum''$ indicates that the sum on h is for $(h, k_1 k_2) = 1$. Note that $\bar{h}$ in the first instance is the inverse of $h \bmod k_1$ and in the second instance it is the inverse of $h \bmod k_2$. Now we have

$$\Sigma_1 \ll_\varepsilon M^{2\eta+\varepsilon} \sum_{k_1,k_2} \left| \sum_h{}'' e\left(\frac{n\bar{h}(k_1-k_2)}{k_1 k_2}\right)\right|$$

since if $h\alpha \equiv 1 \pmod{k_1}$, $h\beta \equiv 1 \pmod{k_2}$, and $h\gamma \equiv 1 \pmod{k_1 k_2}$, then

$$\alpha k_2 - \beta k_1 \equiv \gamma(k_2 - k_1) \pmod{k_1 k_2}.$$

Thus, by R^*,

$$\Sigma_1 \ll_\varepsilon M^{2\eta+\varepsilon}\{M_1 \sum_{k_1 k_2 \leqq M_1} 1 + M_1^{\frac{1}{2}} \sum_{k_1 k_2 \geqq M_1}^* (n(k_1-k_2), k_1 k_2)^{\frac{1}{2}} + M_1 \sum_{k \leqq M_1} 1\}$$

$$\ll_\varepsilon M^{2+2\eta+\varepsilon} + M^{\frac{1}{2}+2\eta+\varepsilon} \sum_{k_1,k_2}^* (n(k_1-k_2), k_2 k_2)^{\frac{1}{2}}$$

where $\sum^*$ indicates that the sum is for $k_1 \neq k_2$. The sum on k_1 and k_2 is

$$\ll \sum_{k_1,k_2}^* (n, k_1 k_2)^{\frac{1}{2}} (k_1 - k_2, k_1 k_2)^{\frac{1}{2}}$$

$$\ll \{\sum_{k_1,k_2} (n, k_1 k_2)\}^{\frac{1}{2}} \{\sum_{k_1,k_2}^* (k_1 - k_2, k_1 k_2)\}^{\frac{1}{2}}$$

by Cauchy's inequality. The first sum is

$$\ll M^\varepsilon \sum_{k \leqq M^2} (n, k) \ll M^\varepsilon \sum_{t|n} t \sum_{k \leqq M^2, t|k} 1$$

$$\ll M^\varepsilon \sum_{t|n} t\left(\frac{M^2}{t}\right) \ll M^{2+\varepsilon} d(n).$$

Since $(k_1 - k_2, k_1 k_2) = (k_1 - k_2, k_2^2)$ we have

$$\sum_{k_1,k_2}^* (k_1 - k_2, k_1 k_2) \leqq \sum_{k_2} \sum_{t|k_2^2} t \sum_{k_1 \equiv k_2 (\bmod t)}^* 1 \ll \sum_{k_2} \sum_{t|k_2^2} t\left(\frac{M}{t}\right)$$

$$\ll M \sum_{k_2} d(k_2^2) \ll M^{2+\varepsilon}.$$

Hence,

$$\Sigma_1 \ll_\varepsilon d(n)\, M^{\frac{5}{2}+2\eta+\varepsilon}$$

which is what was required.

Lemma 7. *Suppose that $a(m)=\mu(m)\,F(m)$, where $F\in\mathscr{F}_M$. Let $1\leqq U, V\leqq M$, where $M\leqq T^{\frac{3}{5}}$. Let $N\geqq 1$, $\eta>0$, and take $s=c+it$, with $c=\eta$ for $UV\geqq TN$ and $c=1+\eta$ otherwise. Then for any $\varepsilon>0$ we have*

$$\mathscr{M}(N,U,V)=\sum_{n\sim N} d(n)\,n^{-s}\sum_{\substack{u\sim U\ v\sim V\\(u,v)=1}}\sum a(ug)\,a(vg)\,(uv)^{s-1}\,e\left(n\frac{\bar{u}}{v}\right)$$

$$\ll_{\varepsilon,\eta}(1+|s|)\,(MN)^{\varepsilon}\,M^{2\eta+\frac{17}{12}}T^{c-\frac{3}{4}}N^{-\eta}$$

uniformly for all t and for all $g\leqq MV^{-1}$.

Proof. We have

$$\mathscr{M}(N,U,V)\ll N^{\varepsilon-c}V^{c-1}\sum_{n\sim N}\sum_{v\sim V}\left|\sum_{\substack{u\sim U\\(u,vg)=1}}\mu(u)\,F(ug)\,u^{s-1}e\left(n\frac{\bar{u}}{v}\right)\right|$$

$$\ll(1+|s|)\,N^{\varepsilon-c}(UV)^{c-1}\sum_{n\sim N}\sum_{v\sim V}\left|\sum_{u\sim U}{}'\,\mu(u)\,e\left(n\frac{\bar{u}}{v}\right)\right|,$$

by partial summation. Here $\sum'_{u\sim U}$ denotes summation for $(u,vg)=1$ with u running over some subinterval of $(U,2U]$. Now we use the following identity, which corresponds to Vaughan's identity for the function $\mu(n)$:

$$\mu(u)=-\sum_{\substack{\alpha\beta\gamma=u\\\alpha,\beta\leqq x}}\mu(\alpha)\,\mu(\beta),\qquad u^{\frac{1}{2}}<x<u.$$

Taking $x=(3U)^{\frac{1}{2}}$, we obtain

$$\mathscr{M}(N,U,V)\ll(1+|s|)\,(MN)^{\varepsilon}\,N^{-c}(UV)^{c-1}\,S,$$

where

$$S=\sum_{n\sim N}\sum_{v\sim V}\left|\sum_{\alpha\sim A}{}'\sum_{\beta\sim B}{}'\sum_{\gamma\sim C}{}'\,\mu(\alpha)\,\mu(\beta)\,e\left(n\frac{\overline{\alpha\beta\gamma}}{v}\right)\right|.$$

Here A, B, C satisfy $U\ll ABC\ll U$ and $A,B\ll U^{\frac{1}{2}}$.

We estimate S in two different ways. Firstly, using Weil's bound for the Kloosterman sum it follows that

$$\sum_{\gamma\sim C}{}'\,e\left(l\frac{\bar{\gamma}}{v}\right)\ll v^{\frac{1}{2}}(vg)^{\varepsilon}\,(l,v)\,(1+Cv^{-1}).$$

Here $(vg)^{\varepsilon}\leqq M^{\varepsilon}$, and $(l,v)=(n\overline{\alpha\beta},v)=(n,v)$, whence

$$S\ll ABM^{\varepsilon}V^{\frac{1}{2}}(1+CV^{-1})\sum_{n\sim N}\sum_{v\sim V}(n,v).$$

The v-sum is $O(Vd(n))$, and so

$$S\ll AB(MN)^{\varepsilon}\,NV^{\frac{1}{2}}(C+V). \tag{1}$$

The second estimate will treat α, β, γ symmetrically. Thus we illustrate the method by writing

$$S \ll \sum_{n\sim N} \sum_{v\sim V} \sum_{\beta\sim B} \sum_{\gamma\sim C} \left| \sum_{\alpha\sim A}{}' \mu(\alpha)\, e\left(n\frac{\overline{\alpha\beta\gamma}}{v}\right)\right|.$$

The treatment would be analogous if we wished to isolate β or γ in preference to α. Let

$$r_{v,m} = \#\{n, \beta, \gamma : n\sim N, \beta\sim B, \gamma\sim C, n\overline{\beta\gamma}\equiv m \pmod v\},$$

whence

$$S \ll \sum_{v\sim V} \sum_{m=1}^{v} r_{v,m} \left| \sum_{\alpha\sim A}{}' \mu(\alpha)\, e\left(m\frac{\bar{\alpha}}{v}\right)\right|.$$

By Hölder's inequality we have

$$S \ll \Big(\sum_{v,m} r_{v,m}\Big)^{\frac{1}{2}} \Big(\sum_{v,m} r_{v,m}^2\Big)^{\frac{1}{4}} \left(\sum_{v,m} \left| \sum_{\alpha\sim A}{}' \mu(\alpha)\, e\left(m\frac{\bar{\alpha}}{v}\right)\right|^4\right)^{\frac{1}{4}}.$$

The first sum on the right is

$$\ll \sum_{n,\beta,\gamma,v} 1 \ll BCNV.$$

Moreover

$$\sum_{v,m} r_{v,m}^2 = \sum_{n_1,n_2} \sum_{\beta_1,\beta_2} \sum_{\gamma_1,\gamma_2} \#\{v;\, v\sim V,\, v \mid n_1\overline{\beta_1\gamma_1} - n_2\overline{\beta_2\gamma_2}\}.$$

Here the condition on v is $v \mid n_1\beta_2\gamma_2 - n_2\beta_1\gamma_1$. If $n_1\beta_2\gamma_2 = n_2\beta_1\gamma_1$ there will be $O(V)$ solutions v, and otherwise $O((BCN)^{\varepsilon})$. However $n_1\beta_2\gamma_2 = n_2\beta_1\gamma_1$ has $O((BCN)^{1+\varepsilon})$ solutions, so that

$$\sum_{v,m} r_{v,m}^2 \ll (MN)^{\varepsilon}\,(BCNV + B^2C^2N^2).$$

Finally

$$\sum_{v,m} \left| \sum_{\alpha\sim A}{}' \mu(\alpha)\, e\left(m\frac{\bar{\alpha}}{v}\right)\right|^4$$

$$= \sum_{\alpha_1} \sum_{\alpha_2} \sum_{\alpha_3} \sum_{\alpha_4} \mu(\alpha_1)\,\mu(\alpha_2)\,\mu(\alpha_3)\,\mu(\alpha_4) \sum_{v\sim V} \sum_{m=1}^{v} e\left(\frac{m(\overline{\alpha_1} + \overline{\alpha_2} - \overline{\alpha_3} - \overline{\alpha_4})}{v}\right).$$

The innermost sum is v if $v \mid \overline{\alpha_1} + \overline{\alpha_2} - \overline{\alpha_3} - \overline{\alpha_4}$, and zero otherwise. Writing

$$Y = \alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_4 - \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_2\alpha_3$$

this condition becomes $v \mid Y$. Hence

$$\sum_{v,m} \left| \sum_{\alpha\sim A}{}' \mu(\alpha)\, e\left(m\frac{\bar{\alpha}}{v}\right)\right|^4 \ll V^2\,\#\{\alpha_1,\dots,\alpha_4 : Y=0\} + A^4M^{\varepsilon}V.$$

However $Y=0$ requires

$$\alpha_1^2\alpha_2^2 = \{\alpha_3(\alpha_1+\alpha_2) - \alpha_1\alpha_2\}\,\{\alpha_4(\alpha_1+\alpha_2) - \alpha_1\alpha_2\}.$$

Thus each pair α_1, α_2 produces $O(M^\varepsilon)$ possible factors $\alpha_3(\alpha_1+\alpha_2)-\alpha_1\alpha_2$, and each factor determines α_3 and α_4. Hence there are $O(A^2 M^\varepsilon)$ solutions of $Y=0$. It now follows that

$$S \ll \{BCNV\}^{\frac{1}{2}} \{(MN)^\varepsilon (BCNV + B^2C^2N^2)\}^{\frac{1}{4}} \{M^\varepsilon (A^2V^2 + A^4V)\}^{\frac{1}{4}}$$

$$(2) \qquad \ll (MN)^\varepsilon (NUV)^{\frac{3}{4}} (A^{-\frac{1}{4}}V^{\frac{1}{2}} + A^{\frac{1}{4}}V^{\frac{1}{4}} + A^{-\frac{1}{2}}N^{\frac{1}{4}}U^{\frac{1}{4}}V^{\frac{1}{4}} + N^{\frac{1}{4}}U^{\frac{1}{4}}).$$

We shall show that

$$S \ll (MN)^\varepsilon \max(TN, UV)\, M^{\frac{17}{12}} T^{-\frac{3}{4}}.$$

Lemma 7 will then follow, with a different value of ε. The estimate (1) suffices, if $C \geqq M^{-\frac{17}{12}} T^{-\frac{1}{4}} U V^{\frac{3}{2}}$, for then

$$S \ll (MN)^\varepsilon (NUV^{\frac{1}{2}} + C^{-1}NUV^{\frac{3}{2}})$$

$$\ll (MN)^\varepsilon (TN)(M^{\frac{3}{2}}T^{-1} + M^{\frac{17}{12}}T^{-\frac{3}{4}}).$$

Here $M^{\frac{3}{2}}T^{-1} \leqq M^{\frac{17}{12}}T^{-\frac{3}{4}}$, since $M \leqq T^{\frac{3}{5}} \leqq T^3$. We now apply (2) with "A" chosen as the largest of A, B, C. We may therefore suppose that

$$U^{\frac{1}{3}} \ll A \ll \max(U^{\frac{1}{2}}, M^{-\frac{17}{12}}T^{-\frac{1}{4}}UV^{\frac{3}{2}}),$$

whence (2) yields

$$S \ll (MN)^\varepsilon (NUV)^{\frac{3}{4}} (U^{-\frac{1}{12}}V^{\frac{1}{2}} + U^{\frac{1}{8}}V^{\frac{1}{4}} + M^{-\frac{17}{48}}T^{-\frac{1}{16}}U^{\frac{1}{4}}V^{\frac{5}{8}} + N^{\frac{1}{4}}U^{\frac{1}{12}}V^{\frac{1}{4}} + N^{\frac{1}{4}}U^{\frac{1}{4}}).$$

Then

$$(NUV)^{\frac{3}{4}} U^{-\frac{1}{12}} V^{\frac{1}{2}} = (TN)^{\frac{3}{4}} (UV)^{\frac{1}{4}} U^{\frac{5}{12}} V T^{-\frac{3}{4}} \leqq \max(TN, UV)\, M^{\frac{17}{12}} T^{-\frac{3}{4}};$$

$$(NUV)^{\frac{3}{4}} U^{\frac{1}{8}} V^{\frac{1}{4}} = (TN)^{\frac{3}{4}} (UV)^{\frac{1}{4}} U^{\frac{5}{8}} V^{\frac{3}{4}} T^{-\frac{3}{4}} \leqq \max(TN, UV)\, M^{\frac{11}{8}} T^{-\frac{3}{4}},$$

here $\frac{11}{8} < \frac{17}{12}$;

$$(NUV)^{\frac{3}{4}} M^{-\frac{17}{48}} T^{-\frac{1}{16}} U^{\frac{1}{4}} V^{\frac{5}{8}} = (TN)^{\frac{3}{4}} (UV)^{\frac{1}{4}} M^{-\frac{17}{48}} T^{-\frac{13}{16}} U^{\frac{3}{4}} V^{\frac{9}{8}}$$

$$\leqq \max(TN, UV)\, M^{\frac{73}{48}} T^{-\frac{13}{16}},$$

here $M^{\frac{73}{48}} T^{-\frac{13}{16}} \leqq M^{\frac{17}{12}} T^{-\frac{3}{4}}$, since $M \leqq T^{\frac{3}{5}}$;

$$(NUV)^{\frac{3}{4}} N^{\frac{1}{4}} U^{\frac{1}{12}} V^{\frac{1}{4}} = (TN)\, T^{-1} U^{\frac{5}{6}} V \leqq \max(TN, UV)\, M^{\frac{11}{6}} T^{-1},$$

where $M^{\frac{11}{6}} T^{-1} \leqq M^{\frac{17}{12}} T^{-\frac{3}{4}}$, since $M \leqq T^{\frac{3}{5}}$; and lastly

$$(NUV)^{\frac{3}{4}} N^{\frac{1}{4}} U^{\frac{1}{4}} = (TN)\, T^{-1} U V^{\frac{3}{4}} \leqq \max(TN, UV)\, M^{\frac{7}{4}} T^{-1},$$

with $M^{\frac{7}{4}} T^{-1} \leqq M^{\frac{17}{12}} T^{-\frac{3}{4}}$, since $M \leqq T^{\frac{3}{4}}$. This completes the proof of the lemma. It may be worth noting that the term $M^{\frac{17}{12}} T^{-\frac{3}{4}}$ arises only when, roughly speaking, $U = V = M$, $A = B = C = M^{\frac{1}{3}}$ and $TN = M^2$.

§ 3. Proof of Theorem 1

We can easily deduce Theorem 1 from the following

Theorem 2. *For $\Delta > 0$, define*

$$g(u) = \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-u)^2\Delta^{-2}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|A\left(\frac{1}{2}+it\right)\right|^2 dt.$$

Then for large T and $\log M \ll \log T$ we have

$$g(u) = \sum_{h,k \leq M}\sum \frac{a(h)\,\overline{a(k)}}{hk}(h,k)\left(\log\frac{u(h,k)^2}{2\pi hk} + b_0 + O\left(\left(\frac{\Delta}{u}\right)^2\right)\right) + \mathscr{E}_0$$

uniformly for $T \leqq u \leqq 2T$ and $\exp(5(\log T)^{\frac{1}{2}}) \leqq \Delta \leqq T(\log T)^{-1}$ where b_0 is a certain constant. Assume $a(m) \ll_\varepsilon m^\varepsilon$ for any $\varepsilon > 0$. Then

(A 1) $$\mathscr{E}_0 \ll_\varepsilon \Delta^{-\frac{7}{2}} T^{\frac{5}{2}+\varepsilon} M^2;$$

assuming conjecture R^* we have

(B 1) $$\mathscr{E}_0 \ll_\varepsilon \Delta^{-\frac{7}{2}} T^{\frac{5}{2}+\varepsilon} M^{\frac{7}{4}};$$

and if $a(m) = \mu(m)\,F(m)$ with $F \in \mathscr{F}_M$, and $M \leqq T^{\frac{3}{5}}$, then

(C 1) $$\mathscr{E}_0 \ll_\varepsilon \Delta^{-\frac{7}{2}} T^{\frac{5}{2}+\varepsilon} \cdot M^{\frac{17}{12}} T^{\frac{1}{4}}$$

for any $\varepsilon > 0$.

Remark. It is possible to replace the $b_0 + O\left(\left(\frac{\Delta}{u}\right)^2\right)$ term above by an asymptotic series $\sum_{n=0}^{N-1} b_n \left(\frac{\Delta}{u}\right)^{2n} + O\left(\left(\frac{\Delta}{u}\right)^{2N}\right)$. It will follow later that $b_0 = 2\gamma$.

We begin by proving Theorem 2. Let $s_0 = \frac{1}{2} + iu$. Then

$$g(u) = \frac{1}{i\Delta\sqrt{\pi}} \int_{(\frac{1}{2})} e^{(s-s_0)^2\Delta^{-2}} \zeta(s)^2 \chi(1-s)\,A(s)\,\bar{A}(1-s)\,ds,$$

where

$$\bar{A}(s) = \sum_{m \leqq M} \overline{a(m)}\, m^{-s}.$$

We move the path of integration to the line from $c - i\infty$ to $c + i\infty$ where $c = 1 + \eta$. The contribution from the residue at $s = 1$ is

$$R = \frac{2\sqrt{\pi}}{\Delta} e^{(1-s_0)^2\Delta^{-2}} \zeta(0)\,A(0)\,\bar{A}(1) \ll_\varepsilon \exp(-\log^2 T)\,\Delta^{-1} M^{1+\varepsilon} \ll T^{-10}.$$

Then $g(u)=I-R$ where I is $(i\Delta\sqrt{\pi})^{-1}$ times the integral on the new path. By Lemma 2,

$$I=\sum_{h,k\leq M}\sum\frac{a(h)\overline{a(k)}}{k}\sum_{n=1}^{\infty}d(n)\,J\left(\frac{nh}{k}\right)$$
$$=\sum_{h,k\leq M}\sum\frac{a(h)\overline{a(k)}}{k}\sum_{n=1}^{\infty}d(n)\int_0^{\infty}v^{s_0}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)\left(e^{-2\pi i v n\frac{h}{k}}+e^{2\pi i v n\frac{h}{k}}\right)\frac{dv}{v}.$$

We express the integral as a sum of two integrals and use Cauchy's theorem to move one path to L_δ and the other to $L_{-\delta}$ where L_δ is the half-line whose points have δ as their argument. We interchange summation and integration and have

$$I=\sum_{h,k\leq M}\sum\frac{a(h)\overline{a(k)}}{k}(I_1+I_2)$$

where

$$I_1=I_1(h,k)=\int_{L_\delta}v^{s_0}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)S\left(\frac{h}{k}(v-1),\frac{h}{k}\right)\frac{dv}{v}$$

and

$$I_2=I_2(h,k)=\int_{L_{-\delta}}v^{s_0}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)S\left(-\frac{h}{k}(v-1),-\frac{h}{k}\right)\frac{dv}{v}.$$

By Lemma 1 we have

$$I_1=M_1+R_1-iE_1$$

where

$$R_1=D\left(0,\frac{h^*}{k^*}\right)\int_{L_\delta}v^{s_0}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)\frac{dv}{v},$$

$$M_1=\int_{L_\delta}v^{s_0}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)\frac{\left(\gamma-\log\left(-2\pi i\frac{h}{k}(v-1)\right)-2\log k^*\right)}{-2\pi i h^*(v-1)}\frac{dv}{v},$$

and

$$E_1=k^*\int_{L_\delta}v^{s_0}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)F_1(v)\frac{dv}{v}$$

with

$$F_1(v)=\int_{(c)}(2\pi)^{-2s}\frac{\Gamma(s)}{\sin\pi s}\left(D\left(s,\frac{\overline{h^*}}{k^*}\right)+(\cos\pi s)\,D\left(s,-\frac{\overline{h^*}}{k^*}\right)\right)(-2\pi i h^* k^*(v-1))^{s-1}\,ds.$$

Similarly,

$$I_2 = M_2 + R_2 - iE_2$$

where

$$R_2 = D\left(0, -\frac{h^*}{k^*}\right) \int_{L-\delta} v^{s_0} \exp\left(\frac{-\Delta^2 \log^2 v}{4}\right) \frac{dv}{v},$$

$$M_2 = \int_{L-\delta} v^{s_0} \exp\left(\frac{-\Delta^2 \log^2 v}{4}\right) \frac{\left(\gamma - \log\left(2\pi i \frac{h}{k}(v-1)\right) - 2\log k^*\right)}{2\pi i h^*(v-1)} \frac{dv}{v},$$

and

$$E_2 = k^* \int_{L-\delta} v^{s_0} \exp\left(\frac{-\Delta^2 \log^2 v}{4}\right) F_2(v) \frac{dv}{v}$$

with

$$F_2(v) = \int_{(c)} (2\pi)^{-2s} \frac{\Gamma(s)}{\sin \pi s}\left(D\left(s, -\frac{\overline{h^*}}{k^*}\right) + (\cos \pi s)\, D\left(s, \frac{\overline{h^*}}{k^*}\right)\right)(2\pi i h^* k^*(v-1))^{s-1}\, ds.$$

By Lemma 3,

$$M_1 + M_2 = \frac{1}{h^*}\left(\log \frac{u(h,k)^2}{2\pi hk} + \gamma + c_0 + O\left(\left(\frac{\Delta}{u}\right)^2\right)\right).$$

By Lemma 4 and Cauchy's theorem,

$$R_1 + R_2 = \frac{4\sqrt{\pi}}{\Delta} \exp\left(\frac{s_0^2}{\Delta^2}\right) \operatorname{Re} D\left(0, \frac{h^*}{k^*}\right)$$
$$\ll \Delta^{-1} \exp(-\log^2 T)\, M(\log M)^2$$

so that

$$\left| \sum_{h,k \leq M}\sum \frac{a(h)\overline{a(k)}}{k} (R_1 + R_2) \right| \ll T^{-10}.$$

The other error term is

$$\sum_{h,k \leq M} \frac{a(h)\overline{a(k)}}{k} (E_1 + E_2)$$

which we write as a sum of four terms of which a typical one is

$$Z = \int_{L_\delta} \int_{(c)} G(v, s_0, \Delta, s)\, \mathcal{M}(s)\, ds\, dv$$

where $c > 1$,

$$G(v, s_0, \Delta, s) = v^{s_0} \exp\left(\frac{-\Delta^2 \log^2 v}{4}\right) \frac{\Gamma(s)}{\sin \pi s} (2\pi)^{-2s} (-2\pi i(v-1))^{s-1} v^{-1},$$

and

$$\mathcal{M}(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \sum_{h,k \leq M} \frac{a(h)\overline{a(k)}}{(hk)^{1-s}} (h,k)^{1-2s}\, e\left(n \frac{\overline{h^*}}{k^*}\right).$$

The trivial estimate for $\mathcal{M}(s)$ is (with $s=1+\eta+it$),

$$\mathcal{M}(s) \ll_{\varepsilon} M^{2+\varepsilon+2\eta}$$

which by Lemma 5 yields

$$Z \ll_{\varepsilon} \Delta^{-\frac{7}{2}-\eta} T^{\frac{5}{2}+\eta+\varepsilon} M^{2+\varepsilon+2\eta}.$$

This leads to part (A 1) of the theorem, on taking $\eta=\varepsilon$ and redefining ε.

Assuming conjecture R^*, we have

$$\mathcal{M}(s) \ll_{\varepsilon} M^{\frac{7}{4}+2\eta+\varepsilon}$$

by Lemma 6; and then by Lemma 5,

$$Z \ll_{\varepsilon} \Delta^{-\frac{7}{2}-\eta} T^{\frac{5}{2}+\eta+\varepsilon} M^{\frac{7}{4}+2\eta+\varepsilon}.$$

This shows part (B 1).

Finally, if $a(m)=\mu(m)F(m)$ with $F\in\mathscr{F}_M$ then we rewrite $\mathcal{M}(s)$ as

$$\mathcal{M}(s)=\sum_{g\leq M}\frac{1}{g}\sum_{U,V,N}\mathcal{M}(N,U,V,g,s)$$

where

$$\mathcal{M}(N,U,V,g,s)=\sum_{n\sim N}\frac{d(n)}{n^s}\sum_{\substack{u\sim U\\(u,v)=1}}\sum_{v\sim V}\frac{a(ug)\,a(vg)}{(uv)^{1-s}}\,e\left(n\frac{\bar{u}}{v}\right)$$

where the notation $x\sim X$ means $X\leq x<2X$, and the sums on U and V have $\ll\log M$ terms with $U, V\ll\frac{M}{g}$ and the sum on N is for $N=2^L$, $L=0,1,2,\ldots$. Now Z is a sum of terms of the shape

$$Z(N,U,V)=\int_{L_\delta}\int_{(c)} G(v,s_0,\Delta,s)\,\mathcal{M}(N,U,V,g,s)\,ds\,dv.$$

If $UV\geqq TN$ then we move the s path of integration to $s=\eta+it$; otherwise we leave it at $s=1+\eta+it$. In moving the path of integration, we cross a pole at $s=1$ with residue

$$\frac{-1}{2\pi^2}\mathcal{M}(N,U,V,g,1)\int_{L_\delta} v^{s_0}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)\frac{dv}{v}\ll M^{2+\varepsilon}\Delta^{-1}\exp(-\log^2 T)\ll T^{-1},$$

by Lemma 4. Thus, by Lemmas 5 and 7,

$$\begin{aligned}Z&=\sum_g\frac{1}{g}\sum_{N,U,V}\mathcal{M}(N,U,V)\\&\ll_{\varepsilon,\eta}\Delta^{-\frac{7}{2}}T^{\frac{5}{2}+2\eta+\varepsilon}\cdot M^{2\eta+\varepsilon+\frac{17}{12}}T^{\frac{1}{4}}\sum_{N,U,V}N^{\varepsilon-\eta}+\sum_{\substack{N,U,V\\NT\leqq UV}}T^{-1}\\&\ll\Delta^{-\frac{7}{2}}T^{\frac{5}{2}+2\varepsilon}\cdot M^{3\varepsilon+\frac{17}{12}}T^{\frac{1}{4}}\end{aligned}$$

on taking $\eta=\frac{\varepsilon}{2}$. This proves (C 1) and completes the proof of Theorem 2 with $b_0=\gamma+c_0$.

We now deduce Theorem 1 from Theorem 2 (as in Heath-Brown [3], § 4). First we note that it suffices to prove that

$$I(T,2T)=\int_T^{2T}\left|\zeta A\left(\frac{1}{2}+it\right)\right|^2 dt = T\sum_{h,k\leqq M}\sum \frac{a(h)\,\overline{a(k)}}{hk}(h,k)\left(\log\frac{T(h,k)^2}{2\pi hk}+(2\gamma-1)+2\log 2\right)+\frac{\mathscr{E}}{\log T} \tag{3}$$

where $\mathscr{E}$ is as in Theorem 1. For it is easy to deduce Theorem 1 from this by replacing T here by $\frac{T}{2^k}$, $1\leqq k \ll \log T$, and summing. To prove this estimate we use Theorem 2 and the following trivial estimates. Let $\frac{T}{2}\leqq T_1<T_2<3T$ and

$$w(t,T_1,T_2)=\Delta^{-1}\pi^{-\frac{1}{2}}\int_{T_1}^{T_2}\exp\left(-(t-u)^2\Delta^{-2}\right)du.$$

Then

$$0\leqq w(t,T_1,T_2)\leqq 1$$

always;

$$w(t,T_1,T_2)\ll \exp(-\log^2 T)$$

if $t<T_1-\Delta\log T$ or $t>T_2+\Delta\log T$; and

$$w(t,T_1,T_2)=1+O(\exp(-\log^2 T))$$

if $T_1+\Delta\log T\leqq t\leqq T_2-\Delta\log T$.

Now by these estimates and Theorem 2,

$$\begin{aligned}
I(T,2T) &\leqq \int_T^{2T} w(t,T-\Delta\log T,2T+\Delta\log T)\left|\zeta A\left(\frac{1}{2}+it\right)\right|^2 dt+O(1)\\
&= \int_{T-\Delta\log T}^{2T+\Delta\log T}(\Delta^{-1}\pi^{-\frac{1}{2}})\int_T^{2T}\exp\left(-(t-u)^2\Delta^{-2}\right)\left|\zeta A\left(\frac{1}{2}+it\right)\right|^2 dt\,du+O(1)\\
&\leqq \int_{T-\Delta\log T}^{2T+\Delta\log T} g(u)\,du+O(1)\\
&= \int_{T-\Delta\log T}^{2T+\Delta\log T}\sum_{h,k\leqq M}\sum\frac{a(h)\,\overline{a(k)}}{hk}(h,k)\left(\log\frac{u(h,k)^2}{2\pi hk}+b_0+O(\Delta^2T^{-2})\right)du\\
&\quad +O(\mathscr{E}_0T)\\
&= \int_T^{2T}\sum_{h,k\leqq M}\sum\frac{a(h)\,\overline{a(k)}}{hk}(h,k)\left(\log\frac{u(h,k)^2}{2\pi hk}+b_0\right)du\\
&\quad +O(\mathscr{E}_0T)+O\Big(\Delta\log^2 T\sum_{h,k}|a(h)\,a(k)|\,(hk)^{-1}(h,k)\Big).
\end{aligned}$$

Here

$$\sum_{h,k} (hk)^{-1} (h,k) \leqq \sum_{t \leqq M} t\Big(\sum_{\substack{h \leqq M \\ t \mid h}} h^{-1}\Big)^2 \ll \sum_{t \leqq M} t(t^{-1} \log M)^2 \ll \log^3 M,$$

so that the final error term above is $O(\Delta T^{\varepsilon})$. Similarly

$$\begin{aligned} I(T, 2T) &\geqq \int_T^{2T} w(t, T+\Delta \log T, 2T-\Delta \log T) \left|\zeta A\left(\frac{1}{2}+it\right)\right|^2 dt \\ &= \int_{T+\Delta\log T}^{2T-\Delta\log T} (\Delta^{-1}\pi^{-\frac{1}{2}}) \int_T^{2T} \exp(-(t-u)^2 \Delta^{-2}) \left|\zeta A\left(\frac{1}{2}+it\right)\right|^2 dt\, du \\ &= \int_{T+\Delta\log T}^{2T-\Delta\log T} g(u)\, du + O(1) \end{aligned}$$

which leads to the same lower bound as our upper bound. The fact that $b_0 = 2\gamma$ follows from these equations, the choices $M=1$, $a(1)=1$, $\Delta = T^{\frac{5}{7}}$, and Ingham's well-known result about the mean square of $\left|\zeta\left(\frac{1}{2}+it\right)\right|$ (see Titchmarsh [8], Chap. 7 for example). Then the choice $\Delta = T^{1-2\varepsilon}$ proves (3) and so Theorem 1.

To prove the first part of the corollary it suffices, by Theorem 1, to show that

$$\sum_{h,k \leqq M}\sum \frac{\mu(h)\,\mu(k)}{hk}\left(1-\frac{\log h}{\log M}\right)\left(1-\frac{\log k}{\log M}\right)(h,k)\left(\log\frac{T(h,k)^2}{2\pi hk}+2\gamma-1\right) \sim 1+\frac{\log T}{\log M}.$$

This can easily be done using, for example, the techniques in Conrey [1], section 6. The second part follows as in Selberg [6]. Briefly, by Littlewood's lemma and the arithmetic-geometric mean inequality,

$$\begin{aligned} \sum_{\substack{\beta > \frac{1}{2} \\ 0<\gamma<T \\ \zeta(\beta+i\gamma)=0}} \left(\beta-\frac{1}{2}\right) &\leqq \sum_{\substack{\beta > \frac{1}{2} \\ 0<\gamma<T \\ A\zeta(\beta+i\gamma)=0}} \left(\beta-\frac{1}{2}\right) = \frac{1}{2\pi}\int_0^T \log\left|\zeta A\left(\frac{1}{2}+it\right)\right| dt + O(\log T) \\ &\leqq \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T \left|\zeta A\left(\frac{1}{2}+it\right)\right|^2 dt\right) + O(\log T) \\ &= \frac{T}{4\pi}\log\left(1+\frac{\log T}{\log M}\right) + o(T) \\ &= \frac{T}{4\pi}\log\frac{26}{9} + o(T) \end{aligned}$$

which gives the result.

References

[1] *J. B. Conrey*, Zeros of derivatives of Riemann's xi-function on the critical line, J. Number Th. **16** (1983), 49—74.
[2] *T. Estermann*, On the representation of a number as the sum of two products, Proc. London Math. Soc. (2) **31** (1930), 123—133.
[3] *D. R. Heath-Brown*, The fourth power moment of the Riemann zeta-function, Proc. London Math. Soc. (3) **38** (1979), 385—422.
[4] *C. Hooley*, On the greatest prime factor of a cubic polynomial, J. reine angew. Math. **303/304** (1978), 21—50.
[5] *H. Iwaniec*, On mean values for Dirichlet's polynomials and the Riemann zeta-function, J. London Math. Soc. (2) **22** (1980), 39—45.
[6] *A. Selberg*, On the zeros of Riemann's zeta-function, Skr. Norske Vid. Akad. Oslo (1942), No. 10.
[7] *E. C. Titchmarsh*, Introduction to the Theory of Fourier Integrals, Oxford 1937.
[8] *E. C. Titchmarsh*, The Theory of the Riemann Zeta-Function, Oxford 1951.

Tata Institute for Fundamental Research, Bombay, India

Oklahoma State University, Stillwater, Okla. 74078, USA

Magdalen College, Oxford, OX1 4AU, England

Eingegangen 20. Juni 1984