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Invited Article

Averages of ratios of the Riemann zeta-function and correlations of divisor sums

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Abstract

Nonlinearity has published articles containing a significant number-theoretic component since the journal was first established. We examine one thread, concerning the statistics of the zeros of the Riemann zeta function. We extend this by establishing a connection between the ratios conjecture for the Riemann zeta-function and a conjecture concerning correlations of convolutions of Möbius and divisor functions. Specifically, we prove that the ratios conjecture and an arithmetic correlations conjecture imply the same result. This provides new support for the ratios conjecture, which previously had been motivated by analogy with formulae in random matrix theory and by a heuristic recipe. Our main theorem generalises a recent calculation pertaining to the special case of two-over-two ratios.

Keywords: Riemann zeta-function, correlations of divisor sums, significant number-theoretic component
Mathematics Subject Classification numbers: 11M50, 11M26, 11N37, 81Q50

1. Introduction

When *Nonlinearity* was established, it was intended that the journal's remit should be non-linear mathematics and physics, interpreted broadly. It was, therefore, a somewhat surprising decision that the subject of the first paper published should be a problem usually considered to be primarily number theoretic in nature: the asymptotic value distribution of incomplete Gauss, or theta sums [BeGo]. That decision turned out to be rather inspired, because the sums in question were later found to have deep and important connections with ergodic theory,

specifically with ergodic properties of geodesic flows on the unit tangent bundle of a certain hyperbolic surface [Mark].

Perhaps more surprising was the publication later in the first volume of *Nonlinearity* of a paper by Michael Berry [Be2] on the statistical properties of the nontrivial zeros of the Riemann zeta-function, which is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

when Re$s > 1$ and then by analytic continuation [T]. These zeros are of central importance in number theory. In this case the motivation was a conjectural connection with the semiclassical theory of quantum chaotic systems [Be1, Ke1], [BourKe].

Montgomery had earlier conjectured that the limiting pair correlation of the zeros should coincide with that of the eigenvalues of large random complex Hermitian matrices drawn from the gaussian unitary ensemble of random matrix theory (RMT) [Mont], and had proved a theorem consistent with this. It is one of the central conjectures of quantum chaos that the energy levels of generic, classically chaotic, non-time-reversal-syemetric systems should, on the scale of the mean level separation, have the same statistics in the semiclassical limit.

It was observed by Odlyzko, in numerical computations of zero-statistics around the height of the 10^{20}th zero, that there are significant deviations from predictions based on Montgomery's conjecture when one looks at a large but finite height up the critical line [O]. This suggests that the random matrix limit is approach slowly as the height T along the critical line tends to infinity. In [Be2], Berry wrote down a formula describing Odlyzko's data uniformly and remarkably accurately. This augments the random-matrix limit with lower order terms which vanish when $T \to \infty$.

These papers proved highly influential, stimulating research in several directions. First, Berry's formula was re-expressed by Bogomolny and Keating [BoKe3] to show that the lower order terms representing the deviations from the random matrix limit for large but finite T are directly related to the lowest zeros; that is, there is a resurgent relationship between the statistics of the high-lying zeros and the positions of the low-lying zeros of the zeta function. Moreover, an additional contribution from the low-lying zeros, not captured in Berry's formula, was identified [BoKe3]. For a review of these formulae, including a comparison with Odlyzko's data, see [BeKe].

Second, Montgomery's paper on the pair correlation of the Riemann zeros was extended to other principal *L*-functions [BL] (published in *Nonlinearity*), [KS, RS] and to all *n*-tuple correlations [BoKe1, BoKe2, RS]. In [BoKe1] and [BoKe2], which were also published in *Nonlinearity*, the goal was to demonstrate how correlations between the zeros relate to correlations between the primes; the zeros and primes being connected by an expression known as the explicit formula. When $n \geqslant 4$, this necessitated identifying certain unexpected contributions in the multiple prime sums involved, contributions that do not arise when $n = 2, 3$. These were called Type-II contributions in [BoKe1, BoKe2]. The Type-II contributions remained somewhat mysterious until recently, but the role they play is now starting to become clearer, as we review below.

In a separate development, the conjectured connection between the statistics of the zeros of the zeta function and RMT was used to predict the asymptotics of the moments of $\zeta(s)$ on the critical line Re$s = 1/2$. It is a long-standing and important conjecture that as $T \to \infty$

$$\frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + \mathrm{i}t)|^{2\lambda} \mathrm{d}t \sim f_\zeta(\lambda) \prod_p \left[(1 - \frac{1}{p})^{\lambda^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(\lambda + m)}{m!\Gamma(\lambda)} \right)^2 p^{-m} \right] (\log \frac{T}{2\pi})^{\lambda^2} \tag{2}$$

for some function $f_\zeta(\lambda)$. Number theoretic calculations using approximations to (1) lead to values for $f_\zeta(1)$ [HL] and $f_\zeta(2)$ [I], and to conjectures for $f_\zeta(3)$ [CG1] and $f_\zeta(4)$ [CG2]. Curiously, these calculations could not be extended straightforwardly to higher moments: they give negative values for f_ζ when clearly this function must be non-negative. Keating and Snaith put forward the idea that the zeta function on the critical line could be modelled by the characteristic polynomials of random unitary matrices [KeSn1]. The moments of random unitary matrices can be calculated, and this lead to a prediction for $f_\zeta(\lambda)$ for all $\mathrm{Re}\,\lambda > -1/2$ that matches the values previously calculated or conjectured. For a review, see [Ke2].

This approach extends to other L-functions [CF, KeSn2], to lower order terms in the moment asymptotics [CFKRS2], and generalises to conjectures for ratios such as

$$\frac{1}{T} \int_0^T \frac{\prod_{\alpha \in A} \zeta(s+\alpha) \prod_{\beta \in B} \zeta(1-s+\beta)}{\prod_{\gamma \in C} \zeta(s+\gamma) \prod_{\delta \in D} \zeta(1-s+\delta)} \, \mathrm{d}t, \tag{3}$$

where $s = 1/2 + \mathrm{i}t$, A and B are sets of complex numbers with real parts smaller than $1/4$, and C and D are sets of complex numbers with positive real parts smaller than $1/4$ [CFZ]. Averages of ratios like this are important for several reasons. First one can use them to derive expressions for correlations between the zeros [CS]. Indeed, the conjectured form for the ratio with two zetas in the numerator and two in the denominator (i.e. the 'two-over-two' ratio) leads to precisely the same formula for the pair correlation of the zeros as that obtained by Bogomolny and Keating [BoKe3], including the resurgent lower order terms. (In this case, the leading order asymptotic as $T \to \infty$ had previously been conjectured by Farmer, who showed that it implied Montgomery's conjecture [F].) Second, the conjectured form of ratios with terms only in the numerator, i.e. averages of the form

$$\frac{1}{T} \int_0^T \prod_{\alpha \in A} \zeta(s+\alpha) \prod_{\beta \in B} \zeta(1-s+\beta) \, \mathrm{d}t, \tag{4}$$

coincide with the leading order formulae for the moments of the zeta function and other L-functions predicted by Keating and Snaith [KeSn1, KeSn2], and in addition with all lower-order terms in the asymptotics [CFKRS2].

This leaves open the problem of finding a firm number-theoretic foundation for the moment conjecture and the ratio conjecture. These conjectures are supported by random matrix models (the analogous formulae can be proved rigorously and unconditionally in random matrix theory [KeSn1, KeSn2, CFKRS1, CFZ]), by formal manipulations of the Dirichlet series (1) called the heuristic recipe [CFKRS2, CFZ], and by extensive numerical computations (see e.g. [CFKRS2]). However, as noted above, systematic number-theoretical calculations for the moments give rigorous results when $\lambda = 1, 2$ and lead to conjectures consistent with the random matrix models when $\lambda = 3, 4$, but lead to answers that are clearly incorrect for higher moments.

The resolution of this mystery has been explained in a recent series of papers [CK1–CK5], where it was shown that contributions similar to the Type-II terms introduced in [BoKe1, BoKe2] account for the discrepancy. These contributions had previously been neglected in moment calculations, but evaluating them gives answers that are fully consistent with those of [CFKRS2] for averages such as (4), i.e. for all terms in the asymptotics identified in [CFKRS2], not just the leading order term of [KeSn1, KeSn2].

We have recently begun to extend our calculations to averages of ratios as in (3). In [CK6] we computed a particular set of contributions to the two-over-two ratio average and showed that the result matches previously conjectured expressions based on the recipe [CFKRS2, CFZ]. Our goal here is to extend that calculation to general ratios. We set out our specific results in the next section.

2. Statement of results

Let A and B be sets of complex numbers with real parts smaller than $1/4$. Let C and D be sets of complex numbers with positive real parts smaller than $1/4$. The purpose of this paper is to investigate the averages

$$\mathcal{R}_{A,B,C,D}(T) := \int_0^\infty \psi\left(\frac{t}{T}\right) \frac{\prod_{\alpha\in A}\zeta(s+\alpha)\prod_{\beta\in B}\zeta(1-s+\beta)}{\prod_{\gamma\in C}\zeta(s+\gamma)\prod_{\delta\in D}\zeta(1-s+\delta)}\,\mathrm{d}t$$

where ψ is a smooth function with compact support, say $\psi \in C^\infty[1,2]$, $s = 1/2 + \mathrm{i}t$, and A, B, C, D are sets of small complex numbers, referred to as the shifts. $\mathcal{R}$ is the subject of the 'ratios conjecture' originally formulated in [CFZ] and studied in [CS]. In these prior studies the perspective was from the point of view of analogy with random matrix theory (RMT). Our new perspective is to study this quantity from an arithmetic point of view. In particular, we identify those parts of the ratios conjecture that arise from a study of the coefficient correlations

$$\sum_{n\leqslant X} I_{A,C}(n)I_{B,D}(n+h)$$

where $I_{A,C}$ is defined implicitly by

$$\sum_{n=1}^\infty \frac{I_{A,C}(n)}{n^s} = \frac{\prod_{\alpha\in A}\zeta(s+\alpha)}{\prod_{\gamma\in C}\zeta(s+\gamma)}.$$

In this paper we will describe this connection explicitly.

Not surprisingly, $\mathcal{R}$ is related to averages of the (analytic continuation of the) Rankin–Selberg convolution

$$\mathcal{B}_{A,B,C,D}(s) := \sum_{n=1}^\infty \frac{I_{A,C}(n)I_{B,D}(n)}{n^s}.$$

In fact, we can state the ratios conjecture in a relatively simple way in terms of $\mathcal{B}$.

Conjecture 1. [[CFZ] and [CS]] Suppose that the sets A, B, C and D are as in the introduction and that the imaginary parts of all of the parameters in this set are $O(T^{1-\xi})$ for some $\xi > 0$. Then

$$\mathcal{R}_{A,B,C,D}(T) = \int_0^\infty \psi\left(\frac{t}{T}\right) \sum_{\substack{U\subset A, V\subset B \\ |U|=|V|}} \left(\frac{t}{2\pi}\right)^{-\sum_{\hat\alpha\in U}\hat\alpha - \sum_{\hat\beta\in V}\hat\beta} \mathcal{B}_{A-U+V^-,B-V+U^-,C,D}(1)\,\mathrm{d}t + O(T^{1-\eta})$$

for some $\eta > 0$.

Here V^- denotes the set obtained from V by replacing every element by its negative. So, the set $A - U + V^-$ may be obtained from the set A by deleting the elements of U and then inserting the negatives of the elements of V. (We assume that the elements of all of these sets are distinct. The situation with repeating elements may be deduced from this case by a limiting argument.)

It is also not surprising that $\mathcal{R}$ is connected to weighted averages over n and h of

$$I_{A,C}(n)I_{B,D}(n+h).$$

It is this connection that we are elucidating.

Using the δ-method [DFI] it may be shown that the weighted averages relevant to the consideration of $\mathcal{R}$ can be expressed in terms of

$$\mathcal{C}_{A,B,C,D}(s) := \frac{1}{(2\pi i)^2} \int_{|w-1|=\epsilon} \int_{|z-1|=\epsilon} \chi(w+z-s-1) \sum_{q=1}^{\infty} \sum_{h=1}^{\infty} \frac{r_q(h)}{h^{s+2-w-z}}$$

$$\times \sum_{m=1}^{\infty} \frac{I_{A,C}(m) e(m/q)}{m^w} \sum_{n=1}^{\infty} \frac{I_{B,D}(n) e(n/q)}{n^z} \, dw \, dz$$

where $r_q(h)$ denotes Ramanujan's sum and where $\chi(s)$ is the factor from the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$. Here and below ϵ is chosen to be larger than the absolute values of the shift parameters $\alpha, \beta, \gamma, \delta$ but smaller than $1/2$.

The main conclusion of this paper is that the arithmetic contributions arising from the averages of $I_{A,C}(n) I_{B,D}(n+h)$ coincide exactly with the terms from the ratios conjecture with $|U| = |V| = 1$, i.e. what are referred to elsewhere as the 'one-swap' terms.

The result that explicates this is encapsulated in the following identity.

Theorem 1. *Assuming the generalized Riemann hypothesis*

$$\mathcal{C}_{A,B,C,D}(s) = \sum_{\substack{U \subset A, V \subset B \\ |U|=|V|=1}} \mathcal{B}_{A-U+V^-, B-V+U^-, C, D}(s+1).$$

It turns out to be convenient to study an average of the ratios conjecture. To this end let

$$\mathcal{I}_{A,C}(s; X) = \sum_{n \leqslant X} I_{A,C}(n) n^{-s}.$$

We are interested in the average over t of $\mathcal{I}_{A,C}(s, X)\overline{\mathcal{I}_{B,D}}(1-s, X)$ (N.B. $s = 1/2 + it$) in the case that $X = T^{\lambda}$ for some $\lambda > 1$. (When $\lambda < 1$ this average is dominated by diagonal terms.) We give two different treatments of the average of 'truncated' ratios:

$$\mathcal{M}_{A,B,C,D}(T; X) := \int_0^{\infty} \psi\left(\frac{t}{T}\right) \mathcal{I}_{A,C}(s, X)\mathcal{I}_{B,D}(1-s, X) \, dt$$

(where again $s = 1/2 + it$) which lead to the same answer. The first is by the ratios conjecture and the second is by consideration of the correlations of the coefficients.

In each case we prove

Theorem 2. *Let A, B, C, D be as above. Then, assuming either a uniform version of the ratios conjecture or a uniform version of the conjectural formula for correlations of values of $I_{\alpha,\gamma}(n)$, we have for some $\eta > 0$ and some $\lambda > 1$,*

$$\mathcal{M}_{A,B,C,D}(T; X)$$
$$= \int_0^{\infty} \psi\left(\frac{t}{T}\right) \frac{1}{2\pi i} \int_{\Re s=2} \sum_{\substack{U \subset A, V \subset B \\ |U|=|V| \leqslant 1}} \left(\frac{t}{2\pi}\right)^{-|U|s - \sum_{\hat{\alpha} \in U} \hat{\alpha} - \sum_{\hat{\beta} \in V} \hat{\beta}} \mathcal{B}_{A-U+V^-, B-V+U^-, C, D}(s+1) \frac{X^s}{s} \, ds \, dt$$
$$+ O(T^{1-\eta}).$$

This shows that the ratios conjecture follows not only from the 'recipe' of [CFZ], but also relates to correlations of values of $I_{A,C}(n)$.

An earlier paper [CK6] had this calculation but in the special case that all of the sets A, B, C, D are singletons. That paper has additional background information and motivation, in particular relating to connections with correlations between the zeros (see [BoKe1, BoKe2]) and with the moments (see [CK1–CK5]) of the zeta function. The first part of the

present calculation follows closely that in [CK6]; we include this so that the narrative is as self-contained as possible.

3. Approach via the ratios conjecture

We have

$$\mathcal{I}_{A,C}(s,X) = \frac{1}{2\pi i} \int_{(2)} \mathcal{I}_{A,C}(s+w) \frac{X^w}{w} \, dw;$$

there is a similar expression for $\mathcal{I}_{B,D}(s,X)$. Inserting these expressions and rearranging the integrations we have

$$\mathcal{M}_{A,B,C,D}(T;X) = \frac{1}{(2\pi i)^2} \int_{\Re w=2} \int_{\Re z=2} \frac{X^{w+z}}{wz} \mathcal{R}_{A_w,B_z,C_w,D_z}(T) \, dw \, dz,$$

where $A_w = \{\alpha + w : \alpha \in A\}$, etc. We note that conjecture 1 implies that $\mathcal{R}_{A_w,B_z,C_w,D_z}$ is, to leading order as $T \to \infty$, a function of $z + w$. We therefore make the change of variable $s = z + w$. The integration in the s variable is now on the vertical line $\Re s = 4$. We retain z as our other variable and integrate over it. This leads to the integral

$$\frac{1}{2\pi i} \int_{\Re z=2} \frac{dz}{z(s-z)} = \frac{1}{s},$$

which may be seen by moving the path of integration to the left, to $\Re z = -\infty$. Thus we have that $\mathcal{M}_{A,B,C,D}(T;X)$ is given to leading order by

$$\frac{1}{2\pi i} \int_{\Re s=4} \frac{X^s}{s} \mathcal{R}_{A_s,B,C_s,D}(T) \, ds.$$

Moving the path of integration to $\Re s = \epsilon$, avoiding any poles, inserting conjecture 1, and noting that

$$\mathcal{B}_{A_s,B,C_s,D}(1) = \mathcal{B}_{A,B,C,D}(s+1),$$

we have that the uniform ratios conjecture implies the conclusion of theorem 2.

4. Approach via coefficient correlations

We follow the approach developed by Goldston and Gonek [GG] in their work on mean-values of long Dirichlet polynomials.

Expanding the sums and integrating term-by-term, we have

$$\mathcal{M}_{\alpha,\beta,\gamma,\delta}(T;X) = T \sum_{m,n \leqslant X} \frac{I_{A,C}(m) I_{B,D}(n)}{\sqrt{mn}} \hat{\psi} \left(\frac{T}{2\pi} \log \frac{m}{n} \right).$$

4.1. Diagonal

The diagonal term is

$$T\hat{\psi}(0) \sum_{m \leqslant X} \frac{I_{A,C}(m) I_{B,D}(m)}{m}.$$

By Perron's formula this sum is

$$\frac{1}{2\pi i} \int_{(2)} \mathcal{B}_{A,B,C,D}(s+1) \frac{X^s}{s} \, ds.$$

4.2. Off-diagonal

For the off-diagonal terms we need to analyze

$$2T \sum_{T \leqslant m \leqslant X} \sum_{1 \leqslant h \leqslant \frac{X}{T}} \frac{I_{A,C}(m) I_{B,D}(m+h)}{m} \hat{\psi}\left(\frac{Th}{2\pi m}\right).$$

We replace the arithmetic terms by their average and express this as

$$2T \int_T^X \sum_{1 \leqslant h \leqslant \frac{X}{T}} \frac{\langle I_{A,C}(m) I_{B,D}(m+h)\rangle_{m \sim u}}{u} \hat{\psi}\left(\frac{Th}{2\pi u}\right) \, du.$$

We now compute the average heuristically via the delta-method [DFI]:

$$\langle I_{A,C}(m) I_{B,D}(m+h)\rangle_{m \sim u} \sim \sum_{q=1}^{\infty} r_q(h) \langle I_{A,C}(m) e(m/q)\rangle_{m \sim u} \langle I_{B,D}(m) e(m/q)\rangle_{m \sim u}$$

where $r_q(h)$ is the Ramanujan sum, a formula for which is $r_q(h) = \sum_{\substack{d|h \\ d|q}} d\mu(\frac{q}{d})$. This may be formalized as a precise conjecture exactly as in section 5 of [CK6]. It is this conjectural formula that we refer to in theorem 2. Now

$$\langle I_{A,C}(m) e(m/q)\rangle_{m \sim u} = \frac{1}{2\pi i} \int_{|w-1|=\epsilon} \sum_{m=1}^{\infty} I_{A,C}(m) e(m/q) m^{-w} u^{w-1} \, dw.$$

The off-diagonal contribution is thus

$$2T \sum_{1 \leqslant h \leqslant \frac{X}{T}} \int_T^X \frac{1}{(2\pi i)^2} \int \int_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^{\infty} r_q(h) \hat{\psi}\left(\frac{Th}{2\pi u}\right) u^{w+z-2}$$

$$\times \sum_{m_1=1}^{\infty} \frac{I_{A,C}(m_1) e(m_1/q)}{m_1^w} \sum_{m_2=1}^{\infty} \frac{I_{B,D}(m_2) e(m_2/q)}{m_2^z} \, dw \, dz \, \frac{du}{u}.$$

We next make the change of variables $v = \frac{Th}{2\pi u}$. The inequality $u \leqslant X$ then implies that $\frac{Th}{2\pi v} \leqslant X$ or $h \leqslant \frac{2\pi vX}{T}$. The above can be re-expressed as

$$2T \int_0^{\infty} \sum_{1 \leqslant h \leqslant \frac{2\pi vX}{T}} \frac{1}{(2\pi i)^2} \int \int_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^{\infty} r_q(h) \hat{\psi}(v) \left(\frac{Th}{2\pi v}\right)^{w+z-2}$$

$$\times \sum_{m_1=1}^{\infty} \frac{I_{A,C}(m_1) e(m_1/q)}{m_1^w} \sum_{m_2=1}^{\infty} \frac{I_{B,D}(m_2) e(m_2/q)}{m_2^z} \, dw \, dz \, \frac{dv}{v}.$$

Using Perron's formula to express the sum over h gives

$$2T \int_0^\infty \frac{1}{(2\pi i)^3} \int_{\Re s=2} \int \int_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^\infty \sum_{h=1}^\infty \frac{r_q(h)}{h^s} \hat\psi(v) \left(\frac{Th}{2\pi v}\right)^{w+z-2} \left(\frac{2\pi vX}{T}\right)^s$$
$$\times \sum_{m_1=1}^\infty \frac{I_{A,C}(m_1)e(m_1/q)}{m_1^w} \sum_{m_2=1}^\infty \frac{I_{B,D}(m_2)e(m_2/q)}{m_2^z} \frac{ds}{s} \, dw \, dz \, \frac{dv}{v}.$$

Now

$$2\int_0^\infty \hat\psi(v) v^A \frac{dv}{v} = \chi(1-A) \int_0^\infty \psi(t) t^{-A} \, dt.$$

Incorporating this formula gives

$$T \int_0^\infty \psi(t) \frac{1}{(2\pi i)^3} \int_{\Re s=2} \int \int_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^\infty \sum_{h=1}^\infty \frac{r_q(h)}{h^{s+2-w-z}} \left(\frac{Tt}{2\pi}\right)^{w+z-2} \left(\frac{2\pi X}{tT}\right)^s \chi(w+z-s-1)$$
$$\times \sum_{m_1=1}^\infty \frac{I_{A,C}(m_1)e(m_1/q)}{m_1^w} \sum_{m_2=1}^\infty \frac{I_{B,D}(m_2)e(m_2/q)}{m_2^z} \frac{ds}{s} \, dw \, dz \, dt.$$

By theorem 1, this is

$$\sum_{\substack{\hat\alpha \in A \\ \hat\beta \in B}} \int_0^\infty \psi\left(\frac{t}{T}\right) \frac{1}{2\pi i} \int_{\Re s=2} \left(\frac{t}{2\pi}\right)^{-\hat\alpha-\hat\beta-s} \mathcal{B}_{A'\cup\{-\hat\beta\},B'\cup\{-\hat\alpha\},C,D}(s+1) \frac{X^s}{s} \, ds \, dt$$

where $A' = A - \{\hat\alpha\}$ and $B' = B - \{\hat\beta\}$. Adding the diagonal and off-diagonal terms, we thus obtain that the conjecture for the correlations of values of $I_{A,C}(n)$ also implies the conclusion of theorem 2.

5. Proof of theorem 1

First of all, we have

$$\sum_{h=1}^\infty \frac{r_q(h)}{h^A} = \sum_{h=1}^\infty \frac{\sum_{\substack{g|q \\ g|h}} g\mu(\frac{q}{g})}{h^A} = \sum_{g|q} g^{1-A}\mu(\frac{q}{g})\zeta(A) = q^{1-A}\Phi(1-A,q)\zeta(A)$$

where

$$\Phi(x,q) = \prod_{p|q}\left(1 - \frac{1}{p^x}\right).$$

Using this and the functional equation for ζ, we have to evaluate

$$\frac{1}{(2\pi i)^2} \int \int_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^\infty q^{w+z-s-1}\Phi(w+z-s-1,q)$$
$$\times \zeta(w+z-s-1) \sum_{m_1=1}^\infty \frac{I_{A,C}(m_1)e(m_1/q)}{m_1^w} \sum_{m_2=1}^\infty \frac{I_{B,D}(m_2)e(m_2/q)}{m_2^z} \, dw \, dz.$$

We identify the polar structure of the Dirichlet series here by passing to characters via the formula

$$\mathrm{e}\left(\frac{m}{q}\right) = \sum_{\substack{d|m \\ d|q}} \frac{1}{\phi\left(\frac{q}{d}\right)} \sum_{\chi \bmod \frac{q}{d}} \tau(\overline{\chi})\chi\left(\frac{m}{d}\right).$$

Assuming GRH, the only poles near $w = 1$ arise from the principal characters $\chi_{\frac{q}{d}}^{(0)}$. Using

$$\tau(\chi_{\frac{q}{d}}^{(0)}) = \mu(\frac{q}{d})$$

we have that the poles of $\sum_{m=1}^{\infty} I_{A,C}(m)\mathrm{e}(m/q)m^{-w}$ are the same as the poles of

$$\sum_{d|q} \frac{\mu\left(\frac{q}{d}\right)}{\phi\left(\frac{q}{d}\right)} \sum_{m=1}^{\infty} I_{A,C}(md)\chi_{\frac{q}{d}}^{(0)}(m)m^{-w}d^{-w}$$

$$= q^{-w} \sum_{d|q} \frac{\mu(d)}{\phi(d)}d^w \sum_{m=1}^{\infty} \frac{I_{A,C}(\frac{mq}{d})\chi_d^{(0)}(m)}{m^w}$$

and the principal parts are the same. We now replace $\chi_d^{(0)}(m)$ by $\sum_{\substack{e|d \\ e|m}} \mu(e)$. This leads to

$$q^{-w} \sum_{d|q} \frac{\mu(d)d^w}{\phi(d)} \sum_{e|d} \mu(e)\mathrm{e}^{-w} \sum_{m=1}^{\infty} \frac{I_{A,C}(\frac{meq}{d})}{m^w}.$$

Now we need the polar structure of

$$\sum_{m=1}^{\infty} I_{A,C}(mr)m^{-w}$$

for $r = qe/d$.

Since $I_{A,C}(n)$ is a multiplicative function of n, $I_{A,C}(nr)/I_{A,C}(r)$ is also a multiplicative function of n. The generating function may therefore be expressed as an Euler product:

$$\sum_{n=1}^{\infty} \frac{I_{A,C}(nr)/I_{A,C}(r)}{n^w} = \sum_{n=1}^{\infty} \frac{I_{A,C}(n)}{n^w} \prod_{p|r} \frac{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^{j+\lambda_r(p)})/I_{A,C}(p^{\lambda_r(p)})}{p^{jw}}}{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^j)}{p^{jw}}}$$

This gives

$$\sum_{n=1}^{\infty} \frac{I_{A,C}(nr)}{n^w} = \frac{\prod_{\alpha \in A} \zeta(w+\alpha)}{\prod_{\gamma \in C} \zeta(w+\gamma)} \prod_{p|r} \frac{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^{j+\lambda_r(p)})}{p^{jw}}}{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^j)}{p^{jw}}}$$

$$= \frac{\prod_{\alpha \in A} \zeta(w+\alpha)}{\prod_{\gamma \in C} \zeta(w+\gamma)} E_{A,C}(w,r),$$

say. In particular, the poles are at $w = 1 - \alpha$ for $\alpha \in A$. Thus, the integral over w and z is

$$\sum_{\substack{\hat{\alpha} \in A \\ \hat{\beta} \in B}} \sum_{q=1}^{\infty} q^{1-\hat{\alpha}-\hat{\beta}-s} \Phi(1-\hat{\alpha}-\hat{\beta}-s,q)\zeta(1-\hat{\alpha}-\hat{\beta}-s)$$

$$\times q^{-1+\hat{\alpha}} \sum_{d_1|q} \frac{\mu(d_1)d_1^{1-\hat{\alpha}}}{\phi(d_1)} \sum_{e_1|d_1} \mu(e_1)e_1^{-1+\hat{\alpha}} q^{-1+\hat{\beta}} \sum_{d_2|q} \frac{\mu(d_2)d_2^{1-\hat{\beta}}}{\phi(d_2)} \sum_{e_2|d_2} \mu(e_2)e_2^{-1+\hat{\beta}}$$

$$\times \frac{\prod_{\alpha \in A'} \zeta(1-\hat{\alpha}+\alpha) \prod_{\beta \in B'} \zeta(1-\hat{\beta}+\beta)}{\prod_{\gamma \in C} \zeta(1-\hat{\alpha}+\gamma) \prod_{\delta \in D} \zeta(1-\hat{\beta}+\delta)} E_{A,C}(1-\hat{\alpha},\frac{qe_1}{d_1})E_{B,D}(1-\hat{\beta},\frac{qe_2}{d_2}).$$

So we have to identify the Dirichlet series

$$\sum_{q=1}^{\infty} q^{-1-s} \Phi(1-\hat{\alpha}-\hat{\beta}-s,q)\mathcal{E}_{A,C}(1-\hat{\alpha},q)\mathcal{E}_{B,D}(1-\hat{\beta},q)$$

$$= \sum_{r=1}^{\infty} \frac{\mu(r)}{r^{2-\hat{\alpha}-\hat{\beta}}} \sum_{q=1}^{\infty} \frac{\mathcal{E}_{A,C}(1-\hat{\alpha},qr)\mathcal{E}_{B,D}(1-\hat{\beta},qr)}{q^{1+s}}$$

where we have made use of $\Phi(\xi,q) = \sum_{r|q} \mu(r)r^{-\xi}$, and where

$$\mathcal{E}_{A,C}(1-\hat{\alpha},q) = \sum_{d|q} \frac{\mu(d)d^{1-\hat{\alpha}}}{\phi(d)} \sum_{e|d} \mu(e)e^{-1+\hat{\alpha}} E_{A,C}(1-\hat{\alpha},\frac{qe}{d}).$$

This itself may be expressed as an Euler product. So, let us assume $q = p^J$ with $J \geqslant 1$ and identify

$$\sum_{d|q} \frac{\mu(d)d^{1-\hat{\alpha}}}{\phi(d)} \sum_{e|d} \mu(e)e^{-1+\hat{\alpha}} E_{A,C}(1-\hat{\alpha},\frac{qe}{d})$$

$$= E_{A,C}(1-\hat{\alpha},p^J) - \frac{p^{1-\hat{\alpha}}}{p-1} E_{A,C}(1-\hat{\alpha},p^{J-1}) + \frac{1}{p-1} E_{A,C}(1-\hat{\alpha},p^J)$$

$$= \frac{p}{p-1} E_{A,C}(1-\hat{\alpha},p^J) - \frac{p^{1-\hat{\alpha}}}{p-1} E_{A,C}(1-\hat{\alpha},p^{J-1}).$$

Now we note the identity

$$I_{A,C}(p^J) = I_{A',C}(p^J) + p^{-\alpha} I_{A,C}(p^{J-1})$$

where $A = A' \cup \{\alpha\}$. Thus

$$\sum_{j=0}^{\infty} \frac{I_{A,C}(p^{j+J})}{p^{jw}} - p^{-\alpha} \sum_{j=0}^{\infty} \frac{I_{A,C}(p^{j+J-1})}{p^{jw}} = \sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+J})}{p^{jw}},$$

and so

$$\mathcal{E}_{A,C}(1-\hat{\alpha},p^J) = \frac{p}{p-1} \frac{\sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+J})}{p^{j(1-\hat{\alpha})}}}{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^j)}{p^{j(1-\hat{\alpha})}}}$$

$$= \frac{p}{p-1} \frac{\prod_{\alpha \in A}(1-\frac{1}{p^{1-\hat{\alpha}+\alpha}})}{\prod_{\gamma \in C}(1-\frac{1}{p^{1-\hat{\alpha}+\gamma}})} \sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+J})}{p^{j(1-\hat{\alpha})}}$$

$$= \frac{\prod_{\alpha \in A'}(1-\frac{1}{p^{1-\hat{\alpha}+\alpha}})}{\prod_{\gamma \in C}(1-\frac{1}{p^{1-\hat{\alpha}+\gamma}})} \sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+J})}{p^{j(1-\hat{\alpha})}}.$$

Now, all that is left to do is to prove that

$$\sum_{\ell=0}^{\infty} \frac{\mu(p^\ell)}{p^{\ell(2-\hat\alpha-\hat\beta)}} \sum_{J=0}^{\infty} \frac{1}{p^J} \sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+\ell+J})}{p^{j(1-\hat\alpha)}} \sum_{k=0}^{\infty} \frac{I_{B',D}(p^{k+\ell+J})}{p^{k(1-\hat\beta)}}$$

$$= \left(1 - \frac{1}{p^{1-\hat\alpha-\hat\beta}}\right) \sum_{\ell=0}^{\infty} \frac{I_{A'\cup\{-\hat\beta\},C}(p^\ell) I_{B'\cup\{-\alpha\},D}(p^\ell)}{p^\ell}.$$

Temporarily let $X = 1/p$, $Y = p^{-\hat\alpha}$, $Z = p^{-\hat\beta}$, $a_j = I_{A,C}(p^j)$, $a'_j = I_{A',C}(p^j)$, $\tilde{a}_j = I_{A'\cup\{-\hat\beta\},C}(p^j)$; and $b_k = I_{B,D}(p^k)$, $b'_k = I_{B',D}(p^k)$, $\tilde{b}_k = I_{B'\cup\{-\hat\alpha\},D}(p^k)$. Then the desired identity follows from the theorem of the next section.

6. The identity

Theorem 3. *Suppose that $a', b', \tilde{a}, \tilde{b}$ are sequences such that*

$$\sum_{J=0}^{\ell} Z^{J-\ell} a'_J = \tilde{a}_\ell \qquad \sum_{K=0}^{\ell} Y^{K-\ell} b'_K = \tilde{b}_\ell.$$

Then

$$\sum_{J=0}^{\infty} \sum_{\ell=0}^{\min(1,J)} (-1)^\ell \frac{X^{2\ell+J}}{Y^\ell Z^\ell} \sum_{j=0}^{\infty} a'_{j+\ell+J} \left(\frac{X}{Y}\right)^j \sum_{k=0}^{\infty} b'_{k+\ell+J} \left(\frac{X}{Z}\right)^k = \left(1 - \frac{X}{YZ}\right) \sum_{\ell=0}^{\infty} \tilde{a}_\ell \tilde{b}_\ell X^\ell.$$

Proof. The left side may be written as

$$\sum_{J=0}^{\infty} X^J \sum_{j=0}^{\infty} a'_{j+J} \left(\frac{X}{Y}\right)^j \sum_{k=0}^{\infty} b'_{k+J} \left(\frac{X}{Z}\right)^k$$

$$- \sum_{J=0}^{\infty} X^J \sum_{j=0}^{\infty} a'_{j+1+J} \left(\frac{X}{Y}\right)^{j+1} \sum_{k=0}^{\infty} b'_{k+1+J} \left(\frac{X}{Z}\right)^{k+1}.$$

This may be rewritten as

$$\sum_{J=0}^{\infty} X^J \left[\left(\sum_{j=0}^{\infty} a'_{j+J} \left(\frac{X}{Y}\right)^j - \sum_{j=0}^{\infty} a'_{j+1+J} \left(\frac{X}{Y}\right)^{j+1} \right) \sum_{k=0}^{\infty} b'_{k+J} \left(\frac{X}{Z}\right)^k \right.$$

$$\left. + \sum_{j=0}^{\infty} a'_{j+1+J} \left(\frac{X}{Y}\right)^{j+1} \left(\sum_{k=0}^{\infty} b'_{k+J} \left(\frac{X}{Z}\right)^k - \sum_{k=0}^{\infty} b'_{k+1+J} \left(\frac{X}{Z}\right)^{k+1} \right) \right]$$

which simplifies to

$$\sum_{J=0}^{\infty} X^J \left[a_J' \sum_{k=0}^{\infty} b_{k+J}' \left(\frac{X}{Z} \right)^k + b_J' \sum_{j=0}^{\infty} a_{j+1+J}' \left(\frac{X}{Y} \right)^{j+1} \right].$$

Now

$$\sum_{J=0}^{\infty} X^J a_J' \sum_{k=0}^{\infty} b_{k+J}' \left(\frac{X}{Z} \right)^k = \sum_{\ell=0}^{\infty} X^\ell b_\ell' \sum_{J=0}^{\ell} a_J' Z^{J-\ell} = \sum_{\ell=0}^{\infty} \tilde{a}_\ell b_\ell' X^\ell.$$

And

$$\sum_{J=0}^{\infty} X^J b_J' \sum_{j=0}^{\infty} a_{j+1+J}' \left(\frac{X}{Y} \right)^{j+1}$$

$$= \sum_{J=0}^{\infty} X^J b_J' \sum_{j=0}^{\infty} a_{j+J}' \left(\frac{X}{Y} \right)^{j} - \sum_{J=0}^{\infty} a_J' b_J' X^J$$

$$= \sum_{\ell=0}^{\infty} a_\ell' \tilde{b}_\ell X^\ell - \sum_{\ell=0}^{\infty} a_\ell' b_\ell' X^\ell.$$

Thus, the left side of the identity is

$$\sum_{\ell=0}^{\infty} \tilde{a}_\ell b_\ell' X^\ell + \sum_{\ell=0}^{\infty} a_\ell' \tilde{b}_\ell X^\ell - \sum_{\ell=0}^{\infty} a_\ell' b_\ell' X^\ell.$$

But $a_\ell' = \tilde{a}_\ell - \frac{\tilde{a}_{\ell-1}}{Z}$ and $b_\ell' = \tilde{b}_\ell - \frac{\tilde{b}_{\ell-1}}{Y}$ so that

$$\tilde{a}_\ell b_\ell' + a_\ell' \tilde{b}_\ell - a_\ell' b_\ell' = \tilde{a}_\ell \tilde{b}_\ell - \frac{\tilde{a}_{\ell-1} \tilde{b}_{\ell-1}}{YZ}.$$

The sum over ℓ of this expression times X^ℓ gives the right side of the identity. $\qquad\square$

The reader may have noticed the similarity between this identity and the corresponding identity that formed the crux of [CK3].

Acknowledgments

References

[Be1]   Berry M V 1986 Riemann's zeta function: a model for quantum chaos? *Quantum Chaos and Statistical Nuclear Physics* (*Springer Lecture Notes in Physics*) ed T H Seligman and H Nishioka pp 1–17

[Be2]   Berry M V 1988 Semiclassical formula for the number variance of the Riemann zeros *Nonlinearity* **1** 399–407

[BeGo]   Berry M V and Goldberg J 1988 Renormalization of curlicues *Nonlinearity* **1** 1–26

[BeKe]   Berry M V and Keating J P 1999 The Riemann zeros and eigenvalue asymptotics *SIAM Rev.* **41** 236–66

[BoKe1]   Bogomolny E B and Keating J P 1995 Random matrix theory and the Riemann zeros I: three- and four-point correlations *Nonlinearity* **8** 1115–31

[BoKe2]   Bogomolny E B and Keating J P 1996 Random matrix theory and the Riemann zeros II: *n*-point correlations *Nonlinearity* **9** 911–35

[BoKe3]   Bogomolny E B and Keating J P 1996 Gutzwiller's trace formula and spectral statistics: beyond the diagonal approximation *Phys. Rev. Lett.* **77** 1472–5

[BL]   Bogomolny E B and Leboeuf P 1994 Statistical properties of the zeros of zeta-functions – beyond the Riemann case *Nonlinearity* **7** 1155–67

[BourKe]   Bourgade P and Keating J P 2013 Quantum chaos, random matrix theory, and the Riemann ζ-function *Chaos: Poincaré Seminar 2010* (*Progress in Mathematical Physics* vol 66) ed B Duplantier *et al* (Berlin: Springer) pp 125–68

[CF]   Conrey J B and Farmer D W 2000 Mean values of *L*-functions and symmetry *Int. Math. Res. Notices* **17** 883–908

[CFKRS1]   Conrey J B, Farmer D W, Keating J P, Rubinstein M O and Snaith N C 2003 Autocorrelation of random matrix polynomials *Commun. Math. Phys.* **237** 365–95

[CFKRS2]   Conrey J B, Farmer D W, Keating J P, Rubinstein M O and Snaith N C 2005 Integral moments of *L*-functions *Proc. Lond. Math. Soc.* **91** 33–104

[CFZ]   Conrey J B, Farmer D W and Zirnbauer M R 2008 Autocorrelation of ratios of *L*-functions *Commun. Number Theory Phys.* **2** 593–636

[CG1]   Conrey J B and Ghosh A 1992 On mean values of the zeta-function, iii *Proc. of the Amalfi Conf. on Analytic Number Theory Università di Salerno*

[CG2]   Conrey J B and Gonek S M 2001 High moments of the Riemann zeta-function *Duke Math. J.* **107** 577–604

[CK1]   Conrey J B and Keating J P 2015 Moments of zeta and correlations of divisor-sums: I *Phil. Trans. R. Soc.* A **373** 20140313

[CK2]   Conrey J B and Keating J P 2015 Moments of zeta and correlations of divisor-sums: II *Proc. of the 13th Conf. of the Canadian Number Theory Association, Fields Institute Communications* (*Advances in the Theory of Numbers*) ed A Alaca *et al* (Berlin: Springer) pp 75–85

[CK3]   Conrey J B and Keating J P 2015 Moments of zeta and correlations of divisor-sums: III *Indagationes Math.* **26** 736–47

[CK4]   Conrey J B and Keating J P 2016 Moments of zeta and correlations of divisor-sums: IV *Res. Number Theory* **2** 24

[CK5]   Conrey J B and Keating J P 2017 Moments of zeta and correlations of divisor-sums: V (arXiv:1701.06651)

[CK6]   Conrey J B and Keating J P 2016 Pair correlation and twin primes revisited *Proc. R. Soc.* A **472** 20160548

[CS]   Conrey J B and Snaith N C 2007 Applications of the *L*-functions ratios conjectures *Proc. London Math. Soc.* **94** 594–646

[DFI]   Duke W, Friedlander J B and Iwaniec H 1994 A quadratic divisor problem *Invent. Math.* **115** 209–17

[F]   Farmer D W 1993 Long mollifiers of the Riemann zeta-function *Mathematika* **40** 71–87 (part 1)

[GG]   Goldston D A and Gonek S M 1998 Mean value theorems for long Dirichlet polynomials and tails of Dirichlet series *Acta Arith.* **84** 155–92

[HL]   Hardy G H and Littlewood J E 1918 Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes *Acta Math.* **41** 119–96

[I]   Ingham A E 1926 Mean-value theorems in the theory of the Riemann zeta-function *Proc. London Math. Soc.* 27 273–300

[KS]   Katz N M and Sarnak P 1999 *Random Matrices, Frobenius Eigenvalues and Monodromy* (*American Mathematical Society Colloquium Publications* vol 45) (Providence, RI: American Mathematical Society)

[Ke1]   Keating J P 1993 The Riemann zeta function and quantum chaology *Quantum Chaos* ed G Casati *et al* pp 145–85 (Amsterdam: North-Holland)

[Ke2]   Keating J P 2005 *L*-functions and the characteristic polynomials of random matrices *Recent Perspectives in Random Matrix Theory and Number Theory* (*London Mathematical Society Lecture Note Series* vol 322) ed F Mezzadri and N C Snaith (Cambridge: Cambridge University Press) pp 251–78

[KeSn1]   Keating J P and Snaith N C 2000 Random matrix theory and $\zeta(1/2 + \mathrm{i}t)$ *Commun. Math. Phys.* **214** 57–89

[KeSn2]   Keating J P and Snaith N C 2000 Random matrix theory and *L*-functions at $s = 1/2$ *Commun. Math. Phys.* **214** 91–110

[Mark]   Marklof J 1999 Limit theorems for theta sums *Duke Math. J.* **97** 127–53

[Mont]   Montgomery H L 1973 The pair correlation of zeros of the zeta function *Proc. Symp. Pure Mathematics* vol 24 pp 181–93

[O]   Odlyzko A M 1989 The 10^{20}th zero of the Riemann zeta function and 70 million of its neighbors. Unpublished

[RS]   Rudnick Z and Sarnak P 1996 Zeros of principal *L*-functions and random matrix theory *Duke Math. J.* **81** 269–322

[T]   Titchmarsh E C 1986 *The Theory of the Riemann Zeta Function* (Oxford: Oxford University Press)