Bull. London Math. Soc. 50 (2018) 349–356
doi:10.1112/blms.12142

On r-gaps between zeros of the Riemann zeta-function

J. Brian Conrey and Caroline L. Turnage-Butterbaugh

Abstract

Under the Riemann Hypothesis, we prove for any natural number r there exist infinitely many natural numbers n such that $(\gamma_{n+r}-\gamma_n)/(2\pi r/\log\gamma_n) > 1+\Theta/\sqrt{r}$ and $(\gamma_{n+r}-\gamma_n)/(2\pi r/\log\gamma_n) < 1-\vartheta/\sqrt{r}$ for explicit absolute positive constants Θ and ϑ, where γ denotes an ordinate of a zero of the Riemann zeta-function on the critical line. Selberg published announcements of this result several times without proof.

1. *Introduction*

Let $\zeta(s)$ denote the Riemann zeta-function, and let $\rho=\beta+i\gamma$ denote a nontrivial zero of $\zeta(s)$. Consider the sequence of ordinates of zeros in the upper half-plane

$$0<\gamma_1\leqslant\gamma_2\leqslant\cdots\leqslant\gamma_n\leqslant\gamma_{n+1}\leqslant\cdots.$$

It is well known that

$$N(T):=\sum_{0<\gamma\leqslant T}1\sim\frac{T}{2\pi}\log T,$$

from which it follows that the average gap between consecutive zeros is $2\pi/\log\gamma_n$. Assuming the Riemann Hypothesis, $\beta=1/2$ and $\gamma\in\mathbb{R}$. The result of this note is a proof of the following theorem.

Theorem. *Assuming the Riemann Hypothesis, for any natural number r there exist infinitely many n such that*

$$\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n}>1+\frac{\Theta}{\sqrt{r}}\quad\text{and}\quad\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n}<1-\frac{\vartheta}{\sqrt{r}}$$

for the absolute positive constants $\Theta=0.574271$ and $\vartheta=0.299856$. Moreover, for r sufficiently large, we may take $\Theta=\vartheta=0.9065$.

There are discrepancies in the literature regarding the correct statement of this result, which we hope to now clarify. In [**10**, p. 199], Selberg announced, without proof, that there exists an absolute positive constant θ such that for all positive integers r

$$\limsup_{n\to\infty}\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n}>1+\theta\quad\text{and}\quad\liminf_{n\to\infty}\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n}<1-\theta.$$

This statement was later updated in the Acknowledgements section of [**8**], with the θ appearing above replaced with $\theta/\sqrt{r}$. Finally, in the errata of volume 1 of his collected papers [**11**, p. 355], Selberg clarified the correct statement of his result.

Received 15 September 2017; revised 18 November 2017; published online 29 January 2018.

2010 *Mathematics Subject Classification* 11M26 (primary).

Conrey is supported in part by a grant from the National Science Foundation. Turnage-Butterbaugh was supported by the National Science Foundation Grant DMS-1440140 while at the Mathematical Sciences Research Institute during the Spring 2017 semester.

Selberg's Announced Result. *There exist an absolute positive constant θ such that for all positive integers r*

$$\limsup_{n\to\infty} \frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n} > 1+\theta r^{-\alpha} \quad \text{and} \quad \liminf_{n\to\infty} \frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n} < 1-\theta r^{-\alpha},$$

where α may be taken as 2/3, and if one assumes the Riemann Hypothesis as 1/2.

Selberg did not give an indication of a proof for either statement, however Heath-Brown in [**12**, pp. 246–249] provides an unconditional proof of Selberg's result in the case $r = 1$ using the work of Fujii [**5**] concerning the mean value of $S(t)$ in short intervals. (Note that $\pi S(t)$ is the argument of $\zeta(s)$ at the point $s = 1/2 + it$.) We remark that Heath-Brown's proof for $r = 1$ shows that the result holds for a positive proportion of integers n.

The goal of this note is to give a proof of Selberg's conditional result for all $r \geqslant 1$ with explicit constants. To prove our theorem, we adapt a method developed by Conrey, Ghosh, and Gonek [**3**] on gaps between consecutive nontrivial zeros of $\zeta(s)$ in the interval $[0, T]$ for T large. The method is conditional on the Riemann Hypothesis. To our knowledge, our proof is the first to appear in the literature for $r > 1$.

For a fixed, positive integer r, let

$$\lambda_r := \limsup_{n\to\infty} \frac{\gamma_{n+r}-\gamma_n}{2\pi/\log\gamma_n} \quad \text{and} \quad \mu_r := \liminf_{n\to\infty} \frac{\gamma_{n+r}-\gamma_n}{2\pi/\log\gamma_n}. \tag{1.1}$$

By definition $\lambda_r \geqslant r$ and similarly $\mu_r \leqslant r$, however random matrix theory predicts that $\lambda_r = \infty$ and $\mu_r = 0$. Following [**3**], we compare averages of a well-chosen polynomial of the form

$$A(t) := \sum_{n\leqslant X} \frac{a^{\pm}(n)}{n^{it}}, \tag{1.2}$$

where $X = T^{1-\delta}$ for some small $\delta > 0$. To adapt for r-gaps, we set

$$M_1 := \int_T^{2T} |A(t)|^2 \, dt$$

and

$$M_2(c_r) := \int_{-\pi c_r/\log T}^{\pi c_r/\log T} \sum_{T\leqslant\gamma\leqslant 2T} |A(\gamma+\alpha)|^2 \, d\alpha,$$

where c_r is some nonzero real number. We see that $M_2(c_r)$ is monotonically increasing and

$$M_2(\mu_r) \leqslant rM_1 \leqslant M_2(\lambda_r).$$

Therefore, if $M_2(c_r) < rM_1$ for some choice of $a^{+}(n)$ and c_r then $\lambda_r > c_r$. Similarly, if $M_2(c_r) > rM_1$ for some choice of $a^{-}(n)$ and c_r then $\mu_r < c_r$.

Connecting their work to a previous result of Montgomery and Odlyzko [**7**], Conrey, Ghosh, and Gonek show

$$\frac{M_2(c_r)}{M_1} = h^{\pm}(c_r) + o(1),$$

where $h(c_r)$ is defined by

$$h^{\pm}(c_r) := c_r \mp \frac{\Re\left(\displaystyle\sum_{kn\leqslant X} \frac{a^{\pm}(n)\overline{a^{\pm}(kn)}g_{c_r}(k)\Lambda(k)}{kn}\right)}{\displaystyle\sum_{n\leqslant X} \frac{|a^{\pm}(n)|^2}{n}} \tag{1.3}$$

and

$$g_{c_r}(k) = \frac{2\sin\left(\pi c_r \frac{\log k}{\log T}\right)}{\pi \log k}$$

so that $|g_{c_r}(k)| \leqslant 2c_r/\log T$. The function $h^{\pm}(c_r)$ was introduced by Montgomery and Odlyzko to study gaps between consecutive zeros of $\zeta(s)$. In particular, they show that if one is able to find c_r such that $h^+(c_r) < r$ then $\lambda_r > c_r$ and such that if $h^-(c_r) > r$ then $\mu_r < c_r$.

Letting $r = 1$ in (1.1), it follows from our theorem that $\lambda_1 > 1$ and $\mu_1 < 1$. Quantitative bounds on λ_1 and μ_1 have been obtained using the above approach, with different choices of $a(n)$ leading to improved results. See [**2**] and subsequently [**4**] for discussions of these choices. The best current quantitative bounds concerning gaps between consecutive zeros of the Riemann zeta-function (under the assumption of the Riemann Hypothesis) are $\lambda_1 > 3.18$, due to Bui and Milinovich [**1**], and $\mu_1 < 0.515396$, due to Preobrazhenskii [**9**]. We note that the method employed in [**1**], which is based on the work of Hall [**6**] and different from the method discussed above, is unconditional if one restricts the analysis to critical zeros.

2. *Proof of the theorem for fixed $r \geqslant 1$*

For large gaps for any fixed $r \geqslant 1$, we choose $a^+(n) = d_\ell(n)$, where d_ℓ is multiplicative and defined on prime powers by

$$d_\ell(p^m) = \frac{\Gamma(m+\ell)}{\Gamma(\ell)m!}.$$

Fix $\ell \geqslant 1$. (In the proof, we will ultimately set ℓ to be an explicit value depending on r.) Similarly, for small gaps for any fixed $r \geqslant 1$, we choose $a^-(n) = \lambda(n)d_\ell(n)$, where $\lambda(n)$ denotes the Liouville function.

We now prove the result for large gaps for any fixed $r \geqslant 1$. Take $a^+(n) = d_\ell(n)$ for $\ell \geqslant 1$ an integer to be determined later. In this case the relevant mean value to compute is well known:

$$\sum_{n \leqslant x} \frac{d_\ell(n)^2}{n} = C_\ell(\log x)^{\ell^2} + O((\log T)^{\ell^2-1})$$

for fixed $\ell \geqslant 1$, uniformly for $x \leqslant T$, where C_ℓ is a constant which will not have an effect in our application. It is shown in [**3**, p. 422] that for this choice of $a^+(n)$, the equation $M_2(c_r)/M_1 = h^+(c_r) + o(1)$ reduces to

$$h^+(c_r) = c_r - 2\ell \int_0^1 \frac{\sin(\pi c_r v(1-\delta))}{\pi v}(1-v)^{\ell^2}\,dv + O(1/\log T), \tag{2.1}$$

where $\delta > 0$ is as in (1.2) and will be taken to be sufficiently small. To detect large gaps, we must show that $h^+(c_r) < r$ for fixed $r \geqslant 1$. By the previous discussion, this will imply $\lambda_r > c_r$. For example, using (2.1) we can compute the values as provided in Table 1.

In general, to prove large gaps of the desired shape, we show that $h^+(c_r) < r$ for fixed $r \geqslant 1$ and $c_r = r + \Theta\sqrt{r}$ with $\Theta > 0$. We estimate the integral appearing in (2.1) as follows. Let

$$\int_0^1 \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv = I_1 + I_2,$$

where

$$I_1 := \int_0^{1/c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv \quad \text{and} \quad I_2 := \int_{1/c_r}^1 \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv.$$

TABLE 1. *For fixed r, the table gives values of ℓ, c_r for which $h^+(c_r) < r$, implying $\lambda_r > c_r$.*

r	ℓ	c_r	$h^+(c_r)$
1	2.2	2.337	0.99965
2	2.8	3.708	1.99937
3	3.3	4.994	2.99975
4	3.7	6.235	3.99950
5	4.0	7.448	4.99978

For I_1, we first observe that the integrand is positive in the range of integration and write

$$I_1 \geqslant I_{1,a} + I_{1,b}, \tag{2.2}$$

say, where

$$I_{1,a} := \int_0^{1/4c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv,$$

$$I_{1,b} := \int_{1/4c_r}^{1/2c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv,$$

and we have discarded the portion of the integral from $1/(2c_r)$ to $1/c_r$. Now we estimate $I_{1,a}$ and $I_{1,b}$. For $I_{1,a}$, we compare $\sin(\pi c_r(1-\delta)v)$ to $2\sqrt{2}c_r(1-\delta)v$ and find

$$I_{1,a} \geqslant \int_0^{1/4c_r} \frac{2\sqrt{2}c_r(1-\delta)v}{\pi v}(1-v)^{\ell^2}\,dv = \frac{2\sqrt{2}c_r(1-\delta)}{\pi(\ell^2+1)}\left(1-\left(1-\frac{1}{4c_r}\right)^{\ell^2+1}\right). \tag{2.3}$$

Similarly for $I_{1,b}$, we compare $\sin(\pi c_r(1-\delta)v)$ to $(4-2\sqrt{2})c_r(1-\delta)v$ and find

$$I_{1,b} \geqslant \frac{(4-2\sqrt{2})c_r(1-\delta)}{\pi(\ell^2+1)}\left(\left(1-\frac{1}{2c_r}\right)^{\ell^2+1}-\left(1-\frac{1}{4c_r}\right)^{\ell^2+1}\right). \tag{2.4}$$

Thus by (2.2), (2.3), and (2.4), we have

$$I_1 \geqslant \frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left(\sqrt{2}-(2\sqrt{2}-2)\left(1-\frac{1}{4c_r}\right)^{\ell^2+1}-(2-\sqrt{2})\left(1-\frac{1}{2c_r}\right)^{\ell^2+1}\right).$$

Furthermore, since $\exp(-x) \geqslant 1-x$ for $x \geqslant 0$, it follows that

$$I_1 \geqslant \frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left\{\sqrt{2}-(2\sqrt{2}-2)\exp\left(\frac{-(\ell^2+1)}{4c_r}\right)-(2-\sqrt{2})\exp\left(\frac{-(\ell^2+1)}{2c_r}\right)\right\}. \tag{2.5}$$

We now estimate the second integral I_2. Since $v \geqslant 0$, we have

$$|I_2| \leqslant \frac{1}{\pi}\int_{1/c_r}^{1}\frac{(1-v)^{\ell^2}}{v}\,dv \leqslant \frac{1}{\pi}\int_{1/c_r}^{1}\frac{\exp(-\ell^2 v)}{v}\,dv.$$

Thus, by the change of variable $u = \ell^2 v$, we find

$$I_2 \geqslant \frac{-1}{\pi}\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\,du. \tag{2.6}$$

Combining the estimates in (2.5) and (2.6), we have

$$h^+(c_r) \leqslant c_r - 2\ell\left\{\frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left(\sqrt{2}-(2\sqrt{2}-2)\exp\left(\frac{-(\ell^2+1)}{4c_r}\right)\right.\right.$$
$$\left.\left.-(2-\sqrt{2})\exp\left(\frac{-(\ell^2+1)}{2c_r}\right)\right)-\frac{1}{\pi}\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\,du\right\}+O\left(1/\log T\right).$$

In this case, $c_r = r+\Theta\sqrt{r}$ where $\Theta>0$, and thus $c_r>1$ for any $r\geqslant 1$. Thus, letting $\ell=\sqrt{bc_r-1}$, where $b>1$ is a real number that will be chosen later, we have

$$\ell=\sqrt{bc_r-1}\geqslant\sqrt{br}\sqrt{1-\frac{1}{b}}$$

for any $r\geqslant 1$. Furthermore, since we always have $c_r>1$, for any $r\geqslant 1$ it follows that

$$\frac{\ell^2}{c_r}>b-1,$$

and thus we may again increase the length of integration in I_2 to write

$$\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\,du<\int_{b-1}^{\infty}\frac{\exp(-u)}{u}\,du.$$

Combining these estimates, we find

$$h^+(c_r)<r+\Theta\sqrt{r}-2\sqrt{br}\sqrt{1-\frac{1}{b}}\left\{\frac{2(1-\delta)}{\pi b}\left(\sqrt{2}-(2\sqrt{2}-2)\exp\left(\frac{-b}{4}\right)\right.\right.$$
$$\left.\left.-(2-\sqrt{2})\exp\left(\frac{-b}{2}\right)\right)-\frac{1}{\pi}\int_{b-1}^{\infty}\frac{\exp(-u)}{u}\,du\right\}+O\left(\frac{1}{\log T}\right).$$

To show $h^+(c_r)<r$ and prove the theorem, we set

$$\Theta=\max_b\left\{2\sqrt{b}\sqrt{1-\frac{1}{b}}\left(\frac{2}{\pi b}\left(\sqrt{2}-(2\sqrt{2}-2)\exp\left(\frac{-b}{4}\right)\right.\right.\right.$$
$$\left.\left.\left.-(2-\sqrt{2})\exp\left(\frac{-b}{2}\right)\right)-\frac{1}{\pi}\int_{b-1}^{\infty}\frac{\exp(-u)}{u}\,du\right)\right\}.$$

The choice $b=5.0107$ yields $\Theta=0.574271$. With δ sufficiently small and T sufficiently large, these choices guarantee that $h^+(c_r)<r$, as desired.

We now prove the result for small gaps for any fixed $r\geqslant 1$. The proof for small gaps is similar to the proof for large gaps, so we indicate the necessary changes. Take $a^-(n)=\lambda(n)d_\ell(n)$ for $\ell\geqslant 1$ fixed. It is given in [**3**, p. 422] that this choice of $a^-(n)$, yields

$$h^-(c_r)=c_r+2\ell\int_0^1\frac{\sin(\pi c_r v(1-\delta))}{\pi v}(1-v)^{\ell^2}\,dv+O\left(1/\log T\right). \tag{2.7}$$

To detect small gaps, we must show that $h^-(c_r)>r$ for fixed $r\geqslant 1$. By the previous discussion, this will imply $\mu_r<c_r$. For example, using (2.7) we can compute the values as provided in Table 2.

In general, to prove small gaps of the desired shape, we show that $h^-(c_r)<r$ for fixed $r\geqslant 1$ and $c_r=r-\Theta\sqrt{r}$ with $\Theta>0$. We estimate the integral appearing in (2.7) as before, however

TABLE 2. *For fixed r, the table gives values of ℓ, c_r for which $h^-(c_r) > r$, implying $\mu_r < c_r$.*

r	ℓ	c_r	$h^-(c_r)$
1	1.1	0.5172	1.00012
2	1.4	1.126	2.00118
3	1.9	1.831	3.00072
4	2.3	2.588	4.00099
5	2.7	3.375	5.00116

for brevity we will perform the calculation without writing I_1 as the sum of two integrals of equal length†. We find

$$h^-(c_r) \geqslant c_r + 2\ell\left\{\frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left(1-\exp\left(\frac{-(\ell^2+1)}{c_r}\right)-\frac{1}{\pi}\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\,du\right\} + O\left(\frac{1}{\log T}\right).$$

Let $\ell = \sqrt{bc_r - 1}$ and $c_r = r - \vartheta\sqrt{r}$, with $\vartheta > 0$. In this case, we do not always have $c_r > 1$. Indeed, since $\vartheta > 0$, if $r = 1$ then $0 < c_r < 1$. However, if we require that $\vartheta \leqslant 0.5$, the estimate

$$\ell = \sqrt{bc_r - 1} > \sqrt{br}\sqrt{\frac{1}{2} - \frac{1}{b}}$$

holds for any $r \geqslant 1$. The requirement that $\vartheta \leqslant 0.5$ also implies

$$\frac{\ell^2}{c_r} \geqslant b - 2$$

for any $r \geqslant 1$, and we may increase the length of integration in I_2 to write

$$\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\,du \leqslant \int_{b-2}^{\infty}\frac{\exp(-u)}{u}\,du.$$

Thus, requiring that $\vartheta \leqslant 0.5$, we may put these estimates together to write

$$h^-(c_r) > r - \vartheta\sqrt{r} + 2\sqrt{br}\sqrt{\frac{1}{2} - \frac{1}{b}}\left\{\frac{2(1-\delta)}{\pi b}\left(1 - \exp(-b)\right)\right.$$
$$\left. -\frac{1}{\pi}\int_{b-2}^{\infty}\frac{\exp(-u)}{u}\,du\right\} + O\left(\frac{1}{\log T}\right).$$

To show $h^-(c_r) > r$ and thus prove the theorem, we set

$$\vartheta = \max_b\left\{2\sqrt{b}\sqrt{\frac{1}{2} - \frac{1}{b}}\left(\frac{2}{\pi b}(1 - \exp(-b)\right) - \frac{1}{\pi}\int_{b-2}^{\infty}\frac{\exp -u}{u}\,du\right)\right\}.$$

The choice $b = 5.17305$ yields $\vartheta = 0.299856$. (We note that the condition $\vartheta < 0.5$ is satisfied.) With δ sufficiently small and T sufficiently large, these choices guarantee that $h^+(c_r) < r$, as desired.

REMARK. In the argument above for large gaps, if we had not divided the remaining portion of I_1 into two smaller integrals and instead compared $\sin(\pi c_r(1-\delta)v)$ to $2c_r(1-\delta)v$ over the interval $[0, 1/2c_r]$, we would have ultimately found that one can take $\Theta = 0.447$. Instead, by carrying out the analysis on $I_1 > I_{1,a} + I_{1,b}$ (see (2.2)) and estimating $I_{1,a}$ and $I_{1,b}$ separately,

†To see how these choices affect the size of ϑ here and in the large gaps setting, please refer to the remark following the proof.

we were able to provide the stronger constant $\Theta = 0.570717$. One could thus slightly improve the absolute constant Θ by breaking up I_1 into smaller pieces over the interval $[0, 1/2c_r]$, and estimating each piece accordingly. For example, writing $I_1 > I_{1,a'} + I_{1,b'} + I_{1,c'} + I_{1,d'}$ where each integral has equal length of integration over the interval $[0, 1/2c_r]$, one can obtain $\Theta = 0.593234$, and comparing I_1 to the sum of sixteen such smaller integrals $\Theta = 0.599648$. Similarly, for small gaps, comparing I_1 to the sum of two smaller integrals of equal length over the interval $[0, 1/2c_r]$ yields $\vartheta = 0.359222$; using sixteen smaller integrals of equal length of integration over the interval $[0, 1/2c_r]$ yields $\vartheta = 0.379674$.

3. *Proof of the theorem for r sufficiently large*

We can improve the constants Θ and ϑ appearing in the theorem if we take r to be large. In fact, we will see that in this setting, we may take $\Theta = \vartheta = 0.9065$.

We first consider large gaps for sufficiently large r. Starting with (2.1), to detect large gaps of the desired size, we must show that $h^+(c_r) < r$ for sufficiently large r and $c_r = r + \Theta\sqrt{r}$ with $\Theta > 0$. Choosing $\ell = B\sqrt{r}$, we have

$$h^+(c_r) < c_r - 2B\sqrt{r}\int_0^1 \frac{\sin(\pi r v(1-\delta))}{\pi v}(1-v)^{B^2 r}\,dv + O(1/\log T)$$

for sufficiently large r. Making the change of variable $rv = w$, the above inequality becomes

$$\begin{aligned} h^+(c_r) &< c_r - 2B\sqrt{r}\int_0^r \frac{\sin(\pi w(1-\delta))}{\pi w}\left(1-\frac{w}{r}\right)^{B^2 r} dw + O(1/\log T) \\ &< c_r - 2B\sqrt{r}\int_0^r \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right)\,dw + O(1/\log T) \qquad (3.1) \\ &= c_r - 2B\sqrt{r}\int_0^\infty \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right)\,dw - 2B\sqrt{r}E(r) + O(1/\log T), \end{aligned}$$

where

$$E(r) = \int_r^\infty \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right)\,dw.$$

Note that as $r \to \infty$, $\sqrt{r}E(r) \to 0$, so for sufficiently large r this term is negligible. Thus we set

$$\Theta = \max_B \left\{2B\int_0^\infty \frac{\sin(\pi w)}{\pi w}\exp\left(-B^2 w\right)\,dw\right\} = \max_B \left\{\frac{2B}{\pi}\arctan\left(\frac{\pi}{B^2}\right)\right\}.$$

The choice $B = 1.502243$. yields $\Theta = 0.9065$. With δ sufficiently small, T and r sufficiently large, these choices guarantee that $h^+(c_r) < r$.

We now consider small gaps for r sufficiently large. We begin with (2.7) and let $\ell = B\sqrt{r - \sqrt{r}}$. If we assume $\vartheta < 1$, then $r - \vartheta\sqrt{r} > r - \sqrt{r}$ for all r, and we have

$$h^-(c_r) > c_r + 2B\sqrt{r-\sqrt{r}}\int_0^1 \frac{\sin(\pi(r-\sqrt{r})v(1-\delta))}{\pi v}(1-v)^{B^2(r-\sqrt{r})}\,dv + O(1/\log T).$$

Using the change of variable $(r - \sqrt{r})v = w$, we follow an analogous argument as in the previous subsection and ultimately set

$$\vartheta = \max_B \left\{2B\int_0^\infty \frac{\sin(\pi w)}{\pi w}\exp\left(-B^2 w\right)\,dw\right\} = \max_B \left\{\frac{2B}{\pi}\arctan\left(\frac{\pi}{B^2}\right)\right\}.$$

As before, the choice $B = 1.502243$ yields $\vartheta = 0.9065$. With δ sufficiently small, T and r sufficiently large, these choices guarantee that $h^-(c_r) > r$.

Acknowledgements. The authors thank D. A. Goldston and M. B. Milinovich for helpful comments on an earlier version of the article. We also thank the anonymous referee for a suggestion that led to improved constants in the main theorem. Discussions for this work began at the American Institute of Mathematics Workshop 'Bounded Gaps Between Primes.'

References

1. H. M. Bui and M. B. Milinovich, 'Gaps between zeros of the Riemann zeta-function', *Q. J. Math. Oxford*, 2017, https://doi.org/10.1093/qmath/hax047.

2. H. M. Bui, M. B. Milinovich and N. C. Ng, 'A note on the gaps between consecutive zeros of the Riemann zeta-function', *Proc. Amer. Math. Soc.* 138 (2010) 4167–4175.

3. J. B. Conrey, A. Ghosh and S. M. Gonek, 'A note on gaps between zeros of the zeta function', *Bull. Lond. Math. Soc.* 16 (1984) 421–424.

4. S. Feng and X. Wu, 'On gaps between zeros of the Riemann zeta-function', *J. Number Theory* 132 (2012) 1385–1397.

5. A. Fujii, 'On the distribution of the zeros of the Riemann zeta function in short intervals', *Bull. Amer. Math. Soc.* 81 (1975) 139–142.

6. R. R. Hall, 'The behaviour of the Riemann zeta-function on the critical line', *Mathematika* 46 (1999) 281–313.

7. H. L. Montgomery and A. M. Odlyzko, 'Gaps between zeros of the zeta function', *Topics in classical number theory (Budapest, 1981)*, vol. I, II, Colloquia of Mathematical Society János Bolyai 34 (North-Holland, Amsterdam, 1984) 1079–1106.

8. J. Mueller, 'On the difference between consecutive zeros of the Riemann zeta function', *J. Number Theory* 14 (1982) 327–331.

9. S. Preobrazhenskiĭ, 'A small improvement in the gaps between consecutive zeros of the Riemann zeta-function', *Res. Number Theory* 2 (2016) Art. 28, 11.

10. A. Selberg, 'The zeta-function and the Riemann hypothesis', *C. R. Dixième Congrès Math. Scandinaves 1946* (Jul. Gjellerups Forlag, Copenhagen, 1947) 187–200.

11. A. Selberg, *Collected papers*, vol. I (Springer, Berlin, 1989), With a foreword by K. Chandrasekharan.

12. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd edn (The Clarendon Press, Oxford University Press, New York, 1986), Edited and with a preface by D. R. Heath-Brown.

J. Brian Conrey
American Institute of Mathematics
600 E. Brokaw Road
San Jose, CA 95112
USA

and

Department of Mathematics
University of Bristol
University Walk
Bristol, BS8 1TW
United Kingdom

conrey@aimath.org

Caroline L. Turnage-Butterbaugh
Department of Mathematics
Duke University
120 Science Drive
Durham, NC 27708
USA

ctb@math.duke.edu