



# Moments of zeta and correlations of divisor-sums: V

Brian Conrey and Jonathan P. Keating

## ABSTRACT

In this series of papers we examine the calculation of the  $2k$ th moment and shifted moments of the Riemann zeta-function on the critical line using long Dirichlet polynomials and divisor correlations. The present paper completes the general study of what we call Type II sums which utilize a circle method framework and a convolution of shifted convolution sums to obtain all of the lower order terms in the asymptotic formula for the mean square along  $[T, 2T]$  of a Dirichlet polynomial of arbitrary length with divisor functions as coefficients.

## 1. Introduction

This paper is part V of a sequence devoted to understanding how to conjecture all of the integral moments of the Riemann zeta-function from a number theoretic perspective. The method is to approximate  $\zeta(s)^k$  by a long Dirichlet polynomial and then to compute the mean square of the Dirichlet polynomial (c.f. [13]). There are many off-diagonal terms and it is the care of these that is the concern of these papers. In particular it is necessary to treat the off-diagonal terms by a method invented by Bogomolny and Keating [1, 2]. Our perspective on this method is that it is most properly viewed as a multi-dimensional Hardy–Littlewood circle method.

In previous papers [5, 6] we have developed a general method to calculate what were called type-I off-diagonal contributions in [1, 2]; these are the off-diagonal terms usually considered in number-theoretic computations, for example, in [13]. In parts II [7] and IV [8] we considered the simplest of the type-II off-diagonal terms (in the terminology of [1, 2]). These, somewhat unexpectedly, give a significant contribution in certain cases. They have not previously been analyzed systematically. Our purpose here is to develop a general method for computing all type-II off-diagonal terms.

The formula we obtain is in complete agreement with all of the main terms predicted by the recipe of [4] (and in particular, with the leading order term conjectured in [14]).

## 2. Shifted moments

Let  $A$  and  $B$  be two sets of cardinality  $k$  containing ‘shifts’ which may be thought of as parameters of size  $\ll \frac{1}{\log T}$  where  $T$  is the basic large parameter near where we want to know the average of a product of  $\zeta$ -functions. Let

$$D_A(s) := \prod_{\alpha \in A} \zeta(s + \alpha) = \sum_{n=1}^{\infty} \frac{\tau_A(n)}{n^s};$$

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this implicitly defines the arithmetic functions  $\tau_A(n)$ . A basic question (the moment problem) is to evaluate

$$I_{A,B}^\psi(T) = \int_0^\infty \psi\left(\frac{t}{T}\right) D_A(s) D_B(1-s) dt$$

where  $\psi$  is a smooth function with compact support, say  $\psi \in C^\infty[1, 2]$ , and  $s = 1/2 + it$ . The technique we are developing in this sequence of papers is to approach the moment problem through long Dirichlet polynomials. To this end we let

$$D_A(s; X) = \sum_{n \leq X} \frac{\tau_A(n)}{n^s}$$

be the truncated Dirichlet series for  $D_A(s)$ . By Perron's formula for  $\Re s > 1/4$  we have

$$\begin{aligned} D_A(s; X) &= \frac{1}{2\pi i} \int_{(1)} D_A(s+z) \frac{X^z}{z} dz \\ &= D_A(s) + R_A(s; X) + \frac{1}{2\pi i} \int_{(-\epsilon)} D_A(s+z) \frac{X^z}{z} dz \end{aligned}$$

where

$$\begin{aligned} R_A(s; X) &= \sum_{\alpha \in A} \operatorname{Res}_{z=1-s-\alpha} D_A(s+z) \frac{X^z}{z} \\ &= \sum_{\alpha \in A} D_{A \setminus \{\alpha\}} (1-\alpha) \frac{X^{1-s-\alpha}}{(1-s-\alpha)}. \end{aligned}$$

In this sequence of papers we consider

$$I_{A,B}^\psi(T; X) := \int_0^\infty \psi\left(\frac{t}{T}\right) (D_A(s; X) - R_A(s; X)) (D_B(1-s; X) - R_B(1-s; X)) dt$$

for various ranges of  $X$ . The expectation is that if  $X > T^k$  then this will be asymptotically equal to  $I_{A,B}^\psi(T)^\dagger$ .

We calculate this average in two different (conjectural) ways: the first is via the recipe and the second is via the delta method applied to the correlations of shifted divisor functions. We show that these two methods produce identical detailed main terms.

To use the recipe of [4] to conjecture a formula for  $I_{A,B}^\psi(T; X)$ , we start with

$$D_A(s; X) - R_A(s; X) = \frac{1}{2\pi i} \int_{(\epsilon)} \frac{X^w}{w} D_{A_w}(s) dw$$

so that

$$I_{A,B}^\psi(T; X) = \frac{1}{(2\pi i)^2} \iint_{(\epsilon), (\epsilon)} \frac{X^{z+w}}{zw} I_{A_w, B_z}^\psi(T) dw dz$$

where

$$A_w := \{\alpha + w : \alpha \in A\},$$

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<sup>†</sup>In the earlier papers we did not include the terms  $R_A$  and  $R_B$  because their contributions were negligible since  $X$  was not large. In general these terms do not create extra difficulties and so they will be ignored here as well.

that is, the set  $A$  but with all its elements shifted by  $w$ . From the recipe [4] we expect that

$$I_{A,B}^\psi(T) = T \int_0^\infty \psi(t) \sum_{\substack{U \subset A, V \subset B \\ |U|=|V|}} \left(\frac{tT}{2\pi}\right)^{-\sum_{\substack{\hat{\alpha} \in U \\ \hat{\beta} \in V}} (\hat{\alpha} + \hat{\beta})} \mathcal{B}(A - U + V^-, B - V + U^-, 1) dt + o(T)$$

where

$$\mathcal{B}(A, B, s) := \sum_{n=1}^\infty \frac{\tau_A(n)\tau_B(n)}{n^s}.$$

We have used an unconventional notation here; by  $A - U + V^-$  we mean the following: start with the set  $A$  and remove the elements of  $U$  and then include the negatives of the elements of  $V$ . We think of the process as ‘swapping’ equal numbers of elements between  $A$  and  $B$ ; when elements are removed from  $A$  and put into  $B$  they first get multiplied by  $-1$ . We keep track of these swaps with our equal-sized subsets  $U$  and  $V$  of  $A$  and  $B$ ; and when we refer to the ‘number of swaps’ in a term we mean the cardinality  $|U|$  of  $U$  (or, since they are of equal size, of  $V$ ). We insert this conjecture and expect that

$$I_{A,B}^\psi(T; X) = T \int_0^\infty \psi(t) \frac{1}{(2\pi i)^2} \iint_{(\epsilon), (\epsilon)} \frac{X^{z+w}}{zw} \sum_{\substack{U \subset A, V \subset B \\ |U|=|V|}} \left(\frac{tT}{2\pi}\right)^{-\sum_{\substack{\hat{\alpha} \in U \\ \hat{\beta} \in V}} (\hat{\alpha} + w + \hat{\beta} + z)} \times \mathcal{B}(A_w - U_w + V_z^-, B_z - V_z + U_w^-, 1) dw dz dt + o(T). \tag{1}$$

We have done a little simplification here: instead of writing  $U \subset A_w$  we have written  $U \subset A$  and changed the exponent of  $(tT/2\pi)$  accordingly.

Note that there is a factor  $(X/T^{|U|})^{w+z}$  in the previous equation. As mentioned above we refer to  $|U|$  as the number of ‘swaps’ in the recipe, and now we see more clearly the role it plays; in the terms above for which  $X < T^{|U|}$  we move the path of integration in  $w$  or  $z$  to  $+\infty$  so that the factor  $(X/T^{|U|})^{w+z} \rightarrow 0$  and the contribution of such a term is 0. Thus, the size of  $X$  determines how many ‘swaps’ we must keep track of. To account for this we introduce a parameter  $\ell$  defined by

$$T^\ell \leq X < T^{\ell+1}.$$

Then the above may be rewritten as

$$I_{A,B}^\psi(T; X) = T \int_0^\infty \psi(t) \frac{1}{(2\pi i)^2} \iint_{(\epsilon), (\epsilon)} \frac{X^{z+w}}{zw} \sum_{\substack{U \subset A, V \subset B \\ |U|=|V| \leq \ell}} \left(\frac{tT}{2\pi}\right)^{-\sum_{\substack{\hat{\alpha} \in U \\ \hat{\beta} \in V}} (\hat{\alpha} + w + \hat{\beta} + z)} \times \mathcal{B}(A_w - U_w + V_z^-, B_z - V_z + U_w^-, 1) dw dz dt + o(T) \tag{2}$$

where we have restricted the sum to at most  $\ell$  swaps.

Now we turn to the second approach via divisor correlations with the goal of obtaining this formula in a completely different way. In [5, 6] we accomplished this in the situations where there were 0 or 1 swaps (that is, when  $X < T^2$ ). In [7] we considered two swaps but in a special case. In [8] we looked at the general case of two swaps. In this paper, which is the final paper of the sequence, we look at the general case with any number of swaps.

As an extension of the ideas in these papers, we have also begun to explore the analogous calculations for averages of ratios of the zeta function, specifically in the context of zero correlations [9, 10].

The second method to obtain a conjecture for  $I_{A,B}^\psi(T)$  will involve an intricate study of convolutions of shifted divisor problems and will occupy the rest of this paper. We begin that

calculation by integrating term-by-term to obtain

$$\begin{aligned} \frac{1}{T} \int_0^\infty \psi\left(\frac{t}{T}\right) D_A(s; X) D_B(1-s; X) dt &= \sum_{m, n \leq X} \frac{\tau_A(m) \tau_B(n) \hat{\psi}\left(\frac{T}{2\pi} \log \frac{m}{n}\right)}{\sqrt{mn}} \\ &=: \mathcal{O}_{A;B}(T; X) \end{aligned} \tag{3}$$

where  $\hat{\psi}(x) = \int_{-\infty}^\infty \psi(t)e(-xt) dt$  ( $= \int_0^\infty \psi(t)e(-xt) dt$  because of the support of  $\psi$ ). Now let us assume that  $\ell \leq k$  where  $\ell$  is defined above. We partition  $A$  and  $B$  into  $\ell$  non-empty sets  $A = A_1 \cup A_2 \cdots \cup A_\ell$  and  $B = B_1 \cup B_2 \cdots \cup B_\ell$ . Then  $\tau_A$  and  $\tau_B$  are convolutions:  $\tau_A = \tau_{A_1} * \tau_{A_2} * \cdots * \tau_{A_\ell}$  and  $\tau_B = \tau_{B_1} * \tau_{B_2} * \cdots * \tau_{B_\ell}$ . For any such partition, the right-hand side of (3) is equal to

$$\begin{aligned} &\mathcal{O}_{A_1, \dots, A_\ell; B_1, \dots, B_\ell}(T; X) \\ &:= \sum_{\substack{m_1 m_2 \dots m_\ell \leq X \\ n_1 n_2 \dots n_\ell \leq X}} \frac{\tau_{A_1}(m_1) \cdots \tau_{A_\ell}(m_\ell) \tau_{B_1}(n_1) \cdots \tau_{B_\ell}(n_\ell)}{\sqrt{m_1 \dots m_\ell n_1 \dots n_\ell}} \hat{\psi}\left(\frac{T}{2\pi} \log \frac{m_1 \dots m_\ell}{n_1 \dots n_\ell}\right). \end{aligned}$$

In other words  $\mathcal{O}_{A;B}(T; X) = \mathcal{O}_{A_1, \dots, A_\ell; B_1, \dots, B_\ell}(T; X)$  as long as  $A$  and  $B$  are the disjoint unions of the  $A_i$  and  $B_j$ . Now we want to define a refinement of this sum. We impose a pairing  $A_j$  with  $B_j$  and analyze this sum according to rational approximations to  $m_j/n_j$ . In this way, the ordering of the sets  $A_i, B_j$  now matters. The eventual evaluation of  $\mathcal{O}_{A;B}(T; X)$  will involve a sum of these pairings, which we describe in detail in the next section.

### 3. Type II convolution sums

There are various ways to decompose  $A$  and  $B$  and various ways to ‘pair’ divisor functions  $\tau_{A_i}$  and  $\tau_{B_j}$  in preparation for the delta method.

More importantly, however, it turns out that there are various stratifications that also present themselves; basically one for each rational ‘direction.’ If we ignore these then a simple application of the expected main terms from the delta-method analysis will lead us to the wrong main terms.

At first sight it seems that when we do this we are counting the same terms repeatedly. However, we believe that our situation is an example of Manin’s stratified subvarieties wherein counting solutions to high-dimensional diophantine equations often involves identifying a collection of subvarieties on each of which the solutions are counted separately (by the delta method for example). The point is that the main terms of the delta method do not always count all of the solutions. This phenomenon was first identified in [12]; see, for example, [3, 15] for reviews of the subject.

Given  $A_1, \dots, A_\ell$  and  $B_1, \dots, B_\ell$ , the number of ways to pair each  $A_n$  with a  $B_m$  so that all are paired off is  $\ell!$ . Let us consider the pairing of  $A_j$  with  $B_j$ . Now we think of  $m_j/n_j$  as being approximated by a rational number  $M_j/N_j$  with a small denominator for each of  $j = 1, 2, \dots, \ell$  where  $(M_j, N_j) = 1$ . In this way we get subvarieties indexed by the rational directions  $M_j/N_j$  with  $1 \leq j \leq \ell$ . We will use all directions  $M_j/N_j$  subject to the natural conditions  $(M_j, N_j) = 1$  and

$$M_1 \dots M_\ell = N_1 \dots N_\ell. \tag{4}$$

We sum over all of the terms with  $m_j/n_j$  close to  $M_j/N_j$ . We introduce variables  $h_j$  where, for a given  $m_j, n_j, M_j$  and  $N_j$ , we define

$$h_j := m_j N_j - n_j M_j.$$

The rapid decay of  $\hat{\psi}$  governs the ranges of all of the variables; see below.

We have

$$n_1 \dots n_\ell M_1 \dots M_\ell = (m_1 N_1 - h_1) \dots (m_\ell N_\ell - h_\ell)$$

so that for  $M_1 \dots M_\ell = N_1 \dots N_\ell$  we have

$$\frac{n_1 \dots n_\ell}{m_1 \dots m_\ell} = \left(1 - \frac{h_1}{m_1 N_1}\right) \dots \left(1 - \frac{h_\ell}{m_\ell N_\ell}\right)$$

and

$$\log \frac{n_1 \dots n_\ell}{m_1 \dots m_\ell} = -\frac{h_1}{m_1 N_1} - \dots - \frac{h_\ell}{m_\ell N_\ell} + O\left(\frac{h_i h_j}{m_1 \dots m_\ell N_1 \dots N_\ell}\right).$$

The error term is negligible so we can arrange the sum as

$$\begin{aligned} & \sum_{\substack{(M_j, N_j)=1 \\ M_1 \dots M_\ell = N_1 \dots N_\ell}} \sum_{h_1, \dots, h_\ell} \sum_{\substack{m_1 \dots m_\ell \leq X \\ (*_1), \dots, (*_\ell)}} \frac{\tau_{A_1}(m_1) \dots \tau_{A_\ell}(m_\ell) \tau_{B_1}(n_1) \dots \tau_{B_\ell}(n_\ell)}{\sqrt{m_1 \dots m_\ell n_1 \dots n_\ell}} \hat{\psi} \\ & \times \left( \frac{Th_1}{2\pi m_1 N_1} + \dots + \frac{Th_\ell}{2\pi m_\ell N_\ell} \right) \end{aligned}$$

where

$$(*_j) : m_j N_j - n_j M_j = h_j.$$

We can replace  $n_j$  in the denominator by  $\frac{m_j N_j}{M_j}$ . Thus we are led to define

$$\begin{aligned} & \mathcal{O}_{A_1, \dots, A_\ell; B_1, \dots, B_\ell}^\dagger(T; X) \\ & := \sum_{\substack{(M_j, N_j)=1 \\ M_1 \dots M_\ell = N_1 \dots N_\ell}} \sum_{h_1, \dots, h_\ell} \sum_{\substack{m_1 \dots m_\ell \leq X \\ (*_1), \dots, (*_\ell)}} \frac{\tau_{A_1}(m_1) \dots \tau_{A_\ell}(m_\ell) \tau_{B_1}(n_1) \dots \tau_{B_\ell}(n_\ell)}{m_1 \dots m_\ell} \\ & \times \hat{\psi} \left( \frac{Th_1}{2\pi m_1 N_1} + \dots + \frac{Th_\ell}{2\pi m_\ell N_\ell} \right). \end{aligned} \tag{5}$$

Also, we define

$$\mathcal{O}_\ell(T; X) := \frac{1}{w_\ell} \sum_{\substack{A=A_1 \cup \dots \cup A_\ell \\ B=B_1 \cup \dots \cup B_\ell}} \mathcal{O}_{A_1, \dots, A_\ell; B_1, \dots, B_\ell}^\dagger(T; X), \tag{6}$$

where the weight factor

$$w_\ell = \ell!^2 \ell^{2k-2\ell}$$

in  $\mathcal{O}_\ell(T; X)$  will be explained in a later section. Note that  $\ell$  is defined in terms of  $T$  and  $X$ , so its inclusion in the notation is redundant. Now we can state

CONJECTURE 1. Suppose  $k \leq \ell + 1$  and  $T^\ell \leq X < T^{\ell+1}$ . Then for some  $\delta > 0$ ,

$$I_{A, B}^\psi(T; X) = T \mathcal{O}_\ell(T; X) + O(T^{1-\delta}).$$

One way to view this paper is that it gives evidence for this conjecture. In particular, in the next few sections we will conjecturally understand  $\mathcal{O}_{A_1, \dots, A_\ell; B_1, \dots, B_\ell}^\dagger(T; X)$  by replacing the shifted divisor sums by what the delta-method leads us to expect for them. Then we evaluate the result and prove the rigorous theorem that our evaluation is precisely the quantity on the right-hand side of (1).

4. The case where  $h_1 \dots h_\ell \neq 0$

Let us first look at the situation where none of the  $h_j$  is 0. The idea is to evaluate  $\mathcal{O}_{A_1, \dots, A_\ell; B_1, \dots, B_\ell}^\dagger(T; X)$  by replacing the  $m_j$  by real variables  $u_j$  while the  $M_j, N_j$  and  $h_j$  remain integer-valued variables (and the  $n_j$  are determined by the equations  $*_j$ ).

To do this we will replace the convolution sums by their averages, that is,

$$\langle \tau_A(m) \tau_B(n) \rangle_{m=u}^{(*)} \sim \frac{1}{M} \sum_{q=1}^\infty r_q(h) \langle \tau_A(m) e(mN/q) \rangle_{m=u} \langle \tau_B(n) e(nM/q) \rangle_{n=\frac{uN}{M}} \tag{7}$$

where  $r_q(h)$  is the Ramanujan sum (usually denoted  $c_q(h)$ ) and

$$\langle \tau_A(m) e(mN/q) \rangle_{m=u} = \frac{1}{2\pi i} \int_{|w-1|=\epsilon} D_A \left( w, e \left( \frac{N}{q} \right) \right) u^{w-1} dw$$

where

$$D_A \left( w, e \left( \frac{N}{q} \right) \right) = \sum_{n=1}^\infty \frac{\tau_A(n) e(nN/q)}{n^s}.$$

(In the above few lines we have replaced a sum  $\sum_{n \leq x} a_n f(n)$  by an integral  $\int_1^x f(u) \langle a_n \rangle_{n=u} du$  where (in the handy physics notation)  $\langle a_n \rangle_{n=u}$  denotes the average of  $a_n$  when  $n = u$  (the instantaneous rate of change of a good approximation to  $\sum_{n \leq u} a_n$  with respect to  $u$ ). In our context this may be expressed using  $A(s) = \sum_{n=1}^\infty a_n n^{-s}$  and defining  $\langle a_n \rangle_{n=u} = \text{Res}_{|s-1|<\epsilon} u^{s-1} A(s)$  where we sum the residues at all of the poles of  $A(s)$  near  $s = 1$ .)

Thus, we believe that

$$\sum_{\substack{M_j, N_j, (M_j, N_j)=1 \\ N_1 \dots N_\ell = M_1 \dots M_\ell}} \sum_{\substack{h_1, \dots, h_\ell \\ h_1 \dots h_\ell \neq 0}} \int_{u_1 \dots u_\ell \leq X} \langle \tau_{A_1}(m_1) \tau_{B_1}(n_1) \rangle_{m_1=u_1}^{(*)_1} \dots \langle \tau_{A_\ell}(m_\ell) \tau_{B_\ell}(n_\ell) \rangle_{m_\ell=u_\ell}^{(*)_\ell} \\ \times \hat{\psi} \left( \frac{Th_1}{2\pi u_1 N_1} + \dots + \frac{Th_\ell}{2\pi u_\ell N_\ell} \right) \frac{du_1}{u_1} \dots \frac{du_\ell}{u_\ell}.$$

is, up to a power savings, equal to

$$\sum_{\substack{M_j, N_j, (M_j, N_j)=1 \\ N_1 \dots N_\ell = M_1 \dots M_\ell}} \frac{1}{M_1 \dots M_\ell} \sum_{q_j, h_j} r_{q_1}(h_1) \dots r_{q_\ell}(h_\ell) \int_{T^\ell \leq u_1 \dots u_\ell \leq X} \\ \times \prod_{j=1}^\ell \frac{1}{2\pi i} \int_{|w_j-1|=\epsilon} D_{A_j} \left( w_j, e \left( \frac{N_j}{q_j} \right) \right) u_j^{w_j-1} \\ \times dw_j \prod_{k=1}^\ell \frac{1}{2\pi i} \int_{|z_k-1|=\epsilon} D_{B_k} \left( z_k, e \left( \frac{M_k}{q_k} \right) \right) \left( \frac{u_k N_k}{M_k} \right)^{z_k-1} dz_k \\ \times \hat{\psi} \left( T \left( \frac{h_1}{2\pi u_1 N_1} + \dots + \frac{h_\ell}{2\pi u_\ell N_\ell} \right) \right) \frac{du_1}{u_1} \dots \frac{du_\ell}{u_\ell}.$$

To further analyze this quantity, we make the changes of variable  $v_j = \frac{T|h_j|}{2\pi u_j N_j}$  and bring the sums over the  $h_j$  to the inside;  $u_1 \dots u_\ell < X$  implies that

$$|h_1 \dots h_\ell| < \frac{(2\pi)^\ell X v_1 \dots v_\ell N_1 \dots N_\ell}{T^\ell}.$$

We detect this condition using Perron’s formula in an integral over  $s$ . Then the above is

$$\begin{aligned} & \int_{0 < v_1, \dots, v_\ell < \infty} \sum_{\substack{M_j, N_j, (M_j, N_j)=1 \\ N_1 \dots N_\ell = M_1 \dots M_\ell}} \frac{1}{M_1 \dots M_\ell} \frac{1}{2\pi i} \int_{(2)} \sum_{q_j, h_j} \frac{r_{q_1}(h_1) \dots r_{q_\ell}(h_\ell)}{|h_1 \dots h_\ell|^s} \\ & \times \prod_{j=1}^\ell \frac{1}{2\pi i} \int_{|w_j-1|=\epsilon} D_{A_j} \left( w_j, e \left( \frac{N_j}{q_j} \right) \right) \left( \frac{T|h_j|}{2\pi v_j N_j} \right)^{w_j-1} dw_j \\ & \prod_{k=1}^\ell \frac{1}{2\pi i} \int_{|z_k-1|=\epsilon} D_{B_k} \left( z_k, e \left( \frac{M_k}{q_k} \right) \right) \left( \frac{T|h_k|}{2\pi v_k M_k} \right)^{z_k-1} dz_k \\ & \times \left( \frac{(2\pi)^\ell X v_1 \dots v_\ell N_1 \dots N_\ell}{T^\ell} \right)^s \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_\ell v_\ell) \frac{dv_1}{v_1} \dots \frac{dv_\ell}{v_\ell} \frac{ds}{s} \end{aligned}$$

where  $\epsilon_j = \text{sgn}(h_j)$ . We simplify this a bit. We combine the middle two lines into a single product over  $j$  and gather together all of the like variables (note that the sums over  $h_j$  below are now restricted to the positive integers):

$$\begin{aligned} \text{LHS}_\ell & := \frac{1}{2\pi i} \int_{(2)} X^s \left( \frac{T}{2\pi} \right)^{-\ell s} \sum_{\substack{(M_1, N_1)=\dots=(M_\ell, N_\ell)=1 \\ N_1 \dots N_\ell = M_1 \dots M_\ell \\ \epsilon_j \in \{-1, +1\}}} \int_{0 < v_1, \dots, v_\ell < \infty} \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_\ell v_\ell) \\ & \prod_{j=1}^\ell \left[ \frac{1}{(2\pi i)^2} \iint_{\substack{|w_j-1|=\epsilon \\ |z_j-1|=\epsilon}} M_j^{-z_j} N_j^{s+1-w_j} \sum_{h_j, q_j} \frac{r_{q_j}(h_j)}{h_j^{s+2-w_j-z_j}} v_j^{s+1-w_j-z_j} \right. \\ & \left. D_{A_j} \left( w_j, e \left( \frac{N_j}{q_j} \right) \right) D_{B_j} \left( z_j, e \left( \frac{M_j}{q_j} \right) \right) \left( \frac{T}{2\pi} \right)^{w_j+z_j-2} dw_j dz_j dv_j \right] \frac{ds}{s} \end{aligned}$$

At this point we can rigorously identify  $\text{LHS}_\ell$  with the terms on the right of (1), through our key identity:

**THEOREM 1.**

$$\begin{aligned} \text{LHS}_\ell & = \frac{1}{2\pi i} \int_{(2)} X^s \int_0^\infty \psi(t) \sum_{\substack{U(\ell) \subset A \\ V(\ell) \subset B}} \left( \frac{Tt}{2\pi} \right)^{-\sum_{\substack{\hat{\alpha} \in U(\ell) \\ \hat{\beta} \in V(\ell)}} (\hat{\alpha} + \hat{\beta} + s)} \\ & \times \mathcal{B}(A_s - U(\ell)_s + V(\ell)^-, B - V(\ell) + U(\ell)_s^-, 1) dt \frac{ds}{s} \end{aligned}$$

where  $U(\ell)$  denotes a set of cardinality  $\ell$  with precisely one element from each of  $A_1, \dots, A_\ell$  and similarly  $V(\ell)$  denotes a set of cardinality  $\ell$  with precisely one element from each of  $B_1, \dots, B_\ell$ .

### 5. Preliminary reductions

**LEMMA 1.**

$$\sum_{\epsilon_j \in \{-1, +1\}} \int_0^\infty \dots \int_0^\infty \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_\ell v_\ell) \prod_{j=1}^\ell v_j^{s+1-w_j-z_j} dv_\ell \dots dv_1$$

$$= \int_0^\infty \psi(t) \prod_{j=1}^\ell \chi(w_j + z_j - s - 1) t^{w_j + z_j - s - 2} dt.$$

*Proof.* The case  $\ell = 1$  of this identity may be found in [5]. We may prove the general case by working our way from the inside out and using the technique of that proof. For example, with fixed  $v_1, \dots, v_{\ell-1}, \epsilon_1, \dots, \epsilon_{\ell-1}$  we have that the integral over  $v_\ell$  is

$$\int_{v_\ell=0}^\infty \int_{t=0}^\infty \psi(t) e(t\epsilon_1 v_1 + \dots + t\epsilon_{\ell-1} v_{\ell-1}) (e(tv_\ell) + e(-tv_\ell)) v_\ell^{s+1-w_\ell-z_\ell} dt dv_\ell$$

We split this into two double integrals, one with  $e(tv_\ell)$  and the other with  $e(-tv_\ell)$ . The first we rotate the  $v_\ell$ -path onto the positive imaginary axis, and the second we rotate the  $v_\ell$  path onto the negative imaginary axis. By absolute convergence, we may now interchange the order of integration to arrive at a sum of two  $v_\ell$ -integrals inside a  $t$ -integral. We evaluate the  $v_\ell$  integrals using the definition of the gamma-function. Then we repeat the process to evaluate the sum over  $\epsilon_{\ell-1}$  of the integral over  $v_{\ell-1}$  for a fixed  $v_1, \dots, v_{\ell-2}, \epsilon_1, \dots, \epsilon_{\ell-2}$ . And so on.  $\square$

### 6. Poles

We have

$$\langle \tau_A(m) e(m/q) \rangle_{m=u} \sim \frac{1}{q} \frac{1}{2\pi i} \int_{|w-1|=\frac{1}{8}} \prod_{\alpha \in A} \zeta(w + \alpha) G_A(w, q) \left(\frac{u}{q}\right)^{w-1} dw,$$

where  $G$  is a multiplicative function for which

$$G_A(1 - \alpha, p^r) = \prod_{\hat{\alpha} \in A'} \left(1 - \frac{1}{p^{1+\hat{\alpha}-\alpha}}\right) \sum_{j=0}^\infty \frac{\tau_{A'}(p^{j+r})}{p^{j(1-\alpha)}}$$

with  $A' = A - \{\alpha\}$ . With  $* : mN - nM = h$ , this leads to

$$\begin{aligned} \langle \tau_A(m) \tau_B(n) \rangle_{m=u}^{(*)} &\sim \frac{1}{M} \sum_{q=1}^\infty r_q(h) \langle \tau_A(m) e(mN/q) \rangle_{m=u} \langle \tau_B(n) e(nM/q) \rangle_{n=\frac{uN}{M}} \\ &\sim \sum_{\substack{\alpha \in A \\ \beta \in B}} u^{-\alpha-\beta} M^{-1+\beta} N^{-\beta} Z(A'_{-\alpha}) Z(B'_{-\beta}) \sum_{d|h} \frac{1}{d^{1-\alpha-\beta}} \\ &\times \sum_q \frac{\mu(q) (qd, M)^{1-\beta} (qd, N)^{1-\alpha}}{q^{2-\alpha-\beta}} G_A\left(1 - \alpha, \frac{qd}{(qd, N)}\right) G_B\left(1 - \beta, \frac{qd}{(qd, M)}\right), \end{aligned}$$

where

$$Z(A) = \prod_{a \in A} \zeta(1 + a).$$

Inserting this into  $LHS_\ell$  we have

$$\begin{aligned} LHS_\ell &= \frac{1}{2\pi i} \int_{(2)} X^s \left(\frac{T}{2\pi}\right)^{-\ell s} \sum_{\substack{(M_1, N_1) = \dots = (M_\ell, N_\ell) = 1 \\ N_1 \dots N_\ell = M_1 \dots M_\ell \\ \epsilon_j \in \{-1, +1\}}} \int_{0 < v_1, \dots, v_\ell < \infty} \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_\ell v_\ell) \\ &\sum_{\substack{U^{(\ell)} \subset A \\ V^{(\ell)} \subset B}} \prod_{j=1}^\ell \left[ Z((A_j)'_{-\alpha_j}) Z((B_j)'_{-\beta_j}) M_j^{\beta_j-1} N_j^{s+\alpha_j} \right] \end{aligned}$$

$$\times \sum_{h_j, q_j, d_j} \frac{1}{h_j^{s+\alpha_j+\beta_j} d_j^{1+s}} \frac{\mu(q_j)(q_j d_j, M_j)^{1-\beta_j} (q_j d_j, N_j)^{1-\alpha_j}}{q_j^{2-\alpha_j-\beta_j}} v_j^{s-1+\alpha_j+\beta_j} G_{A_j} \left( 1 - \alpha_j, \frac{q_j d_j}{(q_j d_j, N_j)} \right) G_{B_j} \left( 1 - \beta_j, \frac{q_j d_j}{(q_j d_j, M_j)} \right) \left( \frac{T}{2\pi} \right)^{-\alpha_j-\beta_j} dv_j \Big] \frac{ds}{s}$$

where  $U(\ell) = \{\alpha_1, \dots, \alpha_\ell\}$  with  $\alpha_j \in A_j$  and  $V(\ell) = \{\beta_1, \dots, \beta_\ell\}$  with  $\beta_j \in B_j$ . Now we sum over the  $h_j$  to get factors  $\zeta(s + \alpha_j + \beta_j)$ . Thus,

$$\begin{aligned} \text{LHS}_\ell &= \frac{1}{2\pi i} \int_{(2)} X^s \left( \frac{T}{2\pi} \right)^{-\ell s} \sum_{\substack{(M_1, N_1)=\dots=(M_\ell, N_\ell)=1 \\ N_1 \dots N_\ell = M_1 \dots M_\ell \\ \epsilon_j \in \{-1, +1\}}} \int_{0 < v_1, \dots, v_\ell < \infty} \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_\ell v_\ell) \\ &\sum_{\substack{U(\ell) \subset A \\ V(\ell) \subset B}} \prod_{j=1}^{\ell} \left[ Z((A_j)'_{-\alpha_j}) Z((B_j)'_{-\beta_j}) M_j^{\beta_j-1} N_j^{s+\alpha_j} \zeta(s + \alpha_j + \beta_j) \right. \\ &\sum_{q_j, d_j} \frac{1}{d_j^{1+s}} \frac{\mu(q_j)(q_j d_j, M_j)^{1-\beta_j} (q_j d_j, N_j)^{1-\alpha_j}}{q_j^{2-\alpha_j-\beta_j}} \\ &\left. v_j^{s-1+\alpha_j+\beta_j} G_{A_j} \left( 1 - \alpha_j, \frac{q_j d_j}{(q_j d_j, N_j)} \right) G_{B_j} \left( 1 - \beta_j, \frac{q_j d_j}{(q_j d_j, M_j)} \right) \left( \frac{T}{2\pi} \right)^{-\alpha_j-\beta_j} dv_j \right] \frac{ds}{s} \end{aligned}$$

If we move the path of integration in  $s$  to the line with  $\Re s = \epsilon$ , then we cross the poles of the  $\zeta(s + \alpha_j + \beta_j)$  at  $s = 1 - \alpha_j - \beta_j$ . These contribute an amount that cancels the contribution of the  $R_{A;B}(T; X)$ .

Next, we apply the lemma of Section 5 to evaluate the integral over the  $v_j$  and obtain a factor of  $\chi(1 - \alpha_j - \beta_j - s)$ . Then using the functional equation for  $\zeta$  we have  $\zeta(1 - s - \alpha_j - \beta_j)$ . Thus, the  $s$ -integrand without the  $\frac{X^s}{s}$  in LHS becomes

$$\begin{aligned} &\int_0^\infty \psi(t) \left( \frac{T}{2\pi} \right)^{-\ell s} \sum_{\substack{(M_1, N_1)=\dots=(M_\ell, N_\ell)=1 \\ N_1 \dots N_\ell = M_1 \dots M_\ell}} \sum_{\substack{U(\ell) \subset A \\ V(\ell) \subset B}} \prod_{j=1}^{\ell} \zeta(1 - s - \alpha_j - \beta_j) t^{-s-\alpha_j-\beta_j} \\ &\left[ Z((A_j)'_{-\alpha_j}) Z((B_j)'_{-\beta_j}) M_j^{\beta_j-1} N_j^{s+\alpha_j} \sum_{q_j, d_j} \frac{1}{d_j^{1+s}} \frac{\mu(q_j)(q_j d_j, M_j)^{1-\beta_j} (q_j d_j, N_j)^{1-\alpha_j}}{q_j^{2-\alpha_j-\beta_j}} \right. \\ &\left. G_{A_j} \left( 1 - \alpha_j, \frac{q_j d_j}{(q_j d_j, N_j)} \right) G_{B_j} \left( 1 - \beta_j, \frac{q_j d_j}{(q_j d_j, M_j)} \right) \left( \frac{T}{2\pi} \right)^{-\alpha_j-\beta_j} \right] dt, \end{aligned}$$

our goal is to prove that this is equal to

$$\begin{aligned} &\int_0^\infty \psi(t) \sum_{\substack{U(\ell) \subset A \\ V(\ell) \subset B}} \left( \frac{Tt}{2\pi} \right)^{-\sum_{\substack{\hat{\alpha} \in U(\ell) \\ \hat{\beta} \in V(\ell)}} (\hat{\alpha} + \hat{\beta} + s)} \\ &\times \mathcal{B}(A_s - U(\ell)_s + V(\ell)^-, B - V(\ell) + U(\ell)_s^-, 1) dt. \end{aligned}$$

This further reduces to proving for each  $U(\ell), V(\ell)$  that

$$\sum_{\substack{(M_1, N_1)=\dots=(M_\ell, N_\ell)=1 \\ N_1 \dots N_\ell = M_1 \dots M_\ell}} \prod_{j=1}^{\ell} \zeta(1 - s - \alpha_j - \beta_j)$$

$$\begin{aligned} & \left[ Z((A_j)'_{-\alpha_j})Z((B_j)'_{-\beta_j})M_j^{\beta_j-1}N_j^{s+\alpha_j} \sum_{q_j, d_j} \frac{1}{d_j^{1+s}} \frac{\mu(q_j)(q_j d_j, M_j)^{1-\beta_j} (q_j d_j, N_j)^{1-\alpha_j}}{q_j^{2-\alpha_j-\beta_j}} \right. \\ & \left. G_{A_j} \left( 1 - \alpha_j, \frac{q_j d_j}{(q_j d_j, N_j)} \right) G_{B_j} \left( 1 - \beta_j, \frac{q_j d_j}{(q_j d_j, M_j)} \right) \right] \\ & = \mathcal{B}(A_s - U(\ell)_s + V(\ell)^-, B - V(\ell) + U(\ell)_s^-, 1). \end{aligned}$$

7. Local considerations

We shall find it convenient to state our main theorem as an identity of the Euler factor at a prime  $p$ . We begin by introducing a set-theoretic notation. First of all, since  $p$  is fixed for this discussion we will often suppress it. In fact we write  $X$  for  $1/p$  and mostly consider power series in  $X$ . We take the unusual step of suppressing not only the prime  $p$  but the divisor function and so we write  $A(n)$  in place of  $\tau_A(p^n)$ . Also, for a set  $A$  we let

$$A_\alpha = \{a + \alpha : a \in A\}.$$

A further piece of notation:  $A^+ = A \cup \{0\}$ . We have two important identities. The first is

$$A^+(d) = A(d) + A^+(d - 1).$$

This is a special case of

$$(A \cup \{-\alpha\})(d - 1) = X^\alpha \left( (A \cup \{-\alpha\})(d) - A(d) \right).$$

The other identity is

$$\sum_{r=0}^R A(r + M) = A^+(R + M) - A^+(M - 1)$$

which follows by repeated application of the first identity.

For arbitrary sets  $A, B$  we let

$$C(A, B) := \sum_{M=0}^\infty A(M)B(M)X^M.$$

Also, we let

$$Z(A) = \sum_{j=0}^\infty A(j)X^j = \prod_{a \in A} (1 - X^{1+a})^{-1}.$$

We begin with sets  $A_j, B_j$  and numbers  $\alpha_j, \beta_j$  for  $j = 1, 2, \dots, \ell$ . We consider

$$\mathcal{Q} := \sum_{\substack{\min(M_j, N_j)=0 \\ \sum_{j=1}^\ell M_j = \sum_{j=1}^\ell N_j}} \prod_{j=1}^\ell \Sigma_{A_j, B_j, \alpha_j, \beta_j}(M_j, N_j) X^{M_j(1-\beta_j) - N_j \alpha_j}$$

where

$$\begin{aligned} \Sigma_{A, B, \alpha, \beta}(M, N) &= \sum_{\substack{d, j, k \\ q \leq 1}} (-1)^q X^{d(\alpha+\beta)} A_{-\alpha}(j + q + d - \min(q + d, N)) \\ &\quad \times B_{-\beta}(k + q + d - \min(q + d, M)) X^{2q+d+j+k - \min(q+d, M) - \min(q+d, N)} \end{aligned}$$

Our identity is

THEOREM 2.

$$\mathcal{Q} = \prod_{j=1}^{\ell} (1 - X^{1-\alpha_j-\beta_j}) \mathcal{C}(A_1 \cup \dots \cup A_{\ell} \cup \{-\beta_1, \dots, -\beta_{\ell}\}, B_1 \cup \dots \cup B_{\ell} \cup \{-\alpha_1, \dots, -\alpha_{\ell}\}).$$

By the results of the previous section, Theorem 1 follows from Theorem 2 with  $(A_j \setminus \{\alpha_j\})_s$  in place of  $A_j$ ,  $B_j \setminus \{\beta_j\}$  in place of  $B_j$  and  $\alpha_j + s$  in place of  $\alpha_j$ .

7.1. Some lemmas

Because of the condition  $\min(M_j, N_j) = 0$  we consider  $\Sigma_{A,B,\alpha,\beta}(M, 0)$  and  $\Sigma_{A,B,\alpha,\beta}(0, N)$ . We have

LEMMA 2.

$$\begin{aligned} \Sigma_{A,B,\alpha,\beta}(M, 0) &= X^{M\beta} \left( \sum_K (B \cup \{-\alpha\})(K) A(K+M) X^K - \sum_K B(K) A(K+M) X^K \right. \\ &\quad \left. + \sum_K B(K) (A \cup \{-\beta\})(K+M) X^K \right) \end{aligned}$$

and

$$\begin{aligned} \Sigma_{A,B,\alpha,\beta}(0, N) &= X^{N\alpha} \left( \sum_K (A \cup \{-\beta\})(K) B(K+N) X^K - \sum_K A(K) B(K+N) X^K \right. \\ &\quad \left. + \sum_K A(K) (B \cup \{-\alpha\})(K+N) X^K \right). \end{aligned}$$

We defer the proof to later.

The result of the lemma leads us to consider

$$\mathcal{F}(A_1, \dots, A_{\ell}; B_1, \dots, B_{\ell}) = \sum_{\substack{\sum M_i = \sum N_i \\ \min(M_i, N_i) = 0 \\ K_i}} \prod_{j=1}^{\ell} A_j(K_j + M_j) B_j(K_j + N_j) X^{K_j + M_j}.$$

We will prove

LEMMA 3. We have

$$\mathcal{F}(A_1, \dots, A_{\ell}; B_1, \dots, B_{\ell}) = \mathcal{C}(A_1 \cup \dots \cup A_{\ell}, B_1 \cup \dots \cup B_{\ell}).$$

The right-hand side of Theorem 2 may be expanded. This leads to

LEMMA 4. For  $J \subset \{1, \dots, \ell\}$  let

$$-\beta_J = \{-\beta_j : j \in J\} \quad -\alpha_J = \{-\alpha_j : j \in J\}.$$

We have

$$\begin{aligned} &\prod_{j=1}^{\ell} (1 - X^{1-\alpha_j-\beta_j}) \mathcal{C}(A_1 \cup \dots \cup A_{\ell} \cup \{-\beta_1, \dots, -\beta_{\ell}\}, B_1 \cup \dots \cup B_{\ell} \cup \{-\alpha_1, \dots, -\alpha_{\ell}\}) \\ &= (-1)^{\ell} \sum_{\substack{J_1, J_2 \subset \{1, \dots, \ell\} \\ J_1 \cap J_2 = \emptyset}} (-1)^{|J_1|+|J_2|} \mathcal{C}(A \cup -\beta_{J_1}, B \cup -\alpha_{J_2}) \end{aligned}$$

where  $A = A_1 \cup \dots \cup A_{\ell}$  and  $B = B_1 \cup \dots \cup B_{\ell}$ .

The combination of these three lemmas easily leads to a proof of Theorem 2.

7.2. Proof of Theorem 2

Proof. By Lemma 1 the left side of the identity in Theorem 2 may be written as

$$\begin{aligned} & \sum_{\substack{\min(M_j, N_j)=0 \\ \sum_{j=1}^{\ell} M_j = \sum_{j=1}^{\ell} N_j}} \prod_{j=1}^{\ell} \Sigma_{A_j, B_j, \alpha_j, \beta_j}(M_j, N_j) X^{M_j(1-\beta_j) - N_j \alpha_j} \\ &= \sum_{\substack{\min(M_j, N_j)=0 \\ \sum_{j=1}^{\ell} M_j = \sum_{j=1}^{\ell} N_j}} X^{M_1 + \dots + M_{\ell}} \prod_{j=1}^{\ell} \left( \sum_K (A_j \cup \{-\beta_j\})(K + M_j) B_j(K + N_j) X^K \right. \\ & \quad - \sum_K A_j(K + M_j) B_j(K + N_j) X^K \\ & \quad \left. + \sum_K A_j(K + M_j) (B_j \cup \{-\alpha_j\})(K + N_j) X^K \right). \end{aligned}$$

By Lemma 2 this is

$$= (-1)^{\ell} \sum_{\substack{J_1, J_2 \subset \{1, \dots, \ell\} \\ J_1 \cap J_2 = \emptyset}} (-1)^{|J_1| + |J_2|} \mathcal{C}(A \cup -\beta_{J_1}, B \cup -\alpha_{J_2})$$

and by Lemma 3 this is

$$= \prod_{j=1}^{\ell} (1 - X^{1-\alpha_j-\beta_j}) \mathcal{C}(A_1 \cup \dots \cup A_{\ell} \cup \{-\beta_1, \dots, -\beta_{\ell}\}, B_1 \cup \dots \cup B_{\ell} \cup \{-\alpha_1, \dots, -\alpha_{\ell}\})$$

which is the right side of the identity in Theorem 2. □

7.3. Proof of first lemma

Proof. Expanding the  $q$ -sum, we have

$$\begin{aligned} &= \sum_{d, j, k} X^{d(\alpha+\beta)} A_{-\alpha}(j+d) B_{-\beta}(k+d-\min(d, M)) X^{d+j+k-\min(d, M)} \\ & \quad - \sum_{d, j, k} X^{d(\alpha+\beta)} A_{-\alpha}(j+1+d) B_{-\beta}(k+1+d-\min(1+d, M)) X^{2+d+j+k-\min(1+d, M)}. \end{aligned}$$

We split this into the terms with  $d < M$  and those with  $d \geq M$ . We have

$$\begin{aligned} \Sigma^-(M, 0) &= \sum_{\substack{j, k \\ d < M}} X^{d(\alpha+\beta)} A_{-\alpha}(j+d) B_{-\beta}(k) X^{j+k} \\ & \quad - \sum_{\substack{j, k \\ d < M}} X^{d(\alpha+\beta)} A_{-\alpha}(j+1+d) B_{-\beta}(k) X^{1+j+k} \\ &= Z(B_{-\beta}) \left( \sum_{\substack{j \\ d < M}} X^{d(\alpha+\beta)} A_{-\alpha}(j+d) X^j - \sum_{\substack{j \\ d < M}} X^{d(\alpha+\beta)} A_{-\alpha}(j+1+d) X^{1+j} \right). \end{aligned}$$

The sum over  $j$  telescopes so that this is

$$\begin{aligned} \Sigma^-(M, 0) &= Z(B_{-\beta}) \sum_{d < M} X^{d(\alpha+\beta)} A_{-\alpha}(d) \\ &= Z(B_{-\beta}) \sum_{d < M} A_{\beta}(d) = Z(B_{-\beta})(A_{\beta})^+(M - 1). \end{aligned}$$

Next we consider

$$\begin{aligned} \Sigma^+(M) &= \sum_{\substack{j, k \\ d \geq M}} X^{d(\alpha+\beta)} A_{-\alpha}(j + d) B_{-\beta}(k + d - M) X^{d+j+k-M} \\ &\quad - \sum_{\substack{j, k \\ d \geq M}} X^{d(\alpha+\beta)} A_{-\alpha}(j + 1 + d) B_{-\beta}(k + 1 + d - M) X^{2+d+j+k-M}. \end{aligned}$$

We replace  $d$  by  $d + M$  and have

$$\begin{aligned} \Sigma^+(M) &= \sum_{j, k, d} X^{(d+M)(\alpha+\beta)} A_{-\alpha}(j + d + M) B_{-\beta}(k + d) X^{d+j+k} \\ &\quad - \sum_{j, k, d} X^{(d+M)(\alpha+\beta)} A_{-\alpha}(j + 1 + d + M) B_{-\beta}(k + 1 + d) X^{2+d+j+k}. \end{aligned}$$

Now the sum over  $j$  and  $k$  telescopes and we have

$$\begin{aligned} \Sigma^+(M) &= \sum_{j, d} X^{(d+M)(\alpha+\beta)} A_{-\alpha}(j + d + M) B_{-\beta}(d) X^{d+j} \\ &\quad + \sum_{k, d} X^{(d+M)(\alpha+\beta)} A_{-\alpha}(d + M) B_{-\beta}(k + d) X^{d+k} \\ &\quad - \sum_d X^{(d+M)(\alpha+\beta)} A_{-\alpha}(d + M) B_{-\beta}(d) X^d. \end{aligned}$$

We recognize a convolution in the first term and rewrite this as

$$\begin{aligned} \Sigma^+(M) &= \sum_r X^{M(\alpha+\beta)} A_{-\alpha}(r + M) (B_{\alpha})^+(r) X^r \\ &\quad + \sum_{k, d} X^{(d+M)(\alpha+\beta)} A_{-\alpha}(d + M) B_{-\beta}(k + d) X^{d+k} \\ &\quad - \sum_d X^{(d+M)(\alpha+\beta)} A_{-\alpha}(d + M) B_{-\beta}(d) X^d. \end{aligned}$$

The middle term here may be written as

$$\begin{aligned} \sum_K B_{-\beta}(K) X^K \sum_{d \leq K} A_{\beta}(d + M) &= \sum_K B_{-\beta}(K) X^K ((A_{\beta})^+(K + M) - (A_{\beta})^+(M - 1)) \\ &= \sum_K B_{-\beta}(K) (A_{\beta})^+(K + M) X^K - Z(B_{-\beta})(A_{\beta})^+(M - 1). \end{aligned}$$

The second term of this cancels with  $\Sigma^-(M, 0)$  and so we have

$$\Sigma_{A, B, \alpha, \beta}(M, 0) = X^{M(\alpha+\beta)} \sum_K (B_{\alpha})^+(K) A_{-\alpha}(K + M) X^K - X^{M(\alpha+\beta)}$$

$$\begin{aligned} & \times \sum_K B_\alpha(K)A_{-\alpha}(K + M)X^K \\ & + \sum_K B_{-\beta}(K)(A_\beta)^+(K + M)X^K. \end{aligned}$$

This may be rewritten as

$$\begin{aligned} \Sigma_{A,B,\alpha,\beta}(M, 0) &= X^{M\beta} \left( \sum_K (B \cup \{-\alpha\})(K)A(K + M)X^K - \sum_K B(K)A(K + M)X^K \right. \\ & \left. + \sum_K B(K)(A \cup \{-\beta\})(K + M)X^K \right). \end{aligned} \quad \square$$

By symmetry

$$\begin{aligned} \Sigma_{A,B,\alpha,\beta}(0, N) &= X^{N\alpha} \left( \sum_K (A \cup \{-\beta\})(K)B(K + N)X^K - \sum_K A(K)B(K + N)X^K \right. \\ & \left. + \sum_K A(K)(B \cup \{-\alpha\})(K + N)X^K \right). \end{aligned}$$

7.4. Proof of second lemma

*Proof.* We prove more generally that

$$\begin{aligned} & \sum_{\substack{M_1 + \dots + M_\ell = N_1 + \dots + N_\ell \\ \min(M_1, N_1) = \dots = \min(M_\ell, N_\ell) = 0 \\ K_1, \dots, K_\ell}} A_1(K_1 + M_1)B_1(K_1 + N_1) \dots A_\ell \\ & \times (K_\ell + M_\ell)B_\ell(K_\ell + N_\ell)X^{K_1 + \dots + K_\ell + M_1 + \dots + M_\ell} \\ & = \sum_{N=0}^\infty (A_1 * \dots * A_\ell)(N)(B_1 * \dots * B_\ell)(N)X^N \end{aligned}$$

where the  $A_i$  and  $B_i$  are any functions on the natural numbers (that is, sequences) and  $*$  just means the usual Cauchy convolution one encounters when multiplying power series together. It suffices to prove

$$\begin{aligned} & \sum_{\substack{\min(M_1, N_1) = \dots = \min(M_\ell, N_\ell) = 0 \\ K_1, \dots, K_\ell}} A_1(K_1 + M_1)B_1(K_1 + N_1) \dots A_\ell(K_\ell + M_\ell)B_\ell(K_\ell + N_\ell) \\ & \times Y^{2(K_1 + \dots + K_\ell) + M_1 + \dots + M_\ell + N_1 + \dots + N_\ell} e(\theta(M_1 + \dots + M_\ell - N_1 - \dots - N_\ell)) \\ & = \sum_{M, N=0}^\infty (A_1 * \dots * A_\ell)(M)(B_1 * \dots * B_\ell)(N)Y^{M+N} e(\theta(M - N)) \end{aligned}$$

as then our desired result follows upon integrating  $\theta$  from 0 to 1 upon taking  $X = Y^2$ . But now the left-hand side is a product

$$\prod_{j=1}^\ell \sum_{\substack{K, M, N \\ \min(M, N) = 0}} A_j(M + K)B_j(N + K)Y^{2K+M+N} e(\theta((M + K) - (N + K)))$$

and the right-hand side is a product

$$\prod_{j=1}^{\ell} \sum_{R=0}^{\infty} A_j(R)(Ye(\theta))^R \sum_{S=0}^{\infty} B_j(S)(Ye(-\theta))^S.$$

Therefore, it suffices to prove that

$$\sum_{\substack{K,M,N \\ \min(M,N)=0}} A(M+K)B(N+K)Y^{K+M}Z^{K+N} = \sum_{R=0}^{\infty} A(R)Y^R \sum_{S=0}^{\infty} B(S)Z^S.$$

To do this, we consider the right-hand side and order the double sum according to the minimum, call it  $K$ , of  $R$  and  $S$ . The right-hand side may be rewritten as

$$\sum_{K=0}^{\infty} \sum_{\substack{R,S \\ \min(R,S)=K}} A(R)Y^R B(S)Z^S.$$

Replacing  $R$  by  $M + K$  and  $S$  by  $N + K$ , we see that this is exactly the left-hand side. □

### 7.5. Proof of third lemma

*Proof.* Recall that

$$X^{-\beta}(A \cup \{-\beta\})(n) = (A \cup \{-\beta\})(n+1) - A(n+1).$$

Using this we see that

$$\begin{aligned} & (1 - X^{1-\alpha-\beta}) \sum_{m,n=0}^{\infty} (A \cup \{-\beta\})(m)X^{\frac{m}{2}} (B \cup \{-\alpha\})(n)X^{\frac{n}{2}} \\ &= \sum_{m,n=0}^{\infty} (A \cup \{-\beta\})(m)X^{\frac{m}{2}} (B \cup \{-\alpha\})(n)X^{\frac{n}{2}} \\ &\quad - \sum_{m,n=0}^{\infty} X^{\frac{(m+1)}{2} + \frac{(n+1)}{2}} \left( (A \cup \{-\beta\})(m+1) - A(m+1) \right) \\ &\quad \times \left( (B \cup \{-\alpha\})(n+1) - B(n+1) \right). \end{aligned}$$

In the last line we can replace  $m + 1$  and  $n + 1$  by  $m$  and  $n$  since  $(A \cup \{-\beta\})(0) = A(0) = 1$  and similarly for  $B$ . Multiplying out the last line and combining it with the line above we have

$$\begin{aligned} & \sum_{m,n=0}^{\infty} (A \cup \{-\beta\})(m)X^{\frac{m}{2}} B(n)X^{\frac{n}{2}} + \sum_{m,n=0}^{\infty} A(m)X^{\frac{m}{2}} (B \cup \{-\alpha\})(n)X^{\frac{n}{2}} \\ &\quad - \sum_{m,n=0}^{\infty} A(m)B(n)X^{\frac{m+n}{2}}. \end{aligned}$$

Now the idea is to apply this to each  $A_j, B_j, \alpha_j, \beta_j$ . We have

$$\begin{aligned} & (1 - X^{1-\alpha_1-\beta_1}) \dots (1 - X^{1-\alpha_\ell-\beta_\ell}) \mathcal{C}(A_1 \cup \{-\beta_1\} \cup \dots \cup A_\ell \cup \{-\beta_\ell\}, B_1 \cup \{-\alpha_1\} \cup \dots \\ & \quad \times \cup B_\ell \cup \{-\alpha_\ell\}) \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \sum_{m_1, n_1} (1 - X^{1-\alpha_1-\beta_1})(A_1 \cup \{-\beta_1\})(m_1)(X^{\frac{1}{2}}e(\theta))^{m_1}(B_1 \cup \{-\alpha_1\})(n_1)(X^{\frac{1}{2}}e(-\theta))^{n_1} \\
 &\dots \sum_{m_\ell, n_\ell} (1 - X^{1-\alpha_\ell-\beta_\ell})(A_\ell \cup \{-\beta_\ell\})(m_\ell)(X^{\frac{1}{2}}e(\theta))^{m_\ell}(B_\ell \cup \{-\alpha_\ell\})(n_\ell)(X^{\frac{1}{2}}e(-\theta))^{n_\ell} d\theta.
 \end{aligned}$$

We end up with

$$\begin{aligned}
 &\int_0^1 \prod_{j=1}^{\ell} \sum_{m_j, n_j=0}^{\infty} (X^{\frac{1}{2}}e(\theta))^{m_j}(X^{\frac{1}{2}}e(-\theta))^{n_j} \left( (A_j \cup \{-\beta_j\})(m_j)B_j(n_j) \right. \\
 &\quad \left. + A_j(m_j)(B_j \cup \{-\alpha_j\})(n_j) - A_j(m_j)B_j(n_j) \right) d\theta,
 \end{aligned}$$

which is equal to

$$(-1)^\ell \sum_{\substack{J_1, J_2 \subset \{1, \dots, \ell\} \\ J_1 \cap J_2 = \emptyset}} (-1)^{|J_1|+|J_2|} \mathcal{C}(A \cup -\beta_{J_1}, B \cup -\alpha_{J_2}).$$

□

8. Terms with some  $h_j = 0$

Suppose that we are in the situation where

$$h_1 \dots h_{\ell'} \neq 0 \quad \text{and} \quad h_{\ell'+1} = \dots = h_\ell = 0.$$

Then for each  $j > \ell'$  we have

$$m_j N_j = n_j M_j.$$

Since  $(M_j, N_j) = 1$  this implies that

$$m_j = \kappa_j M_j \quad \text{and} \quad n_j = \kappa_j N_j$$

for some  $\kappa_j$ . Then, our sum is

$$\begin{aligned}
 &\sum_{\substack{M_j, N_j \leq Q \\ (M_j, N_j) = 1 \\ M_1 \dots M_\ell = N_1 \dots N_\ell \\ \kappa_{\ell'+1} M_{\ell'+1} \dots \kappa_\ell M_\ell \leq X}} \frac{\tau_{A_{\ell'+1}}(\kappa_{\ell'+1} M_{\ell'+1}) \tau_{B_{\ell'+1}}(\kappa_{\ell'+1} N_{\ell'+1}) \dots \tau_{A_\ell}(\kappa_\ell M_\ell) \tau_{B_\ell}(\kappa_\ell N_\ell)}{\kappa_{\ell'+1} \dots \kappa_\ell M_{\ell'+1} \dots M_\ell} \\
 &\sum_{h_1 \dots h_{\ell'} \neq 0} \sum_{\substack{m_1 \dots m_{\ell'} \leq \frac{X}{\kappa_{\ell'+1} M_{\ell'+1} \dots \kappa_\ell M_\ell} \\ (*_1), \dots, (*_{\ell'})}} \times \frac{\tau_{A_1}(m_1) \dots \tau_{A_{\ell'}}(m_{\ell'}) \tau_{B_1}(n_1) \dots \tau_{B_{\ell'}}(n_{\ell'})}{m_1 \dots m_{\ell'}} \hat{\psi} \left( \frac{Th_1}{2\pi m_1 N_1} + \dots + \frac{Th_{\ell'}}{2\pi m_{\ell'} N_{\ell'}} \right)
 \end{aligned}$$

where

$$(*_j) : m_j N_j - n_j M_j = h_j.$$

Now, as before, we replace the convolution sums  $(*)$  by their averages, that is,

$$\begin{aligned}
 &\int_{u_1 \dots u_{\ell'} \leq X'} \langle \tau_{A_1}(m_1) \tau_{B_1}(n_1) \rangle_{m_1 \sim u_1}^{(*_1)} \dots \langle \tau_{A_{\ell'}}(m_{\ell'}) \tau_{B_{\ell'}}(n_{\ell'}) \rangle_{m_{\ell'} \sim u_{\ell'}}^{(*_{\ell'})} \hat{\psi} \\
 &\quad \times \left( \frac{Th_1}{2\pi u_1 N_1} + \dots + \frac{Th_{\ell'}}{2\pi u_{\ell'} N_{\ell'}} \right) \frac{du_1}{u_1} \dots \frac{du_{\ell'}}{u_{\ell'}}
 \end{aligned}$$

where  $X'$  is defined by

$$X' = \frac{X}{\kappa_{\ell'+1}M_{\ell'+1} \dots \kappa_{\ell}M_{\ell}};$$

here  $*$  :  $mN - nM = h$ . We expect by the delta-method [11] that

$$\langle \tau_A(m)\tau_B(n) \rangle_{m \sim u}^{(*)} \sim \frac{1}{M} \sum_{q=1}^{\infty} r_q(h) \langle \tau_A(m)e(mN/q) \rangle_{m \sim u} \langle \tau_B(n)e(nM/q) \rangle_{n \sim \frac{uN}{M}}$$

with

$$\langle \tau_A(m)e(mN/q) \rangle_{m \sim u} = \frac{1}{2\pi i} \int_{|w-1|=\epsilon} D_A(w, e\left(\frac{N}{q}\right)) u^{w-1} dw.$$

So, we are led to

$$\begin{aligned} & \sum_{\substack{M_j, N_j \leq Q \\ (M_j, N_j)=1 \\ M_1 \dots M_{\ell} = N_1 \dots N_{\ell} \\ \kappa_{\ell'+1}M_{\ell'+1} \dots \kappa_{\ell}M_{\ell} \leq X}} \frac{\tau_{A_{\ell'+1}}(\kappa_{\ell'+1}M_{\ell'+1})\tau_{B_{\ell'+1}}(\kappa_{\ell'+1}N_{\ell'+1}) \dots \tau_{A_{\ell}}(\kappa_{\ell}M_{\ell})\tau_{B_{\ell}}(\kappa_{\ell}N_{\ell})}{\kappa_{\ell'+1} \dots \kappa_{\ell}M_1 \dots M_{\ell}} \\ & \sum_{q_j, h_j} r_{q_1}(h_1) \dots r_{q_{\ell'}}(h_{\ell'}) \int_{T^{\ell'} \leq u_1 \dots u_{\ell'} \leq X'} \prod_{j=1}^{\ell'} \frac{1}{2\pi i} \int_{|w_j-1|=\epsilon} D_{A_j}\left(w_j, e\left(\frac{N_j}{q_j}\right)\right) u_j^{w_j-1} dw_j \\ & \times \prod_{k=1}^{\ell'} \frac{1}{2\pi i} \int_{|z_k-1|=\epsilon} D_{B_k}\left(z_k, e\left(\frac{M_k}{q_k}\right)\right) \left(\frac{u_k N_k}{M_k}\right)^{z_k-1} dz_k \\ & \times \hat{\psi}\left(T\left(\frac{h_1}{2\pi u_1 N_1} + \dots + \frac{h_{\ell'}}{2\pi u_{\ell'} N_{\ell'}}\right)\right) \frac{du_1}{u_1} \dots \frac{du_{\ell'}}{u_{\ell'}}. \end{aligned}$$

We make the changes of variable  $v_j = \frac{T|h_j|}{2\pi u_j N_j}$  for  $1 \leq j \leq \ell'$  and bring the sums over the  $h_j$  to the inside;  $u_1 \dots u_{\ell'} < X'$  implies that

$$|h_1 \dots h_{\ell'}| < \frac{(2\pi)^{\ell'} X' v_1 \dots v_{\ell'} N_1 \dots N_{\ell'}}{T^{\ell'}}.$$

We detect this condition using Perron's formula in an integral over  $s$ . Then the above is

$$\begin{aligned} & \int_{0 < v_1, \dots, v_{\ell'} < \infty} \sum_{\substack{M_j, N_j \leq Q \\ (M_j, N_j)=1 \\ M_1 \dots M_{\ell} = N_1 \dots N_{\ell} \\ \kappa_{\ell'+1}M_{\ell'+1} \dots \kappa_{\ell}M_{\ell} \leq X}} \frac{\tau_{A_{\ell'+1}}(\kappa_{\ell'+1}M_{\ell'+1})\tau_{B_{\ell'+1}}(\kappa_{\ell'+1}N_{\ell'+1}) \dots \tau_{A_{\ell}}(\kappa_{\ell}M_{\ell})\tau_{B_{\ell}}(\kappa_{\ell}N_{\ell})}{\kappa_{\ell'+1} \dots \kappa_{\ell}M_1 \dots M_{\ell}} \\ & \times \frac{1}{2\pi i} \int_{(2)} \sum_{q_j, h_j} \frac{r_{q_1}(h_1) \dots r_{q_{\ell'}}(h_{\ell'})}{(h_1 \dots h_{\ell'})^s} \prod_{j=1}^{\ell'} \frac{1}{2\pi i} \int_{|w_j-1|=\epsilon} D_{A_j}\left(w_j, e\left(\frac{N_j}{q_j}\right)\right) \left(\frac{Th_j}{2\pi v_j N_j}\right)^{w_j-1} dw_j \\ & \prod_{k=1}^{\ell'} \frac{1}{2\pi i} \int_{|z_k-1|=\epsilon} D_{B_k}\left(z_k, e\left(\frac{M_k}{q_k}\right)\right) \left(\frac{Th_k}{2\pi v_k M_k}\right)^{z_k-1} dz_k \\ & \times \left(\frac{(2\pi)^{\ell'} X' v_1 \dots v_{\ell'} N_1 \dots N_{\ell'}}{T^{\ell'}}\right)^s \sum_{\epsilon_j \in \{-1, +1\}} \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_{\ell'} v_{\ell'}) \frac{dv_1}{v_1} \dots \frac{dv_{\ell'}}{v_{\ell'}} \frac{ds}{s} \end{aligned}$$

where the  $\epsilon_j = \text{sgn}(h_j)$  arise from taking account of the signs of the  $h_j$ . We now simplify this a bit. We combine the middle two lines into a single product over  $j$  and we gather together all of the like variables:

$$\begin{aligned} \text{LHS}_{\ell'} &:= \frac{1}{(2\pi i)^2} \iint_{(2)} X^{s+s'} \left(\frac{T}{2\pi}\right)^{-\ell' s} \sum_{\substack{(M_1, N_1)=\dots=(M_{\ell'}, N_{\ell'})=1 \\ N_1 \dots N_{\ell'} = M_1 \dots M_{\ell'} \\ \epsilon_j \in \{-1, +1\}}} \sum_{\kappa_{\ell'+1}, \dots, \kappa_{\ell'}} \prod_{i=\ell'+1}^{\ell'} \frac{\tau_{A_i}(M_i \kappa_i) \tau_{B_i}(N_i \kappa_i)}{(\kappa_i M_i)^{1+s'+s}} \\ &\int_{0 < v_1, \dots, v_{\ell'} < \infty} \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_{\ell'} v_{\ell'}) \prod_{j=1}^{\ell'} \left[ \frac{1}{(2\pi i)^2} \iint_{\substack{|w_j-1|=\epsilon \\ |z_j-1|=\epsilon}} M_j^{-z_j} N_j^{s+1-w_j} \right. \\ &\times \sum_{h_j, q_j} \frac{r_{q_j}(h_j)}{h_j^{s+2-w_j-z_j}} v_j^{s+1-w_j-z_j} \\ &\left. D_{A_j} \left( w_j, e \left( \frac{N_j}{q_j} \right) \right) D_{B_j} \left( z_j, e \left( \frac{M_j}{q_j} \right) \right) \left( \frac{T}{2\pi} \right)^{w_j+z_j-2} dw_j dz_j dv_j \right] \frac{ds}{s} \frac{ds'}{s'} \end{aligned}$$

Now we have another key identity:

**THEOREM 3.**

$$\begin{aligned} \text{Res}_{s'=0} \text{LHS}_{\ell'} &= \frac{1}{2\pi i} \int_{(2)} X^s \int_0^\infty \psi(t) \sum_{\substack{U(\ell') \subset A \\ V(\ell') \subset B}} \left(\frac{Tt}{2\pi}\right)^{-\sum_{\substack{\hat{\alpha} \in U(\ell') \\ \hat{\beta} \in V(\ell')}} (\hat{\alpha} + \hat{\beta} + s)} \\ &\times \mathcal{B}(A_s - U(\ell')_s + V(\ell')^-, B - V(\ell') + U(\ell')_s^-, 1) dt \frac{ds}{s} \end{aligned}$$

where  $U(\ell')$  denotes a set of cardinality  $\ell'$  with precisely one element from each of  $A_1, \dots, A_{\ell'}$  and similarly  $V(\ell')$  denotes a set of cardinality  $\ell'$  with precisely one element from each of  $B_1, \dots, B_{\ell'}$ .

### 9. Preliminary reductions, again

The result of Section 5 implies that

$$\begin{aligned} &\sum_{\epsilon_j \in \{-1, +1\}} \int_{v_1, \dots, v_{\ell'}=0}^\infty \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_{\ell'} v_{\ell'}) \prod_{j=1}^{\ell'} v_j^{s+1-w_j-z_j} dv_j \\ &= \int_0^\infty \psi(t) \prod_{j=1}^{\ell'} \chi(w_j + z_j - s - 1) t^{w_j+z_j-s-2} dt. \end{aligned}$$

### 10. Poles, again

As before we use

$$\langle \tau_A(m) \tau_B(n) \rangle_{m=u}^{(*)} \sim \sum_{\substack{\alpha \in A \\ \beta \in B}} u^{-\alpha-\beta} M^{-1+\beta} N^{-\beta} Z(A'_{-\alpha}) Z(B'_{-\beta}) \sum_{d|h} \frac{1}{d^{1-\alpha-\beta}}$$

$$\times \sum_q \frac{\mu(q)(qd, M)^{1-\beta}(qd, N)^{1-\alpha}}{q^{2-\alpha-\beta}} G_A \left( 1 - \alpha, \frac{qd}{(qd, N)} \right) G_B \left( 1 - \beta, \frac{qd}{(qd, M)} \right).$$

Inserting this into  $\text{LHS}_{\ell'}$  we have

$$\begin{aligned} \text{LHS}_{\ell'} &= \frac{1}{(2\pi i)^2} \iint_{(2)} X^{s+s'} \left( \frac{T}{2\pi} \right)^{-\ell' s} \sum_{\substack{(M_1, N_1)=\dots=(M_{\ell'}, N_{\ell'})=1 \\ N_1 \dots N_{\ell'} = M_1 \dots M_{\ell'} \\ \epsilon_j \in \{-1, +1\}}} \sum_{\kappa_{\ell'+1}, \dots, \kappa_{\ell}} \prod_{i=\ell'+1}^{\ell} \frac{\tau_{A_i}(M_i \kappa_i) \tau_{B_i}(N_i \kappa_i)}{(\kappa_i M_i)^{1+s'+s}} \\ &\int_{0 < v_1, \dots, v_{\ell'} < \infty} \hat{\psi}(\epsilon_1 v_1 + \dots + \epsilon_{\ell'} v_{\ell'}) \prod_{j=1}^{\ell'} \left[ \sum_{\substack{\alpha_j \in A_j \\ \beta_j \in B_j}} v_j^{s-1+\alpha_j+\beta_j} M_j^{\beta_j-1} N_j^{s+\alpha_j} \right. \\ &\times Z((A_j)'_{-\alpha_j}) Z((B_j)'_{-\beta_j}) \sum_{h_j, q_j, d_j} \frac{1}{h_j^{s+\alpha_j+\beta_j} d_j^{1+s}} \frac{\mu(q_j)(q_j d_j, M_j)^{1-\beta_j} (q_j d_j, N_j)^{1-\alpha_j}}{q_j^{2-\alpha_j-\beta_j}} \\ &\left. G_{A_j} \left( 1 - \alpha_j, \frac{q_j d_j}{(q_j d_j, N_j)} \right) G_{B_j} \left( 1 - \beta_j, \frac{q_j d_j}{(q_j d_j, M_j)} \right) \left( \frac{T}{2\pi} \right)^{-\alpha_j-\beta_j} dv_j \right] \frac{ds}{s} \frac{ds'}{s'}. \end{aligned}$$

Now we sum over the  $h_j$  to get factors  $\zeta(s + \alpha_j + \beta_j)$ ; these pair up with the factors  $\chi(w_j + z_j - s - 1)$  which turned into  $\chi(1 - \alpha_j - \beta_j - s)$  after collecting the residues  $w_j = 1 - \alpha_j$  and  $z_j = 1 - \beta_j$  that arose from the integral over  $v_j$ . Then using the functional equation for  $\zeta$  we have  $\zeta(1 - s - \alpha_j - \beta_j)$ . Thus, the  $s$ -integrand without the  $\frac{X^s}{s}$  in  $\text{LHS}_{\ell'}$  becomes

$$\begin{aligned} &\int_0^\infty \psi(t) \frac{1}{2\pi i} \int_{(2)} X^{s'} \left( \frac{T}{2\pi} \right)^{-\ell' s} \sum_{\substack{(M_1, N_1)=\dots=(M_{\ell'}, N_{\ell'})=1 \\ N_1 \dots N_{\ell'} = M_1 \dots M_{\ell'}}} \sum_{\kappa_{\ell'+1}, \dots, \kappa_{\ell}} \prod_{i=\ell'+1}^{\ell} \frac{\tau_{A_i}(M_i \kappa_i) \tau_{B_i}(N_i \kappa_i)}{(\kappa_i M_i)^{1+s'+s}} \\ &\sum_{\substack{U(\ell') \subset A \\ V(\ell') \subset B}} \prod_{j=1}^{\ell'} \left[ Z((A_j)'_{-\alpha_j}) Z((B_j)'_{-\beta_j}) \zeta(1 - s - \alpha_j - \beta_j) t^{-s-\alpha_j-\beta_j} \right. \\ &\times M_j^{\beta_j-1} N_j^{s+\alpha_j} \sum_{q_j, d_j} \frac{1}{d_j^{1+s}} \frac{\mu(q_j)(q_j d_j, M_j)^{1-\beta_j} (q_j d_j, N_j)^{1-\alpha_j}}{q_j^{2-\alpha_j-\beta_j}} \\ &\left. \times G_{A_j} \left( 1 - \alpha_j, \frac{q_j d_j}{(q_j d_j, N_j)} \right) G_{B_j} \left( 1 - \beta_j, \frac{q_j d_j}{(q_j d_j, M_j)} \right) \left( \frac{T}{2\pi} \right)^{-\alpha_j-\beta_j} \right] \frac{ds'}{s'} dt, \end{aligned}$$

where  $U(\ell') = \{\alpha_1, \dots, \alpha_{\ell'}\}$  and  $V(\ell') = \{\beta_1, \dots, \beta_{\ell'}\}$ . Our goal is to prove that the residue of this at  $s' = 0$  is equal to

$$\begin{aligned} &\int_0^\infty \psi(t) \sum_{\substack{U(\ell') \subset A \\ V(\ell') \subset B}} \left( \frac{Tt}{2\pi} \right)^{-\sum_{\substack{\hat{\alpha} \in U(\ell') \\ \hat{\beta} \in V(\ell')}} (\hat{\alpha} + \hat{\beta} + s)} \\ &\times \mathcal{B}(A_s - U(\ell')_s + V(\ell')^-, B - V(\ell') + U(\ell')_s^-, 1) dt. \end{aligned}$$

This further reduces to proving that

$$\begin{aligned} \text{Res}_{s'=0} & \sum_{\substack{(M_1, N_1)=\dots=(M_{\ell'}, N_{\ell'})=1 \\ N_1 \dots N_{\ell'} = M_1 \dots M_{\ell'}}} \sum_{\kappa_{\ell'+1}, \dots, \kappa_{\ell'}} \prod_{i=\ell'+1}^{\ell'} \frac{\tau_{A_i}(M_i \kappa_i) \tau_{B_i}(N_i \kappa_i)}{(\kappa_i M_i)^{1+s'+s}} \prod_{j=1}^{\ell'} \zeta(1-s-\alpha_j-\beta_j) \\ & \left[ M_j^{\beta_j-1} N_j^{s+\alpha_j} \sum_{q_j, d_j} \frac{1}{d_j^{1+s}} \frac{\mu(q_j)(q_j d_j, M_j)^{1-\beta_j} (q_j d_j, N_j)^{1-\alpha_j}}{q_j^{2-\alpha_j-\beta_j}} \right. \\ & \left. Z((A_j)'_{-\alpha_j}) Z((B_j)'_{-\beta_j}) G_{A_j} \left( 1-\alpha_j, \frac{q_j d_j}{(q_j d_j, N_j)} \right) G_{B_j} \left( 1-\beta_j, \frac{q_j d_j}{(q_j d_j, M_j)} \right) \right] \frac{X^{s'}}{s'} \\ & = \mathcal{B}(A_s - U(\ell')_s + V(\ell')^-, B - V(\ell') + U(\ell')_s^-, 1). \end{aligned}$$

11. Local considerations, again

Again we convert the above to an identity about the Euler factor of each side at a prime  $p$ .

With arbitrary sets  $A_j, B_j$  and numbers  $\alpha_j, \beta_j$  for  $j = 1, 2, \dots, \ell'$ , we consider

$$\begin{aligned} \mathcal{Q}' := & \sum_{\substack{\kappa_{\ell'+1}, \dots, \kappa_{\ell'} \\ \min(M_j, N_j)=0 \\ \sum_{j=1}^{\ell'} M_j = \sum_{j=1}^{\ell'} N_j}} \prod_{i=\ell'+1}^{\ell'} A_i(M_i + \kappa_i) B_i(N_i + \kappa_i) X^{M_i + \kappa_i} \\ & \times \prod_{j=1}^{\ell'} \Sigma_{A_j, B_j, \alpha_j, \beta_j}(M_j, N_j) X^{M_j(1-\beta_j) - N_j \alpha_j}, \end{aligned}$$

where

$$\begin{aligned} \Sigma_{A, B, \alpha, \beta}(M, N) = & \sum_{\substack{d, j, k \\ q \leq 1}} (-1)^q X^{d(\alpha+\beta)} A_{-\alpha}(j+q+d-\min(q+d, N)) \\ & \times B_{-\beta}(k+q+d-\min(q+d, M)) X^{2q+d+j+k-\min(q+d, M)-\min(q+d, N)}, \end{aligned}$$

as before.

Our identity is

THEOREM 4.

$$\mathcal{Q}' = \prod_{j=1}^{\ell'} (1 - X^{1-\alpha_j-\beta_j}) \mathcal{C}(A_1 \cup \dots \cup A_{\ell'} \cup \{-\beta_1, \dots, -\beta_{\ell'}\}, B_1 \cup \dots \cup B_{\ell'} \cup \{-\alpha_1, \dots, -\alpha_{\ell'}\}).$$

By the results of the previous section, Theorem 3 follows from Theorem 4 with  $(A_i)_s$  in place of  $A_i$  for  $i = \ell' + 1, \dots, \ell$  and with  $(A_j \setminus \{\alpha_j\})_s, B_j \setminus \{\beta_j\}$  and  $\alpha_j + s$  in place of  $A_j, B_j$  and  $\alpha_j$ , respectively, for  $j = 1, \dots, \ell'$ .

11.1. Recall lemmas

Our earlier lemma implies that if  $\min(M, N) = 0$  then

$$\Sigma_{A, B, \alpha, \beta}(M, N) = X^{M\beta+N\alpha} \left( \sum_K (A \cup \{-\beta\})(K+M) B(K+N) X^K \right)$$

$$\begin{aligned}
 & - \sum_K A(K + M)B(K + N)X^K \\
 & + \sum_K A(K + M)(B \cup \{-\alpha\})(K + N)X^K.
 \end{aligned}$$

So, we can replace the  $\Sigma$ s in the formula for  $\mathcal{Q}'$  by this expression. Thus, we have

$$\begin{aligned}
 \mathcal{Q}' = & \sum_{\substack{\kappa_{\ell'+1}, \dots, \kappa_{\ell} \\ \min(M_j, N_j) = 0 \\ \sum_{j=1}^{\ell} M_j = \sum_{j=1}^{\ell} N_j}} \prod_{i=\ell'+1}^{\ell} A_i(M_i + \kappa_i)B_i(N_i + \kappa_i)X^{M_i + \kappa_i} \\
 & \times \prod_{j=1}^{\ell'} \left[ X^{M_j \beta_j + N_j \alpha_j} \left( \sum_{K_j} (A_j \cup \{-\beta_j\})(K_j + M_j)B_j(K_j + N_j)X^{K_j} \right. \right. \\
 & - \sum_{K_j} A_j(K_j + M_j)B_j(K_j + N_j)X^{K_j} \\
 & \left. \left. + \sum_{K_j} A_j(K_j + M_j)(B_j \cup \{-\alpha_j\})(K_j + N_j)X^{K_j} \right) \right] X^{M_j(1-\beta_j) - N_j \alpha_j}.
 \end{aligned}$$

Now, the critical observations are that

$$\sum_{\substack{K_1, \dots, K_{\ell} \\ \sum M_j = \sum N_j \\ \min(M_j, N_j) = 0}} \prod_{j=1}^{\ell} A_j(K_j + M_j)B_j(K_j + N_j)X^{K_j + M_j} = \mathcal{C}(A_1 \cup \dots \cup A_{\ell}, B_1 \cup \dots \cup B_{\ell}),$$

as before, and

$$\begin{aligned}
 & \prod_{j=1}^{\ell'} (1 - X^{1-\alpha_j-\beta_j}) \mathcal{C}(A_1 \cup \dots \cup A_{\ell} \cup \{-\beta_1, \dots, -\beta_{\ell'}\}, B_1 \cup \dots \cup B_{\ell} \cup \{-\alpha_1, \dots, -\alpha_{\ell'}\}) \\
 & = \sum_{\substack{J_1, J_2 \subset \{1, \dots, \ell'\} \\ J_1 \cap J_2 = \emptyset}} (-1)^{|J_1| + |J_2| + \ell'} \mathcal{C}(A \cup -\beta_{J_1}, B \cup -\alpha_{J_2}),
 \end{aligned}$$

where  $A = A_1 \cup \dots \cup A_{\ell}$  and  $B = B_1 \cup \dots \cup B_{\ell}$ .

These together imply Theorem 4.

## 12. Multiplicities

### 12.1. How many times is a given $\ell$ swap repeated?

Now we need to give an accounting of what we have so far. Each time we split  $A$  and  $B$  up into subsets  $A = A_1 \cup \dots \cup A_{\ell}$  and  $B = B_1 \cup \dots \cup B_{\ell}$  we accumulate terms that correspond to all swaps of  $\alpha_j \in A_j$  and  $\beta_j \in B_j$ . For a fixed decomposition of  $A$  into  $\ell$  subsets we clearly do not get ALL swaps of  $\ell$ -sized subsets of  $A$  and  $B$ . Our solution to this dilemma is that we consider all decompositions of  $A$  into  $\ell$  disjoint non-empty subsets and similarly for  $B$ . Then every pair of  $\ell$  sized subsets will indeed appear in the swaps. However, now two different decompositions will often lead to the same swap. So how do we account for the overcounting?

How many times will a given  $\ell$ -sized swap  $S$  for  $T$  occur? This is equivalent to asking how many ways can  $A$  be split into  $\ell$  subsets where  $A_j$  contains  $\alpha_j$ ? If  $A$  has  $k$  elements then there are  $k - \ell$  elements that can be distributed arbitrarily into  $\ell$  sets. This can happen in  $\ell^{(k-\ell)}$  ways. Similarly for  $B$ . Taking into account permutations we end up with a multiplicity of  $\ell!^2 \ell^{2(k-\ell)}$ .

12.2. *How many times does the same  $(m, n)$  lead to a solution of a  $(*)$ -system?*

Our original problem is to evaluate

$$\sum_{m, n \leq X} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} \hat{\psi} \left( \frac{T}{2\pi} \log \frac{m}{n} \right).$$

Note that if  $A = \{\alpha_1, \dots, \alpha_k\}$  and  $B = \{\beta_1, \dots, \beta_k\}$  then

$$\tau_A(m) = \sum_{\mu_1 \dots \mu_k = m} \mu_1^{-\alpha_1} \dots \mu_k^{-\alpha_k} \quad \tau_B(n) = \sum_{\nu_1 \dots \nu_k = n} \nu_1^{-\beta_1} \dots \nu_k^{-\beta_k}.$$

We split  $A$  into  $A_1 \cup \dots \cup A_\ell$  and  $B$  into  $B_1 \cup \dots \cup B_\ell$ ; this is equivalent to splitting  $\{1, 2, \dots, k\}$  into  $I_1 \cup \dots \cup I_\ell$  where  $A_i = \{\alpha_i : i \in I_i\}$  and also  $\{1, 2, \dots, k\} = J_1 \cup \dots \cup J_\ell$  where  $B_j = \{\beta_j : j \in J_j\}$ . Then  $\tau_{A_i}(m_i) = \prod_{i' \in I_i} \mu_{i'}^{-\alpha_{i'}}$  and  $\tau_{B_j}(n_j) = \prod_{j' \in J_j} \nu_{j'}^{-\beta_{j'}}$ . Now after this splitting we count the  $m_i$  and  $n_j$  according to our  $(*)$ -system:

$$(*_1) : N_1 m_1 = M_1 n_1 + h_1$$

⋮

$$(*_\ell) : N_\ell m_\ell = M_\ell n_\ell + h_\ell$$

where  $M_1 \dots M_\ell = N_1 \dots N_\ell$ . Now let us say we have a solution of the  $(*)$ -system as above and let us take a collection of divisors of the  $m_i$  and  $n_j$ . For simplicity, let us suppose that  $\mu_i \mid m_i$  and  $\nu_j \mid n_j$  for  $1 \leq i, j \leq \ell$ . Let us write

$$m_i = \mu_i \hat{\mu}_i, \dots, m_\ell = \mu_\ell \hat{\mu}_\ell \quad n_1 = \nu_1 \hat{\nu}_1, \dots, n_\ell = \nu_\ell \hat{\nu}_\ell.$$

The question is: How many ways are there to do this? If we multiply the  $j$ th equation in our system by  $\mu_j^* \nu_j^*$ , where  $\prod_{j=1}^\ell \mu_j \mu_j^* = m$  and  $\prod_{j=1}^\ell \nu_j \nu_j^* = n$ , then we have a new equation

$$\tilde{N}_j \tilde{m}_j = \tilde{M}_j \tilde{n}_j + \tilde{h}_j$$

where

$$\tilde{m}_j = \mu_j \mu_j^* \quad \tilde{n}_j = \nu_j \nu_j^* \quad \tilde{M}_j = M_j \mu_j^* \hat{\nu}_j \quad \tilde{N}_j = N_j \hat{\mu}_j \nu_j^* \quad \tilde{h}_j = h_j \mu_j^* \nu_j^*.$$

If  $(\tilde{M}_j, \tilde{N}_j) > 1$  then the common factor can be divided out and out of  $\tilde{h}_j$ . Note that

$$\prod_{j=1}^\ell \tilde{M}_j = \prod_{j=1}^\ell M_j \mu_j^* \hat{\nu}_j = mn \prod_{j=1}^\ell \frac{M_j}{\mu_j \nu_j} = \prod_{j=1}^\ell \tilde{N}_j.$$

Thus, we have a new  $(*)$  system but it corresponds to exactly the same  $m = \mu_1 \dots \mu_k$  and  $n = \nu_1 \dots \nu_k$  as in the old one. The number of ways to construct these  $(*)$ -systems is just the number of ways to compose the  $\hat{\mu}_j$  as products of the available  $\{\mu_{\ell+1}, \dots, \mu_k\}$  and the  $\hat{\nu}_j$  from the  $\{\nu_{\ell+1}, \dots, \nu_k\}$ . But this is exactly  $\ell^{k-\ell}$  for the  $\hat{\mu}_j$  and the same for the  $\hat{\nu}_j$ . Then we take into account the ordering of the  $\mu_1, \dots, \mu_\ell$  and of the  $\nu_1, \dots, \nu_\ell$ ; this gives a factor of  $\ell!^2$ . In this way we arrive at a multiplicity  $\ell!^2 \ell^{2(k-\ell)}$  for each solution of our  $(*)$ -system which is the same as the multiplicity counted in the swaps of  $\ell$ -sets.

Note that the same argument applies whether any of the  $h_j$  are 0 or not. We need to divide out this multiplicity.

This explains the weight factor  $w_\ell$  in (6).

### 12.3. Conclusion

We have found that  $I_{A;B}^\psi(T; X)$  can be conjecturally evaluated by two different methods which produce the same answer. One way is to use the recipe of [4]. The other way is to let  $\ell$  be defined by  $T^\ell \leq X < T^{\ell+1}$ . Then partition  $A$  and  $B$  into  $\ell$  subsets and evaluate a convolution of  $\ell$  shifted divisor sums

$$\sum_{\substack{m_i \leq u_i \\ m_i n_i - n_i M_i = h_i}} \tau_{A_i}(m_i) \tau_{B_i}(n_i)$$

by a conjectural approach that involves the delta-method of [11]. A rigorous theorem identifying two Euler products proves that the result of the above agrees with some of the terms arising from the recipe. The terms with all  $h_i \neq 0$  correspond to  $\ell$ -swap terms from the recipe. The terms with  $\ell'$  of the  $h_i$  non-zero and  $\ell - \ell'$  of the  $h_i$  equal to 0 give  $\ell'$  swap-terms. Finally, if we sum over all possible partitions of  $A$  and  $B$  into non-empty subsets and account for multiplicities we achieve the desired equality between the two approaches.

A natural direction for further research is to consider other families of L-functions, for example quadratic Dirichlet L-functions, and to determine an arithmetic basis for the relevant moment conjectures.

### References

1. E. B. BOGOMOLNY and J. P. KEATING, ‘Random matrix theory and the Riemann zeros I: three- and four-point correlations’, *Nonlinearity* 8 (1995) 1115–1131.
2. E. B. BOGOMOLNY and J. P. KEATING, ‘Random matrix theory and the Riemann zeros II:  $n$ -point correlations’, *Nonlinearity* 9 (1996) 911–935.
3. T. D. BROWNING, *Quantitative arithmetic of projective varieties*, Progress in Mathematics 277 (Birkhäuser, Basel, 2009).
4. J. B. CONREY, D. W. FARMER, J. P. KEATING, M. O. RUBINSTEIN and N. C. SNAITH, ‘Integral moments of  $L$ -functions’, *Proc. Lond. Math. Soc.* 91 (2005) 33–104.
5. J. B. CONREY and J. P. KEATING, ‘Moments of zeta and correlations of divisor-sums: I’, *Phil. Trans. A* 373 (2015) 20140313.
6. J. B. CONREY and J. P. KEATING, ‘Moments of zeta and correlations of divisor-sums: III’, *Indag. Math.* 26 (2015) 736–747.
7. J. B. CONREY and J. P. KEATING, ‘Moments of zeta and correlations of divisor-sums: II’, *Advances in the theory of numbers, Proceedings of the Thirteenth Conference of the Canadian Number Theory Association*, Fields Institute Communications (eds: A. Alaca, S. Alaca and K. S. Williams; Springer, New York, 2015) 75–85.
8. J. B. CONREY and J. P. KEATING, ‘Moments of zeta and correlations of divisor-sums: IV’, *Res. Number Theory* 2 (2016) 24.
9. J. B. CONREY and J. P. KEATING, ‘Pair correlation and twin primes revisited’, *Proc. Math. Phys. Eng. Sci.* 472 (2016) 20160548.
10. J. B. CONREY and J. P. KEATING, ‘Averages of ratios of the Riemann zeta-function and correlations of divisor sums’, *Nonlinearity* 30 (2017) R67–R80.
11. W. DUKE, J. B. FRIEDLANDER and H. IWANIEC, ‘A quadratic divisor problem’, *Invent. Math.* 115 (1994) 209–217.
12. J. FRANKE, YU. I. MANIN and YU. TSCHINKEL, ‘Rational points of bounded height on Fano varieties’, *Invent. Math.* 95 (1989) 421–435.
13. D. A. GOLDSTON and S. M. GONEK, ‘Mean value theorems for long Dirichlet polynomials and tails of Dirichlet series’, *Acta Arith.* 84 (1998) 155–192.
14. J. P. KEATING and N. C. SNAITH, ‘Random matrix theory and  $\zeta(1/2 + it)$ ’, *Comm. Math. Phys.* 214 (2000) 57–89.
15. B. LEHMANN and S. TANIMOTO, On exceptional sets in Manin’s Conjecture, Preprint, 2018, arXiv:1807.07995.

*Brian Conrey*  
*American Institute of Mathematics*  
*360 Portage Ave*  
*Palo Alto, CA 94306*  
*USA*

and

*School of Mathematics*  
*University of Bristol*  
*Bristol BS8 1TW*  
*United Kingdom*

[conrey@aimath.org](mailto:conrey@aimath.org)

*Jonathan P. Keating*  
*School of Mathematics*  
*University of Bristol*  
*Bristol BS8 1TW*  
*United Kingdom*

[j.p.keating@bristol.ac.uk](mailto:j.p.keating@bristol.ac.uk)