MATHEMATICS

Prime Time for the Riemann Hypothesis

Brian Conrey

Mathematicians regard the Riemann hypothesis as the most fundamental of all unsolved problems. Its beauty lies in its simplicity; it reveals a deep connection between addition and multiplication, and it implies an unforeseen symmetry hidden in the unruly prime numbers. In *Prime Obsession*, John Derbyshire has produced a remarkably accessible and deeply researched description of this fascinating problem.

Bernhard Riemann proposed the hypothesis that has come to bear his name in a seminal 1859 paper presented to the Berlin Academy. His objective was to find an exact formula for the quantity of prime numbers smaller than a given bound. There are 4 primes smaller than 10, 25 primes smaller than 100, 168 primes smaller than 1000, and so on. The primes become less frequent, but is it possible to say exactly how they thin out? Riemann reported that the key to understanding how the primes are nestled into the natural numbers is to understand another set of numbers, the zeros of a function now called the Riemann zeta function. He found an amazing formula that dramatically revealed that each zero of the zeta function has a subtle but definite effect on the distribution of prime numbers. Moreover, because of a symmetry he proved between the value of the zeta function at any number s and its value at 1 – s, he was led to hypothesize that each nonreal zero of the zeta function has a real part that is equal to one-half. This conjecture is the famous Riemann hypothesis. Today we know that its study leads to applications far beyond prime number theory to areas that include cryptography, quantum chaos, and network design.

Prime Obsession
Bernhard Riemann and the Greatest Unsolved Problem in Mathematics
by John Derbyshire

Joseph Henry Press, Washington, DC, 2003. 438 pp. $27.95, C$37.95. ISBN 0-309-08549-7.

Derbyshire interweaves a fascinating history—embedded in the political turmoil of western Europe—of how the Riemann hypothesis came to be posed and how it has influenced the work of 20th-century mathematicians. Moreover, he gives us a detailed account of exactly what the hypothesis says mathematically. In doing so, the author assumes only a slight mathematical background (some basic algebra, which he reviews for the reader's convenience). He gently motivates the comprehension of concepts needed to have more than a peripheral view of the Riemann hypothesis, such as infinite series, calculus, complex numbers, and the notion of analytic continuation for complex functions. In explaining how the prime numbers are related to the zeta function, he begins with the Euler product (professional mathematicians will delight in seeing Euler's original proof here) and proceeds all the way to Riemann's remarkable formula in which every zero of the zeta function contributes to the prime number count. Derbyshire, a banker and novelist with an advanced degree in mathematics, is eminently successful at bringing this story to life.

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Riemann in 1863.

Part of the charm of the Riemann hypothesis is provided by the interesting cast of people who became immersed in the problem and by the anecdotes about them, which have been told and retold by generations of mathematicians. For example, in an address to the 1900 International Congress of Mathematicians at Paris, the preeminent mathematician David Hilbert boldly laid out an agenda for 20th-century mathematicians with a list of 23 unsolved fundamental problems, the eighth of which was the Riemann hypothesis. Later, in the 1930s, he declared that if he awoke from a sleep of several centuries (like the medieval German emperor Barbarossa purportedly could), the first question he would ask was whether anyone had solved the Riemann hypothesis. Derbyshire does an especially good job of vividly presenting the personalities and stories surrounding the math.

Two other books on the Riemann hypothesis have recently appeared: Karl Sabbagh's *The Riemann Hypothesis* (*1*), published in the United Kingdom with the more interesting title *Dr. Riemann's Zeros*, and Marcus du Sautoy's *The Music of the Primes* (*2*). If you like accounts of mathematics written for nonmathematicians, you will want to read all three. Sabbagh, a science writer, concentrates more on an outsider's perspective of how mathematics is done by the practitioners; du Sautoy, a mathematics professor at Oxford University, fills his account with anecdotes that the aficionado will relish.

Why are there suddenly three books on the Riemann hypothesis? One answer may have to do with the recent appearances of mathematics in popular culture: movies (*A Beautiful Mind*, *Good Will Hunting*, and *Pi*), plays (*Proof* and *Arcadia*), and even the newspaper comics. More people have begun to realize that mathematics is compelling. They find great interest in Andrew Wiles's dramatic solution of Fermat's Last Theorem (documented so well in the public television program NOVA's production of *The Proof*) and the equally fascinating stories of the legendary peripatetic Hungarian Paul Erdős and the Indian number theorist Srinivasa Ramanujan. People who do mathematics are passionate about their subject in a way that inspires all of us. The increased interest in mathematics is also reflected in the recent establishment in the United

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Highlighting the zeros. By plotting one over the absolute value of zeta, the zeros become poles (truncated here) and so are more prominent. The Riemann hypothesis asserts that all of the zeros fall along the x = 0.5 line.

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States of two privately funded organizations to promote and disseminate mathematical research and its history: the American Institute of Mathematics (founded by Silicon Valley businessmen John Fry and Steve Sorenson), where I work, and the Clay Mathematics Institute (founded by Boston businessman Landon Clay).

Another reason for the sudden proliferation of books on the Riemann hypothesis may be the flurry of activity as we entered the new century. The American Institute of Mathematics has sponsored three international conferences on the problem, and the Clay Mathematics Institute has offered a million-dollar reward for its solution. Programs to solve the Riemann hypothesis through methods of probability theory, physics, algebraic geometry, and through the theory of L-functions (generalizations of the zeta function) have recently been proposed. The three books touch on all of these themes in complementary ways. My hope is to someday read a sequel with a satisfying ending.

References

1. K. Sabbagh, *The Riemann Hypothesis: The Greatest Unsolved Problem in Mathematics* (Farrar, Straus, and Giroux, New York, 2003); *Dr. Riemann's Zeros: The Search for the $1 Million Solution to the Greatest Problem in Mathematics* (Atlantic, London, 2002).
2. M. du Sautoy, *The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics* (HarperCollins, New York, 2003).

NEUROSCIENCE

Thoughts Without a Thinker

Franz Mechsner and Albert Newen

When the 17th-century philosopher René Descartes made his famous statement "I think, therefore I am," he was certain that this intuition could not possibly be doubted. If there are thoughts, there must be someone who thinks. Descartes identified the thinker with "himself," and himself with the immortal soul. Unsatisfied with the Cartesian framework, scientists try to explain human self-consciousness as a natural phenomenon. This "naturalization project" is guided by the complex question: How may conscious selfhood (subjective experience and autonomous agency) emerge from causal chains of events in a physical world? In *Being No One*, the German philosopher Thomas Metzinger addresses this challenge and proposes a framework of how self-consciousness might be naturalized. In a bold, thorough, and thought-provoking synthesis, he combines a huge body of neuroscientific and psychological research data with philosophical considerations and fine-grained phenomenological reflections on real-life experiences.

Metzinger, a professor at Johannes Gutenberg University in Mainz, Germany, maintains that there are actually no autonomous selves in the material world. The perception that one is the source of thoughts and actions is an illusion, emerging from physical processes in neuronal networks where no self can be identified. To put it provocatively, there are experiences, but no one who experiences; there are thoughts, but no thinker; actions, but no actor. Based on this premise, naturalization of self-consciousness means explaining the detailed representational, functional, and computational structure of the selfhood illusion. One must consider its evolutionary advantage, how it emerges from neuronal processes, and how it is related to the puzzling philosophical riddles in connection with consciousness, such as the mind-body problem.

Being No One
The Self-Model Theory of Subjectivity
by Thomas Metzinger

MIT Press, Cambridge, MA, 2003. 713 pp. $55, £35.95. ISBN 0-262-13417-9.

According to Metzinger's "self-model theory of subjectivity," there are two central constituents of human self-consciousness, the phenomenal self-model (PSM) and the phenomenal model of the intentionality relation (PMIR). The first emerges if a representation of the organism itself is embedded into its conscious "mental world model" (the whole of its experience). Every conscious mental content is available for flexible, non-automatic cognition and the control of action, and so is the PSM. The core of the self-model is formed by our awareness of our own body and its motions. Mental contents are experienced as mine—my leg, my thought, my memory of the past—if they are bound to this sensory core according to the binding and attribution rules known from cognitive psychology. For example, Daniel Wegner proposed that the conviction that an ongoing event is my action is based on the well-known principles of causal attribution.

The self-model thus presents the organism as a subjective-objective thing, as part of the world. However, in order for the PSM to be useful, the relations between subject and object—in philosophical terms, the "arrow of intentionality"—must also be represented. This representation is what Metzinger calls the PMIR. The combination of the two components, PSM and PMIR, leads to full-blown subjectivity, i.e., to the experience of being in the world under a first-person perspective, of being someone who acts, perceives, and thinks autonomously and flexibly. This someone is what philosophers usually call a self. The author proposes that many human abilities, such as highly developed abstract cognition and understanding the mind of others, are rooted in the basic characteristics of the subject-world relationship as instantiated by the PSM and PMIR.

Metzinger maintains that this theoretical skeleton contains, in a nutshell, everything necessary for a naturalization of self-consciousness and its function. Using this perspective, he then thoroughly and subtly analyzes a wide variety of data, clinical case studies, and daily experiences, including the phenomenology, psychology, and neurobiology of the human sense of identity and its often dramatic disturbances. These fascinating explanations constitute one of the main rewards of reading the book. For example, he carefully considers Cotard's syndrome, in which patients experience themselves as being nonexistent, obviously contradicting Descartes's claim that the mere presence of thoughts leads to the conviction of existence. Metzinger presents the following analysis: The Cotard patients do not recognize any feelings, and as a result they do not construct an emotional self-model. Nevertheless, they still have a cognitive self-model that enables them to grasp thoughts about themselves. However, they experience themselves only as an object, not as a subject; a "conscious self-model is in place, but it is not a subject-model anymore, only an object-model." Interestingly, considerations of this kind might also serve to explain certain characteristics of spiritual and mystical experiences.

The theory of subjectivity Metzinger presents in *Being No One* seems very promising in that it offers a conceptual framework for explaining many empirical phenomena related to human self-consciousness. His basic strategy is to show that everything of interest regarding self-consciousness can be reduced to phenomenal representations. Under the presupposition that phenomenal representations emerge from neuronal processes, this means that naturalization of self-consciousness is indeed possible. Metzinger's interdisciplinary approach opens a new path toward a scientific theory of consciousness and self-consciousness.

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