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The Riemann Hypothesis

J. B. Conrey
American Institute of Mathematics

1 Note to Reader

This article describes one of the most fundamental unsolved problems in mathematics. The origins of the problem are quite easy and, I hope, are written in a way that anyone can understand the motivation. Then, the basic problem can be quantified once you are acquainted with logarithms; to this end I have written a section giving a perspective about logarithms. When we get to the precise statement of the problem, a little more will be required of you, namely some patience with making sense of infinite series. Finally, to understand the problem, it is necessary to work with complex numbers. I've tried to explain these things as they arise; hopefully you will be able to follow and appreciate the mathematics, and be motivated to learn some higher mathematics. If you lose patience in the middle, please skip to the end to read some anecdotes about this great problem, which is 150 years old this year!

I would like to thank Dr. Helmut Rolfing of the University of Göttingen library for permission to reproduce one of Gauss' amazing tables of primes and I also thank Dr. Andrew Odlyzko for permission to reproduce his correspondence with Pólya.

2 How Many Primes Are There?

The Riemann Hypothesis is one of the seven problems that the Clay Mathematics Institute has offered a one million dollar reward for. To give a sense for it, it is best to go back to its origins. In 1859 Georg Friedrich Bernhard Riemann wrote a paper which basically explained how to use the zeta-function (henceforth called the Riemann zeta-function) to analyze the distribution of prime numbers. The basic question is "How many prime numbers are there less than a given (large) number N?"

Many mathematicians have thought about this question. Legendre, Chebyshev, and Gauss made the most progress, before Riemann, toward ascertaining an answer.

The usual notation for the number of primes less than or equal to N is $\pi(N)$. Thus, $\pi(10) = 4$ because there are four primes 2, 3, 5, and 7 below 10. Also, $\pi(100) = 25$. The notation has nothing to do with the ratio of a circle's circumference to its diameter; think of it more like π is p for prime.

Gauss actually calculated how many primes there are in various intervals up to 5 million or so. Within these tables he saw a pattern and guessed what that pattern was. The Riemann Hypothesis is a statement about the accuracy of Gauss' guess.

Let's cheat a little and see if we can replicate Gauss' guess. The cheating is that we will use Mathematica to give us a table of $\pi(N)$ for $N = 10^n$ with $1 \leq n \leq 9$.

N	10	100	1000	10000	100000	1000000	10000000	100000000	1000000000
$\pi(N)$	4	25	168	1229	9592	78498	664579	5761455	50847534

From this table we see that 40% of the first ten numbers are prime, 25% of the first one-hundred, 16.8% of the first thousand and so on. Clearly the primes are thinning out, but at what rate? Only slightly above 5% of the first billion numbers are prime. We do know, thanks to Euclid, that the list of prime numbers goes on forever. But, judging from this data, it appears that the proportion of primes within the first N natural numbers seems to be a variable quantity that shrinks to 0 as N gets larger and larger.

So, the question is, how do we measure that rate? For example, is the number of primes up to N comparable to $\sqrt{N}$? It's easy to visualize square root by looking at the *length* of a number written out in decimal because the square root of a number has about half as many digits as the number itself. That does not reflect the current situation; 10^9 has 10 digits whereas $\pi(10^9) = 50,847,534$ has 8 digits. It must be some other function. Let's look at the table above but now put in a row with the ratio of N to $\pi(N)$.

N	10	100	1000	10000	100000	1000000	10000000	100000000	1000000000
$\pi(N)$	4	25	168	1229	9592	78498	664579	5761455	50847534
$N/\pi(N)$	2.5	4.0	5.952	8.137	10.425	12.739	15.047	17.357	19.667

We begin to see some regularity in this last row. It seems to be increasing at a steady rate. Let's make a fourth row which indicates the differences of entries in the third row, which we indicate simply with the symbol Δ.

N	10	100	1000	10000	100000	1000000	10000000	100000000	1000000000
$\pi(N)$	4	25	168	1229	9592	78498	664579	5761455	50847534
$N/\pi(N)$	2.5	4.0	5.952	8.137	10.425	12.739	15.047	17.357	19.667
Δ	–	1.5	1.952	2.185	2.288	2.314	2.308	2.310	2.310

The entries in the fourth row seem to be approaching a constant slightly above 2.3. If correct, then the entries in the third row should be about $2.3n$, which means that $\pi(10^n)$ is about $10^n/(2.3n)$. Put simply, the number of primes up to a large number x is proportional to x divided by the number of digits of x. This explains the decrease in the proportion of primes among the first x natural numbers and would predict that for numbers with about 45 digits only 1 in 100 would be expected to be prime.

And what kind of function is it whose value at a large number N is proportional to the number of digits of N? A logarithmic function. A logarithmic function f has the basic property that it transforms multiplication into addition, so that $f(MN) = f(M) + f(N)$. It follows from this basic property that $f(10^n) = nf(10)$ so that $f(N)$ is proportional to the number of digits of N. And it turns out that the natural logarithmic function $f(N) = \ln N$ with the property that $\ln 10 = 2.30258509299\ldots$ is the perfect candidate for the function we are looking for. And this is the conjecture of Legendre and Gauss, that

$$\pi(N) \sim \frac{N}{\ln N}$$

where the symbol $\sim$ is read "is asymptotic to" and means that the ratio of the quantities on either side of the symbol approach 1 as N gets ever larger.

3 A Digression About Logs

Here is a recipe for calculating $\ln N$ to any degree of accuracy. First write N in the form $N = x \cdot 10^n$ where $0 < x \leq 1$. Then $\ln N = n \ln 10 + \ln x$. To calculate $\ln x$ to some desired accuracy, write $x = 1 - t$ and use

$$\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \cdots - \frac{t^R}{R} + E_R(t)$$

where to work out the accuracy, use the inequality $|E_R(t)| \leq \frac{|t|^{R+1}}{(1-|t|)(R+1)}$, valid for $|t| < 1$. Here you are free to choose R; the larger it is, the more accurate your answer will be. To work out ln 10 to any desired accuracy, use $\ln 10 = -\ln \frac{1}{10}$ and then the series above. Thus, for example, we have

$$\begin{aligned}\ln 10 = -\ln \frac{1}{10} &= -\ln(1 - \frac{9}{10}) = \frac{9}{10} + \frac{81}{2 \times 100} + \frac{729}{3 \times 1000} + E_3(.9)\\ &= 0.9 + 0.405 + 0.243 + E_3(.9) = 1.55 + E_3(.9).\end{aligned}$$

This isn't very good because our estimate for $E_3(.9)$ is $E_3(.9) < 0.9^4/(0.1 \times (4)) = 1.64\ldots$. But, we can use the property $\ln ab = \ln a + \ln b$ to do better.

$$\begin{aligned}\ln 10 &= -\ln \frac{1}{10} = -\ln \frac{8}{10} + \ln 8 = -\ln(1 - 1/5) - 3\ln(1 - 1/2)\\ &= \frac{1}{5} + \frac{1}{2 \times 5^2} + \frac{1}{3 \times 5^3} + E_3(.2) + 3\left(\frac{1}{2} + \frac{1}{2 \times 2^2} + \frac{1}{3 \times 2^3} + \frac{1}{4 \times 2^4} + E_4(.5)\right)\\ &= 0.2 + 0.02 + 0.008/3 + 1.5 + 0.375 + 0.125 + 0.375/8 + E_3(.2) + 3 \cdot E_4(.5)\\ &= 2.2695 + E_3(.2) + 3 \cdot E_4(.5).\end{aligned}$$

Since $|E_3(.2) + 3 \cdot E_4(.5)| < .2^4/(4 \times .8) + 3 \cdot .5^5/(5 \times .5) = 0.038$ we are getting close to the actual value $\ln 10 = 2.302585\ldots$.

If x is too close to 0, one could simply try the above procedure but with 10 replaced by some other suitable number, say 2. It is an interesting exercise in identities using the binomial coefficients from Pascal's triangle to verify the addition formula using the series above: in other words check that the series for $\ln(1-a) + \ln(1-b)$ matches the series for $\ln((1-a)(1-b)) = \ln(1 - (a + b - ab))$. Thus we need to check that

$$a + \frac{a^2}{2} + \frac{a^3}{3} + \cdots + b + \frac{b^2}{2} + \frac{b^3}{3} + \ldots$$

matches up with

$$a + b - ab + \frac{(a + b - ab)^2}{2} + \frac{(a + b - ab)^3}{3} + \ldots.$$

Another useful application of knowing about logs is Stirling's formula:

$$\ln n! = (n + 1/2)\ln n - n + \frac{1}{2}\ln(2\pi) + E_n$$

where $|E_n| < 1/(12n)$. Thus, ln 100! is within 0.001 of

$$100.5 \ln 100 - 100 + \frac{1}{2}\ln(2\pi) = 201 \times 2.302585 - 100 + 0.9189 = 363.739.$$

Now $\ln 10^{157.97} = 157.97 \times 2.302585$ is also within 0.001 of 363.739. In particular, 100! is a number with 158 digits.

Another application of logs is to the approximation of a partial sum of the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + \gamma + \frac{1}{2n} + e_n$$

where $|e_n| \le 1/(12n^2)$ and γ is Euler's constant $\gamma = 0.57721\ldots$ (which is believed, but not known, to be an irrational number.) Incidentally, this would give another way to calculate ln 10 to three decimal places.

4 Gauss' Guess

The table below, made by Gauss and scanned from his papers in the University of Göttingen library, contains data on the prime numbers between 1,000,001 and 1,100,000. That span of 100,000 numbers is broken up into intervals of 10,000 which are charted in the ten columns. Thus, the third column represents frequencies of occurrences of primes between 1,130,001 and 1,140,000. These blocks of 10,000 are broken up into one hundred blocks of 100. Within each block of 100 Gauss recorded how many prime numbers there were. These frequencies are recorded in the cells.

Thus, the data for the one-hundred intervals of length 100 between 1,000,001 and 1,010,000 are recorded in the first column. We see that there was one block that had a single prime (namely 1,004,873 is the only prime between 1,004,800 and 1,004,900), no blocks that had exactly 2 or 3 primes, two blocks that had 4 primes (these are the four primes 1,005,701, 1,005,709, 1,005,751, and 1,005,761 between 1,005,700 and 1,005,800, and the four primes 1,005,821, 1,005,827, 1,005,833, and 1,005,883 between 1,005,800 and 1,005,900), eleven blocks that had 5 primes, and so on.

If you notice the bottom right corner of Gauss' table you will see that there are 7216 primes between 1 million and 1.1 million. Below the table Gauss has a calculation that predicts that there are 7212.99 primes in this interval. That is an amazingly accurate prediction. How does he make that prediction? Basically his argument is that the probability that a number n is prime is $1/\ln n$. Thus, the expected number of primes between 1 million and 1.1 million is

$$\frac{1}{\ln 1000001} + \frac{1}{\ln 1000002} + \frac{1}{\ln 1000003} + \cdots + \frac{1}{\ln 1100000} = 7212.99.$$

I re-created Gauss' table using Mathematica and found a few discrepancies in certain cells, but agreement with the overall total of 7216. For example, in the first column Mathematica asserts that the sixth entry should be 13 and the seventh entry 27 whereas Gauss has 14 and 26. I'm not sure whether to trust Gauss or Mathematica!

Among Gauss' papers are many little slips of paper such as the one above. He may have tabulated all of the primes up to a few million. On some of the slips, where there is especially good agreement between Gauss' prediction and the actual number of primes, Gauss writes an exclamation point after

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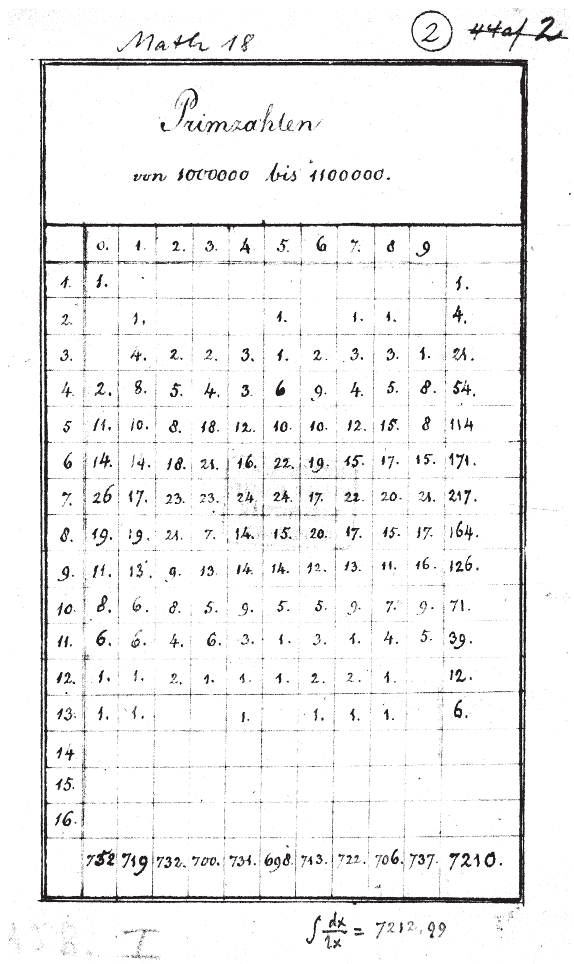

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Primzahlen

von 1000000 bis 1100000.

	0.	1.	2.	3.	4	5.	6	7.	8	9	
1.	1.										1.
2.		1.				1.		1.	1.		4.
3.		4.	2.	2.	3.	1.	2.	3.	3.	1.	21.
4.	2.	8.	5.	4.	3.	6	9.	4.	5.	8.	54.
5	11.	10.	8.	18.	12.	10.	10.	12.	15.	8	114
6	14.	14.	18.	21.	16.	22.	19.	15.	17.	15.	171.
7.	26	17.	23.	23.	24.	24.	17.	22.	20.	21.	217.
8.	19.	19.	21.	7.	14.	15.	20.	17.	15.	17.	164.
9.	11.	13.	9.	13.	14.	14.	12.	13.	11.	16.	126.
10.	8.	6.	8.	5.	9.	5.	5.	9.	7.	9.	71.
11.	6.	6.	4.	6.	3.	1.	3.	1.	4.	5.	39.
12.	1.	1.	2.	1.	1.	1.	2.	2.	1.		12.
13.	1.	1.			1.		1.	1.	1.		6.
14											
15.											
16.											
	752	719	732.	700.	731.	698.	713.	722.	706.	737.	7210.

$\int \frac{dx}{lx} = 7212,99$

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Figure 1: The table at the left, made by Gauss and scanned from his papers in the University of Göttingen library, contains data on the prime numbers between 1,000,001 and 1,100,000. That span of 100,000 numbers is broken up into intervals of 10,000 which are charted in the ten columns. Thus, the third column represents frequencies of occurrences of primes between 1,130,001 and 1,140,000. These blocks of 10,000 are broken up into one hundred blocks of 100. Within each block of 100 Gauss recorded how many prime numbers there were. These frequencies are recorded in the cells

his calculation. He seems to have been fascinated by the accuracy of his prediction, and rightly so. For it is exactly this accuracy that is addressed by the Riemann Hypothesis. In fact, a version of the Riemann Hypothesis is the assertion that Gauss' prediction of

$$\sum_{2 \leq n \leq x} \frac{1}{\ln n}$$

differs from $\pi(x)$ by an amount which is no more than $\sqrt{x} \ln x$. Here is a table illustrating Gauss' prediction, the difference between Gauss' prediction and the actual prime count (labeled 'Error') and the value of $\sqrt{x} \ln x$ for comparison.

n	10	100	1000	10000	100000	1000000	10000000	100000000	1000000000
$\pi(n)$	4	25	168	1229	9592	78498	664579	5761455	50847534
Gauss	6.1	29.99	177.4	1245.9	9629.6	78627.5	664918.6	5762209.6	50849235.2
Error	2	5	9	17	38	129	840	755	1681
$\sqrt{x}\ln x$	7	46	218	921	3641	13816	50970	184207	655327

I think you'll agree that Gauss' prediction is truly remarkable!

5 Riemann

Enough about Gauss. The purpose of this article is to describe Riemann's even more amazing contribution because Riemann figured out the incredible mathematics behind Gauss' assertion. What Riemann discovered is an infinite sequence of numbers that begins

$$
\begin{gathered}
14.134725141734693790457251990 25726\ldots\\
21.022039638771554992628479593896 90\ldots\\
25.010857580145688763213790992562 82\ldots\\
30.424876125859513210311897530584 09\ldots\\
32.935061587739189690662368964074 90\ldots\\
37.586178158825671257217763480705 33\ldots\\
40.918719012147495187398126914633 25\ldots\\
43.327073280914999519496122165406 80\ldots\\
48.005150881167159727942472749427 51\ldots\\
\ldots
\end{gathered}
$$

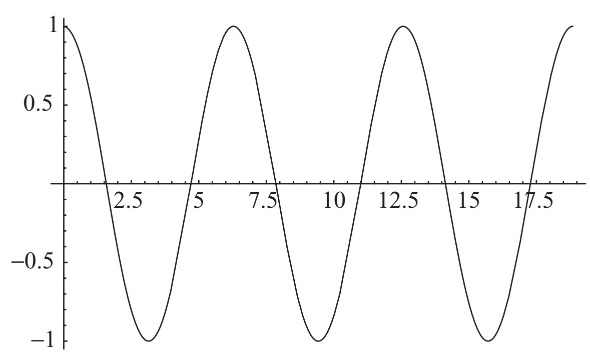

Figure 2: The graph of $y = \cos x$

and which encodes all of the information there is about the primes!

Before going on to illustrate what this means, and then to reveal the million dollar mystery behind these numbers let me tell a few things that we do know for sure about these numbers. We label these numbers using the Greek letter γ; thus $\gamma_1 = 14.134\ldots$, $\gamma_2 = 21.022\ldots$, etc. Or sometimes we just let γ denote a generic number in this set. One thing we know is the number of γ with $|\gamma| \le T$ for any large number T. In fact, the number is

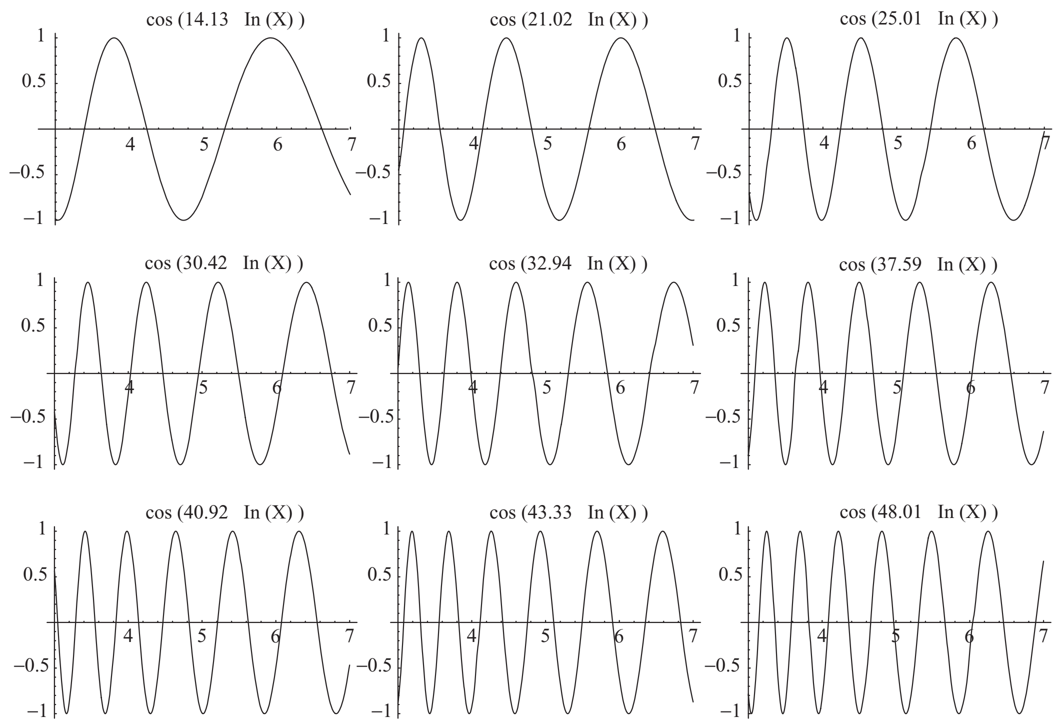

Figure 3: The graphs of $y = \cos(\gamma \ln x)$ for the first nine γ

$$N(T) = \#\{|\gamma| \le T\} = \frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi} + E(T)$$

where $E(T)$ is a quantity which is smaller than $\ln T$. This is a very accurate formula, and from it we see that there are lots of γ. For example, between 1 million and 2 million there are nearly 2 million of these γ. Another thing we know is how to calculate these numbers. We can calculate any one of them to any degree of accuracy. And, if you want to know, say, the $(10^{22}+1)^{\text{st}}$ of these γ we can calculate that. (It is 1370919909931995308226.680160 . . . .)

Now we show how, given these numbers, we can find the primes. To begin with, here is a graph of $\cos(x)$: and here are graphs of $\cos(\gamma \ln x)$ plotted for x between 3 and 7.

Notice the graphs are predominantly negative at 3 and at 7, and also at 5. Figure 4 shows the result of adding these nine graphs:

Notice that there are sharp minima when x is a prime! This feature gets more pronounced the more γ's we include. Actually, smaller sharp peaks also form at the prime powers, 4, 8, 9, 16, 25, 32, ... This shows how the γ's give us the primes and prime powers. How do we get the γ's from the primes? Basically, by integrating the error term in Gauss' guess for $\pi(x)$ against $\cos(t \ln x)$. Now, spikes show up at the γ's.

I hope these pictures and comments convince you that our set of γ's is a very interesting and special set of numbers. I still haven't told you their mystery nor explained how Riemann found them.

I'll go ahead and tell you their mystery right now. The thing we don't know about the γ's is whether they are all *real* numbers or not. That's right, it is possible that some of the γ's have an

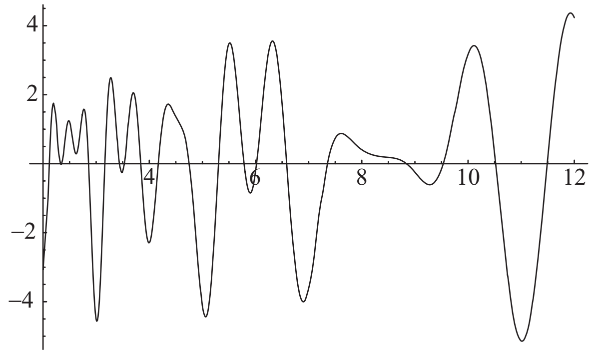

Figure 4: The sum of the nine graphs of $\cos(\gamma \ln x)$

imaginary part. We do know that any imaginary parts, if they exist, cannot be any larger that 1/2; in fact they must be strictly smaller than 1/2. (By the way, this assertion is equivalent to the prime number theorem, that $\pi(x) \sim x/\ln x$, first proved, independently, by Jacques Hadamard and Charles de la Vallée Poussin, in 1896.)

Now we move to Riemann's analysis, and reveal where the γ's come from.

6 The Zeta-Function

The key is a function now called the Riemann zeta-function. It is defined for $s > 1$ by the convergent infinite series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots .$$

For example, the sum of the reciprocals of the squares is

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6},$$

a fact proven first by Euler.

Before we continue, let's briefly think about convergence of an infinite series. Sometimes it's obvious that a series converges. For example

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^N} = \frac{2^N - 1}{2^N} = 1 - \frac{1}{2^N}$$

so that as N gets larger this sum approaches 1. Thus, we should have no trouble believing that the sum of the infinite series is 1:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

Equally easy is to see that the series

$$1 + 1 + 1 + \cdots$$

does not converge; it *diverges*. But other series are trickier to work out. For example, the series of reciprocals of the squares does converge, because, using the inequality $\frac{1}{n^2} \le \frac{2}{n(n+1)}$ and the identity $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we have

$$\begin{aligned}
&\frac{1}{(M+1)^2} + \frac{1}{(M+2)^2} + \cdots + \frac{1}{N^2} + \ldots \\
&\quad \le 2\left(\frac{1}{M+1} - \frac{1}{M+2} + \frac{1}{M+2} - \frac{1}{M+3} + \cdots + \frac{1}{N} - \frac{1}{N+1} + \ldots\right) \\
&\quad < \frac{2}{M+1}.
\end{aligned}$$

Since the *tail* of the series goes to 0, the series converges. On the other hand, we can see that the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ does not converge. That is because

$$\begin{aligned}
&1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\
&\quad > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = \frac{n+1}{2}.
\end{aligned}$$

It can be shown without too much difficulty that the series $1 + \frac{1}{2^s} + \ldots$ for $\zeta(s)$ converges precisely when $s > 1$ and diverges when $s \le 1$.

It was Euler who first noticed the connection between $\zeta(s)$ and prime numbers. He observed that

$$\begin{aligned}
&\left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \ldots\right)\left(1 + \frac{1}{5^s} + \frac{1}{25^s} + \frac{1}{125^s} + \ldots\right)\ldots \\
&\quad = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots
\end{aligned}$$

where, in the top line, there is one factor for each prime number. Basically, to prove this you just have to expand out the product using the distributive property. So, you get the sum of a bunch of terms, each term being a product with one factor for each prime. The trick is that for most of the primes you just take the factor 1. So, for example, the term $1/84^s$ on the bottom line occurs from taking the product

$$\frac{1}{4^s} \times \frac{1}{3^s} \times 1 \times \frac{1}{7^s} \times 1 \times 1 \times \cdots = \frac{1}{84^s}.$$

This formula, known as the Euler product formula, is nothing more than a restatement of the fundamental theorem of arithmetic, that every natural number can be expressed as a product of prime powers in a unique way. With Euler's formula as the motivation, Riemann began his incredibly insightful analysis.

The first critical thing is that he decided to regard s as a function of a complex variable. That is, he took $s = \sigma + it$ where σ and t are real numbers. You may well ask what n^s means when s is complex? The answer is that $n^s = n^\sigma n^{it} = n^\sigma(\cos(t \ln n) + i \sin(t \ln n))$. This uses an identity of Euler. Notice that we haven't left the realm of sines, cosines, and logarithms. What about convergence for the series for a complex number s? Well $|n^s| = n^\sigma$ since $|n^{it}| = |\cos(t \ln n) + i \sin(t \ln n)| = \sqrt{\sin^2(t \ln n) + \cos^2(t \ln n)} = 1$. So, if the series for $\zeta(\sigma)$ converges, then so does that for $\zeta(\sigma + it)$. And this happens precisely when $\sigma > 1$. So, basically there is a half-plane $\sigma + it$ with $\sigma > 1$ where the infinite series definition for $\zeta(s)$ makes sense.

The next amazing thing Riemann found is a way to make sense of the zeta-function for all values of s, except for $s = 1$ (where it is infinite). Here is a formula which will do that when $\sigma > 0$:

$$\zeta(s) = \left(1 + \frac{1}{2^s} + \dots \frac{1}{N^s}\right) + \frac{N^{1-s}}{s-1} + E_N(s)$$

provided that $N > |s|/2$, where

$$|E_N(s)| \leq N^{-\sigma}/2.$$

The point is that when $\sigma > 0$ the error term $E_N(s)$ goes to 0 as N gets larger and larger. Riemann found a general formula that works for all s.

For example, suppose we want to calculate $\zeta(.7 + 5i)$. We could take $N = 10$ and be guaranteed that our approximating sum $0.78 + 0.29i$ is within a distance $10^{-.7}/2 < 0.1$ of the actual value of $\zeta(.7 + 5i)$.

The other thing Riemann found is still more amazing: a formula relating $\zeta(s)$ and $\zeta(1 - s)$. Specifically, he proved that

$$\frac{\zeta(1-s)}{\zeta(s)} = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s).$$

You may well complain that we have introduced *another* unknown function $\Gamma(s)$, and, yes, that is true, but actually $\Gamma(s)$ is fairly simple in the grand scheme of things, much simpler than $\zeta(s)$. This function is Euler's Gamma-function, and has a number of amazing properties of its own. For starters, it interpolates factorials, so that $\Gamma(n + 1) = n!$. Also, it satisfies a recursion formula $\Gamma(s + 1) = s\Gamma(s)$. And it never takes on the value 0 no matter what s is, i.e. $\Gamma(s) = 0$ has no solutions. Also, $\Gamma(s)$ is approximately

$$\frac{(N-1)!}{s(s+1)(s+2)\dots(s+N-1)} N^s,$$

as long as $s \neq 0, -1, -2, \dots$, but, unless $|s| < 1$, you must take N to be pretty large for this to be accurate. A way around this numerical problem is to use the multiplication formula of Gauss and Legendre

$$\Gamma(s) = \Gamma\left(\frac{s}{m}\right) \Gamma\left(\frac{s+1}{m}\right) \dots \Gamma\left(\frac{s+m-1}{m}\right) m^{s-1/2}/(2\pi)^{(m-1)/2}.$$

Speaking of taking on the value 0, that is exactly the thing we are interested in (thanks to Riemann) about the zeta-function. We want to know the solutions of $\zeta(s) = 0$. This is where the γ's turn up!

We know that there are no zeros in the half-plane $\sigma > 1$ because that is where the Euler product converges and that product can only be 0 if one of the factors is 0. But the factor for the prime 3, for example, is

$$1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \dots = \frac{1}{1 - \frac{1}{3^s}}$$

which is never 0. Then by Riemann's functional equation there are no zeros in $\sigma < 0$ (that's not 100% true; there are zeros at the points $s = -2, -4, -6, \dots$. These are known as the *trivial* zeros. But that's it.) Thus, the zeros are in the strip $0 \leq \sigma \leq 1$ also known as the critical strip.

We would like to give a graphical representation of $\zeta(s)$. However, since s is a complex number and $\zeta(s)$ is a complex number, it would require 4 dimensions to graph it in the usual way. So, we have to devise other ways to illustrate it. One way is to use two separate planes and draw a path

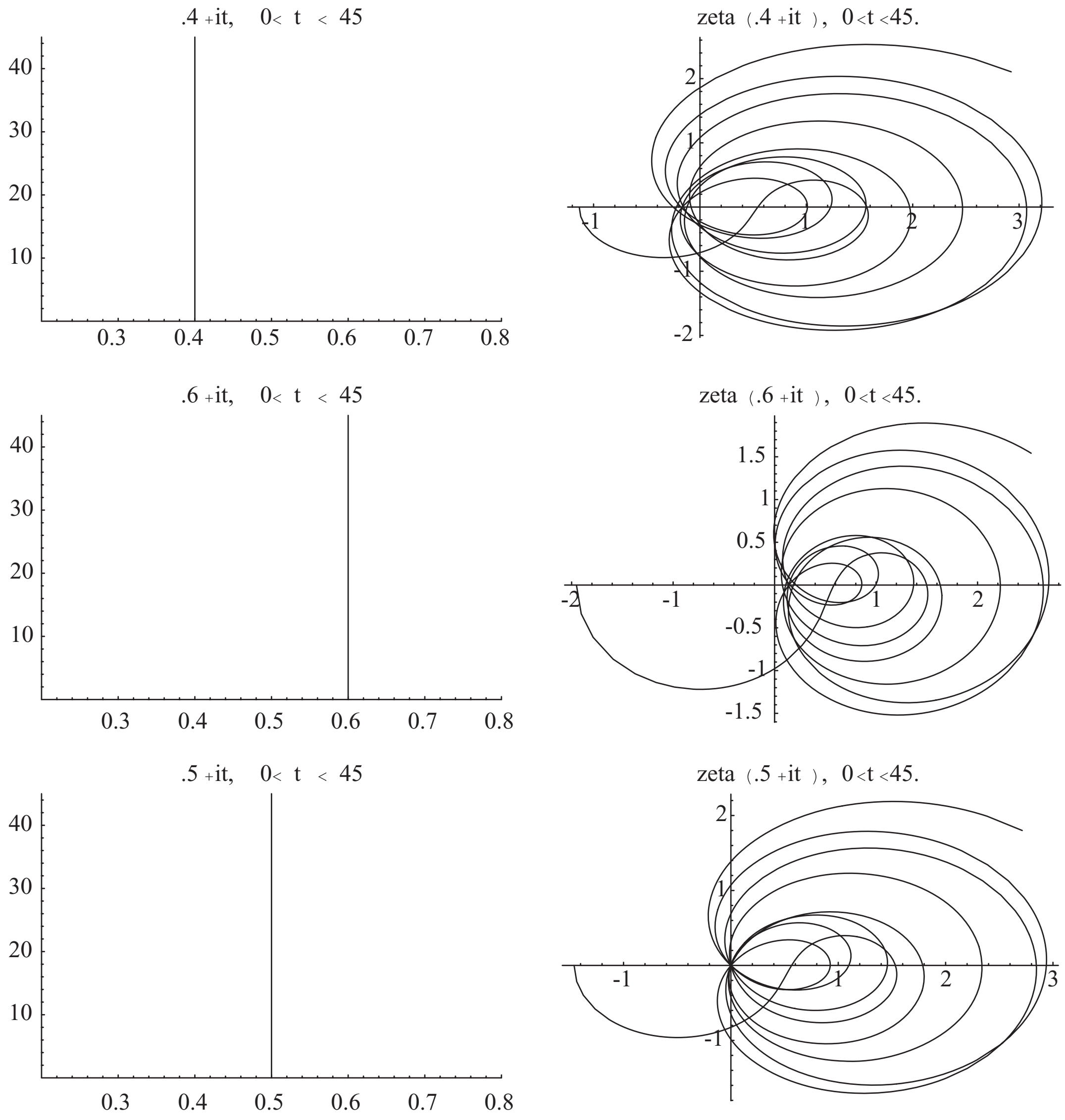

Figure 5: The image of $\zeta(s)$ for three different vertical segments

in one plane and the image of that path in the other. We will now do this for three different line segment paths in the s-plane and for each we will show the image of the path in the $\zeta(s)$ plane. The first path is the line segment stretching from 0.4 to $0.4 + 45i$. The zeta-function maps this segment to the curve pictured in Figure 5; the points of this curve are the points $\zeta(0.4 + it)$ for all t with $0 < t < 45$. The image curve starts at a real point just to the left of -1, then moves over to the right-side of the origin, then spins around the origin lots of times in a clockwise direction.

The second curve shows the image of $\zeta(s)$ as s goes up the 0.6-line, and the third curve shows what happens as s goes up the vertical line with real part $1/2$. Notice that as s makes its way up the

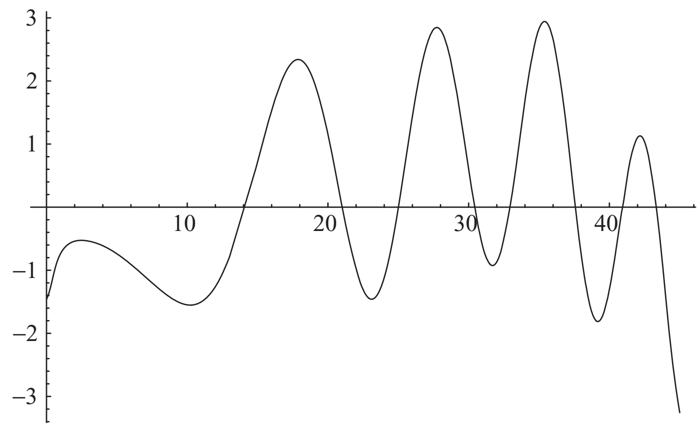

Figure 6: A plot of Hardy's function $Z(t)$ for $0 < t < 45$

1/2-line that there are zeros there and there don't seem to be any on the other lines. This is precisely Riemann's Hypothesis: that all of the zeros in the critical strip are on the 1/2-line!

We can get a better visual of $\zeta(s)$ on the 1/2-line, by looking at the function $Z(t)$ which is just a straightened out version of $\zeta(1/2 + it)$. Figure 6 is a graph of $Z(t)$ for $0 < t < 45$. Notice the zeros at 14.1, 21.0, 25.0, 30.4, ... These are exactly the γ's! Figure 7 is a graph of $Z(t)$ for $1000 < t < 1045$. There are a lot more γ's in this interval, in accordance with Riemann's formula for the number of zeros. What if some γ were not real? What would the $Z(t)$ function look like for t near to a non-real γ? Basically there would be a spot where the graph of $Z(t)$ approaches the real axis, but then turns around before it gets there.

The Z function can behave quite irregularly. For example, here is a plot of $Z(t)$ for $1000325 < t < 1000332$. The irregularity of its behavior gives a reason to expect that understanding its zeros could

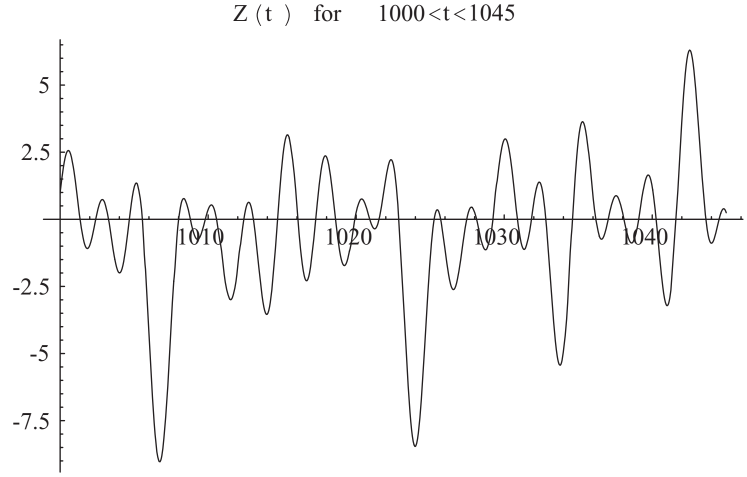

Figure 7: $Z(t)$ for $1000 < t < 1045$

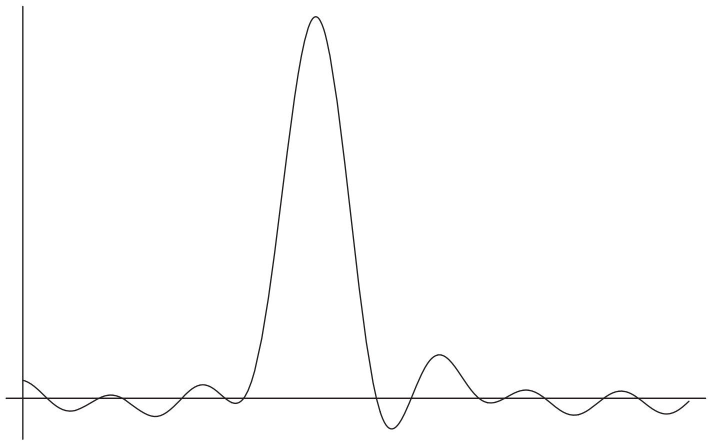

Figure 8: $Z(t)$ for $1000325 < t < 1000332$

be difficult. The Riemann zeta-function is, in some sense, the last basic function that mathematicians do not yet understand.

If you have any ideas, please let me know!

7 The Explicit Formula

One of the most beautiful realizations of Riemann's investigations is the explicit formula for $\psi(x)$, which is a weighted sum of primes and prime powers. To compute $\psi(x)$, for each prime number p, you add up $\ln p$ multiplied by the number of powers of p that are smaller than x. For example,

$$\psi(18) = 4\ln 2 + 2\ln 3 + \ln 5 + \ln 7 + \ln 11 + \ln 13 + \ln 17$$

because there are four powers of 2 smaller than 18, two powers of 3 smaller than 18, and so on.

The explicit formula is this

$$\psi(x) = x - 2\Re \sum_{\gamma \le T} \frac{x^{1/2+i\gamma}}{1/2+i\gamma} - \log(2\pi) - \frac{1}{2}\log(1-x^{-2}) + E(x,T)$$

where $|E(x,T)| \le x/T$ and where $\Re z$ means the real part of z (for example, $\Re(3+2i) = 3$).

Figure 9 shows a graph of $\psi(x)$ for $x < 100$ along with the explicit formula using the first nine γ's.

This figure is followed by a plot of the two graphs together.

8 Anecdotes

There are many, many stories about the Riemann Hypothesis. It was first proposed by Riemann in 1859 in his seminal paper "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" presented when Riemann was 32 upon his election to the Berlin Academy of Sciences.

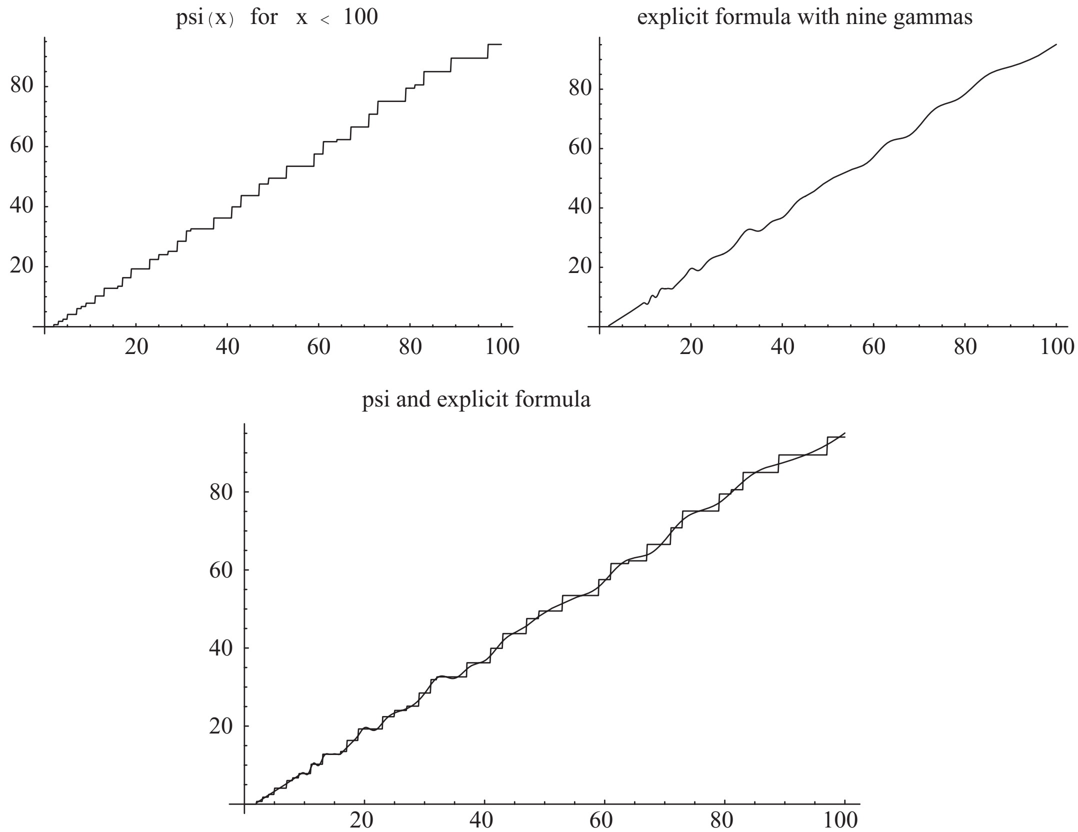

Figure 9: $\psi(x)$ and an approximation using the explicit formula with nine γ's

Hilbert addressed the 1900 International Congress of Mathematicians in Paris, and outlined a list of 23 problems for twentieth century mathematicians to work on. Most of these have been solved. But problem 8, which includes the Riemann Hypothesis, remains open.

Communication was not great between Britain and the continent in the early part of the twentieth century, and British number theorists did not immediately recognize the difficulty of the problem. In fact, in 1906 J. E. Littlewood was assigned the Riemann Hypothesis for his PhD thesis by his advisor E. W. Barnes.

Many stories involve the famous G. H. Hardy, who was the first to prove that infinitely many zeros of $\zeta(s)$ are on the critical line. A facetious list of Hardy's New Year's resolutions included proving the Riemann Hypothesis, as well as being the first man atop Mt. Everest and making 211 not out in the fourth innings of the last test match at the Oval. Hardy used to work with Bohr in Germany but was terrified of crossing the English Channel. Also, he was convinced the God was his personal enemy. Prior to crossing the English channel on one particularly stormy occasion, Hardy wrote a postcard to Bohr saying he had solved the Riemann Hypothesis, knowing full well that God would not allow his boat to sink in circumstances so favorable to Hardy!

Hilbert once said that if he awoke from a sleep of 1000 years, the first thing he would ask is whether the Riemann Hypothesis has been solved.

Earlier in his career, he predicted during a lecture that the Riemann Hypothesis would be solved within a few years, that some people in the room would live to see Fermat's Last Theorem proven, but that we would not know whether $2^{\sqrt{2}}$ was transcendental for thousands of years. Amazingly, $2^{\sqrt{2}}$ was proven transcendental by Gelfond and Schneider, independently, a few years later, Andrew Wiles proved Fermat's Last Theorem in 1996, and the Riemann Hypothesis is still unsolved. Let's hope it won't remain that way for a thousand years!

Hilbert and Pólya are reputed to have suggested that the zeros of $\zeta(s)$ should be interpreted as eigenvalues of an appropriate operator.

In 1942 Selberg proved that a positive proportion of zeros are on the critical-line.

In the 1950s physicists predicted that excited nuclear particles emit energy at levels that are distributed like the eigenvalues of random matrices. This was verified experimentally in the 1970s and 1980s; see Mehta's book.

In 1972 Hugh Montgomery, then a graduate student at Cambridge, delivered a lecture at a symposium on analytic number theory in St. Louis, outlining his work on the spacings between zeros of the Riemann zeta-function. This was the first time anyone had considered such a question. On his flight back to Cambridge he stopped over in Princeton to show his work to Selberg. At afternoon tea at the Institute for Advanced Study, Chowla insisted that Montgomery meet the famous physicist—and former number theorist—Freeman Dyson. When Montgomery explained to Dyson the kernel he had found that seemed to govern the spacings of pairs of zeros, Dyson immediately responded that it was the same kernel that governs pairs of eigenvalues of random matrices.

In 1974 N. Levinson, who was in his seventies, proved that at least 1/3 of the zeros of the zeta-function are on the critical line.

In 1980, Andrew Odlyzko and Schönhage invented an algorithm which allowed for the very speedy calculation of many values of $\zeta(s)$ at once. This led Odlyzko to compile extensive statistics about the zeros at enormous heights—up to 10^{20} and higher. His famous graphs showed an incredible match between data for zeros of $\zeta(s)$ and for the proven statistical distributions for random matrices.

These amazing graphs reminded people of the Pólya and Hilbert philosophy and prompted Odlyzko to write to Pólya. Here is the text of Odlyzko's letter, dated Dec. 8, 1981.

> Dear Professor Pólya:
>
> I have heard on several occasions that you and Hilbert had independently conjectured that the zeros of the Riemann zeta function correspond to the eigenvalues of a self-adjoint hermitian operator. Could you provide me with any references? Could you also tell me when this conjecture was made, and what was your reasoning behind this conjecture at that time?
>
> The reason for my questions is that I am planning to write a survey paper on the distribution of zeros of the zeta function. In addition to some theoretical results, I have performed extensive computations of the zeros of the zeta function, comparing their distribution to that of random hermitian matrices, which have been studied very seriously by physicists. If a hermitian operator associated to the zeta function exists, then in some respects we might expect it to behave like a random hermitian operator, which in turn ought to resemble a random hermitian matrix. I have discovered that the distribution of zeros of the zeta function does indeed resemble the distribution of eigenvalues of random hermitian matrices of unitary type.
>
> Any information or comments you might care to provide would be greatly appreciated.
>
> Sincerely yours,
> Andrew Odlyzko

and Pólya's response, dated January 3, 1982.

Dear Mr. Odlyzko,

Many thanks for your letter of Dec. 8. I can only tell you what happened to me.

I spent two years in Göttingen ending around the beginning of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann Hypothesis should be true?"

This would be the case, I answered, if the non-trivial zeros of the ζ function were so connected with the physical problem that the Riemann Hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.

I never published this remark, but somehow it became known and it is still remembered.

With best regards.

Yours sincerely,
George, Pólya

In 1987 I extended Levinson's method and proved that at least 2/5 of the zeros are on the critical line. This remains the best percentage known.

The American Institute of Mathematics (AIM) sponsored 3 international symposia in the last 15 years devoted to stimulate work on the Riemann Hypothesis: Seattle in 1996, Vienna in 1998, and New York in 2002. At the second of these, at the Schrödinger Institute in Vienna in 1998, Jon Keating and Nina Snaith announced an even more amazing correlation between the distribution of values of $\zeta(s)$ and the values of the characteristic polynomials of unitary matrices. Their work has opened the door to dozens of investigations focused on modeling the zeta-function by random matrix models. Work of Katz and Sarnak around the same time, and preliminarily announced at the 1996 Seattle meeting, indicated that general families of zeta-like functions (called L-functions) had symmetry types associated with them and could be modeled by an ensemble of unitary, orthogonal, or symplectic matrices, depending on the symmetry type.

Three popular books on the Riemann zeta-function appeared shortly after the last of these symposia. Each of the three authors attended at least one of these meetings.

Many people have worked on verifying the Riemann Hypothesis. A few of these include E. C. Titchmarsh who verified in 1935 that the first 1041 zeros are on the critical line, D. H. Lehmer, 1956, the first 25,000, van de Lune, te Riele, and Winter, 1981, the first 3.5 million, and Gourdon and Demichel, 2004, the first 10 trillion. And Odlyzko has verified it in certain intervals high up the critical line, including near the 10^{20}, 10^{21}, and 10^{22} zero.

In addition, many approaches have been developed over the years to attack the Riemann Hypothesis. For example, the analogue of the Riemann Hypothesis for zeta-functions over function fields has been proven by Hasse, Weil (1948), and Deligne (1974).

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