# AN EXAMPLE TO KEEP IN MIND WHEN STUDYING $L$-FUNCTIONS

J. B. CONREY

## 1. Introduction

We give an example of a Dirichlet series with functional equation but no Euler product for which the Riemann Hypothesis badly fails as does the Lindelöf Hypothesis. Moreover, this Dirichlet series is a real linear combination of Dirichlet series, all with the same functional equation, which, conjecturally, has no zeros on the critical line, in contrast to the linear combinations studied by Bombieri and Hejhal. Thus, our example is potentially instructive when studying $L$-functions and their values and zeros. This example is a slight modification of the one presented in [ConGho].

For $q > 1$ and $\Re s > 1$ let

$$D_q(s) = \sum_{n=1}^{\infty} \frac{d(n) \sin \frac{2\pi n}{q}}{n^s}$$

where $d(n)$ is the usual divisor function.

**Theorem 1.** *If $q > 1$, then $D_q(s)$ is an entire function of $s$. It satisfies the functional equation*

$$\left(\frac{q}{\pi}\right)^s \Gamma\left(\frac{s+1}{2}\right)^2 D_q(s) = \left(\frac{q}{\pi}\right)^{1-s} \Gamma\left(\frac{2-s}{2}\right)^2 D_q(1-s).$$

*Moreover, if $q$ is prime, then $D_q(s)$ can be expressed as a linear combination*

$$D_q(s) = \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} -i\tau(\overline{\chi}) L(s,\chi)^2$$

*of functions*

$$D(s,\chi) := -i\tau(\overline{\chi}) L(s,\chi)^2$$

*each of which is expected to satisfy the Riemann Hypothesis and each of which satisfies the same functional equation*

$$\left(\frac{q}{\pi}\right)^s \Gamma\left(\frac{s+1}{2}\right)^2 D(s,\chi) = \left(\frac{q}{\pi}\right)^{1-s} \Gamma\left(\frac{2-s}{2}\right)^2 D(1-s,\overline{\chi}).$$

*Finally, as $q \to \infty$ through the primes, we have the expression*

$$D_q(1/2+it) = \frac{1}{\sqrt{\pi}} \frac{q}{\phi(q)} \left(\frac{\pi}{q}\right)^{1/2+it} \frac{\Gamma(3/4-it/2)}{\Gamma(3/4+t/2)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1/2+it,\chi)|^2$$

*which, shows that for fixed $t$, the estimate $D_q(1/2+it) \gg q^{1/2}\log q$ holds, and so the Lindelöf Hypothesis in $q$-aspect is violated. In particular, the Riemann Hypothesis fails (see [ConGho]) and, in fact, one expects that if $q \geq 7$, then $D_q(s)$ will have no zeros on the 1/2-line.*

The above assertions are all easy to prove by standard methods from analytic number theory. In particular, the functional equation for $D_q(s)$ follows either from the functional equation for the Estermann $D(s,h/k)$ function (for which see [Est]) or, in the case of prime $q$, on emay simply use the functional equation for primitive Dirichlet $L$-functions. Note also that in the Bombieri-Hejhal theorem, the $L$-functions constituents of the linear combination were not allowed too many zeros too close together; clearly our $L$-functions do not satisfy this hypothesis.

## References

[BomHej] Bombieri, E.; Hejhal, D. A. On the distribution of zeros of linear combinations of Euler products. Duke Math. J. 80 (1995), no. 3, 821–862.

[ConGho] Conrey, J. Brian; Ghosh, Amit. Remarks on the generalized Lindelöf hypothesis. Funct. Approx. Comment. Math. 36 (2006), 71–78.

[Est] Ester,mann, T. On the representation of a number as the sum of two products. PLMS (2) 31 (1930), 123–133.