MOMENTS OF $\frac{Z'}{Z}$: SOME CALCULATIONS

We have

$$\int_{U(N)} \left|\frac{Z_A'}{Z_A}(e^{-a})\right|^2 dA = \frac{e^{2a}-e^{2a-2aN}}{(1-e^{2a})^2} = \frac{1-e^{-2\alpha}}{4\alpha^2}N^2 - \frac{1-e^{-2\alpha}}{12} + O(N^{-2})$$

where $a=\frac{\alpha}{N}$. Note that

$$\frac{1-e^{-2\alpha}}{4\alpha^2} = \frac{1}{2\alpha} - \frac{1}{2} + \frac{\alpha}{3} - \frac{\alpha^2}{6} + O\left(\alpha^3\right)$$

Also,

$$\int_{U(N)} \left|\frac{Z_A'}{Z_A}(e^{-a})\right|^4 dA = \frac{2e^{4a}+e^{-2aN}(-e^{2a}N^2+2e^{4a}N^2-e^{6a}N^2-2e^{4a})}{(1-e^{2a})^4} = \frac{1-e^{-2\alpha}(2\alpha^2+1)}{8\alpha^4}N^4 + O(N^2)$$

We have

$$\frac{1-e^{-2\alpha}(2\alpha^2+1)}{8\alpha^4} = \frac{1}{4\alpha^3} - \frac{1}{2\alpha^2} + \frac{2}{3\alpha} - \frac{7}{12} + \frac{11\alpha}{30} - \frac{8\alpha^2}{45} + O\left(\alpha^3\right).$$

The exact calculations get a little complicated but see below for a reasonable way to express these. We have

$$\int_{U(N)} \left|\frac{Z_A'}{Z_A}(e^{-a})\right|^6 dA \sim \frac{e^{-2\alpha}\left(-2\alpha^4+4\alpha^3-12\alpha^2+3e^{2\alpha}-3\right)}{32\alpha^6}N^6$$

and

$$\frac{e^{-2\alpha}\left(-2\alpha^4+4\alpha^3-12\alpha^2+3e^{2\alpha}-3\right)}{32\alpha^6} = \frac{3}{16\alpha^5} - \frac{9}{16\alpha^4} + \frac{1}{\alpha^3} - \frac{9}{8\alpha^2} + \frac{9}{10\alpha} - \frac{11}{20} + \frac{113\alpha}{420} - \frac{61\alpha^2}{560} + O\left(\alpha^3\right).$$

Also

$$\int_{U(N)} \left|\frac{Z_A'}{Z_A}(e^{-a})\right|^8 dA \sim \frac{e^{-4\alpha}\left(-e^{2\alpha}\left(4\alpha^6-24\alpha^5+138\alpha^4-216\alpha^3+306\alpha^2+63\right)+54e^{4\alpha}+9\right)}{576\alpha^8}N^8$$

and

$$\frac{e^{-4\alpha}\left(-e^{2\alpha}\left(4\alpha^6-24\alpha^5+138\alpha^4-216\alpha^3+306\alpha^2+63\right)+54e^{4\alpha}+9\right)}{576\alpha^8}$$
$$= \frac{5}{32\alpha^7} - \frac{5}{8\alpha^6} + \frac{17}{12\alpha^5} - \frac{47}{24\alpha^4} + \frac{15}{8\alpha^3} - \frac{121}{90\alpha^2} + \frac{479}{630\alpha} - \frac{709}{2016} + \frac{6197\alpha}{45360} - \frac{859\alpha^2}{18900} + O\left(\alpha^3\right)$$

Finally, we have

$$\int_{U(N)} \left|\frac{Z_A'}{Z_A}(e^{-a})\right|^{10} dA \sim \frac{e^{-4\alpha}\big(-2e^{2\alpha}\alpha^8+24e^{2\alpha}\alpha^7-244e^{2\alpha}\alpha^6+1044e^{2\alpha}\alpha^5-3270e^{2\alpha}\alpha^4+4140e^{2\alpha}\alpha^3\big)}{4608\alpha^{10}}$$
$$\frac{-90\big(43e^{2\alpha}-5\big)\alpha^2+45\big(-17e^{2\alpha}+12e^{4\alpha}+5\big)\big)}{4608\alpha^{10}}$$

and this is

$$= \frac{35}{256\alpha^9}-\frac{175}{256\alpha^8}+\frac{725}{384\alpha^7}-\frac{575}{192\alpha^6}+\frac{403}{128\alpha^5}-\frac{685}{288\alpha^4}+\frac{775}{576\alpha^3}-\frac{4615}{8064\alpha^2}$$
$$+\frac{775}{4536\alpha}-\frac{2657}{120960}-\frac{12421\alpha}{997920}+\frac{9403\alpha^2}{798336}+O\left(\alpha^3\right)$$

These can be calculated by the ratios theorem [CFZ] (but (22) and (23) of [CS] are more relevant).

1. An alternative formulation

If we make the substitution $y = e^{-2a}$ we get a different perspective. Let us write

$$M_k(y,N) := \int_{U(N)} \left|\frac{Z_A'}{Z_A}(e^{-2a})\right|^{2k} dA_N.$$

The second moment can be rewritten as

$$M_1(y,N) = \frac{y}{(1-y)^2} - \frac{y^N y}{(1-y)^2}.$$

The mean fourth power can be rewritten as

$$M_2(y,N) = \frac{2y^2}{(1-y)^4} - \frac{\left(N^2y^3-\left(2N^2-2\right)y^2+N^2y\right)y^N}{(1-y)^4}.$$

The mean sixth power is

$$M_3(y,N) = \frac{6y^3}{(1-y)^6} - \frac{y^N}{4(1-y)^6}Q_3(y,N)$$

where

$$Q_3(y,N) = (N^4+2N^3+N^2)y^5 - 4(N^4+N^3-5N^2)y^4 + (6N^4-42N^2+24)y^3$$
$$-(4N^4-4N^3-20N^2)y^2 + (N^4-2N^3+N^2)y$$

The eighth moment is

$$M_4(y,N) = \frac{24y^4+4y^4y^{2N}}{(1-y)^8} - \frac{y^N}{36(1-y)^8}Q_4(y,N)$$

where

$$\begin{aligned}Q_4(y,N) = &(N^6+6N^5+13N^4+12N^3+4N^2)y^7 - (6N^6+24N^5-60N^4-168N^3-90N^2)y^6\\ &+(15N^6+30N^5-357N^4-372N^3+828N^2)y^5 - (20N^6-568N^4+1844N^2-1008)y^4\\ &+(15N^6-30N^5-357N^4+372N^3+828N^2)y^3\\ &-(6N^6-24N^5-60N^4+168N^3-90N^2)y^2 + (N^6-6N^5+13N^4-12N^3+4N^2)y\end{aligned}$$

The tenth moment is

$$M_5(y,N) = \frac{120y^5+25(N^2y^6-2(N^2-1)y^5+N^2y^4)y^{2N}}{(1-y)^{10}} - \frac{y^N}{576(1-y)^{10}}Q_5(y,N)$$

where

$$\begin{aligned}Q_5(y,N) = &\left(N^8+12N^7+58N^6+144N^5+193N^4+132N^3+36N^2\right)y^9\\ &-\left(8N^8+72N^7-24N^6-1224N^5-3216N^4-3168N^3-1088N^2\right)y^8\\ &+\left(28N^8+168N^7-1304N^6-6336N^5+3004N^4+19128N^3+11232N^2\right)y^7\\ &-\left(56N^8+168N^7-4072N^6-8424N^5+44048N^4+48288N^3-68544N^2\right)y^6\\ &+\left(70N^8-5700N^6+75270N^4-161800N^2+97920\right)y^5\\ &-\left(56N^8-168N^7-4072N^6+8424N^5+44048N^4-48288N^3-68544N^2\right)y^4\\ &+\left(28N^8-168N^7-1304N^6+6336N^5+3004N^4-19128N^3+11232N^2\right)y^3\\ &-\left(8N^8-72N^7-24N^6+1224N^5-3216N^4+3168N^3-1088N^2\right)y^2\\ &+\left(N^8-12N^7+58N^6-144N^5+193N^4-132N^3+36N^2\right)y\end{aligned}$$

Note that the leading coefficient of $Q_k(y,N)$ seems to be

$$N^2(N+1)^2(N+2)^2\dots(N+k-2)^2y^{2k-1}$$

and there seems to be a symmetry; if we write

$$Q_k(y,N) = \sum_{j=1}^{2k-1} q_k(j,N)y^j$$

then

$$q_k(j,N) = q_k(k-j,-N).$$

Moreover, we can rewrite $Q_5(y,N)$ as y times

$$\begin{aligned}&(y-1)^8N^8 + (y-1)^7N^7(y+1)12 + (y-1)^6N^6(58y^2+372y+58)\\ &+(y-1)^5N^5(y+1)(144y^2+1800y+144)\\ &+(y-1)^4N^4(193y^4+3988y^3+17798y^2+3988y+193)\\ &+(y-1)^3N^3(y+1)(132y^4+3432y^3+25992y^2+3432y+132)\\ &+(y-1)^2N^2(36y^6+1160y^5+13516y^4+94416y^3+13516y^2+1160y+36)\\ &+97920y^4.\end{aligned}$$

In this notation we have

$$\begin{aligned}\lim_{y\to 1}(1-y)M_1(y,N) &= N\\ \lim_{y\to 1}(1-y)^3M_2(y,N) &= 2N\\ \lim_{y\to 1}(1-y)^5M_3(y,N) &= 6N\\ \lim_{y\to 1}(1-y)^7M_4(y,N) &= 20N\\ \lim_{y\to 1}(1-y)^9M_5(y,N) &= 70N\end{aligned}$$

We conjecture that

$$\lim_{y\to 1}(1-y)^{2k-1}M_k(y,N) = \binom{2k-2}{k-1}N.$$

We have verified this to a certain extent for $k=6$. Moreover, it appears that

$$M_k(y,N)\sim c_k(\alpha)N^{2k}$$

where

$$c_k(\alpha)=\frac{\binom{2k-2}{k-1}}{2^{2k-1}\alpha^{2k-1}}-\frac{k\binom{2k-2}{k-1}}{2^{2k-1}\alpha^{2k-2}}+O\left(\frac{1}{\alpha^{2k-3}}\right).$$

Finally, we remark that if we had replaced $M_5(y,N)$ by

$$\tilde{M}_5(y,N):=\frac{120y^5-170y^{N+5}+50y^{2N+5}}{(1-y)^{10}}$$

(this is essentially replacing $Q_5(y,N)$ by $Q_5(y,0)$) we would still have

$$\lim_{y\to 1}(1-y)^9\tilde{M}_5(y,N) = 70N.$$

Similarly with

$$\tilde{M}_4(y,N):=\frac{24y^4-28y^{N+4}+4y^{2N+4}}{(1-y)^8};$$

$$\tilde{M}_3(y,N):=\frac{6y^3-6y^{N+3}}{(1-y)^6};$$

$$\tilde{M}_2(y,N):=\frac{2y^2-2y^{N+2}}{(1-y)^4};$$

$$\tilde{M}_1(y,N):=M_1(y,N)=\frac{y-y^{N+1}}{(1-y)^2}.$$

References

[CFZ] Conrey, Brian; Farmer, David W.; Zirnbauer, Martin R. Autocorrelation of ratios of L-functions. Commun. Number Theory Phys. 2 (2008), no. 3, 593–636.

[CS] Conrey, John Brian; Snaith, Nina Claire Correlations of eigenvalues and Riemann zeros. Commun. Number Theory Phys. 2 (2008), no. 3, 477–536.